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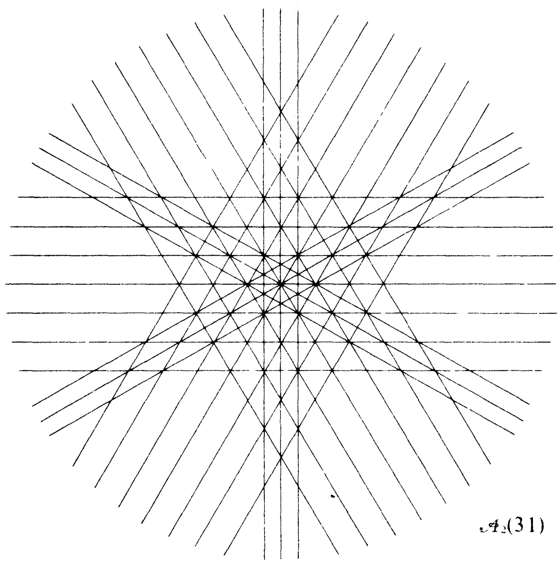

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STATEMENT OF POLICY

Since its beginning, the MONTHLY has been dedicated to the advancement and promotion of collegiate mathematics. The founders of the MONTHLY set out to create a publication that would be neither a research journal nor one devoted primarily to educational and pedagogical topics. Following our predecessors, we again endorse these principles.

More than half of those who receive the MONTHLY teach college mathematics; many others apply mathematics in their daily work. It is our view that our readers can best be served by keeping in mind the following editorial standards.

MAIN ARTICLES. These should be of wide interest and preferably expository. All kinds of mathematical subject matter are welcome, but a special effort will be made to obtain articles dealing with various applications of mathematics. We especially seek articles that describe how mathematics solved, or can solve, an important real-world problem, or was essential in the construction of an actual physical object. Of course we shall also welcome articles describing recent progress in pure mathematics and illustrating the great advances of our science. We hope that, on the average, our articles will be accessible to readers who have had at least a first-year graduate education in mathematics; but this can only be an average, and it is likely that any given reader will sometimes find some articles too difficult, others too trivial, and some more or less interesting than others.

The MONTHLY will definitely not publish research papers of interest only to specialists, and it will try to avoid what is best described as "minor research." However, we shall print (either as a main article or as a mathematical note) an occasional paper containing research that can be followed by non-specialists and can be expected to be of wide appeal.

We ask all authors to write in a good expository style. Beyond the basic principles of good writing (for which consult any standard book on style), this requires above all that the author shows consideration for the reader, who will usually not know as much about the subject as the author does. The prospective reader is especially encouraged if each article has an introduction (from a paragraph to several pages, depending on the nature of the article) saying informally what the article is about and where it fits in with more familiar material. Avoid a tight, formal, "hypothesis-conclusion-proof" format: we will gladly trade brevity for clarity. Too many definitions in the first paragraph or on the first page discourage potential readers. Remember that most people understand words more rapidly than formulas. Long, convoluted sentences, bifurcating into many dependent clauses, especially those with verbs deferred to the end, with the consequent effect of demanding close attention from the reader, as well as comprehension of long and unusual words, or of specialized technical terminology, are rebarbative and to be sedulously avoided. — In short, don't write sentences like the preceding one!

An authors' manual, prepared by Harley Flanders, was published in the MONTHLY, vol. 78 (1971), pp. 1–10. A few reprints are still available (free) from the Washington Office of the Association.

PROGRESS REPORTS. These will be brief reports on recent interesting developments in mathematics. A more detailed description appears on p. 33 of this issue.

MATHEMATICAL NOTES. These should in general be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally) new results that are not too technical. Again, the topics should be of wide current interest.

RESEARCH PROBLEMS. These should be unsolved problems whose statements use only concepts common in undergraduate mathematics, and for which there is some expectation that progress can be made without using terribly advanced methods. The appearance of a research problem in the MONTHLY does *not* mean that its solution will necessarily be a suitable main article or mathematical note — indeed, the opposite is likely to be true.

CLASSROOM NOTES. For this department we hope to obtain brief papers dealing with the subjects currently taught in *undergraduate* mathematics. Occasionally we shall also accept notes dealing with material commonly encountered in the first year of graduate study. Classroom notes should be relevant to some actual classroom situation that can be expected to exist at a good number of institutions.

MATHEMATICAL EDUCATION. There is now a growing and very healthy concern with mathematicians' role as teachers. This department welcomes reports of experiments in novel methods of teaching as well as discussions of all educational aspects of our profession.

PROBLEMS. This department attracts many readers and is an especially valuable feature of the MONTHLY. It will continue to seek and publish interesting elementary and advanced problems in pure and applied mathematics, both classical and modern.

REVIEWS. Through the telegraphic reviews the MONTHLY will continue to provide as complete a coverage as possible of the current textbooks. At present the MONTHLY is the only journal that performs this service for the mathematical community. Extended reviews will continue to appear, and we hope that the number of classroom reviews will increase.

R. P. BOAS, *Editor*

BREAKING RECORDS AND BREAKING BOARDS

NED GLICK

1. Introduction. When I take observations in chronological sequence, how often will the outstanding record value be surpassed? For example, suppose that I register the annual total inches of precipitation or, to take a less gloomy statistic, the total hours of bright sunshine in Vancouver, where I live: what is the probability that next year will be a new maximum?

Breakthroughs are less likely later than early in a sequence of observations. The first observation necessarily must be a "record high." But, prior to observing any values, I know that the second of two numbers in random sequence has equal probability of being smaller or larger than the first. Hence the probability is exactly 50% that a second, independent observation will be a new record high surpassing the initial record, assuming that there cannot be an exact tie (if measurement is arbitrarily precise).

From the same perspective, there is probability 1/3 that a third trial will be a new maximum, since the last of three repeated observations is equally likely to be smallest, middle, or largest.... Similarly, all 10 ranks are equally likely for the tenth observation; so maximum rank for the tenth observation has probability 1/10.

The theoretical expected or average number of record highs in a chronological sequence of n independent observations is the sum of these probabilities: $1+(1/2)+(1/3)+\cdots+(1/n)$.

This result surprises my acquaintances who have no background in probability theory. I have here the beginning of a conversational ploy to hold a person's interest when she or he asks what work I do and I say, "I am a statistician." Otherwise this reply may ruin a fine conversation, as in the dinner scene depicted by William Kruskal [**22**].

My ploy proceeds to verify predictions from the simple probability model with some real weather records. It happens that these weather records excellently illustrate the pathos and problems of work with statistical data.

It is just as interesting to note that the simple model does *not* fit record breaking in athletic competitions. Race times, jump heights, and throw distances have improved over several decades, while rainfall fluctuations from year to year are "random." In fact, the frequencies of record highs and lows can be used to infer whether observations indicate linear trend or random sequence.

After the weather and sports, I talk about traffic. Curiously, the simple model of random record values says something about how cars tend to bunch together behind a slow vehicle.

The same probability model applies in yet another context: a sequential strategy to find the weakest item in a sample of boards or beams, and hence to establish "tolerance limits" for lumber strengths. My interest in record highs and lows actually began in discussions at the Western Forest Products Laboratory of the Canadian Forestry Service. Breaking a random, but usually small, fraction of the available beams can accomplish the same purpose as 100% destructive laboratory testing. This conservation of material illustrates the economic spirit of experimental design.

So one example, the frequency of record highs or lows, ties together several "applied" aspects of a statistician's livelihood. More surprising, the intuitive idea that any record can be beaten also leads to mathematical proof that the harmonic sum $1+(1/2)+(1/3)+\cdots$ grows without bound, becoming bigger than any finite number.

Harmonic divergence is the simplest of many "well known" limit theorems and paradoxes related to record breaking. The mathematical sections comprising the second part of my paper review primarily those results in the theory of record values which do not depend on the particular distribution of the basic sequence of observations. This mathematics is mostly at the level of an undergraduate discrete probability course in the spirit of William Feller's famous text [**12**]. I avoid differentiation and integration as much as possible.

2. Weather records. Weather records in Canada are kept by the Meteorological Branch of the federal Department of Transport. Vancouver's weather records, however, are largely products of "the colourful careers of the two Shearman brothers": T. S. H. Shearman, who "was officially appointed by the government as weather observer for the city of Vancouver" in April, 1905, and E. B. Shearman, who replaced his brother from 1915 to 1948, for which service he received the British Empire Medal. This history I found in a Department of Transport 1964 Annual Meteorological Summary [**25**], from which most of my figures are taken.

Monthly, as well as annual inches of precipitation and hours of sunshine are shown in Data Set 1 and Data Set 2, respectively. Looking down the February column, for example, I find 5 record high years in the precipitation data (1900, 1901, 1902, 1918, 1961) and 5 in the sunshine data. Each sequence includes 65 observations; so $1+(1/2)+(1/3)+\cdots+(1/65)=4.76$ is the theoretical expected number of record highs.

The 12 monthly precipitation sequences actually give an empirical average of 4.17 highs per sequence. Record lows have the same distribution as highs, if the case of absolutely nil precipitation

DATA SET 1.

Monthly & Annual Total Precipitation (Rain + One-Tenth Snow) in Vancouver, B.C., in Inches

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Annual	
1900	7.24	5.95	10.29	4.51	4.20	5.42	1.05	3.60	1.61	9.20	10.00	9.22	72.29	1
1901	11.28	6.31	3.04	5.29	4.38	5.01	.83	.22	2.65	5.20	14.06	8.09	66.36	2
1902	6.68	10.17	7.45	3.11	4.40	1.97	2.37	1.15	3.39	4.72	10.33	9.55	65.29	3
1903	8.23	2.60	6.88	3.78	3.68	3.56	1.12	1.07	8.35	5.72	11.36	4.21	60.56	4
1906	9.66	6.03	2.37	1.04	3.58	3.04	.45	.83	8.87	7.60	8.25	7.33	59.05	5
1907	9.32	8.30	2.39	4.13	1.44	1.43	1.70	1.36	4.51	1.76	13.23	8.02	57.59	6
1908	7.60	6.31	7.14	2.61	4.11	1.86	1.59	1.15	1.46	6.68	13.69	8.41	62.61	7
1909	6.20	8.15	4.31	1.23	3.76	1.69	2.45	1.43	2.23	7.06	15.66	4.28	58.45	8
1910	11.19	5.01	2.91	3.60	2.15	1.98	.24	1.38	2.47	9.04	10.62	8.79	59.38	9
1911	6.11	3.37	3.05	1.96	5.39	2.09	.92	1.23	4.41	2.24	12.68	8.82	52.27	10
1912	8.47	6.25	.89	3.92	2.35	2.28	1.54	5.86	2.84	4.64	9.21	8.80	57.05	11
1913	9.62	4.38	5.38	2.53	4.33	3.81	2.02	.85	3.89	6.19	10.08	3.95	57.03	12
1914	10.56	4.87	3.33	3.28	.74	3.58	.42	.75	6.86	6.37	10.18	2.84	53.78	13
1915	7.13	4.42	4.18	3.04	3.42	1.07	.91	.36	.80	8.83	5.41	10.36	49.93	14
1916	5.96	7.40	14.55	4.07	1.41	1.34	5.25	.58	1.28	2.16	6.37	5.71	56.08	15
1917	9.33	5.87	5.61	8.20	1.69	5.40	.48	.93	3.30	3.49	5.23	11.72	61.25	16
1918	11.05	10.50	7.48	1.70	1.15	1.00	2.29	4.59	.30	7.56	7.02	8.07	62.71	17
1919	7.57	7.35	6.61	4.47	3.60	1.02	.15	1.15	1.16	3.14	11.25	9.74	57.21	18
1920	8.92	1.21	5.20	2.51	1.94	3.06	.67	2.91	10.37	8.34	7.14	11.01	63.28	19
1921	9.39	6.66	2.53	3.62	2.52	3.64	.32	2.84	5.03	10.08	8.99	5.56	61.18	20
1922	3.15	4.75	3.44	2.63	2.46	.17	.02	2.01	5.76	3.26	2.63	10.35	40.63	21
1923	8.73	4.06	3.86	2.14	2.84	2.07	.52	.73	2.97	2.56	6.13	15.88	52.49	22
1924	8.26	8.86	1.16	3.84	.31	.91	.71	1.88	5.81	6.56	5.87	8.35	52.52	23
1925	12.16	6.01	4.24	2.44	2.20	.38	.74	2.36	.44	3.00	4.05	13.05	51.07	24
1926	7.62	6.70	2.48	2.58	4.17	.78	.36	2.10	3.26	6.37	7.80	8.99	53.21	25
1927	9.08	4.73	6.80	1.88	5.12	1.16	.94	3.74	3.07	7.22	8.39	6.92	59.05	26
1928	9.04	1.87	7.01	4.29	2.22	1.93	.47	.20	1.35	7.38	5.36	5.34	46.46	27
1929	3.05	1.64	4.47	4.81	1.25	3.24	1.41	1.50	1.77	3.57	2.54	8.58	37.83	28
1930	2.75	9.42	3.33	4.72	2.86	2.18	.08	.07	2.65	7.50	3.00	5.22	43.78	29
1931	11.24	4.95	6.35	4.57	1.23	5.59	.44	.61	7.14	5.22	8.34	11.92	67.60	30
1932	9.18	6.65	9.15	4.84	1.34	2.08	5.32	2.01	2.62	5.66	10.17	7.37	66.39	31
1933	9.57	6.32	6.43	.53	4.33	2.03	1.76	1.12	5.89	7.94	5.45	12.85	64.22	32
1934	11.90	3.02	5.57	1.18	3.44	.69	1.86	1.24	2.77	4.99	9.56	12.27	58.49	33
1935	20.65	3.75	8.41	2.28	.55	1.06	1.79	1.36	2.57	6.70	4.65	8.46	62.23	34
1936	9.37	4.37	6.12	3.31	5.23	3.13	1.65	2.11	3.51	4.21	1.84	10.63	55.48	35
1937	2.42	8.62	3.64	7.16	3.14	6.14	.47	3.45	1.84	8.14	11.01	10.94	66.97	36
1938	6.59	4.68	4.65	2.93	1.86	.78	.66	.74	1.68	5.74	6.44	13.53	50.28	37
1939	11.91	5.94	3.83	1.71	3.40	3.03	2.34	.68	3.20	6.72	12.35	11.78	66.89	38
1940	5.65	8.11	7.39	4.57	2.85	.30	1.52	1.70	2.15	10.85	4.91	10.47	60.47	39
1941	8.04	6.24	3.82	2.59	5.73	2.19	.48	2.12	8.26	9.86	6.93	9.15	65.41	40
1942	3.56	2.82	4.25	3.88	1.89	5.28	3.36	.47	.79	6.20	6.11	9.59	48.20	41
1943	4.25	4.69	3.94	4.16	3.28	1.20	1.10	2.12	1.82	7.16	3.29	9.16	46.17	42
1944	7.85	4.63	3.41	4.15	1.90	.95	.49	.92	4.86	5.87	8.97	3.76	47.76	43
1945	9.33	7.45	8.54	4.23	2.64	.94	.91	1.27	3.21	7.10	10.07	6.45	62.14	44
1946	11.30	9.36	8.79	7.54	.43	4.87	1.63	.78	1.94	5.91	6.19	8.40	67.14	45
1947	10.68	7.02	6.52	5.62	1.74	2.08	2.34	.49	2.18	10.29	5.22	13.32	67.50	46
1948	3.76	10.31	2.67	3.59	6.05	11.82	2.16	4.03	2.83	4.69	14.57	9.59	66.07	47
1949	.84	7.58	5.05	2.22	1.73	1.76	2.75	1.46	1.45	6.83	11.89	7.62	51.18	48
1950	6.48	10.07	9.42	5.01	2.61	1.48	2.15	2.93	1.65	10.24	5.01	10.50	67.55	49
1951	11.10	10.28	6.40	2.82	3.50	.37	.01	.90	3.64	5.66	6.71	6.17	57.56	50
1952	8.23	5.55	5.99	2.72	2.23	3.92	.40	1.27	1.08	2.13	2.34	7.97	43.83	51
1953	14.08	4.79	4.93	2.38	2.31	2.26	1.29	1.83	4.76	5.26	10.44	11.28	65.61	52

1954	10.00	8.39	2.58	3.92	2.13	2.79	2.11	4.35	4.45	3.15	16.10	9.39	69.36	53
1955	5.09	4.17	7.41	4.32	3.22	3.03	2.97	.31	1.87	7.08	13.59	7.14	60.20	54
1956	7.10	6.47	7.72	1.12	.84	6.57	.89	2.52	5.65	13.69	13.86	13.71	70.14	55
1957	3.52	3.79	6.96	2.87	1.72	2.99	2.81	1.84	1.72	3.88	4.59	9.24	45.93	56
1958	13.48	6.75	3.51	3.60	1.33	1.51	Nil	2.35	3.10	5.93	9.03	9.44	60.03	57
1959	8.51	6.83	8.97	3.66	2.78	3.94	.97	.80	7.03	4.99	8.09	8.24	64.81	58
1960	7.43	7.04	5.49	3.13	5.64	2.05	.02	4.15	2.08	11.00	7.06	7.13	62.22	59
1961	13.28	15.26	6.04	2.98	3.96	1.42	1.34	3.56	1.94	8.82	7.70	10.04	76.34	60
1962	7.57	2.37	4.35	5.02	2.87	1.42	1.20	5.35	3.12	5.77	11.55	12.49	63.08	61
1963	1.76	7.53	4.29	4.09	1.81	2.24	3.08	.83	1.86	8.71	9.33	12.92	58.45	62
1964	11.62	3.82	6.44	3.27	2.75	2.58	3.56	2.33	7.41	3.38	10.17	6.71	64.04	63
1965	9.54	10.67	2.30	2.48	2.89	0.62	0.51	2.57	0.67	8.67	6.68	7.43	55.03	64
1966	9.28	4.53	4.87	1.75	2.89	2.18	3.36	1.66	3.17	7.43	9.47	15.34	65.93	65

DATA SET 2.

Monthly and Annual Total Hours of Bright Sunshine in Vancouver, B.C.

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Annual	
1909	55.3	56.4	143.7	252.4	230.3	251.8	224.4	262.8	294.9	99.2	58.1	51.4	1880.7	1
1910	45.5	91.5	117.7	181.2	276.5	186.5	312.8	224.1	181.2	103.6	49.9	31.0	1801.5	2
1911	14.4	84.8	154.0	248.2	174.9	217.7	272.7	225.1	142.2	140.0	41.6	48.3	1763.9	3
1912	44.2	81.4	233.8	113.8	215.4	229.0	198.1	186.6	185.9	110.7	19.4	26.0	1644.3	4
1913	43.3	97.1	125.6	149.5	178.2	186.6	270.2	228.0	173.2	110.6	62.1	27.6	1652.0	5
1914	38.7	46.4	104.1	155.8	270.2	250.5	313.5	280.7	79.4	101.6	39.0	67.1	1747.0	6
1915	57.7	44.1	131.8	203.0	163.7	266.7	261.7	221.9	141.3	84.6	70.2	36.4	1683.1	7
1916	65.9	84.8	86.4	141.5	216.9	222.9	145.5	284.4	190.5	150.9	84.7	23.5	1697.9	8
1917	53.9	56.7	153.5	90.4	220.6	188.5	358.7	348.2	153.8	104.5	50.9	23.1	1802.8	9
1918	45.3	95.8	103.5	257.5	253.0	318.8	297.4	232.4	236.2	77.1	48.4	58.4	2023.8	10
1919	50.6	58.2	128.2	147.2	257.5	250.1	341.8	265.5	202.9	122.7	60.7	73.5	1958.9	11
1920	44.8	148.4	108.0	196.1	249.0	206.5	308.7	308.6	138.2	91.4	77.8	20.0	1897.5	12
1921	31.6	85.9	137.2	180.1	275.0	137.4	317.2	221.1	172.1	123.7	42.9	57.5	1781.7	13
1922	77.4	98.5	116.5	143.7	243.2	257.0	267.5	193.2	163.4	106.7	71.7	23.3	1762.1	14
1923	52.3	75.2	163.4	194.7	181.4	228.6	289.1	291.7	226.4	142.8	49.9	30.0	1925.5	15
1924	24.6	59.2	157.8	168.0	250.3	238.0	273.9	238.0	217.8	118.7	54.3	62.4	1863.0	16
1925	49.4	49.7	137.3	168.7	261.0	254.3	327.7	240.7	205.2	116.0	59.7	10.7	1880.4	17
1926	27.4	56.1	187.7	220.0	186.4	288.6	300.3	236.9	221.7	93.8	60.5	30.5	1909.9	18
1927	39.9	96.3	128.8	164.5	199.9	219.6	305.1	178.4	124.2	78.3	28.8	56.5	1720.3	19
1928	42.6	102.7	116.2	168.4	278.4	195.7	285.1	258.1	199.2	94.3	45.0	38.9	1824.6	20
1929	50.5	109.7	119.3	169.3	249.0	187.9	322.2	304.9	225.2	113.7	61.7	25.1	1938.5	21
1930	100.7	85.8	169.5	142.7	254.2	219.0	320.6	305.6	192.0	125.9	54.1	40.7	2010.8	22
1931	28.5	80.2	109.8	215.2	274.7	187.9	381.2	334.1	118.5	124.9	96.8	23.2	1975.0	23
1932	65.4	97.0	94.6	147.4	246.3	297.0	195.2	237.7	235.3	115.1	37.8	63.0	1831.8	24
1933	36.8	86.1	117.8	223.4	146.1	198.3	336.3	302.3	123.7	112.4	42.9	30.4	1756.5	25
1934	56.7	118.8	136.5	237.5	236.2	301.0	236.2	270.2	155.3	126.5	34.2	43.8	1952.9	26
1935	44.8	94.3	84.8	246.9	277.5	194.7	247.3	257.1	213.2	109.0	59.1	26.9	1855.6	27
1936	54.6	70.5	121.6	168.5	174.0	212.9	313.3	295.0	181.4	130.1	52.3	36.6	1810.8	28
1937	87.1	69.7	93.7	79.0	221.1	201.6	292.6	203.0	164.6	121.2	37.1	34.0	1604.7	29
1938	30.7	78.0	110.5	175.9	313.1	292.1	314.6	253.9	161.6	132.7	68.4	52.6	1984.1	30
1939	20.8	75.9	116.2	166.5	206.7	143.9	278.7	320.6	193.6	108.7	36.8	30.1	1698.5	31
1940	42.9	63.4	109.0	167.2	242.6	329.2	236.5	266.1	180.1	73.9	48.6	37.6	1798.0	32
1941	38.7	93.0	161.4	184.6	191.8	164.2	326.0	231.2	119.2	80.1	71.2	55.9	1717.3	33
1942	51.2	75.7	101.6	120.6	140.0	192.7	236.4	291.0	175.1	116.3	61.3	44.8	1606.7	34
1943	52.8	104.3	107.7	149.5	177.1	234.7	251.5	203.4	201.0	126.0	42.1	43.2	1693.3	35
1944	50.9	80.7	155.7	121.4	213.0	201.2	259.1	179.9	167.0	108.0	38.5	42.2	1617.6	36
1945	51.9	75.5	77.6	129.9	200.2	191.5	283.3	252.1	154.0	124.0	33.5	32.8	1606.3	37
1946	38.2	42.0	95.2	88.4	287.1	135.7	232.9	267.7	144.9	122.6	66.5	17.8	1539.0	38

DATA SET 2 (continued).

Monthly and Annual Total Hours of Bright Sunshine in Vancouver, B.C.

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Annual	
1947	35.8	94.6	149.0	169.5	223.1	199.3	259.5	264.5	193.2	72.9	54.5	15.9	1731.8	39
1948	45.9	62.2	134.7	115.0	143.0	244.1	207.4	130.6	184.5	118.5	63.2	30.5	1479.6	40
1949	83.3	91.9	105.2	128.5	271.7	237.9	247.3	165.6	209.6	136.6	50.6	43.5	1771.7	41
1950	58.5	48.0	63.0	158.4	231.5	223.0	309.4	294.3	222.6	55.0	33.9	20.1	1747.7	42
1951	35.9	82.2	121.7	290.0	203.7	300.2	316.6	276.8	201.2	81.5	33.3	29.1	1972.2	43
1952	8.9	80.4	83.3	157.1	221.0	183.0	332.2	248.4	215.1	153.1	52.3	21.8	1756.6	44
1953	17.3	62.8	94.3	118.9	189.9	101.6	278.3	195.5	169.0	107.4	37.0	20.0	1392.0	45
1954	30.7	49.9	155.6	143.3	218.2	170.9	240.7	134.1	116.3	100.0	38.6	23.4	1421.7	46
1955	12.8	78.7	106.8	191.2	182.9	162.0	196.5	309.1	148.2	81.3	51.5	37.1	1558.1	47
1956	35.0	46.6	102.2	237.4	312.2	124.6	333.5	278.5	147.6	75.5	40.5	24.8	1758.4	48
1957	69.1	79.1	96.6	159.5	248.1	190.7	201.7	245.8	227.1	135.2	66.3	36.2	1755.4	49
1958	22.6	38.0	131.8	197.4	308.9	220.5	365.0	286.7	164.4	114.6	51.3	10.9	1912.1	50
1959	43.7	56.2	77.3	182.9	236.3	219.9	331.9	220.3	128.5	92.8	71.6	32.1	1693.5	51
1960	50.9	124.9	120.2	165.8	135.6	199.4	388.1	181.6	210.7	115.6	75.3	81.3	1849.4	52
1961	45.0	30.3	80.4	119.9	203.5	322.6	310.9	293.8	152.8	137.8	80.8	20.9	1798.7	53
1962	42.1	77.7	116.8	96.0	119.5	232.8	245.5	148.8	170.2	87.0	35.4	28.3	1400.1	54
1963	69.2	62.6	98.2	91.3	277.6	146.1	177.5	225.3	166.5	78.6	42.0	18.5	1453.4	55
1964	18.9	99.7	67.4	154.9	212.5	124.0	210.8	229.5	169.1	151.6	82.4	54.0	1574.8	56
1965	39	76	222	168	253	330	310	225	179	111	46	52	2010	57
1966	30	84	95	189	263	181	258	281	153	112	46	26	1718	58
1967	42	83	123	151	177	275	319	340	215	74.3	74	47	1920	59
1968	46	156	77	175	254	224	334	216.1	177	73.6	43	60	1836	60
1969	70	103	152	106	280	247	311	215.8	131	165	71	21	1873	61
1970	46	132	163	176	228	262	288	297	192.0	149	75	40	2048	62
1971	37	86	127	152	264	112	352	260	150	127	42	41	2013	63
1972	61	72	75	136	232	168	351	325	165	182	73	64	1904	64
1973	59	88	106	219	259	207	280	263	192.4	74.1	39	29	1815	65

SUMMARY OF RECORD BREAKTHROUGHS IN 65-YEAR SEQUENCES

	Jan.	Feb.	Mar.	Apr.	May	June	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	Average
Precipitation													
Record Highs	4	5	2	3	6	4	5	2	6	4	4	5	4.17
Record Lows	10	3	4	4	6	7	8	4	4	4	7	5	5.50
Reverse Highs	7	3	5	8	$4\frac{1}{2}$	5	3	4	4	6	5	2	4.71
Reverse Lows	3	5	4	3	$7\frac{1}{2}$	5	4	5	4	5	5	6	4.71
Sunshine													
Record Highs	4	5	3	3	4	5	6	4	1	7	5	4	4.25
Record Lows	4	6	7	5	7	5	3	5	4	6	4	7	5.25
Reverse Highs	6	3	5	3	5	4	4	4	6	2	5	3	4.17
Reverse Lows	6	4	4	6	5	4	5	6	7	3	5	5	5.00

can be excluded (an almost valid assumption in Vancouver); and these 12 monthly precipitation sequences give an average of 5.42 lows. Similar counts of "backward" highs and lows are obtained reading the 12 sequences in reverse, from most recent back to the oldest observations. (Since the May precipitation for 1965, recorded with two decimals, ties the figure for 1966, I treat May 1965 as 1/2 in the counts of reverse order highs and lows.) Considering highs and lows, chronological and reverse order, the 48 sequences give an overall empirical average of 4.77 record breaking values per sequence (compare to the theoretical expectation of 4.76).

All these counts can be repeated for the monthly sunshine sequences in Data Set 2.

The accompanying Summary of Record Breakthroughs shows all 96 counts (24 monthly data sets, considering highs and lows, forward and backward sequences). More than half of these 65-year-long sequences have exactly 4 or 5 breakthroughs.

People I ask usually guess that there will be more than 5 record values in 65 years. In predicting the number of record highs for a much longer sequence, say 1,000 or 1,000,000 observations, human intuition, even among mathematicians and statisticians, is definitely extravagant compared to the simple model's expectations of 7.49 and 14.39, respectively.

Over a long time, say a hundred or a thousand or ten thousand years, there may be shifts or cycles of climate; so the basic model of "interchangeable" weather years may not fit well. If climate trends are not negligible, then the actual probabilities for record breaking will differ from what the simple model predicts ..... In fact, the observed frequencies of record highs and lows can be used to infer whether or not data are a random sample.*

3. Tests of randomness. At a meeting of the Royal Statistical Society about twenty-five years ago, F. G. Foster and A. Stuart **[14]** pointed out that record low and record high annual rainfalls at Oxford were much more rare than record breaking performances (low times or high distances) in annual track and field competitions of the British Amateur Athletic Association.

This contrast is not surprising: athletic recruiting and training have intensified over the past century; but no one has done much about the weather. Although athletic performances do fluctuate, there is an average trend over decades for national competitors (and, therefore, winners) to run faster, jump higher, or throw farther; while weather fluctuations over a century are more intuitively random, without dramatic linear trend.

Of course it is possible for 100 random observations to be ordered so that the sequence has as many as 10 or 50 or 100 record highs. But detailed calculation **[6]** shows that the *probability* of 10 or more record highs in a 100-long random sequence is less than 5%. Therefore, in a situation where data are less familiar than rainfalls or race times, the mere finding of many record highs or lows suggests that the data are not a simple random sample; that is, an alternative hypothesis should be sought to fit the data better.

Foster and Stuart **[14]** gave formal procedures using the sum or the difference of record high and record low frequencies to fit or to test the hypothesis of randomness. Other statisticians have also considered such inference procedures [**3**, **4**, **14**, **15**].

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* Returning to the Shearman brothers, I note that their locale of weather observation moved several times. Nonetheless, official records report precipitation for "Vancouver (City)" or "Port Meteorological Office" through 1966. Beyond that date there are reports for 10 separate city locations, plus the University of British Columbia and Vancouver Airport. Since rainfalls at nearby city locations often differ by more than an inch in a single month, I have not extended the precipitation sequences in Data Set 1 beyond 1966. Also I omit 1903 and 1904 because some months of precipitation data are missing from official records for those years.

For years since 1964 the monthly sunshine figures in Data Set 2 are taken from annual reports by the British Columbia provincial Department of Agriculture [**5**], because I could not find relevant annual publications by the federal Department of Transport. Although the provincial figures are derived from the same source, these monthly sunshine totals are rounded to the nearest whole hour, dropping one decimal. The provincial practice of first rounding and then adding the monthly figures may yield a different annual total than the federal total obtained by adding the monthly figures and then rounding-off. Moreover, other discrepancies which I noticed in comparing federal and provincial figures for 1960–1964 confirm that there are fairly frequent copying or typographic errors in one or the other publication (or both). Also the federal 1964 summary makes a substantial mistake in summing its own monthly sunshine figures to obtain its 1964 annual total. Apart from this addition error (which I noticed by accident) I have made no attempt in Data Sets 1 and 2 to find and correct the various errors carried over from my sources.

To count record highs and lows in the monthly sunshine sequences, I have broken several ties between rounded-off figures by finding more exact values from *daily* weather summaries.

... And there you have examples of the complications which arise in compiling real data, no matter how easy the job seems before you do it.

It can be shown that the number of forward and backward record highs, say R_n and R'_n, for any sequence have exactly the same *joint* probability distribution as the forward record highs and lows, say R_n and L_n [**4**]. And, as sample size $n \to \infty$, the counts of record highs and lows become approximately independent [**14**].

4. Car caravans in a one-lane tunnel. When traffic moving in one direction is confined to a single lane, a slow car is likely to be followed closely by a queue of vehicles whose drivers wish to go faster, but who cannot pass. If there is no exit from this lane, then more and more following vehicles will catch up and be added to the slow moving "platoon" or "caravan" ... until there happens to be a following vehicle travelling at a lower speed. This vehicle will not catch up, but will accumulate its own caravan.

Thus cars whose drivers all desire different speeds in fact will travel in caravans at *actual* speeds determined by record lows in the sequence of *desired* speeds.

Applying the simple probability model to a random sequence of drivers, the frequency of record lows corresponds to the *number of caravans* formed by n drivers. And the numbers of trials between successive record breaking low values correspond to the *lengths* of caravans.

Since caravans will be successively slower, separations between caravans will increase as time passes. G. F. Newell [**29**] used this reasoning to explain why cars near the exit of a long tunnel tend to travel faster and in smaller bunches more widely separated than cars in the tunnel near the entrance. This model of traffic flow also has been mentioned by other authors [**13, 19, 29, 38**].

5. Sequential strategy for destructive testing. Many products fail under stress. For example, a wood beam breaks when sufficient perpendicular force is applied to it; an electronic component ceases to function in an environment of too high temperature; and a battery dies under the stress of time. But the precise breaking stress or failure point varies even among "identical" items.

Suppose that I can observe an item's exact failure point in a laboratory by gradually increasing stress (force, temperature, time, etc.). From such destructive testing of 100 items I could find all their failure points, say $X_1, X_2, \ldots, X_{100}$. But now suppose that I only need to find the *weakest* item in my sample: I only want the *minimum* value among failure stresses $X_1, X_2, \ldots, X_{100}$. Then I need not stress most of the items to their failure points.

The minimum failure stress among any batch of items can be determined sequentially. Test the first item until it fails, and record its failure stress X_1. Stop the next test (short of failure) if the second specimen survives this amount: so the second specimen's failure stress X_2 is determined exactly if $X_2 < X_1$; otherwise obtain only the "censored" information that $X_2 > X_1$, and hence $X_1 = \min(X_1, X_2)$. In either case proceed to the third specimen and stop the test if this item survives a stress equal to $\min(X_1, X_2)$: so X_3 is observed only if $X_3 < \min(X_1, X_2)$; but $\min(X_1, X_2, X_3)$ is always determined .... In general, the i^{th} item survives its stress test if $X_i > \min(X_1, \ldots, X_{i-1}) = \min(X_1, \ldots, X_i)$; or the test concludes with stress-to-failure if $X_i = \min(X_1, \ldots, X_i) < \min(X_1, \ldots, X_{i-1})$. In either case, the value $\min(X_1, \ldots, X_i)$ is known after the i^{th} trial.

The items destroyed in this sequential procedure are those with "record low" failure points. The frequency of such record lows fits the same probability model as the lows in a sequence of weather records. For a sample of n items, the expected number of items destroyed is $1 + (1/2) + (1/3) + \cdots + (1/n)$. This harmonic sum grows very slowly compared to sample size n. For example, the sum is only 5.19 when $n = 100$ and is only 7.49 when $n = 1000$ (see Table C).

The sequential strategy to find the minimum value generalizes easily to find the $2, 3, \ldots,$ or j smallest values among $X_1, X_2, \ldots, X_n$. To begin, test j items until they fail, at stresses $X_1, X_2, \ldots, X_j$. Thereafter stop the i^{th} trial if the item survives the j lowest failure stresses among all $i-1$ previous specimens. The probability of stress-to-failure for the i^{th} item $(i > j)$ is the probability that it is among the j smallest of i independent observations from the same continuous distribution: all ranks are equally likely for X_i, so the desired probability is j/i. The expected number of items destroyed is the sum of failure probabilities over all trials:

$$\underbrace{1+1+\cdots+1}_{j \text{ terms}}+\frac{j}{j+1}+\frac{j}{j+2}+\cdots+\frac{j}{n}=j\left(1+\frac{1}{j+1}+\frac{1}{j+2}+\cdots+\frac{1}{n}\right)$$

$$\leqq j\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

If j is much less than the sample size n, then so is the expected number of failures. For example, to find the weakest 4 items in a sample of 1000, I expect to destroy only about 26; and to find the weakest 8 items I expect to destroy less than 50.

For some sorts of failure distributions, notably the exponential ([**13**] page 41), the minimum or the j smallest failure stresses from a sample can be used to estimate the theoretical mean of the failure distribution. Usually, however, the sample minimum is used to estimate a low percentile rather than the mean or median of the failure distribution.

6. Tolerance limits for failure distributions. A building code may prohibit use of a particular type of structural component unless it has probability at least .95 of surviving some severe stress x. In other words, the failure distribution's fifth percentile should satisfy $x_{.05} > x$. It is safer to under-estimate the percentile $x_{.05}$ than to over-estimate; so consider the lowest failure point observed in laboratory testing. In a very large sample it is highly probable that the breaking strength of the single weakest item is below the $x_{.05}$ value. It can be calculated, for example, that sample size $n = 90$ is sufficiently large to assure that $x_{.05} > \min(X_1, X_2, \ldots, X_{90})$ with probability .99; so when this sample minimum exceeds x, the inference $x_{.05} > x$ has ".99 confidence."

The smallest value in a random sample of size $n \geqq 90$ also is called a level .99 *tolerance limit* for $x_{.05}$, the fifth percentile of the sampled distribution.

The tolerance limit interpretation of small order statistics is about thirty-five years old [**26, 41, 45**]. I encountered this subject when I joined a Statistics Committee of the Study Group on Wood Stresses under auspices of the Canadian Standards Association. (Vancouver's lumber export is important enough to be illustrated on the cover of the Annual Meteorological Summary [**25**] cited before.) The 1970 "Tentative Method for Evaluating Allowable Properties for Grades of Structural Lumber" [**1**] mentioned the sample minimum ($n = 90$) as a level .99 tolerance limit, although it did not suggest the sequential strategy for finding it.

The 2$^{\text{nd}}$ smallest sample value is a level .99 tolerance limit for $x_{.05}$ when sample size $n \geqq 130$; and the 3$^{\text{rd}}$ smallest sample value is a level .99 tolerance limit when $n \geqq 165$. In general, the j^{th} smallest sample value is a level .99 tolerance limit for the fifth percentile when $n \geqq n_j$, where values n_j are given in Table A for $j = 1, 2, 3, \ldots, 15$. The greater the value of j, the less the variability of the corresponding tolerance limit and the less its conservative bias.

This table also gives values m_j, the mean number of failures (items destroyed) in sequential examination of n_j items to find the j weakest among them. Because the ratio m_j/n_j is about 1/10, I can afford a sample size about 10 times the number of items I can actually afford to break. For example, Table A shows that I must sample at least 228 items to use the 5$^{\text{th}}$ weakest as a level .99 tolerance limit for $x_{.05}$; but I expect to destroy only 23.63 of the 228 sample items.

Table A can be used in finding not only level .99, but also level .75, .90, or .95 tolerance limits for the fifth percentile. As the confidence level attached to the sample minimum is increased from .75 to .99, notice that sample size must be tripled, from 27 to 90; but the increase in expected failures, from 3.89 to 5.08, is slight. Hence, level .99 may cost little more than level .75 tolerance limits.

Table B similarly gives sample sizes n_j and expected failure numbers m_j for finding percentile $x_{.01}$ tolerance limits, at confidence level .75, .90, .95, or .99.

By now my writing has gone past the point I could ever carry a conversation. Often it is just as well to leave some air of mystery about how to calculate the minimum sample sizes in Tables A and B. But, of course, for a mathematician this computation is quite elementary.

TABLE A. *Non-parametric Tolerance Limits for Fifth Percentile, $x_{.05}$*

The j^{th} smallest order statistic from a sample of n items is a non-parametric tolerance limit for the sampled distribution's fifth percentile if sample size $n \geqq n_j$, where n_j is given in the following tables for confidence level $\gamma = .75, .90, .95$, or $.99$; that is, $X_{(j)}^{(n_j)} < x_{.05}$ with probability $\simeq \gamma$. The sequential strategy to find the j^{th} order statistic is expected to destroy m_j of the n_j items sampled.

$\gamma = .75$

j	n_j	m_j	m_j/n_j
1	27*	3.89	0.144
2	53	8.11	0.153
3	75*	11.45	0.153
4	101*	15.66	0.155
5	124*	20.59	0.166
6	147*	24.73	0.168
7	170	28.86	0.170
8	192*	32.96	0.172
9	215	37.09	0.173
10	237	41.18	0.174
11	259	45.28	0.175
12	281	49.37	0.176
13	303	53.46	0.176
14	325	57.55	0.177
15	346*	61.60	0.178

$\gamma = .90$

j	n_j	m_j	m_j/n_j
1	45	4.39	0.098
2	76*	8.83	0.116
3	105	12.46	0.119
4	132	17.52	0.133
5	158	21.80	0.138
6	183*	26.04	0.142
7	208*	30.27	0.146
8	233*	34.50	0.148
9	257*	38.69	0.151
10	281*	42.88	0.153
11	305*	47.07	0.154
12	329*	51.26	0.156
13	353	55.44	0.157
14	376*	59.59	0.158
15	399*	63.74	0.160

$\gamma = .95$

j	n_j	m_j	m_j/n_j
1	58*	4.65	0.080
2	93	9.22	0.099
3	124	13.70	0.111
4	153	18.11	0.118
5	180*	22.45	0.125
6	207*	26.77	0.129
7	234	31.09	0.133
8	260	35.38	0.136
9	285*	39.62	0.139
10	311	43.90	0.141
11	336	48.14	0.143
12	361	52.37	0.145
13	385*	56.57	0.147
14	409*	60.77	0.149
15	434	65.00	0.150

$\gamma = .99$

j	n_j	m_j	m_j/n_j
1	90	5.08	0.056
2	130	9.90	0.076
3	165	14.56	0.088
4	197*	19.12	0.097
5	228*	23.63	0.104
6	258*	28.09	0.109
7	287*	32.52	0.113
8	315*	36.91	0.117
9	343*	41.29	0.120
10	371	45.66	0.123
11	398	50.00	0.126
12	425	54.33	0.129
13	451	58.63	0.130
14	477*	62.92	0.132
15	503*	67.21	0.134

* Asterisk indicates sample size n_j such that $P\{X_{(j)}^{(n_j)} < x_{.05}\} < \gamma$, but the probability is as close as possible to γ (i.e., for sample size $n_j + 1$, the probability $P\{X_{(j)}^{(n_j)} < x_{.05}\} > \gamma$ and the excess over γ is of greater magnitude than the deficit for sample size n_j). Absence of an asterisk means sample size n_j is such that $P\{X_{(j)}^{(n_j)} < x_{.05}\} \geqq \gamma$ and the probability is as close as possible to γ.

For any proportion p, the p^{th} *percentile* of a continuous probability distribution F can be defined as the unique value x_p such that $F(x_p) = p$. In terms of the distribution's inverse function, the p^{th} percentile $x_p = F^{-1}(p)$. It is not difficult (see [8], page 7) to show that the j^{th} smallest order statistic $X_{(j)}^{(n)}$ from n sample observations will satisfy $X_{(j)}^{(n)} < x_p$ with probability

$$P\{X_{(j)}^{(n)} < x_p\} = \sum_{i=j}^{n} \binom{n}{i} [F(x_p)]^i [1 - F(x_p)]^{n-i} = \sum_{i=j}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1 - \sum_{i=0}^{j-1} \binom{n}{i} p^i (1-p)^{n-i}.$$

TABLE B. *Non-parametric Tolerance Limits for First Percentile*, $x_{.01}$

The j^{th} smallest order statistic from a sample of n items is a non-parametric tolerance limit for the sampled distribution's first percentile if sample size $n \geqq n_j$, where n_j is given in the following tables for confidence level $\gamma = .75, .90, .95$, or $.99$; that is $X_{(j)}^{(n_j)} < x_{.01}$ with probability $\simeq \gamma$. The sequential strategy to find the j^{th} order statistic is expected to destroy m_j of the n_j items sampled.

$\gamma = .75$

j	n_j	m_j	m_j/n_j
1	138	5.51	0.040
2	268*	11.34	0.042
3	391*	17.14	0.044
4	510	22.92	0.045
5	626*	28.67	0.046
6	741*	34.42	0.046
7	855	40.15	0.047
8	967*	45.87	0.047

$\gamma = .90$

j	n_j	m_j	m_j/n_j
1	229*	6.01	0.026
2	388	12.08	0.031
3	531	18.06	0.034
4	666*	23.98	0.036
5	797*	29.88	0.037
6	925*	35.75	0.039

$\gamma = .95$

j	n_j	m_j	m_j/n_j
1	298*	6.28	0.021
2	473	12.47	0.026
3	627*	10.56	0.030
4	773	24.58	0.032
5	913	30.56	0.033

$\gamma = .99$

j	n_j	m_j	m_j/n_j
1	458*	6.71	0.015
2	661	13.14	0.020
3	837*	19.42	0.023
4	1001	25.62	0.026
5	1157	31.74	0.027

* Asterisk indicates sample size n_j such that $P\{X_{(j)}^{(n_j)} < x_{.01}\} < \gamma$, but the probability is as close as possible to γ (i.e., for sample size $n_j + 1$, the probability $P\{X_{(j)}^{(n_j)} < x_{.01}\} > \gamma$ and the excess over γ is of greater magnitude than the deficit for sample size n_j). Absence of an asterisk means sample size n_j is such that $P\{X_{(j)}^{(n_j)} < x_{.01}\} \geqq \gamma$ and the probability is as close as possible to γ.

The subtracted sum is a binomial tail probability which can be evaluated using binomial tables (or a normal approximation) or a calculator. Since this sum vanishes as sample size $n \to \infty$, one can find, for any fraction γ, a sample size $n_j(\gamma)$ so large that the event $x_p > X_{(j)}^{(n)}$ has probability $\geqq \gamma$ for all $n \geqq n_j(\gamma)$.

Clearly this construction of tolerance limits is *distribution-free*: sample size $n_j(\gamma)$ does not depend on the form of the continuous distribution function $F(x) = P\{X_i < x\}$.

7. Persistence of record breaking and divergence of the harmonic series. No matter what the present precipitation record may be, it is certain that eventually there will occur a year with more rain. No matter how many record breaking years have been counted to date, there will be always one more. Such unit increments make the count of record breaking years arbitrarily large as the years of observation increase indefinitely.

These statements about rainfall interpret a specific mathematical theorem (and suggest its proof). From this point onward I emphasize formal limit results rather than applications of the theory of record values.

Formally, suppose that identically distributed random variables $X_1, X_2, \ldots, X_n$ are *exchangeable* (in particular, it suffices that the sequence of random variables be *independent*). The observation X_i is called a *record high* or *upper record value* or *ladder value** if X_i strictly exceeds all previous values in the sequence. For example, there are 3 record highs in the first 10 years of January sunshine figures

* A term used by Feller [12].

(Data Set 2):

55.3, 45.5, 14.4, 44.2, 43.3, 38.7, 57.7, 65.9, 53.9, 45.3.

In this sequence the record highs are $X_1 = 55.3$, $X_7 = 57.7$, and $X_8 = 65.9$. Assume that exact ties have zero probability (arbitrarily precise measurement from a continuous distribution). Then X_i is a record high if and only if $X_i = \max(X_1, \ldots, X_i)$. Since all i ranks are equally likely for X_i, an upper record value (maximum rank) has probability $1/i$ at the i^{th} trial.

Let R_n denote the number of record highs among the first n observations. A formal version of the statement at the beginning of this section is the following theorem:

The count of record highs $R_n \to \infty$ with probability one as sample size $n \to \infty$.

To prove this proposition, consider an initial sequence of n_1 observations $X_1, X_2, \ldots, X_{n_1}$ and a further batch of n_2 observations $X_{n_1+1}, \ldots, X_{n_1+n_2}$. The probability that this additional batch contains no new record value is

$$P\{R_{n_1} = R_{n_1+n_2}\} = P\{\max(X_1, \ldots, X_{n_1}) = \max(X_1, \ldots, X_{n_1+n_2})\} = \frac{n_1}{n_1+n_2},$$

the ratio of the initial sample size divided by the total. By taking batch size n_2 sufficiently large, I can make this probability as small as I wish. And I can repeat this procedure for many successive batches, choosing sizes $n_2, n_3, \ldots, n_r$ to satisfy

$$\frac{n_1}{n_1+n_2} < \varepsilon/r$$

$$\frac{(n_1+n_2)}{(n_1+n_2)+n_3} < \varepsilon/r$$

$$\vdots$$

$$\frac{(n_1+\cdots+n_{r-1})}{(n_1+\cdots+n_{r-1})+n_r} < \varepsilon/r$$

so that, for arbitrary $\varepsilon > 0$,

$$\sum_{k=1}^{r} P\{\text{no record value in } k^{\text{th}} \text{ batch}\} \leqq r(\varepsilon/r) = \varepsilon.$$

But the probability of obtaining at least r record highs in a sequence with length $n \geqq n_1 + n_2 + \cdots + n_r$ is then

$$\begin{aligned} P\{R_n \geqq r\} &\geqq P\{\text{at least one record high in each of } r \text{ batches}\} \\ &= 1 - P\{\text{at least one of } r \text{ batches has no record high}\} \\ &\geqq 1 - \sum_{k=1}^{r} P\{\text{no record high in } k^{\text{th}} \text{ batch}\} \\ &> 1 - \varepsilon. \end{aligned}$$

Since ε is arbitrarily small, conclude that $P\{R_n \geqq r\} \to 1$, for arbitrarily large integer r. That is, the number of record highs $R_n \to \infty$ in probability as sequence length $n \to \infty$. Because the sequence $R_1 \leqq R_2 \leqq \cdots$ is monotone, it follows that also $R_n \to \infty$ with probability one.

Viewed in the theory of recurrent events [12], record breaking is a *persistent* phenomenon.

That the expectation $E(R_n)\to\infty$ is immediate: for any integer r and $n\geqq n_1+n_2+\cdots+n_r$ as above,

$$E(R_n)=\sum_{k=1}^{n} kP\{R_n=k\}\geqq\sum_{k=r}^{n} kP\{R_n=k\}$$
$$\geqq rP\{R_n\geqq r\}>r(1-\varepsilon).$$

In fact, the preceding argument is an obvious modification of the classic harmonic divergence proof with batch sizes $n_k=2^{k-1}$ and $n=n_1+n_2+\cdots+n_r$. The n^{th} partial harmonic sum is:

$$\begin{aligned}1+\tfrac12+\tfrac13+\cdots+1/n&=1+(\tfrac12+\tfrac13)\\&\quad+(\tfrac14+\tfrac15+\tfrac16+\tfrac17)\\&\quad+\cdots\\&\quad+\underbrace{\left(\frac{1}{2^{r-1}}+\cdots+\frac{1}{2^r-1}\right)}_{2^{r-1}\text{ terms}}\\&>1+2/4+4/8+\cdots+2^{r-1}/2^r=(r+1)/2.\end{aligned}$$

The other harmonic divergence demonstration which is popular in calculus texts shows $\Sigma_{i=1}^n i^{-1}\approx\int_1^n x^{-1}dx=\ln(n)$.

Several arguments in subsequent sections exploit these properties of the harmonic series.

8. The frequency of record breaking. Define binary variates to indicate the trials at which record highs occur in the original sequence:

$$Y_i=\begin{cases}1 & \text{if } X_i=\max(X_1,X_2,\ldots,X_i)\\ 0 & \text{otherwise,}\end{cases}$$

with expected value and variance given by

$$\begin{aligned}E(Y_i)&=P\{Y_i=1\}=P\{X_i=\max(X_1,\ldots,X_i)\}=1/i,\\ V(Y_i)&=E(Y_i^2)-[E(Y_i)]^2 \qquad\qquad =1/i-1/i^2.\end{aligned}$$

Any distinct pair Y_i and Y_j (say $i<j$) are uncorrelated since

$$\begin{aligned}E(Y_iY_j)&=P\{Y_i=1\quad\text{and}\quad Y_j=1\}\\&=P\{X_i=\max(X_1,\ldots,X_i)\quad\text{and}\quad X_j=\max(X_1,\ldots,X_j)\}\\&=P\{X_i=\max(X_1,\ldots,X_i)<\max(X_{i+1},\ldots,X_j)=X_j\}\\&=P\{X_i=\max(X_1,\ldots,X_i)\}\\&\qquad\times P\{\max(X_1,\ldots,X_i)<\max(X_{i+1},\ldots,X_j)\}\\&\qquad\times P\{X_j=\max(X_{i+1},\ldots,X_j)\}\\&=\frac1i\,\frac{j-i}{j}\,\frac{1}{j-i}=\frac{1}{ij}\\&=P\{Y_i=1\}\;P\{Y_j=1\}=E(Y_i)E(Y_j).\end{aligned}$$

Thus the random number of record highs among $X_1,X_2,\ldots,X_n$ is $R_n=\Sigma_{i=1}^n Y_i$ with expectation

and variance given by

$$E(R_n)=\sum_{i=1}^{n} E(Y_i)=\sum_{i=1}^{n} 1/i,$$

$$V(R_n)=\sum_{i=1}^{n} V(Y_i)=\sum_{i=1}^{n} 1/i-\sum_{i=1}^{n} 1/i^2.$$

Since

$$\sum_{i=1}^{n}\frac{1}{i}-\ln(n)\to \text{Euler's constant} = .5772\ldots \text{ and } \sum_{i=1}^{n}\frac{1}{i^2}\to\frac{\pi^2}{6}=1.6449\ldots,$$

logarithm tables (or electronic calculators) easily give approximations for $E(R_n)$ and $V(R_n)$. Also both expressions are evaluated numerically in Table C for selected values of n: I find that the actual numbers surprise mathematicians as well as people who know little about logarithms.

TABLE C

Expectation, variance, and standard deviation of R_n, the number of record values in a random sequence of n independent and identically distributed observations

	$E(R_n)=\sum_{i=1}^{n} 1/i$	$V(R_n)=\sum_{i=1}^{n} 1/i-\sum_{i=1}^{n} 1/i^2$	
n	$E(R_n)$	$V(R_n)$	$\sqrt{V(R_n)}$
2	1.50	.25	.50
3	1.83	.47	.69
4	2.08	.66	.81
5	2.28	.82	.91
6	2.45	.96	.98
7	2.59	1.08	1.04
8	2.72	1.19	1.09
9	2.83	1.29	1.14
10	2.93	1.38	1.17
20	3.60	2.00	1.41
30	3.99	2.38	1.54
40	4.28	2.66	1.63
50	4.50	2.87	1.70
60	4.68	3.05	1.75
65	4.76	3.13	1.77
70	4.83	3.20	1.79
80	4.97	3.33	1.83
90	5.08	3.45	1.86
100	5.19	3.55	1.88
200	5.88	4.24	2.06
300	6.28	4.64	2.15
400	6.57	4.93	2.22
500	6.79	5.15	2.27
600	6.97	5.33	2.31
700	7.13	5.49	2.34
800	7.26	5.62	2.37
900	7.38	5.74	2.40
1000	7.49	5.84	2.42
1,000,000	14.39	12.75	3.57

The variance $V(R_n)$ gives bounds on the probability of destroying too many items in a sequential strategy for destructive testing. In particular, a one-sided Chebyshev inequality ([**13**] page 152) implies that

$$P\{R_n \geqq r\} \leqq \frac{V(R_n)}{V(R_n)+[r-E(R_n)]^2}.$$

For example, the values of $E(R_n)$ and $V(R_n)$ in Table C imply that $P\{R_{60} \geqq 9\} \leqq .14$ and $P\{R_{1000} \geqq 18\} \leqq .05$ (and actually these bounds are quite conservative since detailed calculation [**6**] shows $P\{R_{60} \geqq 9\} = .022$).

In a later section I show that the binary variates $Y_1, Y_2, Y_3, \ldots$ are *independent* as well as pairwise uncorrelated. Because of this independence, general limit theorems can be invoked to show more precisely how the random sum $R_n = \Sigma_{i=1}^n Y_i$ grows probabilistically like $\ln(n)$. I have noted that

(i) $$E(R_n) = \sum_{i=1}^n E(Y_i) = \sum_{i=1}^n \frac{1}{i} \to \infty$$

and that $\ln(n)$ approximates the expectation $E(R_n)$ for large n, so that

$$\frac{V(Y_n)}{[E(R_n)]^2} \approx \frac{1}{n[\ln(n)]^2}$$

and consequently

(ii) $$\sum_{n=1}^{\infty} \left\{ \frac{V(Y_n)}{\left[\sum_{i=1}^n E(Y_i)\right]^2} \right\} \approx \sum_{n=1}^{\infty} \frac{V(Y_n)}{[E(R_n)]^2} < \infty.$$

Invoking Kolmogorov's convergence criterion for sums of independent random variables ([**24**] page 238), conditions (i) and (ii) imply that the sequence $R_n/E(R_n) \to 1$; and hence Alfréd Renyi [**30**] obtained the following "strong law of large numbers" for the frequency of record highs (since $E(R_n)/\ln(n) \to 1$):

$$R_n/\ln(n) \to 1 \textit{ with probability one as sample size } n \to \infty,$$

which implies the divergence $R_n \to \infty$ discussed in Section 7.

Similarly, the criterion of Liapounov ([**24**] page 275) gives a "central limit theorem" for the frequency of record highs: as sample size $n \to \infty$, the limit distribution of $(R_n - \ln(n))/\sqrt{\ln(n)}$ is normal with mean $= 0$ and variance $= 1$. (Renyi also gave a "law of the iterated logarithm." Resnick [**33**] later derived all these limit theorems from results concerning counts in a continuous-time Poisson process; see the end of Section 12, below.) But "the asymptotic distribution is a very poor approximation for $n \leqq 1000$ say, which is the only region of much statistical importance" [**4**].

The exact probability that $R_n = r$ is complicated. Instead, consider $R_{2n} - R_n$, the number of record highs over trials $n+1, n+2, \ldots, 2n$. In particular,

$$\begin{aligned} P\{R_{2n} - R_n = 0\} &= P\{\text{no record high for } n+1 \leqq i \leqq 2n\} \\ &= P\{\max(X_1, \ldots, X_n) = \max(X_1, \ldots, X_{2n})\} \\ &= n/2n = 1/2. \end{aligned}$$

In general, the count $R_{2n} - R_n$ has asymptotically a Poisson distribution with mean $= \ln(2)$: that is, for $k = 0, 1, 2, \ldots,$

$$P\{R_{2n} - R_n = k\} \to \frac{[\ln(2)]^k}{2k!} \quad \text{as } n \to \infty.$$

One proof uses the probability generating functions $g_i(s) = E(s^{Y_i}) = 1 + (s-1)/i$ associated with the binary variates Y_i. Since $R_{2n} - R_n = Y_{n+1} + Y_{n+2} + \cdots + Y_{2n}$, and the binary variates are independent, this sum has generating function given by

$$h_n(s) = \prod_{i=n+1}^{2n} g_i(s) = \prod_{i=n+1}^{2n} [1 + (s-1)/i].$$

Take logarithms of both sides and use the Taylor series expansion $\ln(1+x) = x - x^2/2 + x^3/3 - \cdots$ to obtain

$$\begin{aligned}\ln[h_n(s)] &= \sum_{i=n+1}^{2n} \ln[1 + (s-1)/i] \\ &= \sum_{i=n+1}^{2n} \left[\frac{s-1}{i} - \frac{(s-1)^2}{2i^2} + \frac{(s-1)^3}{3i^3} - \cdots\right] \\ &= (s-1) \sum_{i=n+1}^{2n} \frac{1}{i} - \frac{(s-1)^2}{2} \sum_{i=n+1}^{2n} \frac{1}{i^2} + \frac{(s-1)^3}{3} \sum_{i=n+1}^{2n} \frac{1}{i^3} - \cdots.\end{aligned}$$

As $n \to \infty$, all of the above series tails vanish except for the harmonic initial term, which converges to $(s-1)[\ln(2n) - \ln(n)] = (s-1)\ln(2)$. This limit is the logarithm of a Poisson generating function with parameter $\ln(2)$.... The argument can be generalized to prove the following theorem first stated by Dwass **[10]**:

As sample size $n \to \infty$, the frequency of record highs among observations indexed by $an < i \leqq bn$ (for any $b > a > 0$) is asymptotically a Poisson count with mean $\ln(b/a)$.

9. Serial numbers of record breaking trials. Rather than R_n, the random number of record highs in n trials, consider now the random serial number N_r of the trial at which the r^{th} record high occurs. The number N_r is called simply the r^{th} *record value time*. Since I count the initial observation as a record value, $R_1 = 1$ and $N_1 = 1$. In general $N_r \geqq r$ and, for $n \geqq r$,

$$P\{N_r \leqq n\} = P\{R_n \geqq r\}.$$

In particular, the record time N_2 has probability distribution, over the integers $i = 2, 3, \ldots$, given by

$$P\{N_2 = i\} = P\{X_1 = \max(X_1, \ldots, X_{i-1}) < X_i\} = \frac{1}{(i-1)i}.$$

Since this probability is strictly decreasing for $i = 2, 3, \ldots$, the *mode* $= 2$; but the *mean* is

$$E(N_2) = \sum_{i=2}^{\infty} iP\{N_2 = i\} = \sum_{i=2}^{\infty} \frac{1}{i-1} = \infty.$$

Since $N_2 < N_3 < \cdots$, it follows that $E(N_r) = \infty$ for all $r \geqq 2$. More surprising, an argument in the next section shows that $E(N_{r+1} - N_r) = \infty$.

To study joint distribution of N_r, N_{r+1} I must indicate why the binary variates $Y_1, Y_2, \ldots, Y_n$ defined in the preceding section are *independent* (a condition needed earlier to prove the Rényi and Dwass limit theorems). Consider the event that $Y_1, Y_2, \ldots, Y_n$ include exactly r ones, at trials $1 = i_1 < i_2 < \cdots < i_r \leqq n$. This event corresponds to

$$\begin{array}{ccccc}\max(X_1, \ldots, X_{i_2-1}) & < & \max(X_{i_2}, \ldots, X_{i_3-1}) & < \cdots < & \max(X_{i_r}, \ldots, X_n), \\ \| & & \| & & \| \\ X_1 & & X_{i_2} & & X_{i_r}\end{array}$$

which is the intersection of independent events of the form

$$\max(X_1,\ldots,X_{i-1}) < \max(X_i,\ldots,X_j) = X_i,$$

with $i = i_{k-1}$ and $j = i_k - 1$ or $j = n$. This component probability is

$$\frac{j-(i-1)}{j}\,\frac{1}{j-(i-1)} = \frac{1}{j};$$

so the desired joint probability is the product

$$\frac{1}{(i_2-1)(i_3-1)\cdots(i_r-1)n}.$$

It is easy to check that the product of marginal probabilities for $Y_2 = 0$, $Y_3 = 0,\ldots$, $Y_{i_2-1} = 0$, $Y_{i_2} = 1$, $Y_{i_2+1} = 0,\ldots$ gives the same expression:

$$\frac{1}{2}\,\frac{2}{3}\,\frac{3}{4}\cdots\frac{i_2-2}{i_2-1}\cdot\frac{1}{i_2}\,\frac{i_2}{i_2+1}\cdots = \frac{1}{(i_2-1)}\,\frac{1}{(i_3-1)}\cdots.$$

The joint distribution of serial numbers $N_2, N_3,\ldots, N_r$ is therefore given by

$$P\{N_2 = i_2, N_3 = i_3,\ldots, N_r = i_r\} = \frac{1}{(i_2-1)(i_3-1)\cdots(i_r-1)i_r},$$

for $1 < i_2 < i_3 < \cdots < i_r$. The general marginal distribution is complicated, but expressions have been given by various authors [**7**, **19**, **30**, **37**]:

$$P\{N_r = n\} = |S_{n-1}^{r-1}|/n!,$$

where the numerator uses Stirling numbers of the first kind. Rényi [**30**] and David and Barton [**7**] used Stirling numbers to show that

$$\begin{aligned} P\{R_n = r\} &= \sum_{2\le i_2<i_3<\cdots<i_r\le n} P\{R_n = r;\ N_2 = i_2, N_3 = i_3,\ldots, N_r = i_r\} \\ &= \frac{1}{n}\sum_{2\le i_2<i_3<\cdots<i_r\le n}\frac{1}{(i_2-1)(i_3-1)\cdots(i_r-1)} \\ &= \frac{|S_n^{r-1}|}{(r-1)!} \approx \frac{[\ln(n)]^{r-1}}{n(r-1)!} \end{aligned}$$

for large sample size n. But this approximation also can be derived using generating functions as in the preceding section. (Or see [**21**], page 267.)

Rényi [**30**] treated limit theorems for N_r as duals of theorems for R_n. First, there is a "strong law of large numbers":

$$\frac{\ln(N_r)}{r} \to 1 \textit{ with probability one as } r \to \infty;$$

or, equivalently,

$$N_r^{1/r} \to e \textit{ with probability one.}$$

This last result says that almost every sequence of chance observations $X_1, X_2, X_3,\ldots$ from a continuous distribution will give an arbitrarily precise estimate of the mathematical constant e, even when the distribution function is unknown to me! I need only the serial numbers of trials at which record highs occur. (This approach differs somewhat from statistical determinations of e or of π in experiments such as "matching" or "Buffon's needle problem" [**12**]. In those situations, a particular event probability is a function of e or of π; and this probability is estimated by the relative frequency in repeated trials.)

Also Rényi [30] stated a "central limit theorem" for the random variables N_r: as $r \to \infty$, the distribution of $(\ln(N_r) - r)/\sqrt{r}$ is asymptotically normal with mean $= 0$ and variance $= 1$.

Rényi also gave a "law of the iterated logarithm" for N_r. Resnick [33] again derived Rényi's limit theorems for record times from a more sophisticated result. And Vervaat ([44] page 323) gave a Wiener process generalization of Rényi's N_r central limit theorem.

Returning to the distribution of the second record time N_2, described at the beginning of this section, notice that for $n = 2, 3, 4, \ldots,$

$$P\left\{\frac{1}{N_2} < \frac{1}{n}\right\} = P\{N_2 > n\} = \sum_{i=n+1}^{\infty} P\{N_2 = i\}$$
$$= \sum_{i=n+1}^{\infty} \frac{1}{(i-1)i} = \sum_{i=n+1}^{\infty} \left(\frac{1}{i-1} - \frac{1}{i}\right) = \frac{1}{n}.$$

A similar result holds for record time N_3 and *conditional* probability of the event $N_3 > n$, given that the preceding record time $N_2 = m$. More generally, for $n > m \geqq r$,

$$P\left\{\frac{N_r}{N_{r+1}} < \frac{m}{n} \,\middle|\, N_r = m\right\} = P\{N_{r+1} > n \mid N_r = m\}$$
$$= P\{\text{no record high for } m + 1 \leqq i \leqq n \mid N_r = m\}$$
$$= P\{\max(X_1, \ldots, X_m) = \max(X_1, \ldots, X_n) \mid N_r = m\}$$
$$= m/n.$$

Any real x in the interval $(0, 1)$ can be approximated by a rational ratio m/n; and Tata [43] gave the following limit theorem:

The distribution of the ratio N_r/N_{r+1} is asymptotically uniform over the unit interval: that is, for $0 < x < 1$,

$$P\left\{\frac{N_r}{N_{r+1}} < x\right\} \to x \quad \textit{as } r \to \infty.$$

The proof suggests duality between this result and the theorem of Dwass in the preceding section. Given that the r^{th} record high occurs at trial N_r, the conditional probability

$$P\left\{\frac{N_r}{N_{r+1}} < \frac{1}{2} \,\middle|\, N_r\right\} = P\{N_{r+1} > 2N_r \mid N_r\}$$
$$= P\{\text{no record high for } N_r + 1 \leqq i \leqq 2N_r \mid N_r\}$$
$$= P\{R_{2N_r} - R_{N_r} = 0 \mid N_r\} = N_r/2N_r$$
$$= 1/2,$$

independent of N_r. Similarly, for $0 < x < 1$,

$$P\left\{\frac{N_r}{N_{r+1}} < x \,\middle|\, N_r\right\} = P\left\{N_{r+1} > \frac{N_r}{x} \,\middle|\, N_r\right\} = \frac{N_r}{[N_r/x]} \to x$$

as $r \to \infty$, independent of N_r. (Here $[N_r/x]$ means "the largest integer $\leqq N_r/x$.")

The argument also indicates, as Shorrock [37] and later Resnick [33] proved, that successive ratios $N_r/N_{r+1}, N_{r+1}/N_{r+2}, \ldots$ are *asymptotically independent* uniform variates. In other words, Shorrock's theorem shows that an *unknown* continuous distribution can be used to approximate a *uniform* random number generator! I need only the numbers of the trials at which record highs occur in the original randomly sampled $X_1, X_2, X_3, \ldots$.

Tata's argument requires the asymptotic probability that the count $(R_{[n/x]} - R_n) = 0$. In the preceding section, however, the theorem of Dwass gives more: the asymptotic Poisson probability that $(R_{[n/x]} - R_n) = k$, for $k = 0, 1, 2, \ldots$. Therefore, for any positive integer s and real x in the unit interval,

$$P\left\{\frac{N_r}{N_{r+s}} < x \,\middle|\, N_r\right\} = P\left\{N_{r+s} > \frac{N_r}{x} \,\middle|\, N_r\right\} = P\{R_{[N_r/x]} - R_{N_r} < s \mid N_r\}$$

$$= \sum_{k=0}^{s-1} P\{R_{[N_r/x]} - R_{N_r} = k \mid N_r\} \to x \sum_{k=0}^{s-1} (-\ln(x))^k / k!$$

as $r \to \infty$, independent of N_r. Thus a dual of the Dwass theorem generalizes Tata's result: for $s = 1, 2, 3, \ldots$ and $0 < x < 1$,

$$P\left\{\frac{N_r}{N_{r+s}} < x\right\} \to x \sum_{k=0}^{s-1} (-\ln(x))^k / k! \quad \text{as } r \to \infty.$$

10. Waiting times between record breaking trials. Now consider the wait from the r^{th} to the $(r+1)^{\text{st}}$ record breaking trial: denote this *inter-record waiting time* by $W_r = N_{r+1} - N_r$.

Record breaking must occur infinitely often; but it turns out that the inter-record waiting time distributions have mean $= \infty$, although mode $= 1$. These results, first proved by Chandler [6], can be deduced from independence of the binary variates $Y_1, Y_2, \ldots$ via the following arguments.

$$P\{W_r = k \mid N_r\} = P\{N_{r+1} = N_r + k \mid N_r\}$$

$$= P\{Y_i = 0 \quad \text{for } N_r + 1 \leqq i \leqq N_r + k - 1,\ Y_{N_r+k} = 1 \mid N_r\}$$

$$= \frac{N_r}{N_r+1}\,\frac{N_r+1}{N_r+2}\cdots\frac{N_r+k-2}{N_r+k-1}\,\frac{1}{N_r+k} = \frac{N_r}{(N_r+k-1)(N_r+k)}.$$

This expression is monotone decreasing as k increases; so, for any value of N_r, $P\{W_r = 1 \mid N_r\} \geqq P\{W_r = k \mid N_r\}$, for $k = 1, 2, 3, \ldots$. Taking expectations, $P\{W_r = 1\} \geqq P\{W_r = k\}$; that is, the most probable value or mode $= 1$.

$$P\{N_r = i, W_r = k\} = P\{N_r = i, N_{r+1} = i + k\}$$

$$\geqq P\{N_2 = 2, N_3 = 3, \ldots, N_{r-1} = r-1, N_r = i, N_{r+1} = i + k\}$$

$$= \frac{1}{(2)(3)\cdots(r-2)(i-1)(i+k-1)(i+k)}$$

$$\geqq \frac{1}{r!(i-1)}\left(\frac{1}{k+i-1} - \frac{1}{k+i}\right)$$

and hence, for any integers m and $i \geqq r$,

$$P\{N_r = i, W_r \geqq m\} \geqq \frac{1}{r!(i-1)} \sum_{k=m}^{\infty} \left(\frac{1}{k+i-1} - \frac{1}{k+i}\right)$$

$$= \frac{1}{r!(i-1)(i-1+m)} = \frac{1}{r!m}\left(\frac{1}{i-1} - \frac{1}{i-1+m}\right).$$

The marginal distribution of W_r satisfies

$$P\{W_r \geqq m\} = \sum_{i=r}^{\infty} P\{N_r = i, W_r \geqq m\} \geqq \frac{1}{r!m} \sum_{i=r}^{\infty} \left(\frac{1}{i-1} - \frac{1}{i-1+m}\right)$$

$$\geqq \frac{1}{r!m} \sum_{i=r}^{r+m-1} \frac{1}{i-1};$$

and therefore, for arbitrarily large integers m,

$$E(W_r)=\sum_{k=1}^{\infty} kP\{W_r=k\}\geqq mP\{W_r\geqq m\}$$
$$\geqq \frac{1}{r!}\sum_{i=r}^{r+m-1}\frac{1}{i-1}.$$

It follows from the divergence of the harmonic series that $E(W_r)=\infty$.

In other words, if I pay a fixed fee and receive in return a variable dollar amount equal to the waiting time W_r, then my expected gain is positive and my gamble is not a "fair game" no matter how high the fee.

An alternate demonstration that $E(W_r)=\infty$ assumes that observations $X_1, X_2, X_3, \ldots$ have exponential distribution $F(x)=P\{X_i<x\}=1-e^{-x}$, for $x>0$. This assumption is harmless because the preceding section shows that record times N_r, and hence inter-record waiting times W_r, have distributions which do not depend on F (only that it be continuous). The *conditional* probability that waiting time $W_r>m$, given that the r^{th} record value $X_{N_r}=x$, is just the probability that independent $X_i<x$ for indices $N_r+1\leqq i\leqq N_r+m$; viz., conditional probability is $[F(x)]^m=(1-e^{-x})^m$ for exponential sampling. Moreover, Section 11 shows that the record value X_{N_r} from exponential sampling has a gamma density. Hence the *unconditional* probability that $W_r>m$ can be represented (independent of the distribution F) as the integral of $(1-e^{-x})^m$ with respect to a gamma density:

$$P\{W_r>m\}=\int_0^{\infty} P\{W_r>m\mid X_{N_r}=x\}\,dP\{X_{N_r}=x\}$$
$$=\int_0^{\infty}(1-e^{-x})^m x^{r-1}e^{-x}\,dx/(r-1)!$$

or

$$P\{W_r=m\}=P\{W_r>m-1\}-P\{W_r>m\}$$
$$=\int_0^{\infty}[(1-e^{-x})^{m-1}-(1-e^{-x})^m]x^{r-1}e^{-x}\,dx/(r-1)!.$$

Notice that the geometric series

$$\sum_{m=1}^{\infty} m[(1-e^{-x})^{m-1}-(1-e^{-x})^m]=\sum_{m=1}^{\infty}(1-e^{-x})^{m-1}$$
$$=\frac{1}{1-(1-e^{-x})}=e^x;$$

so (as already showed by a different argument) the expectation

$$E(W_r)=\sum_{m=1}^{\infty} mP\{W_r=m\}=\int_0^{\infty}\frac{x^{r-1}\,dx}{(r-1)!}=\infty.$$

(I am trying to avoid integral calculus in this paper, but the preceding argument is too ingenious to omit.)

For large values of r, the gamma can be approximated by a normal density. Neuts [**27, 28**] used such approximation in the integral expression for $P\{W_r>m\}$ to prove a "law of large numbers" for waiting times: as $r\to\infty$,

$$\frac{\ln(W_r)}{r}\to 1$$

in probability (also: with probability one [20]). By the same method, Neuts gave a "central limit theorem": as $r \to \infty$, the distribution of $(\ln(W_r) - r)/\sqrt{r}$ is asymptotically normal with mean = 0 and variance = 1. There is also a "law of the iterated logarithm" for waiting times [42]. Shorrock ([36] page 222) and Resnick ([33] page 867) include these limit results in more general theorems for inter-record waiting times.

The remarkable similarity between limit theorems for record times and for inter-record waiting times suggests that the wait W_r for the *next* record value is somehow comparable in scale to the sum of all past waiting times $W_1 + W_2 + \cdots + W_{r-1} = N_r - 1$. Resnick [33] connected the limit behaviour of W_r and N_r in the following theorem: with probability one,

$$\limsup_{r\to\infty} \frac{|\ln(W_r) - \ln(N_r)|}{\ln(r)} = 1.$$

Since $W_r \approx e^r$ in some probabilistic limit sense, successive waits must have increasing *median* values, although they all have mode = 1 and mean = ∞. Numeric computations [19] show that

$$\frac{\text{median}(W_{r+1})}{\text{median}(W_r)} \approx e = 2.718\ldots$$

even for $r = 4, 5, 6, 7, 8$. Note that "small r" does not mean "small sample size": Table C shows that fewer than 8 record highs are expected in a sample of size $n = 1000$.

r	2	3	4	5	6	7	8
median(W_r)	4	10	26	69	183	490	1316
med(W_r)/med (W_{r-1})		2.50	2.60	2.65	2.65	2.68	2.69

The end of the preceding section found independent uniform limit distributions for N_r/N_{r+1} and N_{r+1}/N_{r+2}; and hence, as $r \to \infty$, the waiting time ratio

$$\frac{W_{r+1}}{W_r} = \frac{N_{r+2} - N_{r+1}}{N_{r+1} - N_r} = \frac{(N_{r+2}/N_{r+1}) - 1}{1 - (N_r/N_{r+1})}$$

asymptotically has the same distribution as

$$\frac{(1/V) - 1}{1 - U} \quad \text{or} \quad \frac{(1/V) - 1}{U},$$

where U and V are independent random variables uniform over the interval (0, 1). For any $x > 0$,

$$P\left\{\frac{(1/V) - 1}{U} > x\right\} = P\{V < (xU + 1)^{-1}\} = \int_{u=0}^{1} \int_{v=0}^{(xu+1)^{-1}} dv\,du$$

$$= \frac{1}{x}\int_0^1 \frac{x\,du}{xu + 1} = \frac{\ln(1 + x)}{x}.$$

Thus Shorrock [37] obtained the following limit result for ratios of waiting times: for any $x > 0$,

$$P\left\{\frac{W_{r+1}}{W_r} > x\right\} \to \frac{\ln(1 + x)}{x} \quad \text{as} \quad r \to \infty.$$

11. The record value sequence. Little has been said so far about actual record values, i.e., magnitudes. Rather I have focused on frequencies and serial positions of record values.

Results concerning the record frequency R_n in a random sample $X_1, X_2, \ldots, X_n$ or concerning the

index number N_r of the r^{th} record value trial are *distribution-free* results: they presume identically distributed and independent (or exchangeable) random variables sampled from a continuous probability distribution; but the distribution function F never appears in the theorems.

The actual record values $X_1 < X_{N_2} < X_{N_3} < X_{N_4} < \cdots$ present a quite contrary situation. Results concerning this record value sequence *per se* definitely depend on the particular distribution F, but do not involve positions in the original random sequence.

Many properties of the record value sequence from continuous distribution F can be stated most conveniently in terms of the transform

$$G(x) = -\ln[1 - F(x)].$$

The derivative $G'(x) = F'(x)/[1 - F(x)]$ is the *failure rate* or *hazard rate* which plays a central role in mathematical theory of reliability (see **[2]** p. 53). Both F and G are strictly increasing functions over their support intervals, so there exist respective inverses F^{-1} and G^{-1},

$$G^{-1}(x) = F^{-1}(1 - e^{-x}).$$

Specific examples will illustrate the distinction between distribution-free results for record value times in random sampling and distribution-dependent results for the record value sequence itself. The theorem of Dwass at the end of Section 8 asserts that, for $0 < \boldsymbol{a} < \boldsymbol{b}$, the count of record value trials indexed between $\boldsymbol{an}$ and $\boldsymbol{bn}$ has asymptotically a Poisson distribution with mean $\ln(\boldsymbol{b}/\boldsymbol{a})$. This result clearly is asymptotic (valid as $n \to \infty$) since there can be at most $(\boldsymbol{b} - \boldsymbol{a})n$ record indices between $\boldsymbol{an}$ and $\boldsymbol{bn}$, while Poisson distribution permits an arbitrarily large frequency. By contrast, for any $\alpha < \beta$ such that $0 < F(\alpha) < F(\beta) < 1$, arbitrarily many of the records $X_1 < X_{N_2} < X_{N_3} < \cdots$ can take values between α and β. Dwass **[10]** proved that this count is *exactly* Poisson distributed, but with mean $G(\beta) - G(\alpha) = -\ln\{[1 - F(\beta)]/[1 - F(\alpha)]\}$, depending on the distribution function F.

The fact that exact distributions are obtained more easily than most limit results for record values is a further contrast to results for record value *times.*

The distribution of the r^{th} record value was required in the preceding section's calculus argument concerning waiting time W_r. Let positive integers $w_1, w_2, \ldots, w_{r-1}$ denote fixed waits, and specify fixed trial numbers $n_2 = 1 + w_1$, $n_3 = n_2 + w_2, \ldots$, $n_r = n_{r-1} + w_{r-1}$. Consider a random sample $X_1, \ldots$, $X_{n_2}, \ldots$, $X_{n_3}, \ldots$ such that

$$\begin{aligned} &X_1 = x_1 > \text{next } (w_1 - 1) \text{ observations} \\ &X_{n_2} = x_2 > \text{next } (w_2 - 1) \text{ observations} \\ &\vdots \\ &X_{n_r} = x_r. \end{aligned}$$

Since $X_1, X_2, X_3, \ldots$ are independent and identically distributed with $P\{X_i < x\} = F(x)$, the event above has probability

$$\begin{aligned} &dF(x_1)[F(x_1)]^{(w_1-1)} \\ \times\ &dF(x_2)[F(x_2)]^{(w_2-1)} \\ &\vdots \\ \times\ &dF(x_{r-1})[F(x_{r-1})]^{(w_{r-1}-1)} \\ \times\ &dF(x_r), \end{aligned}$$

where $dF(x) = F'(x)dx$ and the derivative F' is a probability density function. For increasing values $x_1 < x_2 < \cdots < x_r$, the event above is equivalent to

$$X_1 = x_1, \quad W_1 = w_1,$$
$$X_{N_2} = x_2, \quad W_2 = w_2,$$
$$\vdots$$
$$X_{n_{r-1}} = x_{r-1}, \quad W_{r-1} = w_{r-1},$$
$$X_{N_r} = x_r.$$

Hence summation of the foregoing probability element over all possible waiting times gives the joint probability that $X_1 = x_1, X_{N_2} = x_2, \ldots, X_{N_r} = x_r$. This sum can be expressed as a product of geometric series:

$$\sum_{w_1=1}^{\infty}\sum_{w_2=1}^{\infty}\cdots\sum_{w_{r-1}=1}^{\infty} dF(x_1)[F(x_1)]^{(w_1-1)}\cdots dF(x_{r-1})[F(x_{r-1})]^{(w_{r-1}-1)}dF(x_r)$$
$$= dF(x_1)\sum_{w_1=0}^{\infty}[F(x_1)]^{w_1}\cdots dF(x_{r-1})\sum_{w_{r-1}=0}^{\infty}[F(x_{r-1})]^{w_{r-1}}dF(x_r)$$
$$= \frac{dF(x_1)}{1-F(x_1)}\cdots\frac{dF(x_{r-1})}{1-F(x_{r-1})}dF(x_r)$$
$$= dG(x_1)\cdots dG(x_{r-1})\,dF(x_r).$$

Iterated integration with respect to $x_1, x_2, \ldots, x_{r-1}$ (over the region $x_1 < x_2 < \cdots < x_r$) shows that X_{N_r} has probability element

$$dP\{X_{N_r} = x_r\} = \frac{[G(x_r)]^{r-1}}{(r-1)!}dF(x_r).$$

This argument was suggested by Karlin in a textbook exercise ([**21**] pages 267–268; see also [**31**] page 69).

If observations $X_1, X_2, X_3, \ldots$ have *exponential* distribution $F(x) = 1 - e^{-x}$ and $G(x) = -\ln[1 - F(x)] = x$, then the r^{th} record value has *gamma* probability density $x^{r-1}e^{-x}/(r-1)!$, as asserted in the preceding section.

Moreover, for X from any continuous distribution F, it is easy to show that the transformed variable $F(X)$ has uniform distribution on the unit interval ([**21**] page 237) and $G(X)$ has standard exponential distribution. Since G is strictly increasing over its support, the record value trials in random sampling $X_1, X_2, X_3, \ldots$ from F correspond to record value trials in a sample $G(X_1)$, $G(X_2)$, $G(X_3), \ldots$ from the exponential distribution. Thus transformed record value $G(X_{N_r})$ has gamma distribution with r degrees of freedom. For large r the gamma distribution is approximately normal. Thus Resnick [**31**] obtained a "central limit theorem": as $r \to \infty$, the distribution of $(G(X_{N_r}) - r)/\sqrt{r}$ is asymptotically normal with mean $= 0$ and variance $= 1$. The corresponding "strong law of large numbers" asserts that

$$G(X_{N_r})/r \to 1 \quad \textit{with probability one as } r \to \infty.$$

Resnick [**31**] (also see Shorrock [**37**]) gave explicit conditions on the function G which are necessary and sufficient for convergences in probability

$$X_{N_r} - G^{-1}(r) \to 0, \qquad X_{N_r}/G^{-1}(r) \to 1.$$

But also there are conditions under which the last ratio converges to a non-degenerate random variable rather than to a constant. Indeed, Resnick [**31**] characterized the types of limit distributions for record values X_{N_r}; depending on the sampled distribution function F, a record value sequence satisfies exactly one of the following convergences in distribution as $r \to \infty$:

(i) $$P\left\{\frac{X_{N_r}-G^{-1}(r)}{G^{-1}(r+\sqrt{r})-G^{-1}(r)}<x\right\}\rightarrow \mathcal{N}(x)$$

(ii) $$P\left\{\frac{X_{N_r}}{G^{-1}(r)}<x\right\}\rightarrow\begin{cases}0, & x<0\\ \mathcal{N}[\alpha \ln(x)], & x\geqq 0\end{cases}$$

(iii) $$P\left\{\frac{X_{N_r}-\bar{x}}{\bar{x}-G^{-1}(r)}<x\right\}\rightarrow\begin{cases}\mathcal{N}[-\alpha \ln(-x)], & x<0\\ 1, & x\geqq 0\end{cases}$$

where $\mathcal{N}$ denotes the standard normal distribution function, α is a positive constant depending on F, and $\bar{x}$ in (iii) is the upper (necessarily finite) endpoint of the support interval of distribution F.

I omit more precise statements of limit theorems for record value sequences because these results involve complicated conditions on F or G. For details see Resnick [**31**] and also [**9**, **16**, **32**, **34**, **35**, **36**, **37**, **38**, **39**, **40**, **43**, **44**].

12. Extreme values and extremal processes. Every random sample $X_1, X_2, \ldots, X_n$ has a *sample maximum* or *upper extreme value* $M_n = \max(X_1, X_2, \ldots, X_n)$. Clearly $M_1 = X_1$ and subsequently every *new* maximum is a record value, i.e., record breaking corresponds to a jump or strict inequality $M_n < M_{n+1}$ in the sequence of sample maxima $X_1 \leqq M_2 \leqq M_3 \leqq M_4 \leqq \cdots$.

Thus a record value sequence can be extracted from a maximal sequence which is already removed from the random sequence; but the converse path is impossible. So maximal sequences might logically be studied before record value sequences and such was the historical precedence, contrary to my arrangement of topics. Gumbel [**18**] has given the early history and extensive bibliography on statistics of extremes, including distinctive applications: for example, using a river's past flood levels (annual maxima of daily observations) to plan dams, etc., with sensible allowance for worse floods in the future (see [**18**], page 236, for analysis of Mississippi River flooding at Vicksburg).

Historically, studies of sample maxima also have been concomitant to the more general subject of *order statistics* [**8**], viz., randomly sampled data filed or ranked from smallest value to largest value. In the terms of statistical decision theory, ranked data constitute "sufficient statistics" both for "classical" procedures and for most "nonparametric" methods (of which the tolerance limit construction in Section 6 is one example). Unlike the frequency of record breaking, common sample statistics, such as mean or median annual rainfall, can be computed from ordered data as well as from values in their random sampling sequence (in fact, ranking is generally the fastest way to find the median or other sample percentiles).

In 1943 B. V. Gnedenko [**17**] showed that there are precisely three types of limit distributions for sequences of sample maxima. Gnedenko's three types of *extreme value distributions* correspond exactly to the three limit laws for record values published by Resnick [**31**] in 1973 and mentioned briefly in the preceding section. The limit laws for maxima and for record values from continuous distribution F are linked by Resnick in a duality theorem. That is, the different extreme value distributions partition the space of all continuous distribution functions into disjoint "domains of attraction," and record value distributions determine exactly the same partition. See [**32**] for further comparison of record values and maxima.

* * *

A sequence of sample maxima M_n plotted as a function of sample size n can be regarded (see Shorrock [**39**]) as a discrete-time Markov process with jumps at record value times N_r. Common techniques in stochastic process theory can be used to construct an analogous continuous-time

Markov jump process $M(t)$ such that, for any times $0 \leqq t_1 < t_2 < \cdots < t_n$, the variables $M(t_1) \leqq M(t_2) \leqq \cdots \leqq M(t_n)$ have the same joint distribution as sample maxima $M_1 \leqq M_2 \leqq \cdots \leqq M_n$. The study of such continuous-time *extremal processes* seems to have been originated by Dwass [**10, 11**] and Lamperti [**23**] in the mid-1960s.

Dwass used record value theorems to "motivate some of the results" for extremal processes. But, conversely, Shorrock [**38, 39**] and Resnick [**33, 34, 35**] used continuous-time processes to obtain results for record values. In particular, Resnick [**33**] showed that a continuous-time extremal process $M(t)$ with jump times according to a non-homogeneous Poisson process (with intensity t^{-1}) has a "discrete skeleton" $M(n)$, $n = 1, 2, 3, \ldots$, whose jump times behave precisely as record value times (for n sufficiently large).

Thus Resnick [**33**] used the framework of extremal processes "to give a unified explanation of known limit laws" for record frequencies R_n, for record value trial numbers N_r, and for inter-record waiting times W_r.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the thirty-seventh William Lowell Putnam Mathematical Competition, held on December 4, 1976, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, was awarded to the Department of Mathematics of the **California Institute of Technology**, Pasadena, California. The members of its winning team were Christopher L. Henley, Karl W. Heuer, and Albert L. Wells, Jr.; each was awarded a prize of one hundred dollars.

The second prize, four hundred dollars, was awarded to the Department of Mathematics of

Washington University, St. Louis, Missouri. The members of its team were Philip I. Harrington, Tim J. Steger, and Philip N. Strenski; each was awarded a prize of seventy-five dollars.

The third prize, three hundred dollars, was awarded to the Department of Mathematics of **Princeton University,** Princeton, New Jersey The members of its team were Eric S. Lander, Adam N. Rosenberg, and David J. Rusin; each received a prize of fifty dollars.

The fourth and fifth prizes were shared by the Departments of Mathematics of **Case Western Reserve University** and the **Massachusetts Institute of Technology;** each was awarded one hundred and fifty dollars. The members of the C.W.R.U. team were Paul M. Herdeg, Russell D. Lyons, and Daniel L. Stock, and the members of the M.I.T. team were Ernest S. Davis, Miller S. Puckette, and Dean G. Sturtevant; each of these six students was awarded fifty dollars.

The six highest ranking individual contestants, in alphabetical order, were **Philip I. Harrington,** Washington University; **Christopher L. Henley,** California Institute of Technology; **Paul M. Herdeg,** Case Western Reserve University; **Nathaniel S. Kuhn,** Harvard University; **Steven T. Tschantz,** University of California–Berkeley; and **David J. Wright,** Cornell University. Each of these students was designated as a Putnam Fellow by the Mathematical Association of America and awarded a prize of two hundred and fifty dollars by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were *Eric S. Lander,* Princeton University; *Stephen W. Modzelewski,* Harvard University; *Joseph H. Silverman,* Brown University; *Douglas B. Tyler,* California Institute of Technology; and *Paul A. Vojta,* University of Minnesota–Minneapolis. Each of these five students was awarded a prize of one hundred dollars.

The following teams, named in alphabetical order, received honorable mention: *Brown University* with team members David B. Rudofsky, Edward M. Scheinerman, and Joseph H. Silverman; *University of California–Berkeley* with team members Andrew Z. Fire, N. Christopher Phillips, and Steven T. Tschantz; *University of Chicago* with team members Peter L. Dordal, John R. Conlon, and Leslie F. Reid; *Harvard University* with team members Nathaniel S. Kuhn, Stephen W. Modzelewski, and James P. Sethna; and *Michigan State University* with team members Mark E. Carson, Mark P. Merriman, and Ian H. Redmount.

Honorable mention was achieved by the following twenty-nine individuals, named in alphabetical order: *Christopher S. Bretherton,* University of Colorado–Boulder; *F. Michael Christ,* Harvey Mudd College; *John R. Conlon,* University of Chicago; *Ernest S. Davis,* Massachusetts Institute of Technology; *Peter L. Dordal,* University of Chicago; *Andrew Z. Fire,* University of California–Berkeley; *Dale J. Fixsen,* Pacific Lutheran University; *Alan S. Geller,* Princeton University; *David J. Groisser,* Harvard University; *Mark D. Haiman,* Massachusetts Institute of Technology; *Karl W. Heuer,* California Institute of Technology; *Thomas R. Hurd,* Queen's University; *John O. Lamping,* San Diego State University; *H. Turner Laquer,* University of New Mexico; *Silver Lee,* University of California–Berkeley; *Russell D. Lyons,* Case Western Reserve University; *Mark P. Merriman,* Michigan State University; *Yuko Okamoto,* Brown University; *Douglas W. Oman,* Harvard University; *Arthur S. Parker,* University of Kansas; *Miller S. Puckette,* Massachusetts Institute of Technology; *Adam N. Rosenberg,* Princeton University; *David J. Rusin,* Princeton University; *Tim J. Steger,* Washington University; *Philip N. Strenski,* Washington University; *Dean G. Sturtevant,* Massachusetts Institute of Technology; *Spencer W. Thomas,* Oberlin College; *Albert L. Wells, Jr.,* California Institute of Technology; and *Brian C. White,* Yale University.

The other individuals who achieved ranks among the top 102, in alphabetical order of their schools, were: University of Akron, *Kenneth C. Graf*; University of Alberta, *Stefan J. Jungkind, Stephen G. Nash*; Armstrong State College, *John B. Zipperer*; Bethel College, *Andrew A. Rich*; Brigham Young University, *James B. Howes*; Brooklyn College, *Eric R. Jablow*; California Institute of Technology, *Michael P. Chandler, Bruce G. Cortez, James D. Morrow, Charles W. Schlindwein*; University of California–Berkeley, *Gregory S. Lee*; Case Western Reserve University, *Daniel L. Stock*; University of Chicago, *Niels F. Otani*; University of Delaware, *Jordan I. Levy*; Drexel University, *Michael A. Carchidi*; Grinnell College, *Dale R. Worley*; Harvard University, *Richard*

Anders, Mikhail G. Katz, Joseph C. Keller, Mark S. Manasse, Philip E. Moore, Vladislav G Rutenburg, Stephen E. Schneider, Alan S. Stern, Daniel H. Ullman; Haverford College, *William A. Huber*; Massachusetts Institute of Technology, *Daniel V. D'Eramo, Thomas H. Spencer, John A. Stark, Richard E. Stone, David B. Tuckerman, Albert B. Zisook*; New Mexico State University, *Milton G. Hickman*; Ohio State University, *Bruce D. Lucas*; Pennsylvania State University, *J. Eric Brosius*; Princeton University, *Steven Alexander, Constantin P. Bachas, Theodore S. Goodman, Nathaniel S. Hellerstein, Frank T. Leighton*; Reed College, *William C. Beall*; Rice University, *Matthew T. Delevoryas, John W. Myre*; University of Santa Clara, *David J. Kadlecek*; University of Southern California, *Niles D. Ritter*; University of Southern Colorado, *Peter W. Li*; Southern Illinois University, *James N. Bellinger*; Stanford University, *Greg W. Anderson, Nicholas E. Baxter, Philip L. Wadler*; University of Toronto, *Guy E. Moorhouse, Mark A. Reimers*; Washington University, *George T. Gilbert, Richard P. Mattione*; University of Waterloo, *Rajiv Gupta, Randolph R. Morrison*; West Virginia University, *Robert G. Caflisch*; University of Wisconsin-Madison, *Dennis M. Mancl*; Worcester Polytechnic Institute, *Wayne J. Noss*; Yale University, *James E. Boyce*; Yeshiva University, *Moshe Koppel.*

There were 2131 individual contestants from 344 colleges and universities in Canada and the United States in the competition of December 4, 1976. Teams were entered by 264 institutions.

The winners for the 34th, 35th, and 36th competitions of the annual William Lowell Putnam Prize Scholarship for study at Harvard University were Angelos J. Tsirimokos of Princeton University, Thomas G. Goodwillie of Harvard University, and Christopher L. Henley of California Institute of Technology, respectively.

The Questions Committee, consisting of R. T. Bumby, G. D. Chakerian (Chairman), and J. D. E. Konhauser, prepared the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS, PART A

A–1. P is an interior point of the angle whose sides are the rays **OA** and **OB**. Locate X on **OA** and Y on **OB** so that the line segment $\overline{XY}$ contains P and so that the product of distances $(PX)(PY)$ is a minimum.

A–2. Let $P(x,y) = x^2y + xy^2$ and $Q(x,y) = x^2 + xy + y^2$. For $n = 1,2,3,\ldots$, let $F_n(x,y) = (x+y)^n - x^n - y^n$ and $G_n(x,y) = (x+y)^n + x^n + y^n$. One observes that $G_2 = 2Q$, $F_3 = 3P$, $G_4 = 2Q^2$, $F_5 = 5PQ$, $G_6 = 2Q^3 + 3P^2$. Prove that, in fact, for each n either F_n or G_n is expressible as a polynomial in P and Q with integer coefficients.

A–3. Find all integral solutions of the equation

$$|p^r - q^s| = 1,$$

where p and q are prime numbers and r and s are positive integers larger than unity. Prove that there are no other solutions.

A–4. Let r be a root of $P(x) = x^3 + ax^2 + bx - 1 = 0$ and $r+1$ be a root of $y^3 + cy^2 + dy + 1 = 0$, where a, b, c, and d are integers. Also let $P(x)$ be irreducible over the rational numbers. Express another root s of $P(x) = 0$ as a function of r which does not explicitly involve a, b, c, or d.

A–5. In the (x,y)-plane, if R is the set of points inside and on a convex polygon, let $D(x,y)$ be the distance from (x,y) to the nearest point of R. (a) Show that there exist constants a, b, and c, independent of R, such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-D(x,y)}dx\,dy = a + bL + cA,$$

where L is the perimeter of R and A is the area of R. (b) Find the values of a, b, and c.

A–6. Suppose $f(x)$ is a twice continuously differentiable real valued function defined for all real numbers x and satisfying $|f(x)| \leqq 1$ for all x and $(f(0))^2 + (f'(0))^2 = 4$. Prove that there exists a real number x_0 such that $f(x_0) + f''(x_0) = 0$.

PROBLEMS, PART B

B–1. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\left[\frac{2n}{k}\right] - 2\left[\frac{n}{k}\right] \right)$$

and express your answer in the form $\log a - b$, with a and b positive integers.

Here $[x]$ is defined to be the integer such that $[x] \leqq x < [x]+1$ and $\log x$ is the logarithm of x to base e.

B–2. Suppose that G is a group generated by elements A and B, that is, every element of G can be written as a finite "word" $A^{n_1}B^{n_2}A^{n_3}\cdots B^{n_k}$, where $n_1, \ldots, n_k$ are any integers, and $A^0 = B^0 = 1$ as usual. Also, suppose that $A^4 = B^7 = ABA^{-1}B = 1$, $A^2 \neq 1$, and $B \neq 1$.

(a) How many elements of G are of the form C^2 with C in G?

(b) Write each such square as a word in A and B.

B–3. Suppose that we have n events $A_1, \ldots, A_n$, each of which has probability at least $1-a$ of occurring, where $a < 1/4$. Further suppose that A_i and A_j are mutually independent if $|i-j| > 1$, although A_i and A_{i+1} may be dependent. Assume as known that the recurrence $u_{k+1} = u_k - au_{k-1}$, $u_0 = 1$, $u_1 = 1-a$, defines positive real numbers u_k for $k = 0, 1, \ldots$. Show that the probability of all of $A_1, \ldots, A_n$ occurring is at least u_n.

B–4. For a point P on an ellipse, let d be the distance from the center of the ellipse to the line tangent to the ellipse at P. Prove that $(PF_1)(PF_2)d^2$ is constant as P varies on the ellipse, where PF_1 and PF_2 are the distances from P to the foci F_1 and F_2 of the ellipse.

B–5. Evaluate

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^n.$$

B–6. As usual, let $\sigma(N)$ denote the sum of all the (positive integral) divisors of N. (Included among these divisors are 1 and N itself.) For example, if p is a prime, then $\sigma(p) = p+1$. Motivated by the notion of a "perfect" number, a positive integer N is called "quasiperfect" if $\sigma(N) = 2N+1$. Prove that every quasiperfect number is the square of an odd integer.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqq n \geqq 0$ is the number of students among the top 202 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A–1. $(69, 44, 13, 0, 3, 1, 1, 5, 5, 10, 27, 24)$

Let μ be the angle bisector of $\measuredangle AOB$ and λ be the perpendicular to μ through P. Then the intersections of λ with **OA** and **OB** are chosen as X and Y respectively.

This construction makes $OX = OY$ and there is a circle Γ tangent to **OA** at X and to **OB** at Y. Let $\overline{X_1Y_1}$ be any other segment containing P with X_1 on **OA** and Y_1 on **OB**. Let X_2 and Y_2 be the intersections of $\overline{X_1Y_1}$ with Γ. A theorem of Euclidean geometry states that $(PX)(PY) = (PX_2)(PY_2)$. Clearly $(PX_2)(PY_2)$ is less than $(PX_1)(PY_1)$. Hence $(PX)(PY)$ is a minimum.

One can also locate X and Y by saying that $(\pi - \measuredangle AOB)/2$ should be chosen as the measure of $\measuredangle OXP$ or $\measuredangle OYP$.

A–2. $(31, 10, 3, 3, 0, 2, 1, 5, 0, 4, 16, 127)$

One easily verifies that

$$(x+y)^n = (x+y)^{n-2}Q + (x+y)^{n-3}P,$$

$$x^n + y^n = (x^{n-2} + y^{n-2})Q - (x^{n-3} + y^{n-3})P.$$

Subtracting or adding corresponding sides gives

(R) $$F_n = QF_{n-2} + PG_{n-3}, G_n = QG_{n-2} + PF_{n-3}.$$

The desired results now follow by strong mathematical induction using the given results for G_2, F_3, G_4, F_5, and G_6 and (R).

A–3. (5, 3, 1, 1, 1, 3, 14, 7, 65, 16, 19, 67)

We show that the only solutions are given by $3^2 - 2^3 = 1$, i.e., $(p, r, q, s) = (3,2,2,3)$ or $(2,3,3,2)$.

Clearly either p or q is 2. Suppose $q = 2$. Then p is an odd prime with $p^r \pm 1 = 2^s$. If r is odd, $(p^r \pm 1)/(p \pm 1)$ is the odd integer $p^{r-1} \mp p^{r-2} + p^{r-3} \mp p^{r-4} + \cdots + 1$, which is greater than 1 since $r > 1$; this contradicts the fact that 2^s has no such factor.

Now we try r as an even integer $2t$. Then $p^r + 1 = 2^s$ leads to

$$2^s = (p^t)^2 + 1 = (2n+1)^2 + 1 = 4n^2 + 4n + 2,$$

which is impossible since $4|2^s$ for $s > 1$ and $4 \nmid (4n^2 + 4n + 2)$.

Also $r = 2t$ and $p^r - 1 = 2^s$ leads to $(p^t)^2 - 1 = (2n+1)^2 - 1 = 4n^2 + 4n = 4n(n+1) = 2^s$. Since either n or $n+1$ is odd, this is only possible for $n = 1$, $s = 3$, $p = 3$, and $r = 2$.

A–4. (11, 4, 6, 0, 0, 0, 0, 0, 0, 5, 29, 147)

We show that one answer is $s = -1/(r+1)$ and another answer is $s = -(r+1)/r = -1 - (1/r)$. Since $P(x)$ is irreducible, so is $M(x) = P(x-1)$. Hence $M(x)$ is the only monic cubic over the rationals with $r+1$ as a zero, i.e., $M(x) = x^3 + cx^2 + dx + 1$. If the zeros of P are r, s, and t, the zeros of M are $r+1$, $s+1$, and $t+1$. Now the coefficients -1 and 1 of x^0 in P and M, respectively, tell us that $rst = 1$ and $(r+1)(s+1)(t+1) = -1$. Then

$$st = \frac{1}{r},\ s + t = (s+1)(t+1) - st - 1 = -\frac{1}{r+1} - \frac{1}{r} - 1 = -\frac{r^2+3r+1}{r(r+1)}.$$

Hence s is either root of

$$x^2 + \frac{r^2+3r+1}{r(r+1)}x + \frac{1}{r} = 0.$$

Using the quadratic formula, one finds that s is $-1/(r+1)$ or $-(r+1)/r$.

A–5. (81, 3, 5, 3, 3, 2, 0, 0, 0, 3, 17, 85)

It is shown below that $a = 2\pi$, $b = 1$, and $c = 1$. We use $I[S]$ to denote the integral of $e^{-D(x,y)}$ over a region S. Since $D(x,y) = 0$ on R, $I[R] = A$. Now let σ be a side of R, s be the length of σ, and $S(\sigma)$ be the half strip consisting of the points of the plane having a point on σ as the nearest point of R. Changing to (u, v)-coordinates with u measured parallel to σ and v measured perpendicular to σ, one finds that $I[S(\sigma)] = \int_0^s \int_0^\infty e^{-v} dv du = s$. The sum Σ_1 of these integrals for all the sides of R is L.

If v is a vertex of R, the points with v as the nearest point of R lie in the inside $T(v)$ of an angle bounded by the rays from v perpendicular to the edges meeting at v; let $\alpha = \alpha(v)$ be the measure of this angle. Using polar coordinates, one has

$$I[T(v)] = \int_0^\alpha \int_0^\infty re^{-r} dr d\theta = \alpha.$$

The sum Σ_2 of the $I[T(v)]$ for all vertices v of R is 2π. Now the original double integral equals $\Sigma_2 + \Sigma_1 + A = 2\pi + L + A$. Hence $a = 2\pi$ and $b = 1 = c$.

A–6. (7, 4, 1, 1, 1, 0, 0, 0, 2, 9, 48, 129)

Let $G(x) = [f(x)]^2 + [f'(x)]^2$ and $H(x) = f(x) + f'(x)$. Since H is continuous, it suffices to show

that H changes sign. We assume that either $H(x)>0$ for all x or $H(x)<0$ for all x and obtain a contradiction.

Since $|f(0)|\leqq 1$ and $G(0)=4$, either $f'(0)\geqq\sqrt{3}$ or $f'(0)\leqq-\sqrt{3}$. We deal with the case in which $H(x)>0$ for all x and $f'(0)\geqq\sqrt{3}$; the other cases are similar.

Assume that the set S of positive x with $f'(x)<1$ is nonempty and let g be the greatest lower bound of S. Then $f'(0)\geqq\sqrt{3}$ and continuity of $f'(x)$ imply $g>0$. Now $f'(x)\geqq 0$ and $H(x)\geqq 0$ for $0\leqq x\leqq g$ lead to

$$G(g)=4+(\tfrac{1}{2})\int_0^g f'(x)[f(x)+f''(x)]\,dx\geqq 4.$$

Since $|f(g)|\leqq 1$, this implies $f'(g)\geqq\sqrt{3}$. Then continuity of $f'(x)$ tells us that there is an $a>0$ such that $f'(x)\geqq 1$ for $0\leqq x<g+a$. This contradicts the definition of g and hence S is empty. Now $f'(x)\geqq 1$ for all x and this implies that $f(x)$ is unbounded, contradicting $|f(x)|\leqq 1$. This contradiction means that $H(x)$ must change sign and so $H(x_0)=0$ for some real x_0.

Alternately, we use the Mean Value Theorem to deduce the existence of a and b with $-2<a<0<b<2$ and

$$|f'(a)|=\frac{|f(0)-f(-2)|}{2}\leqq\frac{|f(0)|+|f(-2)|}{2}\leqq\frac{1+1}{2}=1$$

and similarly $|f'(b)|\leqq 1$. Then $G(a)=[f(a)]^2+[f'(a)]^2\leqq 1+1=2$ and also $G(b)\leqq 2$. Since $G(0)=4$, $G(x)$ attains its maximum on $a\leqq x\leqq b$ at an interior point x_0 and hence $G'(x_0)=f'(x_0)H(x_0)=0$. But $f'(x_0)\neq 0$ since otherwise $[f(x_0)]^2=G(x_0)\geqq 4$ and $|f(x_0)|>1$. Thus $H(x_0)=0$.

B–1. (23, 2, 2, 1, 0, 0, 1, 5, 9, 28, 14, 117)

It is shown below that $a=4$ and $b=1$. Let $f(x)=[2/x]-2[1/x]$. Then the desired limit L equals $\int_0^1 f(x)dx$. For $n=1,2,\ldots,f(x)=0$ on $2/(2n+1)<x\leqq 1/n$ and $f(x)=1$ on $1/(n+1)<x\leqq 2/(2n+1)$. Hence

$$L=\left(\frac{2}{3}-\frac{2}{4}\right)+\left(\frac{2}{5}-\frac{2}{6}\right)+\cdots=-1+2\left(1-\frac{1}{2}+\frac{1}{3}-\cdots\right)$$

$$=-1+2\int_0^1\frac{dx}{1+x}=-1+2\ln 2=\ln 4-1.$$

B–2. (74, 3, 8, 7, 1, 2, 2, 0, 4, 2, 22, 77)

The answers are (a) 8; (b) $1, A^2, B, B^2, B^3, B^4, B^5, B^6$. Since $B=(B^4)^2$, $B^3=(B^5)^2$, $B^5=(B^6)^2$, the elements in the answer to (b) are all squares in G. They are distinct since B has order 7 and A has order 4. To show that there are no other squares, we first note that $ABA^{-1}B=1$ implies $AB=B^{-1}A$. Then

$$AB^2=(B^{-1}A)B=B^{-1}(AB)=B^{-1}(B^{-1}A)=B^{-2}A.$$

Similarly $AB^n=B^{-n}A$ for the other n's in $\{0,1,\ldots,6\}$ and so for all integers n. With this, one obtains

(P) $$(B^iA^j)(B^hA^k)=B^uA^v \text{ with } u=i+(-1)^jh,\ v=j+k.$$

Thus the set S of elements of the form B^iA^j is closed under multiplication. S is finite since i and j may be restricted to $0\leqq i\leqq 6$ and $0\leqq j\leqq 3$. Hence S is a group and so $S=G$. It then follows from (P) that the squares in G are the B^uA^v with $u=i[1+(-1)^j]$ and $v=2j$. If j is odd, $u=0$ and $v\equiv 2 \pmod 4$. If j is even, $v\equiv 0 \pmod 4$. Thus there are no squares other than those listed above.

B–3. (0, 0, 0, 0, 0, 0, 0, 0, 1, 6, 44, 151)

The statement to be proved is false for $n\geqq 5$ unless the hypothesis is strengthened to state that A_i is independent of the conjunction of $A_1, A_2,\ldots,A_{i-2}$ for $3\leqq i\leqq n$.

The following counterexample with $n=5$ was furnished by Professor David M. Bloom of Brooklyn College. Let $h=33/37$ and $k=1/(64+h)$. Let $P(A_i)$ be the sum of the numbers in the second row of the following table for which A_i appears in the heading:

$A_1A_2A_4A_5$	A_4	$A_1A_3A_4A_5$	$A_1A_2A_3A_4A_5$	$A_1A_2A_3A_4$	$A_2A_3A_4A_5$
$12k$	$3k$	$6k$	$7k$	$12k$	$6k$

$A_1A_3A_5$	$A_1A_2A_3A_5$	$A_2A_3A_5$	$A_3A_4A_5$
$3k$	$9k$	$3k$	$3k$

Then each $P(A_i)$ is $49k$ and $P(A_i \wedge A_j)=37k$ for all i, j with $|i-j|>1$. Since $(49k)^2=37k$, the original independence hypothesis holds. Also, $P(A_i)=1-a$, where $a=(15+h)k<1/4$. However, for any $a \leqq 1/4$, we have $u_5 \geqq 7/64$ and $P(A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge A_5)=7k<7/64$.

B-4. (41, 2, 5, 0, 0, 0, 0, 6, 5, 8, 43, 92)

We let $P=(x,y)$ and the ellipse have the equation $b^2x^2+a^2y^2=a^2b^2$, with $a>b>0$. Then $F_1=(-c,0)$ and $F_2=(c,0)$ with $c^2=a^2-b^2$. Let $r_1=PF_1$ and $r_2=PF_2$. Then $r_1+r_2=2a$ and

$$\begin{aligned} r_1r_2 &= (\tfrac{1}{2})\left[(r_1+r_2)^2-r_1^2-r_2^2\right] \\ &= (\tfrac{1}{2})\left[4a^2-(x+c)^2-y^2-(x-c)^2-y^2\right] \\ &= 2a^2-x^2-y^2-c^2=a^2+b^2-x^2-y^2. \end{aligned}$$

A point (u,v) on the tangent to the ellipse at P satisfies

$$\frac{xu}{a^2}+\frac{yv}{b^2}=1.$$

Putting this in the form $u\cos\theta+v\sin\theta=d$, one finds that

$$d^2=\frac{1}{(x/a^2)^2+(y/b^2)^2}=\frac{a^4b^4}{b^4x^2+a^4y^2}.$$

But $b^4x^2+a^4y^2=b^2(a^2b^2-a^2y^2)+a^2(a^2b^2-b^2x^2)=a^2b^2(a^2+b^2-x^2-y^2)=a^2b^2r_1r_2$. Hence $d^2r_1r_2=a^4b^4r_1r_2/a^2b^2r_1r_2=a^2b^2$, a constant.

B-5. (12, 5, 3, 2, 0, 0, 2, 1, 2, 8, 65, 102)

The sum is $n!$ since it is an nth difference of a monic polynomial, x^n, of degree n.

B-6. (2, 1, 0, 0, 0, 1, 6, 62, 28, 3, 11, 88)

Let $N=2^\alpha p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$ where α and the β_i are nonnegative integers and the p_i are distinct odd primes. Then

$$\sigma(N)=\sigma(2^\alpha)\sigma(p_1^{\beta_1})\cdots\sigma(p_k^{\beta_k}).$$

Since $\sigma(N)=2N+1$ is odd, it follows that $\sigma(p_i^{\beta_i})$ is odd, $1\leqq i\leqq k$. But

$$\sigma(p_i^{\beta_i})=1+p_i+p_i^2+\cdots+p_i^{\beta_i}$$

is odd if and only if β_i is even; for if β_i were odd, the right hand side would be the sum of an even number of odd numbers and hence even. It follows that the odd part of N must be a square, so that we may write

(1) $$N=2^\alpha M^2,\ \alpha\geqq 0.$$

where M is odd. It remains to show that $\alpha=0$.

Since N is quasiperfect, $\sigma(N)=2^{\alpha+1}M^2+1$, while from (1) we deduce $\sigma(N)=\sigma(2^\alpha)\sigma(M^2)=(2^{\alpha+1}-1)\sigma(M^2)$. Hence $2^{\alpha+1}M^2+1=(2^{\alpha+1}-1)\sigma(M^2)$ so that

$$M^2+1\equiv 0\,(\operatorname{mod} 2^{\alpha+1}-1). \tag{2}$$

If $\alpha>0$, $2^{\alpha+1}-1\equiv 3\ (\operatorname{mod} 4)$. Consequently $2^{\alpha+1}-1$ has a prime divisor $p\equiv 3\ (\operatorname{mod} 4)$. Equation (2) implies

$$M^2+1\equiv 0 \quad (\operatorname{mod} p). \tag{3}$$

But since -1 is a quadratic non-residue modulo p whenever $p\equiv 3\ (\operatorname{mod} 4)$, (3) is impossible. Thus, $\alpha=0$.

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PROGRESS REPORTS

EDITED BY P. R. HALMOS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

FOURIER SERIES

P. R. HALMOS

It is a historical misfortune (which was responsible for almost 200 years of barking up the wrong tree) that Fourier series were discovered before convergence. Fourier series are a vital part of much of both classical and modern analysis; they are important for both abstract theory and concrete

applications. They arise in topological groups and in operator theory; they have their origins in problems about vibrating strings and about heat conduction.

In their most classical manifestation, Fourier series have to do with numerical-valued functions (it is best to let them be complex-valued) on the line $(-\infty, +\infty)$ that are periodic of period 2π and integrable on the interval $[0, 2\pi]$. The Fourier series of such a function f is the series

$$\sum_{n=-\infty}^{+\infty} \alpha_n e^{inx},$$

where

$$\alpha_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx.$$

(Since $e^{inx} = \cos nx + i \sin nx$, there is an alternative way of writing Fourier series; it uses sines and cosines and has the index running in one direction only. This alternative, real, version is geometrically more intuitive, but the complex exponential version given above is algebraically simpler to manipulate.)

Trigonometric polynomials (in either real or complex form) are familiar objects and are computationally accessible; surely nothing but good could come out of representing more difficult functions as limits of such polynomials. It seemed natural, therefore, to hope that the "sum" of the Fourier series associated with a function f would be "equal" to f, and, in any event, to ask for which functions that would occur. The answer, it was hoped, was that good functions have good series, and the history of the subject has been strongly influenced by that hope.

When limits began to be understood, "sum" and "equal" were interpreted in the sense of pointwise convergence; the more fruitful and usable concepts of weak convergence and convergence with respect to a norm came along only after the mathematical community was irretrievably committed to research in the pointwise direction.

How good does a good function have to be? Differentiability is good enough, but, it turns out, continuity is not; there are continuous functions whose Fourier series diverges at a point, and, in fact, at many points. If convergence is replaced by summability in the sense of Cesàro averages, then Fejér's theorem saves the day; in *that* sense the Fourier series of every continuous function f converges to f at every point. Assertions such as these are considered relatively easy nowadays; they occur in most textbooks on the subject.

How bad can an integrable function be? Answer: very bad. Kolmogorov showed that if all that is assumed is that $f \in L^1[0, 2\pi]$ (i.e., that f is integrable on $[0, 2\pi]$), then it could happen that the Fourier series of f diverges almost everywhere (1923), or even everywhere (1926).

The biggest question along these lines was asked by Lusin, and it remained unanswered for 50 years: if $f \in L^2[0, 2\pi]$ (note the exponent 2 in place of 1), does it follow that the Fourier series of f converges to f almost everywhere? Repeated failure to prove the affirmative answer led to the official state religion among the cognoscenti in the 1950's and 1960's: the answer must be no.

The answer is yes. The first proof is due to Carleson (1966). A remarkable feature of Carleson's achievement is that it uses no unknown techniques; it just uses the known ones better. It depends on an ingenious push-me-pull-you way of selecting subintervals. It is as if Carleson had power enough to replace everyone else's ε by ε^2, and that did the trick.

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MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

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AN INCREASING CONTINUOUS SINGULAR FUNCTION

LAJOS TAKÁCS

In this note we are concerned with a nondecreasing function $F(x)$ defined in the interval $[0,1]$ such that $F(0)=0$, and if

$$x=\sum_{r=0}^{\infty} 2^{-a_r} \tag{1}$$

where $a_0<a_1<\cdots<a_r<\cdots$ are positive integers, then

$$F(x)=\sum_{r=0}^{\infty} \rho^r(1+\rho)^{-a_r} \tag{2}$$

where ρ is a positive real number. Clearly, $F(1)=1$.

We shall prove the following theorem for $F(x)$.

THEOREM. *If $\rho\neq 1$, then the function $F(x)$ is strictly increasing, continuous, and its derivative is zero almost everywhere in the interval* $[0,1]$.

Proof. Let $0<\rho<\infty$. First, we prove that $F(x)$ is strictly increasing in the interval $[0,1]$. If $0<x\leqq 1$, then by (2) we have $F(0)<F(x)$. If $0<x<y\leqq 1$, where x is given by (1) and

$$y=\sum_{r=0}^{\infty} 2^{-b_r} \tag{3}$$

where $b_0<b_1<\cdots<b_r<\cdots$ are positive integers, then there is a smallest integer s such that $a_s\neq b_s$. Since $b_s<a_s$, the inequalities

$$\sum_{r=s}^{\infty} \rho^r(1+\rho)^{-a_r}\leqq \rho^s(1+\rho)^{-a_s+1}\leqq \rho^s(1+\rho)^{-b_s}<\sum_{r=s}^{\infty} \rho^r(1+\rho)^{-b_r} \tag{4}$$

imply that $F(x)<F(y)$.

If $x\in(0,1]$ is given by (1), then we define

$$x_n=\sum_{a_r\leqq n} 2^{-a_r} \quad\text{and}\quad y_n=x_n+2^{-n} \tag{5}$$

for $n\geqq 0$. Obviously $x_n<x\leqq y_n$ and

$$F(y_n)-F(x_n)=\rho^{k_n}(1+\rho)^{-n}, \tag{6}$$

where k_n is the number of subscripts $r=0,1,\ldots,n$ for which $a_r\leqq n$.

In (6) $0\leqq k_n\leqq n+1$, and therefore $\lim_{n\to\infty}[F(y_n)-F(x_n)]=0$. This implies that $F(x)$ is continuous in the interval $(0,1]$. Obviously, $\lim_{x\to 0}F(x)=F(0)$.

If $\rho=1$, then $F(x)=x$ for $0\leqq x\leqq 1$. Now we shall prove that if $F(x)$ has a finite nonzero derivative for some $x\in(0,1]$, then necessarily $\rho=1$.

If $x\in(0,1]$ and if $F'(x)$ exists, then by (5) and (6) we obtain that

$$F'(x) = \lim_{n\to\infty} \frac{F(y_n) - F(x_n)}{y_n - x_n} = \lim_{n\to\infty} \left(\frac{2}{1+\rho}\right)^n \rho^{k_n}. \tag{7}$$

If $F'(x)$ is finite and nonzero, then by (7) it follows that

$$\lim_{n\to\infty} \rho^{k_n - k_{n-1}} = (1+\rho)/2. \tag{8}$$

Thus either $\rho = 1$, or $\lim_{n\to\infty}(k_n - k_{n-1}) = k$. In the latter case $\lim_{n\to\infty} k_n/n = k$ and $0 \leqq k_n \leqq n+1$ imply that $k = 0$ or $k = 1$. If $k = 0$ or $k = 1$, then from $\rho^k = (1+\rho)/2$ we can conclude that $\rho = 1$ again.

Accordingly, if $\rho \neq 1$, then $F'(x) = 0$ everywhere where $F(x)$ has a finite derivative. Since by a theorem of H. Lebesgue [**15** p. 128] every nondecreasing function has a finite derivative at every point with the possible exception of the points of a set of measure zero, therefore, if $\rho \neq 1$, then $F'(x) = 0$ almost everywhere in the interval $[0, 1]$. This completes the proof of the theorem.

For the proof of Lebesgue's theorem we refer to H. Lebesgue [**16** pp. 185–188], G. Faber [**9**], W. H. Young and G. Ch. Young [**31**], F. Riesz [**19**], [**20**] and F. Riesz and B. Sz.-Nagy [**21** pp. 5–9], [**22** pp. 5–9].

The results of this paper follow from the results of R. Salem [**25**] and F. Riesz and B. Sz.-Nagy [**21** pp. 48–49]. (See also B. Sz.-Nagy [**27** pp. 198–200] and E. Hewitt and K. Stromberg [**12** pp. 278–282].) They defined an increasing continuous singular function as the limit of a sequence of continuous functions. If we choose $t = (1-\rho)/(1+\rho)$, where $0 < \rho < 1$, in the example of F. Riesz and B. Sz.-Nagy [**21**], then we can demonstrate that the limit function has the explicit form (2). R. Salem [**25**] and E. Hewitt and K. Stromberg [**12**] considered more general functions which can also be reduced to (2) by choosing the parameters $r_k = t_k = (1-\rho)/(1+\rho)$ where $0 < \rho < 1$.

In conclusion we remark that there are several examples for nondecreasing continuous singular functions, but it seems (2) is the simplest one. The best known examples were given in 1904 by H. Lebesgue [**15**], [**16** p. 56] and H. Minkowski [**17**]. Concerning Lebesgue's function we refer to E. Hille and J. D. Tamarkin [**13**], E. Hewitt and K. Stromberg [**12** p. 113], and R. Salem [**24**]. Concerning Minkowski's function we refer to G. Faber [**9** pp. 395–400], and A. Denjoy [**6**], [**7**], [**8**]. See also T. Brodén [**2**], E. Cesàro [**3**], W. Sierpiński [**26**], H. Rademacher [**18**], H. Bohr [**1** pp. 112–117], B. Jessen and A. Wintner [**14** pp. 61–62], E. R. Van Kampen and A. Wintner [**28**], Ph. Hartman and R. Kershner [**11**], N. Wiener and A. Wintner [**29**], [**30**], S. Saks [**23** pp. 100–101], G. De Rham [**4**], [**5**] and G. Freilich [**10**].

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

DO MAXIMAL LINE-GENERATED TRIANGULATIONS OF THE PLANE EXIST?

BRANKO GRUNBAUM AND G. C. SHEPHARD

Let G be any finite set of great circles on a two-dimensional sphere S^2, such that $S^2 \backslash G$ is the union of (a finite number of) disjoint open spherical triangles. Then we shall say that G gives rise to a *circle-generated triangulation* of S^2. Many such triangulations are known. For example, we can take

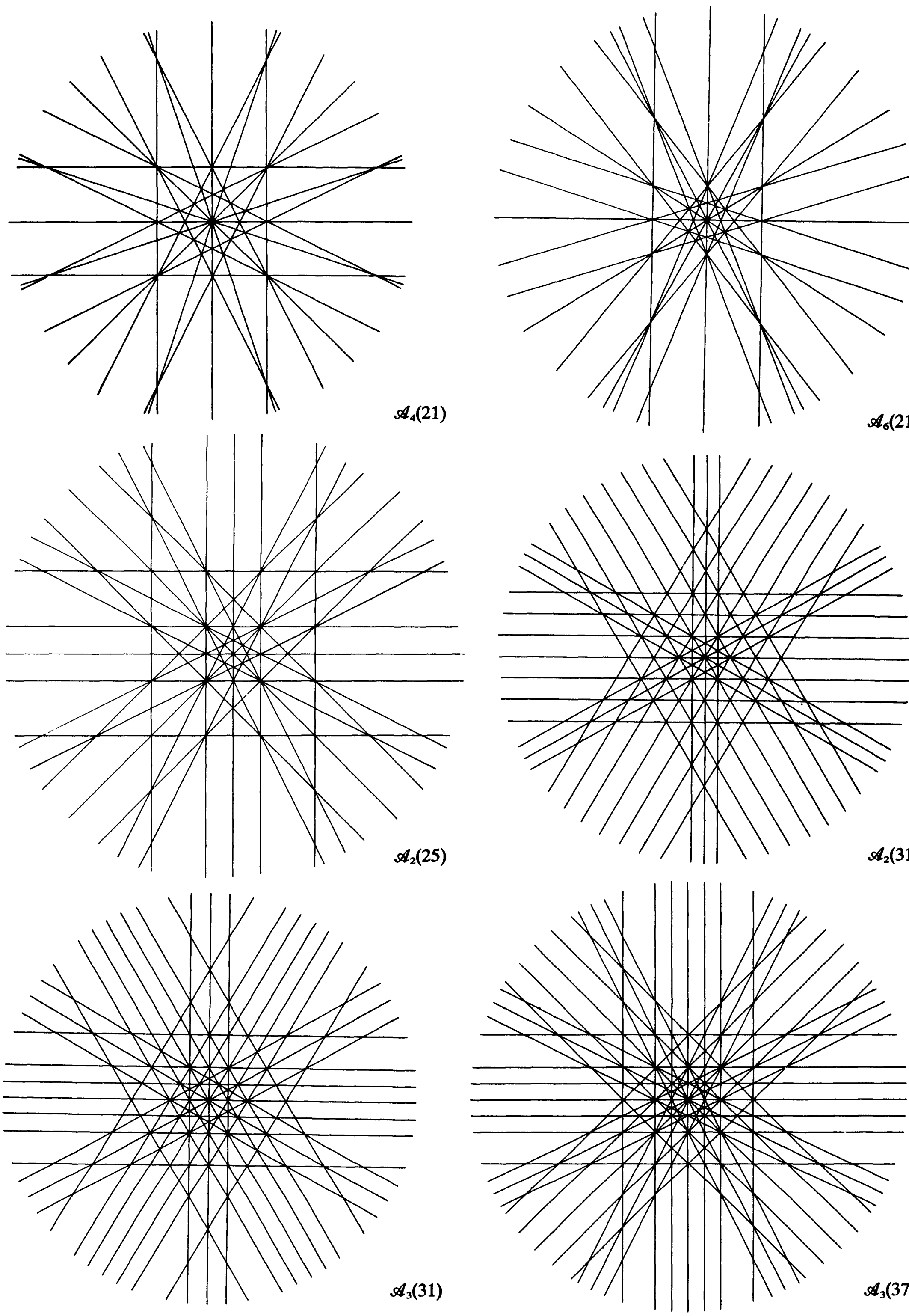


FIG. 1 Six line-generated triangulations of the real projective plane, conjectured to be maximal. The line at infinity is included in each so that there are 21, 21, 25, 31, 31, and 37 lines. The notation is taken from the catalogue [1] of simplicial arrangements. The other four such triangulations that are conjectured to be maximal are there denoted by $\mathcal{A}_7(21)$, $\mathcal{A}_5(25)$, $\mathcal{A}_1(31)$ and $\mathcal{A}_2(37)$.

for G the equator and any finite set of lines of longitude on the surface of the earth, or the set of great circles that result from central projection onto a concentric sphere of the edges of a cuboctahedron or icosidodecahedron.

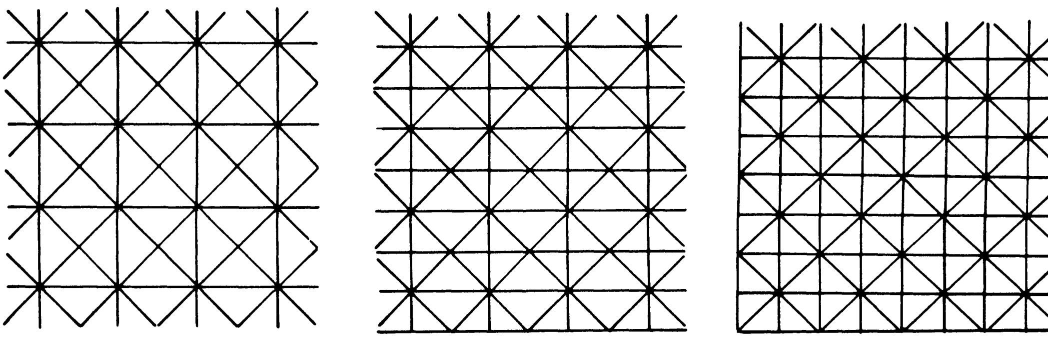
The circle-generated triangulations of the sphere are clearly in one-to-one correspondence with the *line-generated triangulations* of the real projective plane, that is, with triangulations $\mathcal{T}$ in which each edge of $\mathcal{T}$ belongs to a straight line composed entirely of edges of $\mathcal{T}$. Because of easier representability we shall henceforth discuss the latter, and for simplicity of expression we shall often say "triangulation" instead of "line-generated triangulation." Examples of such triangulations appear in Figure 1.

Two triangulations are said to be *equivalent* if they are combinatorially isomorphic, and a triangulation $\mathcal{T}_1$ is called a *refinement* of a triangulation $\mathcal{T}_2$ if $\mathcal{T}_1$ is equivalent to a subdivision of $\mathcal{T}_2$. In other words, triangulations equivalent to all refinements of $\mathcal{T}_2$ can be obtained by drawing additional lines in a diagram representing $\mathcal{T}_2$. Clearly the relation of being a refinement is a partial ordering on the set of all line-generated triangulations of the real projective plane.

In **[1]** and **[2]** it was conjectured that, apart from three well defined infinite families of very special triangulations, there exist only a finite number of line-generated triangulations of the projective plane. If that conjecture were established, it could be shown that there exist *maximal* triangulations, that is to say, triangulations for which no further refinement is possible.

CONJECTURE 1. *The ten line-generated triangulations of the projective plane listed in Figure* 1 *are maximal.*

CONJECTURE 2. *Any maximal line-generated triangulation of the projective plane is equivalent to one of the* 10 *listed in Figure* 1.



(a) (b) (c)

FIG. 2. Line-generated triangulations of the Euclidean plane. The triangulation in (b) is a refinement of (a), that in (c) is a refinement of (b) (and of (a)), and is equivalent to (a).

If we extend these ideas to Euclidean or affine planes, the situation becomes entirely different. We restrict attention to locally finite triangulations; examples are given in Figures 2, 3 and 4. It is easy to see that every such triangulation necessarily contains an infinite number of lines, and that an uncountable infinity of such triangulations exist. The relation of being a refinement is no longer a partial ordering. This is illustrated by the triangulations of Figures 2(a) and 2(b) which are not equivalent, yet each is equivalent to a refinement of the other. Also, a triangulation may be equivalent to a proper refinement of itself, as shown by the examples in Figures 2(a) and 2(c). Consequently, there is no reason why any maximal triangulations should exist. In fact, we propose

CONJECTURE 3. *There are no maximal line-generated triangulations of the Euclidean plane.*

A proof of this conjecture would probably be facilitated if one could establish

CONJECTURE 4. *Every line-generated triangulation of the Euclidean plane has a proper refinement which is a periodic line-generated triangulation.*

This conjecture is non-trivial even if the starting triangulation is itself periodic. It appears that in this case the following stronger statement is true:

CONJECTURE 5. *Every periodic line-generated triangulation of the Euclidean plane is equivalent to a proper refinement of itself.*

As illustrations of this situation we mention that for the triangulation in Figure 3 a contraction with center at the marked vertex in the ratio 2:1 produces a proper subdivision of the original triangulation. Analogously, in the case shown in Figure 4, except that here a contraction in ratio 3:1 or 5:1 is needed.

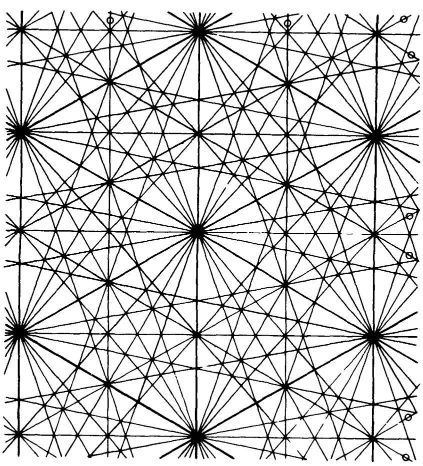


FIG. 3. A line-generated periodic triangulation of the Euclidean plane. Contraction in ratio 2:1 towards the vertex marked * yields a proper refinement of the triangulation. The marked vertex belongs to 12 lines of the triangulation. The deletion of any of the lines marked by a small circle leads to a triangulation.

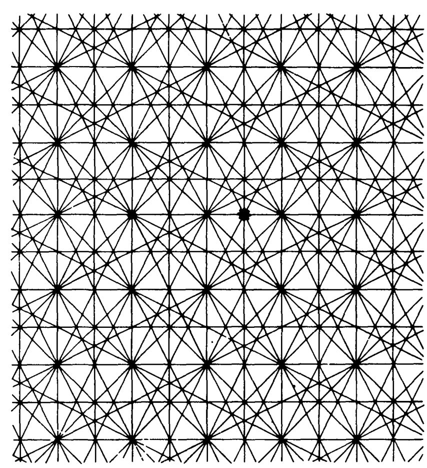


FIG. 4. A line-generated periodic triangulation of the Euclidean plane. Contraction in ratio 3:1 towards the vertex marked * as well as in ratio 5:1 towards the vertex marked *, yields a proper refinement.

In the periodic triangulation $\mathcal{T}$ of Figure 3 the removal of any among the lines marked by small circles leads to a triangulation that is equivalent to a proper refinement of $\mathcal{T}$; hence $\mathcal{T}$ has uncountably many inequivalent refinements. Possibly every triangulation has uncountably many refinements, although this seems unlikely; even for the triangulation of Figure 4 we were unable to settle this question.

Although the triangulations appear to be intricate in many respects, they seem all rather simple in the following sense:

CONJECTURE 6. *Each vertex of a line-generated triangulation of the Euclidean plane belongs to at most* 12 *lines.*

The example in Figure 3 shows that 12 cannot be replaced by any smaller number.

There are many possible variations of these ideas and conjectures. We mention *pseudoline-generated triangulations* of the real projective plane; these coincide with the simplicial arrangements

of pseudolines discussed in [1] and [2]. The corresponding notions in the Euclidean plane also deserve investigation. Then there are analogous problems about *plane-generated triangulations* of Euclidean or projective space of 3 or more dimensions. Another possibility is to consider triangulations on 2-manifolds, generated by simple closed curves any two of which are either disjoint or cross each other precisely twice. Beyond the fact that non-trivial triangulations exist in each case, nothing appears to be known on these topics.

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CLASSROOM NOTES

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AN IMPORTANT FUNCTOR IN ANALYSIS AND TOPOLOGY

DONALD HARTIG

1. The language and viewpoint of categories have become firmly implanted in the mathematics curriculum at the graduate level, and following the normal trend, have now begun to filter into the undergraduate program. First contact with the unifying and generalizing concepts of this theory is usually made in a course in algebra where the relevant mappings become **morphisms** and the student is expected to develop a facility with and (we hope) an appreciation of the new art of commutative diagrams, universal elements and "natural mappings." Surely at this level the most important aspect of Category Theory is its universal applicability and its power to highlight the analogies that exist among structures that arise in so many contexts.

Modern Functional Analysis has evolved from a shift from the study of sets of points to the study of sets of functions (which are mappings between sets of points). Category Theory represents a similar step from the study of sets of functions to a study of mappings between sets of functions. Such mappings are called **functors** and are the morphisms of Category Theory. In what follows we shall investigate some properties of one of the most important functors in analysis.

2. If A is a closed subset of a compact (includes Hausdorff) space X, then we call the ordered pair (X, A) a **compact pair**. The compact pair $(X,\varnothing)$ is referred to simply as X. A morphism $\alpha: (X,A)\rightarrow(Y,B)$ is a continuous function taking X into Y so that $\alpha(A)\subset B$. Such mappings constitute the category of compact pairs, denoted by *CompPr* for short. Contained in this category is the family of continuous maps $\alpha: X\rightarrow Y$ (X and Y compact) which we denote by *Comp.*

Given a compact pair (X,A) we let $C(X,A)$ denote the set of all continuous scalar-valued

functions on X that vanish on A; by $C(X)$ we denote $C(X,\varnothing)$. The scalars will be either the real or complex numbers. Given f in $C(X,A)$ its *norm* is the number $\|f\|=\sup\{|f(x)|: x\in X\}$. Pointwise addition, multiplication and scalar multiplication then make $C(X,A)$ a Banach space, Banach algebra or a number of other objects depending upon the scalar field and how much of the structure one wishes to analyze. For concreteness we regard $C(X,A)$ as a Banach algebra. The relevant morphisms between Banach algebras are the continuous algebraic homomorphisms. (We ask that the morphism preserve the multiplicative identity if there is one present.) The family of all such morphisms is called the category of Banach Algebras, *BanAlg*.

The functor referred to at the end of Section 1 is the mapping that assigns to each morphism $\alpha: (X,A)\to(Y,B)$ in the category of compact pairs the morphism $\alpha^*: C(Y,B)\to C(X,A)$ in the category of Banach algebras defined by $\alpha^*(f)=f\circ\alpha$ for each f in $C(Y,B)$. It is readily verified that α^* qualifies for membership in *BanAlg* since it preserves all algebraic structure and is continuous (it's even norm decreasing). To be perfectly clear, a functor is simply a mapping from the morphisms of one category to the morphisms of another category. To be of interest we must demand that this mapping preserve the ambient structure. For morphisms this structure consists of *composition* and the special role played by the *identity mappings*. Thus to qualify as a functor we ask that

1. The image of a composition of two morphisms is the composition of their images, and
2. The image of an identity morphism is again an identity morphism.

For our situation this translates into the requirement that, for α and β in *CompPr*, $(\alpha\circ\beta)^*=\beta^*\circ\alpha^*$ and if $\iota: (X,A)\to(X,A)$ is the identity map then so also is $\iota^*: C(X,A)\to C(X,A)$.

The functor in question would reverse the mapping arrows and α and β are reversed in composition upon passage to morphisms in *BanAlg*. For this reason our functor is called **contravariant**. Most functors arising in analysis are in fact contravariant. Observe that the simple ploy of writing all mappings on the right, i.e., $(x)f$ instead of $f(x)$, will have no effect on the reversal of α and β. Contravariance is more than skin-deep.

3. In this section we shall show how the Tietze Extension Theorem of general topology translates into a statement about the functor described above. Recall that the extension theorem states that if (X,A) is a compact pair, then any continuous scalar-valued function on A has a continuous extension to all of X. (The theorem actually characterizes a more general topological space called a **normal** space.) You will notice immediately that if $\iota: A\to X$ denotes the natural embedding of A into X, the extension theorem simply states that ι^* is onto ($\iota^*(f)=f\circ\iota=f\,|\,A\ldots f\,|\,A$ denotes the restriction of f to A). Since the spaces we are dealing with are compact, a one-to-one map will be a homeomorphic embedding and the following result could be called a categorial version of the Tietze Extension Theorem:

THEOREM 1. *Let $\alpha: X\to Y$ be a morphism in Comp; then α is one-to-one if and only if α^* is onto.*

We leave it to you to verify that if α is not one-to-one then α^* is not onto.

A question arises concerning the truth of Theorem 1 in the category of compact pairs. That is, given $\alpha: (X,A)\to(Y,B)$, will α^* be onto when α is one-to-one, or vice versa? A moment's reflection will reveal that the answer is "no". If α would take a point of $X\backslash A$ into B then α^* cannot be an onto map even if it is one-to-one. However, α^* will be onto provided α takes no point of $X\backslash A$ into B and α is one-to-one on $X\backslash A$.

4. This section is devoted to a proof of the above statement.

THEOREM 2. *Let $\alpha: (X,A)\to(Y,B)$ be a morphism in CompPr. The following are equivalent:*

(a) *α^* is onto.*

(b) *$\alpha(X\backslash A)\subset Y\backslash B$ and $\alpha\,|\,X\backslash A$ is one-to-one.*

(c) *α embeds $X\backslash A$ homeomorphically into $Y\backslash B$.*

The proof will follow from three lemmas. The reader is asked to supply the proofs of Lemmas 1 and 2.

LEMMA 1. *Let E and F be topological spaces (not necessarily compact) and let $f: E \to F$ be continuous and closed, i.e., f takes closed sets in E to closed sets in F. If $B \subset F$ and $\varnothing \neq A = f^{-1}(B)$ then $f|A: A \to B$ is also closed.*

LEMMA 2. *Let X, A and B be closed sets in the compact space Y with $A = X \cap B$. If $\iota: (X, A) \to (Y, B)$ is the natural embedding then ι^* is onto.*

LEMMA 3. *Let $\alpha: (X, A) \to (Y, B)$ be a morphism in CompPr with α onto. If $\alpha(X \backslash A) \subset (Y \backslash B)$ and α is one-to-one on $X \backslash A$ then α^* is onto.*

Proof. Let $f \in C(X, A)$; we wish to find a function $g \in C(Y, B)$ such that $\alpha^*(g) = f$. The hypotheses of the lemma imply that α establishes a one-to-one correspondence between $X \backslash A$ and $Y \backslash B$. We may define a scalar-valued function g on Y as follows

$$g(y) = \begin{cases} f(x) & \text{if} \quad y \in Y \backslash B \quad \text{and} \quad \alpha(x) = y, \\ 0 & \text{if} \quad y \in B. \end{cases}$$

Because α is a quotient map (continuous, closed, onto) the function g is continuous if and only if $g \circ \alpha$ is continuous. But $g \circ \alpha = f$ so $g \in C(Y, B)$ and $\alpha^*(g) = f$.

We now give a proof of Theorem 2.

(a) $\Rightarrow$ (b). For either case of failure of (b) one can easily construct a function in $C(X, A)$ not of the form $g \circ \alpha$ for any $g \in C(Y, B)$.

(b) $\Rightarrow$ (c). The hypotheses of (b) imply that $X \backslash A = \alpha^{-1} \circ \alpha(X \backslash A)$. Assuming $X \backslash A \neq \varnothing$ (the only case of interest here) we may apply Lemma 1 to assert that $\alpha | X \backslash A: X \backslash A \to \alpha(X \backslash A)$ is closed.

(c) $\Rightarrow$ (b). This is clear.

(b) $\Rightarrow$ (a). Assume (b) holds. Let $\alpha_1: (X, A) \to (\alpha(X), \alpha(A))$ be given by $\alpha_1(x) = \alpha(x)$ for each x in X and let $\iota: (\alpha(X), \alpha(A)) \to (Y, B)$ be the natural embedding. Then $\alpha = \iota \circ \alpha_1$ so $\alpha^* = \alpha_1^* \circ i^*$. Lemma 2 applies to show that ι^* is onto while Lemma 3 implies that α_1^* is onto. Thus α^*, being the composition of two onto maps, is also onto.

5. The reader will have no difficulty proving the following result which is similar to Theorem 1.

THEOREM 3. *Let $\alpha: X \to Y$ be a morphism in Comp. Then α is onto if and only if α^* is one-to-one.*

We also leave it to the reader to formulate and prove the analogous theorem regarding a morphism in *CompPr*.

Anyone interested in more information about this and other functors of analysis, as well as a succinct introduction to categories in general, would do well to consult the monograph **[4]** by Semadeni. The functor we have discussed here is given a more careful analysis in **[1]** where certain of its natural properties are shown to characterize it. The two papers **[2]** and **[3]** by Semadeni are also recommended.

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MATHEMATICAL EDUCATION

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A RATIONAL APPROACH TO INSTRUCTIONAL GROUPING

P. M. EASTMAN AND C. H. DIETZ

The interest of college mathematics instructors in individualizing instruction is indicated by the number of articles on this topic which have appeared in this MONTHLY during the past several years. It is assumed that individualizing instruction, i.e., tailoring instruction to each student as much as possible, will improve the overall quality of instruction. This article will focus on two common ways of attempting to individualize mathematics instruction with emphasis on one of them, the study of aptitude-treatment interactions. We will suggest that a combination of the aptitude-treatment interaction (ATI) approach and approaches emphasizing organizational formats can help the teacher of college and university mathematics decide how to individualize his instruction.

One approach to individualization emphasizes the organization of courses or programs. This approach assumes that instruction which meets the requirements of the individual might be achieved by allowing the student to choose different kinds of instruction or by varying instruction enough so that a student has a good chance of finding a form of instruction under which he can best learn. Proctorial systems of instruction and/or self-pacing programs are examples of this approach; see Maltbie *et al.* (1974) and Young *et al.* (1974). The latter contains an excellent description of the "Keller Plan" system of personalized instruction.

Another approach to individualization, less familiar to most college instructors, focuses on the characteristics of each student, called aptitudes, and attempts to match instruction to those aptitudes. Studies taking this approach are often termed aptitude-treatment interaction (ATI) studies since they seek interactions of students' aptitudes with the instruction (treatment) they receive. By looking at how achievement under a particular form of instruction is related to a certain aptitude or set of aptitudes, and comparing this relation to another under a different type of instruction, it is hoped that eventually we shall be able to match types of instruction to the aptitudes of each student.

Any characteristics which affect the way students learn could be relevant aptitudes. These include general and specific cognitive abilities such as intelligence and mathematical, verbal, or visualization abilities, cognitive style variables that measure how students approach learning situations, and affective characteristics involving a student's attitudes and moods. Previous instruction and experience also may be useful predictors of student achievement and thus be important aptitudes for consideration in ATI studies.

The ideas behind the ATI approach to individualization are neither new nor unfamiliar to teachers of mathematics. Henri Poincaré (1958) in *The Value of Science* discusses two types of mathematical minds. "... two sorts of minds are equally necessary for the progress of science; both the logicians and the intuitionalists have achieved great things that others could not have done." (p. 17). Jacques Hadamard (1945) in *The Psychology of Invention in the Mathematical Field* pursued Poincaré's observations in greater detail, providing examples from the works and letters of numerous mathematicians. Most teachers of mathematics have encountered students for whom one type of explanation seldom proved successful but another often did. Perhaps geometric arguments were more successful with one group of students, whereas algebraic arguments were more successful with others.

That these differences apparently exist, both among great mathematicians and among students, provides evidence for the existence of different patterns of thinking on which one might base differences in instruction.

Jerry P. Becker (1970) suggests that we take advantage of these observed differences in designing instruction.

> ... I do say there is another important question to be asked, one that is more pertinent, namely, "For which students is a particular method of instruction most effective?" I would argue that we should direct some of our efforts toward the question of how students with different ability patterns respond to different instructional techniques. Further, we may expect that an effective instructional technique for one type of objective (e.g., knowledge of formulas) might be inappropriate for another objective (e.g., inventing new formulas), and we may find that the significant effects of teaching derive from interaction among methods of teaching, aptitude variables, and goal of instruction.

According to Cronbach and Snow (1969), the major guide in ATI research, "The task of research is to formulate more precisely the ways in which instruction can be varied so as to fit pupil characteristics." (Cronbach and Snow, p. 1) It is this direction that ATI research takes.

The ATI approach has two kinds of support. The first is the general intuitive support offered by many teachers of mathematics, and the second is empirical support offered by mathematics educators and by medical research into the processing functions which take place in the human brain.

The general intuitive support offered by mathematics teachers takes many different forms. Probably one of the most common is represented by observations such as the following. Particular students may do well learning the definition of a limit in calculus when the concept is presented in a visual way, but may have difficulty with a purely symbolic presentation. When college teachers notice this they seldom follow up their observations with any investigation, because they often have several classes to teach and research to do and just do not have time to investigate further. A plausible explanation for their observations is offered by an ATI hypothesis. It is very possible that the difference in achievement of students in learning the concept of a limit is caused by the differences in how this topic is presented in the calculus textbooks. The definition is, by its nature, very symbolic and consequently textbooks seldom provide graphic material or visual representations of the concept. Hence any student who has a low aptitude for symbolic reasoning may find the concept of a limit difficult to understand. On the other hand, this same student may have a high aptitude for spatial visualization and may find it very easy to learn the concept of a limit if it is presented with a diagram or a pictorial representation so that he can apply his ability for spatial visualization.

The second kind of support for the ATI approach is provided by empirical studies conducted by mathematics educators and in results of medical research on the human brain.

Behr (1970) in an experiment involving 228 college students found several interactions between different aptitudes when students were learning modulus seven arithmetic. Some of the aptitudes involved were spatial visualization, verbal abilities and memory. Eastman and Carry (1975) conducted an experiment involving 80 high school students using quadratic inequalities as content material. By using aptitude measures for spatial visualization and general reasoning ability, he found a significant interaction. Salhab (1973) also described an interaction between the aptitudes of spatial visualization and general reasoning ability and the study of linear absolute value equations. In each of these cases, the aptitude measure which was most congruent to the method of instruction was the better predictor of success. Further experimentation in the area will continue. As more evidence accumulates to support an ATI approach to the problem, we should be prepared to adjust our teaching to fit individual differences in student characteristics, by teaching particular topics in more than one way.

Another important line of research which supports the ATI approach to these problems is research on the human brain. Wittrock (1974) has summarized some results of Bogen (1968) which are of particular importance here. These results indicate that there is a differentiation of functions between the right and left hemispheres of the human brain.

Figure 1 (based on Wittrock, 1974, p. 192) summarizes some of the relevant functions of the brain.

Left Hemisphere	Right Hemisphere
propositional thought	appositional thought
language	spatial relations
verbal	perceptual
symbolic	visiospatial
temporal processing	part-whole processing
linear	(Gestalt perception)
logical or analytic	analogical or relational
	visual imagery
	nonlinear

FIG. 1. Functions that seem to be lateralized in the brain

It is clear from the figure that thought concerning spatial relations and visualization problems occurs in the hemisphere of the brain which does not include propositional thought and symbolic reasoning. This research may offer some insight into how spatial and verbal processes are involved in learning mathematics and lends further support to the empirical studies conducted by Behr (1970), Eastman and Carry (1975), and Salhab (1973). One possible implication of the research on the brain indicates that we might want to offer more than one mode of instruction to students, measure aptitudes which might determine in which mode they are most successful, and then use these findings in assigning students to instruction in regular classrooms.

Have studies dealing either with organizational schema or the search for ATI's been of any use in solving the problem of how best to individualize instruction? The answer is somewhat discouraging. Each of these approaches has had its weaknesses. Organizational studies have not directly considered the characteristics of learners. They have not been based upon theory, but instead have frequently been based on local needs or cost efficiency. Success of an approach in one school has often not been replicable in another. ATI studies, on the other hand, have often had a theoretical basis, but they have been done under carefully controlled and atypical classroom situations. The hypotheses which have been tested in such studies have often not been found valid by classroom teachers. We need to find ways of overcoming these problems.

There are evidently problems in finding ways to provide individualized instruction that will help students learn mathematics and teachers teach mathematics. One possible solution is to combine approaches instead of using them separately as we have done in the past. It is imperative that we solve these problems, and it also seems clear that it is time to consider the two approaches together since the problems in one approach are the strengths in the other. By dealing with both approaches simultaneously we may gain insight into how different students learn mathematics, improve our own teaching by using different kinds of instruction, obtain a greater understanding of the reasons for students' difficulties, and determine effective ways to group students for instruction.

It would be helpful in dealing with the problems of organizing classrooms and dealing with individual differences if we would be alert to aptitudes that might be correlated with success in mathematics learning and that could influence the effectiveness of instruction. Through the combined use of organizational schema and an ATI approach we may be able to attack these problems in a new and better way. Then we can make some advance in the area of how to best organize instruction so that each student can learn better. Also we might try to use aptitude variables when conducting studies and program evaluations. The article by Mayes, Scasta, and Conoley (May 1975) reports a study which uses a form of the combined approach. Many other combinations of aptitudes, instructional forms, and grouping are possible.

This paper has tried to offer an alternate way of solving some traditional problems in the teaching of mathematics. In the past, large group instruction in mathematics has been on a first come, first served basis, and no improvement in mathematics learning has been evident. Maybe if we grouped students according to their aptitudes and then taught them mathematics in ways based on those aptitudes, we would be more successful.

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PROBLEMS AND SOLUTIONS

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All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before April 30, 1978.

E 2689. *Proposed by L.-S. Hahn, University of New Mexico*

Is there a non-empty finite set S of positive integers such that
(1) $n \in S \Rightarrow n-1 \in S$ or $n+1 \in S$, and
(2) $\Sigma_{n \in S} 1/n$ is an integer?

E 2690. *Proposed by Anthony J. Quinzi, Temple University*

Let $S_1, S_2, \ldots, S_k$ be a list of all non-empty subsets of $\{1, 2, \ldots, n\}$. Thus $k = 2^n - 1$. Let $a_{ij} = 0$ if $S_i \cap S_j = \varnothing$ and $a_{ij} = 1$ otherwise. Show that the matrix $A = (a_{ij})$ is non-singular.

E 2691. *Proposed by Živojin M. Mijalković, Pirot, Yugoslavia, and J. B. Keller, Courant Institute*

If $x_i > 0$ $(1 \leqq i \leqq n)$ show that

$$(\Pi x_i)^{\Sigma x_i/n} \leqq \Pi x_i^{x_i} \leqq \left(\frac{\Sigma x_i^2}{\Sigma x_i}\right)^{\Sigma x_i}$$

E 2692. *Proposed by Donald R. Woods, Stanford University*

Show that the sequence of increasingly complex fractions

$$\frac{1}{2}, \left(\frac{1}{2}\right)\Big/\left(\frac{3}{4}\right), \frac{\left(\frac{1}{2}\right)\Big/\left(\frac{3}{4}\right)}{\left(\frac{5}{6}\right)\Big/\left(\frac{7}{8}\right)}, \frac{\left(\frac{1}{2}\right)\Big/\left(\frac{3}{4}\right)}{\left(\frac{5}{6}\right)\Big/\left(\frac{7}{8}\right)} \Bigg/ \frac{\left(\frac{9}{10}\right)\Big/\left(\frac{11}{12}\right)}{\left(\frac{13}{14}\right)\Big/\left(\frac{15}{16}\right)}, \ldots,$$

approaches a limit, and find that limit.

What can be said about the more general sequence

$$\frac{x}{x+1}, \left(\frac{x}{x+1}\right)\Big/\left(\frac{x+2}{x+3}\right), \frac{\left(\frac{x}{x+1}\right)\Big/\left(\frac{x+2}{x+3}\right)}{\left(\frac{x+4}{x+5}\right)\Big/\left(\frac{x+6}{x+7}\right)}, \ldots ?$$

E 2693. *Proposed by Alexandru Lupaş, Cluj-Napoca, Romania*

Find a rational function $f(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials with integral coefficients of degree at most 6, which is a good approximation to arctan x on $[0, 1]$. More precisely, we want $g(x) = \arctan x - f(x)$ to satisfy $0 \leqq g(x) < \varepsilon$ for $x \in [0, 1]$ and ε to be small (such approximations exist if $\varepsilon = 0.000033$).

E 2694. *Proposed by I. J. Schoenberg, University of Wisconsin*

Let Π be a prism inscribed in the sphere S of unit radius and center O. The base of Π is a regular n-gon of radius r. For each face F of Π drop a directed perpendicular from O and let A_F be the point where it intersects S. Let Π^* be the polyhedron obtained by adding to Π, for each face F, the pyramid of base F and apex A_F.

For which values of r is Π^* convex?

SOLUTIONS OF ELEMENTARY PROBLEMS

Union of an Open and a Compact Set

E 2614 [1976, 657]. *Proposed by Frank Siwiec, John Jay College of Criminal Justice, New York*

A set $A \subset \mathbb{R}^n$ is called a g-set if there is a countable family $\{U_n \mid n = 1, 2, \ldots\}$ of open sets containing A with the property: For each open set $G \supset A$ there is a U_n with $A \subset U_n \subset G$. Which subsets of $\mathbb{R}^n$ are g-sets?

Solution by R. F. Dickman and R. A. McCoy, Virginia Polytechnic Institute and State University.

Let (X, d) be a metric space. We shall prove that a subset A of X is a g-set iff $A = G \cup K$ where G is open and K is compact.

If G is open and K compact then each of them is a g-set and so is their union.

Hence it suffices to show that if $A \subset X$ and $B = A \setminus \mathrm{int}(A)$ is not compact then A is not a g-set. To prove this, let U_n $(n = 1, 2, \ldots)$ be any sequence of neighborhoods of A. Since B is not compact there exists a sequence x_n $(n = 1, 2, \ldots)$ of points of B which has no cluster points in B. It follows that this sequence has no cluster points in A.

For each n choose $y_n \in U_n \setminus A$ such that $d(x_n, y_n) < 1/n$. This can be done because B is contained in the boundary of A. Every cluster point z of the sequence (y_n) is also a cluster point of the sequence (x_n) and so $z \notin A$. Thus if Z is the closure of the set $\{y_n \mid n = 1, 2, \ldots\}$ then $X \setminus Z$ is a neighborhood of A which contains no U_n. Hence, A is not a g-set.

Also solved by J. H. Carruth & R. J. Daverman, Eli Isaacson, O. P. Lossers (Netherlands), Mark Meyerson, Stephen Noltie, Edward Ordman, Roy Olson, Paul Vojta, and the proposer.

A System with Trivial Solutions Only

E 2615 [1976, 657]. *Proposed by D. Rameswar Rao, Secunderabad, India*

Show that the system of Diophantine equations

$$x^2 + y^2 = u^2 + v^2, \qquad x^3 + y^3 = u^3 + v^3 \tag{1}$$

has no solutions in positive integers. Prove the same for the system

$$x^2 + y^2 = u^2 + v^2, \qquad x^5 + y^5 = u^5 + v^5. \tag{2}$$

Solution by Ivan Niven, University of Oregon. The same result holds not only in these special cases but also in a much broader setting. (Of course, there are the obvious solutions: $x = u$, $y = v$ and $x = v$, $y = u$, but there are no others.) Much more generally we prove the following result. Let a and b be distinct positive real numbers. The only solutions of the system

$$x^a + y^a = u^a + v^a, \quad x^b + y^b = u^b + v^b \tag{3}$$

in positive real numbers x, y, u, v are the obvious solutions listed above. To prove this we assume that $a < b$ and that there is a solution with $x \geqq y$, $u \geqq v$, say $x > u$ so that $x > u \geqq v > y > 0$. We define

$$x_1 = x^a, y_1 = y^a, u_1 = u^a, v_1 = v^a, c = b/a > 1,$$

so that (3) can be written as

$$x_1 - u_1 = v_1 - y_1 \neq 0, \quad x_1^c - u_1^c = v_1^c - y_1^c.$$

Applying the mean value theorem to the function x^c we see that the last equation implies

$$(x_1 - u_1)r^{c-1} = (v_1 - y_1)s^{c-1}$$

and hence $r^{c-1} = s^{c-1}$ where r and s are intermediate values satisfying $u_1 < r < x_1$, $y_1 < s < v_1$, so that $r > s$. This implies $r^{c-1} > s^{c-1}$ and we have a contradiction.

Also solved by John Boxall (England), Robert Breusch, Kenneth Burke, Romae Cormier, Irving Gerst, Marguerite Gerstell, Eli Isaacson, L. Kuipers (Switzerland), L. E. Mattics, Ken Yocom, and the proposer.

Comments. A. Makowski (Poland) notes that the proposer stated the same problem in Math. Gazette 50 (1966), 166–167. The system (1) the proposer also submitted to Canad. Math. Bull. and it appeared there as P.245, 18 (1975), p. 616.

Approximation by Algebraic Integers

E 2616 [1976, 657]. *Proposed by Andrew Odlyzko and Lloyd Welch, Jet Propulsion Laboratory*

Three Parallel Sections of a Convex Body

E 2617 [1976, 740]. *Proposed by Eugene Ehrhart, Institut de Recherches Mathématiques avancées, Strasbourg, France*

A convex body is cut by three parallel planes. If the three sections thus produced have the same area, show that the portion of the body lying between the two outside planes is a cylinder. Does the same conclusion follow if instead we are given that the three sections have the same perimeter?

The theorem of Brunn–Minkowski (see T. Bonnesen and W. Fenchal, *Theorie der konvexen Körper*, Chelsea, New York 1948, §41, p. 71) is relevant for this problem.

I. *Solution by Rolf Schneider, Freiburg, Germany (communicated by the proposer).* Let S_1, S_2, S_3 be the three planar sections (S_2 lying in between S_1 and S_3). Let L be a line which intersects S_i in a point P_i, $i = 1, 2, 3$. Let $\lambda = P_1P_2/P_1P_3$ and define the Minkowski linear combination

$$S_\lambda = \lambda S_1 + (1-\lambda)S_3 = \{\lambda x + (1-\lambda)y \mid x \in S_1, y \in S_3\}.$$

By convexity, we have $S_\lambda \subset S_2$.

The theorem of Brunn–Minkowski states that the areas of these figures satisfy the inequality

$$A(S_\lambda)^{1/2} \geqq \lambda A(S_1)^{1/2} + (1-\lambda)A(S_3)^{1/2}$$

and equality holds if and only if S_1 and S_3 are homothetic. Since $S_2 \supset S_\lambda$ and we have $A(S_1) = A(S_2) = A(S_3)$ by hypothesis, it follows that S_3 is a parallel translate of S_1 and $S_2 = S_\lambda$. The convexity now implies that the portion of the body between S_1 and S_3 is a cylinder.

II. *Solution by Michael Goldberg, Washington, D.C.* The answer to the second part of the problem is negative. For instance, every section of a regular tetrahedron, parallel to two opposite edges of the tetrahedron, has a constant perimeter.

Also solved (II) by the proposer.

Triangular-Square-Pentagonal Numbers

E 2618 [1976, 740]. *Proposed by Amy J. Phelps, student, West Side Middle School, Rocky Face, Georgia*

The numbers of the form $\frac{1}{2}k(k+1)$, k^2, $\frac{1}{2}k(3k-1)$, where k is a positive integer, are called triangular, square and pentagonal numbers, respectively. Find all triangular-square-pentagonal numbers.

Solution by S. C. Locke, graduate student, University of Waterloo, Ontario. Suppose that for positive integers p, q, r

$$p^2 = \tfrac{1}{2}q(q+1) = \tfrac{1}{2}r(3r-1).$$

Then

$$(2q+1)^2 = 1 + 8p^2, \tag{1}$$

$$(6r-1)^2 = 1 + 24p^2, \tag{2}$$

and $(6r-1)^2 = (4p)^2 + (2q+1)^2$. Since $(2q+1, 4p) = 1$ by (1), and $(6r-1, 4p) = 1$ by (2), it follows that $(4p, 2q+1, 6r-1)$ is a primitive Pythagorean triple. Thus

$$4p = 2mn,\ 2q+1 = m^2 - n^2,\ 6r - 1 = m^2 + n^2$$

where m, n are positive integers, $(m, n) = 1$, $m - n$ being odd. Substituting these in (1), we find that

$$m^4 - 4m^2n^2 + n^4 = 1. \tag{3}$$

Theorem 9, p. 270 in L. Mordell, *Diophantine Equations*, Academic Press, 1969, states that the equation

$$y^2 = Dx^4 + 1,$$

where $D > 0$ and D is not a perfect square, has at most two solutions in positive integers. We can write (3) as

$$(m^2 - 2n^2)^2 = 3n^4 + 1.$$

Since the equation $y^2 = 3x^4 + 1$ has only solutions $(x, y) = (2, 7)$ or $(1, 2)$ in positive integers we find that the only solution of (3) with $m > n > 0$ is $m = 2$, $n = 1$.

Thus 1 is the only positive triangular-square-pentagonal number.

Also solved by K. Inkeri (Finland), A. Makowski (Poland), L. E. Mattics, Paul Monsky, David Penney, Les Reid, Scoll Smith, R. P. Steiner, and R. J. Stroeker (Netherlands).

Editor's Comment. The solution to this problem was published in 1965 by A. Makowski in his note *Remarque à une note de M. A. Rotkiewicz sur les nombres à la fois triangulaires, carrés et pentagonaux*, Bull. Soc. Royale Science Liège, 34 (1965), 27.

The equation (3) in a more general form $m^4 - 4m^2n^2 + n^4 = z^2$ has been solved by Paraira (see R. D. Carmichael, *Diophantine Analysis*, Dover, New York, 1959, Exercise 15, p. 84) and also by M. C. Pocklington in his paper *Some Diophantine impossibilities*, Proc. Cambridge, Phil. Soc. 17 (1914), p. 108–121.

Squares in a Recursive Sequence

E 2619 [1976, 740]. *Proposed by Thomas C. Brown, Simon Fraser University, Burnaby, B. C., Canada*

Let $a_1 = 1$ and $a_{n+1} = a_n + [\sqrt{a_n}]$ for $n = 1, 2, \dots$. Show that a_n is a square if and only if $n = 2^k + k - 2$ for some positive integer k.

I. *Solution by R. Sherman Lehman, University of California, Berkeley.* We will show by induction on $k \geqq 1$ that $a_n = (2^{k-1})^2$ for $n_k = 2^k + k - 2$ and that a_m is not a square for $n_{k-1} < m < n_k$ $(k \geqq 2)$.

This is true for $k = 1$ and let us assume that it is true for some $k \geqq 1$. By induction on i one obtains

$$a_{n+2i+1} = (2^{k-1} + i)^2 + 2^{k-1} - i \qquad (0 \leqq i \leqq 2^{k-1}), \tag{1}$$

$$a_{n+2i} = (2^{k-1} + i - 1)^2 + 2^k \qquad (1 \leqq i \leqq 2^{k-1}), \tag{2}$$

where $n = n_k$. For $i = 2^{k-1}$ we have $n + 2i + 1 = n + 2^k + 1 = n_{k+1}$ and

$$a_{n_{k+1}} = (2^{k-1} + 2^{k-1})^2 = (2^k)^2.$$

The formulas (1) and (2) show that a_m is not a square for $n_k < m < n_{k+1}$.

II. *Generalization by Allan Wm. Johnson, Jr., Defense Communications Agency, Washington, D. C. (revised by the editor.)* Let r $(\geqq 2)$ be an integer, $a_1 = 1$ and $a_{n+1} = a_n + [\sqrt{a_n}]$ for $n \geqq 1$.

THEOREM. *If*

$$n = n_k = 1 + k + \sum_{i=1}^{2^k - 1} \sum_{j=1}^{r-1} \binom{r}{j} i^{j-1} \qquad (k \geqq 0)$$

then $a_n = (2^k)^r$. Moreover, a_n is an r-th power only if $n = n_k$ for some k.

Proof. We shall prove by induction on k that $a_{n_k} = (2^k)^r$ and that a_m for $n_{k-1} < m < n_k$ is not an rth power. This is true for $k = 0$, and assume that it is true for some $k \geqq 0$. Put $n = n_k$, $b = 2^k$. Thus $a_n = b^r$. For $0 \leqq t < b$ we define

$$s_t = \sum_{i=1}^{t} \frac{(b+i)^r - (b+i-1)^r - 1}{b+i-1}$$

Then we have

$$a_{n+s_t+j+1} = (b+t)^r + b - t + j(b+t) \tag{3}$$

for

$$0 \leqq t < b, \quad 0 \leqq j \leqq \frac{(b+t+1)^r - (b+t)^r - 1}{b+t} \tag{4}$$

This can easily be proved by double induction on j and t.

We have

$$a_{n+s_t+j+1} \geqq a_{n+s_t+1} = (b+t)^r + b - t > (b+t)^r$$

for j and t as in (4). On the other hand we have

$$a_{n+s_t+j+1} \leqq (b+t)^r + b - t + [(b+t+1)^r - (b+t)^r - 1]$$
$$= (b+t+1)^r + b - (t+1) \leqq (b+t+1)^r$$

and the inequality is strict unless both t and j take maximal values permitted by (4).

When both t and j take these maximal values then

$$n + s_t + j + 1 = n_{k+1}$$

and we obtain $a_{n_{k+1}} = (2b)^r = (2^{k+1})^r$.

Thus for $n_k < m < n_{k+1}$ the above inequalities show that a_m is not an rth power.

Comments. R. J. Stroeker (Netherlands) proves the following generalization: Let $a_1 = x^2 + x - 1$ where x is a positive integer, and define a_n recursively by $a_{n+1} = a_n + [\sqrt{a_n}]$. Given a positive integer, say m, he shows that there exists a unique x such that m occurs in the sequence (a_n). He also shows that a_n is a square if and only if $n = k - 1 + (2x-1)(2^k - 1)$ for some positive integer k and then $a_n = 2^{2(k-1)}(2x-1)^2$.

Another generalization was found by Richard Pollack.

Also solved by fifty other readers.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before April 30, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6186. *Proposed by Ronald Evans, University of California, San Diego, La Jolla*

Let $\mathbb{N}$ denote the natural numbers. Let $r, k \in \mathbb{N}$, where r is fixed. Fix $\beta > 1$. Let $F_r(k) = \Sigma (j_1 j_2 \cdots j_r)^{\beta-1}$, where the sum is over all vectors $(j_1, j_2, \ldots, j_r) \in \mathbb{N}^r$ for which $j_1 + j_2 + \cdots + j_r = k$. Prove that

$$F_r(k) \sim \frac{\Gamma^r(\beta)}{\Gamma(r\beta)} k^{\beta r - 1} \quad \text{as} \quad k \to \infty.$$

6187. *Proposed by Ronald Evans, University of California, San Diego, La Jolla*

Let $X_1, X_2, \ldots$ be a sequence of random numbers, uniformly distributed in $[0,1]$ and let N be minimal such that $\Sigma_{1 \leqq i \leqq N} X_i^2 > 1$. Show that the expected value of N is

$$e^{\pi/4}\left(1 + \int_0^1 e^{-\pi t^2/4} dt\right).$$

6188. *Proposed by F. S. Cater, Portland State University.*

Do there exist complementary subsets A and B of the set of irrational numbers such that for any

open intervals I and J in the real line, (1) $A \cap I$ and $B \cap J$ are not homeomorphic in the Euclidean topology, and (2) there is a one-to-one continuous function mapping $A \cap I$ onto $B \cap J$?

6189*. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Canada*

Prove or disprove that for each natural number $n \geqq 2$, one can arrange the numbers $1, 2, \ldots, n$ in a sequence such that the sum of any two adjacent terms is a prime.

6190. *Proposed by D. E. Daykin, Reading University, England, and D. J. Kleitman, Massachusetts Institute of Technology*

Let n be a square free integer that is not prime. Let F be a set of divisors of n such that neither the product of two elements of F nor n^2 divided by such a product is in F. What is the maximal proportion of the divisors of n that may lie in F?

6191*. *Proposed by Harry D. Ruderman, Hunter College Campus School*

Let $P(z)$ be a monic polynomial with complex coefficients, in the complex variable z. Let $P(z_1)$ and $P(z_2)$ be in opposite quadrants I and III or II and IV. Let $z_3 = (z_1 + z_2)/2$. What is an upper bound (least, if possible) on r which will guarantee that a zero of $P(z)$ will be within a distance r from z_3?

SOLUTIONS OF ADVANCED PROBLEMS

Common Divisors and Square Free Integers

6086 [1976, 292]. *Proposed by Raymond M. Redheffer, University of California, Los Angeles*

Let d_{ij} be the number of divisors common to i and j. Then the determinant $|d_{ij}|$ for $2 \leqq i, j \leqq n$ equals the number of square free integers from 1 to n.

Solution by Gary Kennedy, University of New Mexico. Let $e_{11} = 1$, $e_{i1} = 0$ for $i > 1$, and $e_{ij} = d_{ij}$ for $j \neq 1$. Then $E = (e_{ij})$, $1 \leqq i, j \leqq n$, is an $n \times n$ matrix whose determinant is $|d_{ij}|$. Denote by R_i the ith row of E. Beginning with $i = n$ and proceeding towards $i = 2$, replace each R_i by

$$\sum_{k\ i} \mu\left(\frac{i}{k}\right) R_k,$$

where μ is the Möbius function. Let ψ_{ij} be defined by

$$\psi_{ij} = \begin{cases} 1 & \text{if } \ i \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Sigma_{k \mid i} \psi_{kj} = d_{ij}$, we may apply the Möbius inversion formula to get

$$\sum_{k \mid i} \mu\left(\frac{i}{k}\right) d_{kj} = \psi_{ij}.$$

Therefore the matrix $F = (f_{ij})$ obtained from E by these elementary row operations is characterized by $f_{11} = 1$, $f_{i1} = \mu(i)$ for $i > 1$, and $f_{ij} = \psi_{ij}$ for $j \neq i$.

Denote by S_i the ith row of F. Let $G = (g_{ij})$ be the matrix obtained from F by replacing S_1 by $\Sigma_{i \leqq n} \mu(i) S_i$. Then for $j > 1$, $g_{1j} = \Sigma_{i \leqq n} \mu(i) \psi_{ij} = \Sigma_{i \mid j} \mu(i) = 0$,

$$g_{11} = \sum_{i \leqq n} \mu(i) \cdot \mu(i) = (\text{number of square free integers from 1 to } n)$$

$$= M.$$

Also (g_{ij}), for $2 \leqq i, j \leqq n$, is an upper triangular matrix with 1's on the diagonal.

Therefore $|d_{ij}| = \det G = M$.

Also solved by Robert Breusch, M. G. Greening (Australia), A. A. Jagers (Netherlands), J. C. Lagarias, L. E. Mattics, R. W. K. Odoni & J. B. Wilker (England), I. J. Schoenberg, K. R. P. Singh (India), Allen Stenger, and the proposer.

The Equation $f^{-1}=f'$

6088 [1976, 293]. *Proposed by Nathaniel Grossman, University of California, Los Angeles*

The functional-differential equation $f'=f^{-1}$ has a solution satisfying $f(0)=0$ and $f'(x)>0$ for $x>0$, namely $f(x)=(1/\alpha)^{1/\alpha}x^{\alpha}$, where $\alpha=(1+\sqrt{5})/2$, the golden section. Is this the only solution satisfying the given conditions for $x\geqq 0$?

Editorial Note. This is identical with Problem E 2105. See the solution by A. C. Hindmarsh (this MONTHLY Vol. 76 (1969), p. 696), proving there are no other such functions. However, in line 5 of the printed solution, $f'(f(x))=x$ should be replaced by $f(f'(x))=x$. See also Problem 71-1 in *SIAM Review*, V. 14 (1972), p. 169, solution also by Hindmarsh.

Additional solutions were received from Gustaf Gripenberg (Finland), O. P. Lossers (Netherlands), J. B. Wilker & R. W. K. Odoni (England), and John Williams (Australia).

Mapping from Γ^n to $\mathbb{C}^m$

6091 [1976, 385]. *Proposed by H. S. Shapiro, Royal Institute of Technology, Stockholm, Sweden*

Let Γ denote the set of complex numbers of modulus one, and consider for positive integers m,n the map $T:\Gamma^n\to\mathbb{C}^m$ defined by

$$\begin{aligned} w_1 &= z_1+z_2+\cdots+z_n,\\ w_2 &= z_1^2+z_2^2+\cdots+z_n^2,\\ &\cdots\cdots\\ w_m &= z_1^m+z_2^m+\cdots+z_n^m, \end{aligned}$$

where each z_i ranges over Γ. Prove that for any m, and any positive R, the range of T contains the ball $\|w\|\leqq R$ for all sufficiently large n.

Solution by J. B. Wilker and R. W. K. Odoni, University of Exeter, England. To cover the ball

$$B^m(R)=\{(w_1,w_2,\ldots,w_m):|w_j|\leqq R,\ j=1,2,\ldots,m\}$$

we use $n=N(m,R)=\sum_{j=1}^m jf(j)$ variables z_i. At the jth stage we combine $f(j)$ free unimodular parameters $\lambda_k(j)$ with the jth roots of unity $\omega_l(j)$ to obtain $jf(j)$ unimodular variables $z_i=z_{jkl}=\lambda_k(j)\omega_l(j)$. For any positive integer p, $\sum_{l=1}^j\omega_l^p(j)=j$ or 0 according as $j\mid p$ or $j\nmid p$, and this allows us to reduce the transformation equations $w_p=\sum_{i=1}^n z_i^p$ to the lower triangular system $w_p=\sum_{j\mid p}j\sum_{k=1}^{f(j)}\lambda_k^p(j)$ which we may rewrite as

$$(*)\qquad p\sum_{k=1}^{f(p)}\lambda_k^p(p)=w_p-\sum_{\substack{j<p\\ j\mid p}}j\sum_{k=1}^{f(j)}\lambda_k^p(j).$$

If $w_1=2\cos\theta e^{i\varphi}$, $w_1\in B^1(2)$, then $w_1=z_1+z_2$ for $z_1=e^{i(\varphi+\theta)}$, $z_2=e^{i(\varphi-\theta)}$, and thus $B^1(2)=\Gamma+\Gamma$. If $w_1\in B^1(n)$ for $n>2$, then $w_1=w_1/|w_1|+(w_1-w_1/|w_1|)$, where $|w_1-w_1/|w_1||\leqq n-1$, so $B^1(n)=B^1(n-1)+\Gamma=\Gamma+\cdots+\Gamma$ (n terms).

It follows that for suitable $\lambda_k^p(p)$ (i.e., suitable $\lambda_k(p)$) the left side of $(*)$ represents any complex number of modulus $\leqq pf(p)$. The right side of $(*)$ has modulus $\leqq R+\sum_{j<p,\,j\mid p}jf(j)$ and hence the transformation equations can be inverted if f is chosen to satisfy $2pf(p)\geqq R+\sum_{j\mid p}jf(j)$. One possible choice is $f(j)=Aj$, where $A=\max\{2,[R]+1\}$, and this leads to $N(m,R)=\sum_{j=1}^m Aj^2=$

$Am(m+1)(2m+1)/6$. To verify that this f is admissible we estimate

$$R+\sum_{j\mid p} jf(j) < A + \sum_{j\mid p} \frac{p}{j} f\left(\frac{p}{j}\right)$$

$$< Ap^2\left(\frac{1}{p^2}+\sum_{j=1}^{\infty} j^{-2}\right) < 2pf(p).$$

With more effort these techniques can be made to yield $N(m,R)=O(Rm^{\theta})$ for some $2<\theta<3$.

Finally, if $n=N(m,R)+j$ with $j \geqq m+1$, then the image of $T_n\colon \Gamma^n \to \mathbf{C}^m$ contains $B^m(R)$ because we may "waste" the last j components of z by setting them equal to the jth roots of unity. This remark completes our solution of the problem.

As an amusing corollary we note that if $B \geqq 0$ and $n \geqq N'(m,B)$ then the first m coefficients of the polynomial $p(z)=z^n+a_1z^{n-1}+\cdots+a_n$ may be specified arbitrarily subject only to $|a_i| \leqq B$, $i=1,2,\ldots,m$ and then the remaining coefficients a_i, $i=m+1,m+2,\ldots,n$ may be chosen so that the roots satisfy $|z_i|=1$, $i=1,2,\ldots,n$. This is because the power sums of the roots, w_i, are integer polynomials in the symmetric functions of the roots, $(-1)^i a_i$, so a bound on the first m $|a_i|$ implies a bound on the first m $|w_i|$.

Also solved by Paul Bruckman, L. E. Mattics, J. Denmead Smith (England), Luis Verde-Star, and the proposer.

Continuous Linear Functionals

6093 [1976, 386]. *Proposed by Richard Johnsonbaugh, Chicago State University*

Let X be a completely regular Hausdorff space and let $C(X)$ denote all real-valued continuous functions on X with the topology of uniform convergence on compact sets. Let F be a continuous non-zero linear functional on $C(X)$. Prove that there exists a smallest compact set K with the property that if $f=0$ on K, then $F(f)=0$.

Solution by F. Michael Christ, student, Harvey Mudd College. We say that $K \subset X$ has property (*) if K is compact, non-empty, and if for any $f \in C(X)$, $f=0$ on K implies $F(f)=0$. $K \subset X$ has property (**) if K is compact, non-empty, and if there exists $\delta>0$ such that for $f \in C(X)$, $|f(x)|<\delta$ for $x \in K$, then $|F(f)|<1$.

If K has property (**), define $\Delta(K)$ to be the supremum of all $\delta>0$ which satisfy the definition of (**). Note that $\Delta(K)<\infty$ for all K, since F is non-zero.

The proof of the result will follow from three lemmas, the first of which is well known.

LEMMA 1. *If A and B are disjoint compact subsets of X, and if $\varepsilon>0$, then there exists $f \in C(X)$, with $0 \leqq f \leqq 1$, $f(A)=1$, and for every $x \in B$, $|f(x)|<\varepsilon$.*

LEMMA 2. *If K has property (**), then K has property (*).*

Proof. Let $n>0$, suppose K has property (**), and $f \in C(X)$ with $f=0$ on K. Then $\sup_{x\in K}|nf(x)|=\sup\{0\}=0<\Delta(K)$, so $|F(nf)|<1$. By the linearity of F, $|F(f)|<1/n$, and the proof is complete upon letting $n\to\infty$.

LEMMA 3. *If K_1 has property (*) and K_2 has property (**), then $K_1 \cap K_2$ has property (**) and $\Delta(K_1\cap K_2) \geqq \Delta(K_2)$. Moreover, $K_1 \cap K_2 \neq \emptyset$.*

Proof. Suppose $h \in C(X)$ and $|h(x)|<\Delta(K_2)$ for every $x \in K_1 \cap K_2$. Then, there exists an open set $V \supset K_1 \cap K_2$, such that $|h(x)|<\Delta(K_2)$ on V. Let $A=K_2-V$. Then A and K_1 are disjoint compact sets, and by Lemma 1, there exists $f \in C(X)$ such that $0 \leqq f \leqq 1$, $f(K_1)=1$, and $|f(x)|<\Delta(K_2)/C$ for all $x \in A$, where $C=\sup_{x\in K_1}|h(x)|$.

Therefore, $|h \cdot f(x)| < \Delta(K_2)$ on A, and $|h \cdot f(x)| \leqq |h(x)|$, for every $x \in X$. Now, $h = h \cdot f + h \cdot (1-f)$, so $F(h) = F(h \cdot f) + F(h \cdot (1-f))$. Also $(h \cdot (1-f))(x) = h(x) \cdot 0 = 0$ for $x \in K_1$, so, since K_1 has property (*), $F(h \cdot (1-f)) = 0$. Since $|h \cdot f(x)| < \Delta(K_2)$ for $x \in A$, and since $|h \cdot f(x)| \leqq |h(x)| < \Delta(K_2)$ for $x \in V$, then $|h \cdot f| < \Delta(K_2)$ on $A \cup V \supset K_2$. Therefore, since K_2 has property (**), $|F(h \cdot f)| < 1$. Combining these results we get $|F(h)| < 1$. Thus, $K_1 \cap K_2$ has property (**) and $\Delta(K_1 \cap K_2) \geqq \Delta(K_2)$.

Finally, if $K_1 \cap K_2 = \emptyset$, then from the above argument, $F = 0$. Therefore, $K_1 \cap K_2 \neq \emptyset$.

Proof of the Result. Let S be the intersection of all K with property (**). (This is a non-empty class: there exists some basic open neighbourhood V of the constant function 0 such that $f \in V$ implies $|F(f)| < 1$. By definition, there exists a compact set K and a $\delta > 0$ such that $f \in V$ if and only if $|f(x)| < \delta$ for all $x \in K$. Then K has property (**).)

S is compact and $S \neq \emptyset$, by the finite intersection property. We claim S has property (**). Let K_0 have property (**) and let $K_0 \supset S$, let $\delta < \Delta(K_0)$ and suppose that $|g(x)| < \delta$, for all $x \in S$. Then there is an open set $V \supset S$ such that $|g(x)| < \delta$ for all $x \in V$. $K_0 - V$ is compact, therefore there is a finite set $\{K_i\}_{i=1}^n$ such that K_i has property (**) for each i, and such that $T = \bigcap_0^n K_i \subset V$. So, $|g(x)| < \delta$ for all $x \in T$. By Lemma 3, T has property (**) and $\Delta(T) \geqq \Delta(K_0)$. Therefore, $|g(x)| < \Delta(K_0) \leqq \Delta(T)$ for all $x \in T$, and by the definition of $\Delta(T)$, $|F(g)| < 1$. Thus S has property (**), and it follows from Lemma 3 that $\Delta(S) = \sup\{\Delta(K)\}$: K has property (**).

Now we claim that S is the smallest compact set with property (*). (Recall by Lemma 2, (**) $\Rightarrow$ (*).) Suppose K has property (*), then by Lemma 3, $K \cap S$ has property (**). So $S \subset K \cap S$, hence $S \subset K$, and we are done.

Also solved by A. A. Jagers (Netherlands), L. R. King & B. G. Klein, C. D. Shannon, E. K. van Douwen & D. J. Lutzer, and the proposer.

"Acquainted" Primes

6094 [1976, 386]. *Proposed by Francis Cald, Robertsbridge, England*

A pair of primes, P and Q, such as 7 and 17, is said to be acquainted if the set of quadratic residues and the set of quadratic nonresidues of P are, respectively, a subset of the set of residues and the set of nonresidues of Q. Is there a positive constant C such that infinitely many pairs of acquainted primes exist for which $Q - P \leqq C$?

Solution by G. Kolesnik and E. G. Strauss, University of California, Los Angeles. The answer to the problem is negative in the following strong sense:

THEOREM 1. *For every $\varepsilon > 0$ there exists a positive integer $c = c(\varepsilon)$ so that for all acquainted primes p, q with $p < q$ we have*

$$q > cp^{4\sqrt{e}-1-\varepsilon} \tag{1}$$

Proof. The condition that p and q are acquainted can be expressed as $(n/p) = (n/q)$, or equivalently $(n/pq) = 1$ for all $n = 1, 2, \ldots, p-1$ where $(n/p), (n/q)$ are the Legendre symbols and (n/pq) is a Jacobi symbol. Theorem 1 is therefore a special case of the following more general theorem:

THEOREM 2. *The pair of squarefree integers p, q with $p < q$, $(p, q) = 1$ for which there exists a character χ belonging properly to pq with $\chi(n) = 1$ for $n = 1, 2, \ldots, p-1$ satisfy the conclusion of Theorem* 1. (A character belongs properly to pq if it is not a product of characters χ_1, χ_2 belonging to proper divisors p_1 and q_1, $p_1 q_1 = pq$ where one of these characters is principal.)

Proof. We use the following result of D. A. Burgess, *On character sums and L-series*, Proc. London Math. Soc. 3, 12 (1962), 193–206.

If χ is a character belonging properly to K where K is square-free, then

$$\left|\sum_{n=1}^{N}\chi(M+n)\right| \ll N^{1-1/(r+1)}K^{1/4r+\varepsilon}, \quad \varepsilon>0 \tag{2}$$

for all integers M and all positive integers N, r. (The implied constant depends on ε and r alone.)

By the hypothesis of Theorem 2 we have $\chi(n)=1$ for all n which have only prime divisors less than p. If we set $N=p^{\eta}\leqq p^2$ then the number of integers $n\leqq N$ which have a prime divisor $s>p$ is

$$\sum_{\substack{s=\text{prime}\\ p<s\leqq N}}\left[\frac{N}{s}\right]=N\sum\frac{1}{s}+O(\pi(N))$$

$$=N(\log\log N-\log\log p)+o(N)$$

$$=N(\log\eta+o(1)).$$

Therefore

$$\left|\sum_{n=1}^{N}\chi(n)\right|>N-2N\log\eta+o(N) \tag{3}$$

$$>(1-2\log\eta-\varepsilon)N$$

for all $\varepsilon>0$ and all sufficiently large N.

If we choose $\eta<\sqrt{e}$ and combine (2) and (3) with $K=pq$ we get $N\ll N^{1-(1/(r+1))}(pq)^{(1/4r)+\varepsilon_1}$ or $N^{4r/(r+1)}\ll(pq)^{1+4r\varepsilon_1}$. For large integers r this yields

$$N^{4-\varepsilon_2}\ll(pq)^{1+\varepsilon_3},$$

that is

$$p^{4\eta-1-\varepsilon_2\eta-\varepsilon_3}\ll q^{1+\varepsilon_3}$$

which implies

$$p^{4\eta-1}\ll q \quad \text{for every } \eta<\sqrt{e}.$$

Remarks. If we set T equal to 4 times the product of all primes $<p$ then the condition that $q>p$ is a prime acquainted with p means that q belongs to one of $\varphi(T)/2^{\pi(p-1)}$ residue classes (mod T). Theorem 1 shows that none of these classes contains a prime s with $p<s<cp^{4\sqrt{e}-1-\varepsilon}$

The proof of Theorem 2 yields the fact that if χ_1 is a nonprincipal character (mod p) and χ_2 is a nonprincipal character (mod q), where p, q are squarefree numbers with $(p,q)=1$, then there exists a positive integer n with

$$n\ll(pq)^{(1/4\sqrt{e})+\varepsilon}, \quad \varepsilon>0$$

for which $\chi_1(n)\neq\chi_2(n)$.

More generally, if K is squarefree and χ is a character properly belonging to K, then $\chi(n)\neq 1$ for some n with

$$n\ll K^{(1/4\sqrt{e})+\varepsilon}, \quad \varepsilon>0.$$

The case $K=$ prime was obtained by D. A. Burgess.

Also solved by R. W. K. Odoni (England), Bruce Ferrero, and L. E. Mattics.

Note. Odoni proves that for any $\alpha>0$, there are only finitely many acquainted primes with $p<q<p^{3-\alpha}$.

Further, the set $Q_p = \{q: q > p;\ p, q \text{ acquainted}\}$ is infinite and $\min Q_p < c_1 \exp(c_2 p(1 + o(1)))$, c_1, c_2 independent of p. See also the papers:

(1) D. A. Burgess, On character sums and L-series II, Proc. London Math. Soc. 13 (1963), 524–536.

(2) R. J. Miech, A number theoretical constant, Acta Arithmetica 15 (1968), 119–137.

Positive Definite Matrices

6095 [1976, 386]. *Proposed by Anon, Erewhon-upon-Thames*

Let P, Q, B be $m \times m, n \times n, n \times m$ resp. complex matrices with P and Q positive definite. Show that $P - B^*Q^{-1}B$ is positive definite if and only if $Q - BP^{-1}B^*$ is positive definite.

I. *Solution by Thomas Foregger, Bell Telephone Laboratories, Murray Hill, New Jersey.* Let $P = S^*S, Q = TT^*$ where S and T are nonsingular. Put $Z = T^{-1}BS^{-1}$. The relations

$$S^{-1*}(P - B^*Q^{-1}B)S^{-1} = I_m - Z^*Z \quad \text{and} \quad T^{-1}(Q - BP^{-1}B^*)T^{-1*} = I_n - ZZ^*,$$

together with the fact that Z^*Z and ZZ^* have the same nonzero eigenvalues, show that $P - B^*Q^{-1}B$ is positive definite if and only if the eigenvalues of Z^*Z are less than 1 if and only if the eigenvalues of ZZ^* are less than 1 if and only if $Q - BP^{-1}B^*$ is positive definite.

II. *Solution by M. F. Smiley, State University of New York at Albany.* Both conditions are equivalent to the requirement that the matrix

$$R = \begin{bmatrix} P & B^* \\ B & Q \end{bmatrix}$$

be positive definite. In fact, if

$$S = \begin{bmatrix} I_m & 0 \\ -Q^{-1}B & I_n \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} I_m & -P^{-1}B^* \\ 0 & I_n \end{bmatrix}$$

then $S^*RS = \operatorname{diag}(P - B^*Q^{-1}B, Q)$ and $T^*RT = \operatorname{diag}(P, Q - BP^{-1}B^*)$.

Also solved by K. V. Bhagwat & R. Subramanian (India), A. J. Bosch (Netherlands), Red Cougar, L. S. de Jong (Netherlands), Peter Gibson, Clark Givens, A. A. Jagers (Netherlands), Edward Keller, Henry Lieberman, Thomas Markham, Carl Meyer, Jr., James Modeer & John Burns, Ingram Olkin, Robert Singleton, Denmead Smith (England), Gerald Stinson, Tavan Trent, William Watkins, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Linear Algebra and its Applications. By Gilbert Strang. Academic Press, New York, 1976. xi + 374 pp. \$11.95. (Telegraphic Review, June-July 1976.)

"I believe that the teaching of linear algebra has become too abstract," says the author in the first

sentence of his preface. He has clearly thought deeply about this problem and how to address it, and has admirably succeeded with this book.

For many years now the instructor of an introductory linear algebra course has been faced with a choice among ten to twenty books, all of which were more or less abstract and rigorous: the Russian translations (Shilov, Mal'cev) and the American texts which followed upon the heels of Halmos's classic *Finite Dimensional Vector Spaces.* Having taught linear algebra several times, I was always faced with a difficult choice: which among these many abstract-oriented texts could the students best understand?

This question is an important one because many texts seem to be written more for one's colleagues than one's students; furthermore, in the last ten or twelve years, students have accelerated their lamentable tendency to shy away from abstraction. One has only to observe that many, perhaps most, students of engineering seem to be getting their whole exposure to such important topics as linear algebra (not to mention complex variables, Laplace transforms, Fourier series...) solely from their engineering courses and solely in the context of applications, to realize that perhaps we mathematicians are partly to blame.

Now Strang's book comes along to fill the gap, and it is refreshingly original—the first really new linear algebra text in many years. Once the book has been completed, all the standard topics have been covered, to be sure. Some of the material that a purist might consider the heart of the course has been relegated to two appendices: one on linear transformations and their matrices with respect to a basis, and the other on the Jordan form. The book does not begin as most do, with an abstract definition of a linear vector space, and nowhere does one appear. The treatment of vector spaces actually begins Chapter 2 (page 44), and it is only after two pages of examples that Strang formally cites closure under addition and scalar multiplication. All discussion is in the context of Euclidean n-space rather than a more abstract setting.

Do not confuse lack of abstraction with lack of rigor, however. This book has its share of theorems and proofs which I find amply rigorous, though so well motivated by preceding examples and helped along by concrete discussion that they seem almost effortless.

The first three chapters thoroughly study the theory and practice of solving simultaneous linear equations: Gaussian elimination for $n \times n$ systems, underdetermined systems and the four fundamental subspaces, and orthogonal projections and least squares. Chapter 4 on determinants leads nicely into Chapters 5 and 6 on eigenvalues and eigenvectors, and positive definite matrices, respectively. There is a brief but lucid chapter on computational methods, then a final chapter on linear programming and game theory.

The text is heavily larded with examples and applications, some of them highly interesting. One wishes that there were more, but the book takes eight quarter hours or about six semester hours to cover as it is.

My criticisms are few. There should be many more problems; I had to assign virtually every one to give my students enough to do, and in some spots (Jordan forms and linear programming, for instance) I had to make up supplementary problems. And finally, some examples were overly trivial (in Appendix B, he reduces a matrix to Jordan form which is already in that form). I had to devise slightly more complicated examples for my class.

In general, however, Strang writes well and lucidly. The book is well edited and has no more than a few misprints. My students thoroughly enjoyed the book and found it eminently readable. I think it possible that some of my more theoretical-minded colleagues might not appreciate the approach, but I unhesitatingly recommend Strang's book to any who have grown jaded by the sameness of so many linear algebra texts.

ZANE C. MOTTELER, Michigan Technological University

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook   P = professional reading
S = supplementary reading   L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL. *Jahrbuch Überblicke Mathematik 1977*. Benno Fuchssteiner, *et al.* Bibliographisches Inst, 1977, 180 pp, 28 DM (P). Third annual volume of exemplary expository essays, with introductory articles on control theory, perfect codes, and universal algebra. Also included are reports on mathematical education in computer science, plus brief notes commemorating solution of the four color problem and the anniversaries of Euler squares and of Gauss. PJC

GENERAL, T(13-16: 2), S, L. *Mathematics: The Alphabet of Science, Third Edition*. Margaret F. Willerding, Harold S. Engelsohn. Wiley, 1977, xiii + 651 pp, $14.50. Text for students who "wish to know what mathematics is about, but who have no desire to become mathematicians." (First edition, TR November 1968; Second Edition, TR October 1972.) Many chapters have been revised, and a new chapter on the use of hand-held and programmable calculators has been added. JEG

GENERAL, L. *Science and Technology in East Asia*. Ed: Nathan Sivin. Sci History Pub, 1977, xxiv + 260 pp, $5.95 (P); $9.95. A collection of 19 articles from *ISIS* dealing primarily with aspects of Chinese science and technology under the headings of Quantitative Sciences (mathematics), Qualitative Sciences (dietetics, astrology, alchemy), Technology (basically weaponry) and Cultural Interaction. Of special interest to mathematicians is a biography of Hua Lo-Keng by S. Salaff which appeared in the June 1972 issue of *ISIS*. SG

GENERAL, P, L. *20 Lectures Delivered at the International Congress of Mathematicians in Vancouver, 1974*. Amer. Math. Soc. Transl., Ser. 2, V. 109. AMS, 1977, iv + 129 pp, $20.

GENERAL. *Mathematics Encyclopedia*. Ed: Max S. Shapiro. Doubleday, 1977, ix + 289 pp, $5.95 (P). Covers math through the college level. Provides a detailed discussion of most math terms, biographies of many famous mathematicians, and some historical references. Lack of references and bibliography severely hampers it from being of real use to the serious answer-seeker. Many illustrations and special reference tables. TLS

PRECALCULUS, T(13). *Algebra for College Students*. Ignacio Bello. Saunders, 1977, x + 701 pp, $14.95; *Student's Study Guide to Accompany Bello's Algebra for College Students*, James R. Gard. viii + 214 pp, $4.95 (P). Intended for students with little or no background in the subject. Covers the real number system, linear equations, polynomials, fractions, exponents, quadratic equations, the Cartesian plane, systems of equations, relations and functions, exponential and logarithmic sequences and series. Each section is followed by a progress test and many exercises. SG

PRECALCULUS, T(13: 1). *Essentials of Trigonometry, Second Edition*. Irving Drooyan, Walter Hadel, Charles C. Carico. Macmillan, 1977, viii + 328 pp, $12.50. Methodical presentation. Many examples and exercises. Applications to vectors, circular motion, simple harmonic motion. Discusses inverse functions and complex numbers. LH

PRECALCULUS, T*(13: 1). *Algebra and Trigonometry*. Bernard J. Rice, Jerry D. Strange. Prindle, 1977, xii + 519 pp, $13.95. Connects elementary mathematical ideas to practical physical problems. Large number of worked examples and exercises, many involving applications. A test of manipulative skills and a test of concepts at end of each chapter. LH

EDUCATION, P. *Mathematics Projects Handbook*. Adrien L. Hess. NCTM, 1977, i + 46 pp, $2 (P). A skimpy, dated outline of ideas, criteria and resources for secondary school projects. References are mostly quite old: very few articles or books published since 1970 are mentioned. LAS

EDUCATION, T(13: 1), P*. *Mathematics: A Good Beginning, Strategies for Teaching Children*. Andria P. Troutman, Betty K. Lichtenberg. Brooks/Cole, 1977, x + 598 pp, $12.95 (P). Teachers' manual approach emphasizing concrete and pictorial representation for preservice and in-service elementary teachers; discussions of concepts and skills are augmented by material sheets and suggested activities. Lists sources of additional materials and numerous research references. JNC

HISTORY, P, L. *Selected Mathematical Papers of Axel Thue*. Ed: Trygve Nagell, *et al.* Universitetsforlaget (Oslo), 1977, lviv + 592 pp, $40. Papers on number theory and logic, supplemented by a biographical sketch (in English) written by Viggo Brun, an introduction (in German) by Carl Ludwig Siegel, a reprint of Siegel's 1970 analysis of Thue's seminal work on approximation of algebraic numbers and Diophantine equations, a bibliography of Thue's work (primarily on geometry and mechanics) not included in the present volume, and brief English summaries of the Norwegian papers reprinted in this volume. Includes, of course, Thue's paper (in Norwegian) proving his famous theorem on the solutions of $F(x,y) = c$ in integers for polynomials F of degree greater than or equal to 3. LAS

HISTORY, S*, L. *Adventures of a Mathematician*. S.M. Ulam. Scribner's, 1976, xi + 317 pp, $4.95 (P). Paperback edition of the well-known 1976 hard-cover volume. LAS

FOUNDATIONS, T(16-17), S, L. *Foundations of Mathematical Logic*. Haskell B. Curry. Dover, 1977, viii + 408 pp, $6 (P). A reprinting of the 1963 text, incorporating several minor corrections. MU

FOUNDATIONS, S(13). *Logic & Boolean Algebra with Computer Applications.* Mary L. Leach. Wills Pub, 1977, viii + 96 pp, $3.50 (P). Two chapters on symbolic logic: an elementary but thorough treatment. Chapter on logic, set theory and Boolean algebra. Computer applications of logic in circuits, number bases, and algorithms. Many diagrams, examples, and exercises. Easy to use in a self-study format. Short bibliography. Answers to selected exercises. Index. RJA

FOUNDATIONS, T(15: 1), S, L. *Beginning Model Theory, The Completeness Theorem and Some Consequences.* Jane Bridge. Clarendon Pr, 1977, viii + 143 pp, $10. Traditional formal presentation of completeness (Henkin style), compactness, and standard applications in model theory. Intended for undergraduates with previous introduction to propositional and predicate calculus and formalized proofs. Fair number of examples and exercises scattered in text. Does well to begin with relational structures, but proceeds quickly to formal languages. One wonders whether the formal approach is best suited for an undergraduate introduction. GM

FOUNDATIONS, S(18), P. *Choice Sequences, A Chapter of Intuitionistic Mathematics.* A.S. Troelstra. Clarendon Pr, 1977, ix + 170 pp, $10.95. Lecture notes for a series delivered at Oxford, 1973. MU

FOUNDATIONS, P. *Non-Classical Logics, Model Theory and Computability.* Ed: A.I. Arruda, N.C.A. da Costa, R. Chuaqui. Stud. in Logic and Found. of Math., V. 89. North-Holland, 1977, xviii + 307 pp, $28.75. Proceedings of the Third Latin-American Symposium on Mathematical Logic, Campinas, Brazil, 1976. Research reports in the three areas indicated by the title. One survey article by M. Benda on some new areas of research in model theory. GM

FOUNDATIONS, P. *Implication, Endometry, Universe of Discourse.* Franco Spisani. Centro Superiore di Logica, 1977, 174 pp, (P). A dialectical discourse leading to the construction of endometric structures. Bilingual, Italian and English on opposing pages. MU

FOUNDATIONS, T*(14-16: 1, 2). *Elements of Set Theory.* Herbert B. Enderton. Acad Pr, 1977, xiv + 279 pp, $12.95. An introductory text covering the usual topics fairly thoroughly in nonpretentious and very readable language. The progression of material is well motivated. Includes a generous mixture of interesting exercises. MU

COMBINATORICS, P. *Combinatorial Surveys: Proceedings of the Sixth British Combinatorial Conference.* Ed: Peter J. Cameron. Acad Pr, 1977, vii + 226 pp, $13.65. Seven of the nine invited survey papers published in advance of the July 1977 conference at Royal Holloway College. Prevailing themes: designs, matroids and projective spaces. LAS

NUMBER THEORY, T*(17: 1, 2), S, P, L*. *Fermat's Last Theorem: A Genetic Introduction to Algebraic Number Theory.* Harold M. Edwards. Grad. Texts in Math., V. 50. Springer-Verlag, 1977, xv + 410 pp, $19.80. This superbly written text traces the development of algebraic number theory by looking at the work of Fermat, Euler, Lagrange, Legendre, Gauss, Dirichlet, Kummer, and others. Lots of good problems. Reads like a novel in many places. An outstanding book. CEC

NUMBER THEORY, P. *Small Fractional Parts of Polynomials.* Wolfgang M. Schmidt. CBMS Reg. Conf. in Math., No. 32. AMS, 1977, v + 41 pp, $7.20 (P). Lecture notes from a conference which took place in July, 1976. The fractional parts of certain polynomials with real coefficients evaluated on integers are studied. CEC

ALGEBRA, P. *Computers in Nonassociative Rings and Algebras.* Ed: Robert E. Beck, Bernard Kolman. Acad Pr, 1977, ix + 297 pp, $14. 10 papers from a Special Session held at the San Antonio AMS meeting in January 1976 describing various uses of computer methods in research on, e.g., the structure of Lie algebras. LAS

ALGEBRA, P. *Modules with Cores and Amalgamations of Indecomposable Modules.* Robert Gordon, Edward L. Green. Memoirs No. 187. AMS, 1977, viii + 145 pp, $8 (P). The authors study a class of indecomposable modules over a ring, namely modules with cores: they classify rings whose indecomposable modules have cores. Their work ties up with that of Gabriel, Dlab, Ringel, Auslander, Reiten, and Nazarova. SG

ALGEBRA, P. *Lecture Notes in Mathematics-579: Combinatoire et Représentation du Groupe Symétrique.* Ed: D. Foata. Springer-Verlag, 1977, iv + 399 pp, $13.70 (P). A collection of 16 papers falling under four general headings: Young tableaux, symmetric functions, characters of permutation groups, combinatorial algorithms and partitions. Of interest to combinatorists and group theorists. SG

ALGEBRA, T(16-17). *Modern Abstract Algebra.* Yutze Chow. Gordon, 1976. *V. I, Monoids, Groups and Rings,* xx + 407 pp; *V. II, Modules, Linear Endormorphisms and Algebras,* xx + 333 pp, $60 set. A two-volume introduction at a formidable price. Each volume has three chapters (I: monoids and semigroups, rings; II: modules, module homomorphisms, algebras). The approach is in the categorical spirit stressing "equivalence of viewpoints, conceptual generalities." Not many exercises. SG

ALGEBRA, P**, L*. *Linear Representations of Finite Groups.* Jean-Pierre Serre. Grad. Texts in Math., V. 42. Trans: Leonard L. Scott. Springer-Verlag, 1977, x + 170 pp, $12.80. Translation of French edition, *Réprésentations Linéaires des Groupes Finis,* 1971. Section one, written for quantum chemists, requires only minimal knowledge of groups and linear algebra. Serre's lucid style makes this a superior introduction to the subject. Sections two and three require some familiarity with graduate algebra. JEG

ALGEBRA, P. *Combinatorial Group Theory.* Roger C. Lyndon, Paul E. Schupp. Ergebnisse der Math., B. 89. Springer-Verlag, 1977, xiv + 339 pp, $33.20. An in-depth study of free groups, including sections on generators and relations, geometric complexes, free products, word problems and small cancellation theory. Extensive bibliography. JEG

FINITE MATHEMATICS, T(13: 1). *Elementary Systems Mathematics: Linear Programming for Business and the Social Sciences.* Robert E. Machol. McGraw, 1976, xviii + 267 pp, $14.95. Topics include Gaussian elimination, linear programming, duality, and the transportation problem. Nicely written with careful attention to problem sets. LLK

FINITE MATHEMATICS, T(13: 1). *A Computer Approach to Introductory College Mathematics.* Frank Scalzo, Rowland Hughes. Petrocelli/Charter, 1977, 366 pp, $13.95. This text covers the Basic programming language, topics from algebra, matrices and linear programming, symbolic logic, and elementary probability and statistics. Every topic includes prepackaged computer programs to solve related problems. LLK

FINITE MATHEMATICS, T(13: 1). *Mathematics, Tools and Models.* Dalton R. Hunkins, Thomas L. Pirnot. A-W, 1977, xiv + 393 pp, $13.95. Basic ideas from graph theory, linear programming, systems of equations, probability and statistics introduced as models of well-chosen practical problems. Elementary level, e.g., only graphical solutions to linear programming problems. Lively style. LH

FINITE MATHEMATICS, T(13: 2). *College Mathematics for Students of Business and the Social Sciences.* Dennis G. Zill, *et al.* Wadsworth, 1977, xii + 487 pp, $14.95. Text broken into very short sections each followed by many exercises. Relies more on examples than in explanation. Light on word problems. Systems of equations, matrices, sequences and series, linear programming, finance, probability, calculus. LH

FINITE MATHEMATICS, T(13-14: 1, 2). *Finite Mathematics Applied.* Clement E. Falbo. Wadsworth, 1977, xii + 574 pp, $14.95. The selection of topics (including linear programming, probability, Markov chains, statistics, and game theory) is adaptable to a number of different one-term courses or may be covered *in toto* in two terms. Multiple regression is treated in some detail. The recommended prerequisite material of one year of high-school algebra (or equivalent in college) should be fresh in the students' minds. PJC

CALCULUS, S(13), P, L. *Calculus: Readings from the Mathematics Teacher.* Ed: Louise S. Grinstein, Brenda Michaels. NCTM, 1977, vii + 230 pp, $6 (P). 41 reprinted articles plus bibliographic references, arranged in topics paralleling the MAA volume *Selected Papers on Calculus.* A valuable source of ideas for the ubiquitous calculus class. LAS

CALCULUS, T(13-14: 2). *Calculus With Applications in the Management and Social Sciences.* William W. Thompson, Jr. P-H, 1977, vi + 505 pp, $13.95. Explanations clear, well-illustrated, but applications removed from and follow a more complete mathematical development than title suggests. Few word problems. Includes multivariate calculus applied to joint density functions. LH

CALCULUS, T(13). *Introductory Calculus for Business, Economics, and Social Science.* Dennis G. Zill. Wadsworth, 1977, x + 377 pp, $13.95. Little justification of methods, many examples, straightforward problems. Most applications are to economics. Includes partial derivatives and constrained extrema. LH

CALCULUS, T(13-14: 2). *Finite Mathematics and Calculus, Applications in Business and the Social and Life Sciences.* Hugh G. Campbell, Robert E. Spencer. Macmillan, 1977, xi + 714 pp, $13.95. Examples demonstrate concern for motivation. Large section on calculus including application of calculus to statistics. Includes probability, matrices, linear programming, statistics, game theory. LH

REAL ANALYSIS, P. *Vector Measures.* J. Diestel, J.J. Uhl, Jr. Math. Surveys, No. 15. AMS, 1977, xiii + 322 pp, $35.60. A survey of recent work on extension of classical real variable theory to functions from a Euclidean space to a Banach space. Generalizations of the Radon-Nikodým theorem play a central role. LAS

COMPLEX ANALYSIS, P. *Bibliography of Schlicht Functions, Part II (1966-1975).* S.D. Bernardi. Courant Inst, 1977, x + 168 pp, (P). A continuation of Part I (TR, August-September 1967), adding 1563 references on analytic univalent and multivalent mappings of singly and multiply connected domains. Papers are listed and numbered alphabetically, then grouped (by number only) into ninety subtopics; each alphabetical listing includes codes to appropriate subtopics, and reference to *Mathematical Reviews.* LAS

DIFFERENTIAL EQUATIONS, P. *Boundary Value Problems for Linear Evolution Partial Differential Equations.* Ed: H.G. Garnir. Reidel, 1977, xiv + 473 pp, $39.50. Ten major survey papers from the NATO Advanced Study Institute held in Liège, Belgium, in September 1976. LAS

DIFFERENTIAL EQUATIONS, P. *Proceedings of the Conference on Stochastic Differential Equations and Applications.* Ed: J. David Mason. Acad Pr, 1977, ix + 253 pp, $10.50. Eleven papers from a conference held in Park City, Utah in February, 1976. Reproduced from typescript. LAS

DIFFERENTIAL EQUATIONS, P. *Boundary Value Problems of Mathematical Physics, IX.* Ed: O.A. Ladyženskaja. Proc. of Steklov Inst. of Math., No. 127. AMS, 1977, 179 pp, $25.60 (P). Translation of the 1975 volume: nine research papers on partial differential equations. LAS

DIFFERENTIAL EQUATIONS, P. *Control Theory of Systems Governed by Partial Differential Equations.* Ed: A.K. Aziz, J.W. Wingate, M.J. Balas. Acad Pr, 1977, ix + 278 pp, $13. Proceedings of the conference on the title topic held at the Naval Surface Weapons Center (White Oak) in Silver Spring, Maryland, May 3-7, 1976. JAS

DIFFERENTIAL EQUATIONS, P. *Stability Theory by Liapunov's Direct Method.* N. Rouche, P. Habets, M. Laloy. Appl. Math. Sci., V. 22. Springer-Verlag, 1977, xii + 396 pp, $14.80 (P). Techniques for determining stability of solutions to ordinary differential equations. JEG

DIFFERENTIAL EQUATIONS, T*(17-18: 1), S, P. *Principles of Differential and Integral Equations.* C. Corduneanu. Chelsea, 1977, xii + 205 pp, $9.50. A well-written sophisticated introduction to differential and integral equations. Although the author claims advanced calculus is the only prerequisite, it is difficult to imagine the average undergraduate handling this book. It contains many good theoretical exercises but essentially no examples or applications. CEC

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-596: Ordinary Differential Equations in Banach Spaces.* Klaus Deimling. Springer-Verlag, 1977, vi + 136 pp, $8 (P). Expanded from the author's lectures in a graduate course, this is a study of countable systems of ordinary differential equations in the context of functional analysis. Well-motivated and replete with examples, informal remarks, and references to the literature (199 sources in the bibliography). TRS

DIFFERENTIAL EQUATIONS, P. *Qualitative Analysis of Large Scale Dynamical Systems.* Anthony N. Michel, Richard K. Miller. Math. in Sci. and Eng., V. 134. Acad Pr, 1977, xv + 289 pp, $22.50. A qualitative study (considering aspects such as Lyapunov stability, Lagrange stability, trajectory behavior) of a variety of dynamical systems such as those representable by ordinary differential equations, ordinary difference equations, stochastic differential equations, functional differential equations, Volterra integrodifferential equations, and certain classes of partial differential equations. Some background material together with fairly extensive notes and bibliography are included. SG

DIFFERENTIAL EQUATIONS, P. *The Hamilton-Jacobi Equation, A Global Approach.* Stanley H. Benton, Jr. Math. in Sci. and Eng., V. 131. Acad Pr, 1977, xi + 147 pp, $13.50. A comprehensive study of the equation $u_t + H(t,x,u_x) = 0$, where $x \in R^n$, $t \in R$, H is a given function of $2n + 1$ variables, and u_x is the gradient of the unknown function $u(x,t)$. Topics: classical methods, existence and uniqueness, applications, and numerical methods. A background in partial differential equations is assumed. Exercises. Bibliography contains 273 sources. TRS

NUMERICAL ANALYSIS, T(15-16: 1, 2), S, L. *Numerical Analysis.* Lee W. Johnson, R. Dean Riess. A-W, 1977, xi + 370 pp, $14.95. Includes linear systems of equations, eigenvalue problems, nonlinear equations, interpolation and approximation, numerical integration and differentiation, ordinary differential equations, and some optimization techniques. Clear and precise mathematical development. Many examples and exercises. A few computer programs. Index. Prerequisites: calculus and some understanding of the use of matrices. RJA

FUNCTIONAL ANALYSIS, P. *A Functional Calculus for Subnormal Operators II.* John B. Conway, Robert F. Olin. Memoirs No. 184. AMS, 1977, vii + 61 pp, $6.80 (P). The functional calculus studied goes like this: if S is a subnormal operator on a Hilbert space H, then for f in the weak*-closure of the polynomials in $L^1(m)^*$, where m is a scalar spectral measure of the minimal normal extension N of S, one defines $f(S) = f(N)|H$. Several properties, including a spectral mapping theorem, are carefully developed. Short list of unsolved problems. TRS

FUNCTIONAL ANALYSIS, T(18: 2), S, P, L. *Perturbation Theory for Linear Operators, Second Edition.* Tosio Kato. Grund. math. Wissenschaften, B. 132. Springer-Verlag, 1976, xxi + 619 pp, $39.80. Little change from first edition (TR, August 1967). A few paragraphs rewritten, minor errors corrected, supplementary notes added, bibliography expanded. LH

FUNCTIONAL ANALYSIS, T*(18: 1), P. *Completeness and Basis Properties of Sets of Special Functions.* J.R. Higgins. Tracts in Math., No. 72. Cambridge U Pr, 1977, x + 134 pp, $19.95. A splendid exposition of practical methods for testing sets of special functions for completeness and basis properties, mostly in L^p and L^2. Topics include: completeness criteria of Vitali and of Dalzell, stability of bases, eigenfunctions and boundary value problems. Many exercises, applications, and references. Written for, and accessible to, a broad audience. TRS

FUNCTIONAL ANALYSIS, T(18: 1), S, P. *Spectral Synthesis.* John J. Benedetto. Pure and Appl. Math., V. 66. Acad Pr, 1975, 278 pp, $27.50. Reconstruction of a c-valued integrable function on a Hausdorff locally compact Abelian group from the elementary "waves" in its spectrum. Surveys development of the field, discusses related areas and outstanding problems. Large bibliography. LH

FUNCTIONAL ANALYSIS, P. *Fourier Analysis and Function Spaces.* Hans Triebel. B.G. Teubner, 1977, 168 pp, 17,50 M (P). Selected topics. MU

FUNCTIONAL ANALYSIS, T(18), S, P. *Interpolation Spaces, An Introduction.* Jöran Bergh, Jörgen Löfström. Grund. math. Wissenschaften, B. 233. Springer-Verlag, 1976, 207 pp, $24.60. An extensive treatment of interpolation spaces including many exercises and historical notes. MU

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-582: Induced Representations and Banach *-Algebraic Bundles.* J.M.G. Fell. Springer-Verlag, 1977, 349 pp, $13.70 (P).

FUNCTIONAL ANALYSIS, P. *Banach Spaces of Analytic Functions and Absolutely Summing Operators.* Aleksander Pelczynski. CBMS Reg. Conf. in Math., No. 30. AMS, 1977, 91 pp, $7.60 (P). A revision of the author's lectures presented at Kent State, July 11-16, 1976. The disc algebra and its analogue in several complex variables are examined from the point of view of general Banach spaces and operator theory. Unsolved problems and extensive bibliography. TRS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Paul J. Campbell, Beloit; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Steven Galovich, Carleton; Jay E. Goldfeather, Carleton; Loren Haskins, Carleton; Lorraine L. Keller, St. Olaf; George Mills, St. Olaf; Thomas R. Savage, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Carleton College: Associate Professor David Appleyard, Chairman of the Mathematics Department, has been promoted to Professor and named Dean of Students; Professor F. L. Wolf has been named Chairman of the Mathematics Department; Assistant Professor S. P. Galovich has been promoted to Associate Professor.

University of Akron: Assistant Professor D. C. Buchthal has been promoted to Associate Professor; Dr. Louis Ross retired in June 1977 with the title of Professor Emeritus; Dr. Joseph Hintz, Gonzaga University, has been appointed Assistant Professor; Dr. W. H. Beyer, Head of the Department of Mathematics and Statistics, has been elected Chairman of the MAA Ohio Section for 1977–78.

Dr. Victorino S. Blanco, University of South Alabama, Mobile, retired in June 1977 with the title of Associate Professor Emeritus.

Assistant Professor D. C. Cathcart, Salisbury State College, has been promoted to Associate Professor.

Professor Shiing-Shen Chern has been elected as one of two Faculty Research Lecturers at the University of California, Berkeley, for the academic year 1977–78.

Professor William E. Ekman, University of South Dakota, Vermillion, has retired with the title of Professor Emeritus.

Associate Professor J. F. Firkins, Gonzaga University, Spokane, received the Distinguished Teacher of the Year Award and has been promoted to Professor.

Associate Professor Morton Goldberg, Broome Community College, has been promoted to Professor.

Dr. C. R. Hadlock, Bowdoin College, has been appointed to the Physical Systems Research Section at Arthur D. Little, Inc., Cambridge.

Associate Professor Norman Locksley, Prince George's Community College, Largo, Maryland, has been appointed Chairman of the Mathematics Department.

Instructor Alice G. Meissner, Chatham College, has been promoted to Assistant Professor.

Visiting Assistant Professor E. T. Ordman, Memphis State University, has been appointed Assistant Professor at New England College, Henniker, New Hampshire.

R. J. Serfling, Professor of Statistics at Florida State University, Tallahassee, will be on leave from September, 1977, until September, 1978, to serve as Statistical Advisor to the National Science Foundation, Division of Science Resources Studies, Washington, D. C.

Professor V. L. Shapiro has been elected *the* Faculty Research Lecturer at the University of California, Riverside, for the academic year 1977–78.

Associate Professor Gene Zirkel, Nassau Community College, has been promoted to Professor.

Professor Emeritus Edward W. Chittenden, University of Iowa, died on June 16, 1977, at the age of 91. He was a Charter Member of the Association.

Dr. Charles Fox, Montreal, Quebec, Canada, died on April 30, 1977. He was a member of the Association for 20 years.

Mr. Joseph B. Friedman, Albany, New York, died on April 3, 1977, at the age of 48. He was a member of the Association for 7 years.

Dr. Cornelius Gouwens, Iowa State University, died on July 26, 1975, at the age of 86. He was a Charter Member of the Association.

Mr. Joseph P. Heckl, U. S. Naval Ordinance Lab, Silver Spring, Maryland, died on April 4, 1977, at the age of 38. He was a member of the Association for 15 years.

Professor Franz E. Hohn, University of Illinois at Urbana-Champaign, died on July 10, 1977, at the age of 61. He was a member of the Association for 36 years.

Professor Emeritus Marston Morse, Institute for Advanced Study, died on June 22, 1977, at the age of 85. He was a member of the Association for 57 years.

Dr. Irwin Roman, Baltimore, Maryland, died on May 29, 1976, at the age of 83. He was a Charter Member of the Association.

Professor Richard J. Semple, Carleton University, Ottawa, Canada, died on February 18, 1977, at the age of 47. He was a member of the Association for 3 years.

MARSTON MORSE, 1892–1977

In order to memorialize Marston Morse some of his friends are planning to make contributions to the Institute for this purpose, and hope that others will join them. One idea being considered is to have one or more memorial lectures. Checks payable to the Institute for Advanced Study may be sent to Caroline Underwood, School of Mathematics Administrative Officer, Institute for Advanced Study, Princeton, NJ 08540.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The 1976 Fall meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at Virginia Commonwealth University on November 20, 1976. There were 86 persons in attendance, including 74 MAA members.

The principal speaker was Carolyn A. Maher of Rutgers University. The title of Professor Maher's presentation was "The Unification of Mathematics, Fiction or Reality?"

The participants were welcomed to the campus of Virginia Commonwealth University by Dr. P. D. Minton, Dean of the School of Arts and Sciences. Several members of the mathematics department of the host institution presided at the various sessions.

The chairman of the section, Professor Ronald Davis, presided over a brief business section. He announced that the spring meeting would be held at the University of Maryland on April 30, 1977.

The contributed papers included:

1. *Back-to-the-wall team play in the World Series*, by B. L. Schwartz, Analytical Services, Inc.
2. *A solution to a placement problem*, by Carla B. Oviatt and K. S. Weiner, Montgomery College.
3. *Mathematics in today's world*, by B. A. Fusaro, Salisbury State College.
4. *Applications of test surfunctions*, by John Schmeelk, Virginia Commonwealth University.
5. *The triangle is rigid, a direct proof*, by C. J. Maloney, Department of Health, Education, and Welfare.
6. *The hypervolume enclosed by the unit n-sphere goes to zero as the dimension goes to infinity*, by W. E. Hoover, U.S. Naval Air Test Center (Patuxent).
7. *Maximal separable subfields*, by Bonnie Page Danner, Virginia Commonwealth University.
8. *Continuous sup*, by G. M. Bryce, Randolph-Macon College.
9. *On the history of the generalized Stokes theorem*, by V. J. Katz, Federal City College.
10. *An intuitive approach to L'Hospital's rule*, by R. E. Allen, Virginia Commonwealth University.
11. *Which second-order linear integral recurrences have at most all prime divisors?* by Lawrence Somer, Washington, D.C.
12. *The solution of congruences by Euclid's algorithm*, by R. H. Anglin, Dan River, Inc.
13. *Generalized Hermite-Birkhoff interpolation*, by Jerrold Rosenbaum, Virginia Commonwealth University.

REUBEN C. DRAKE, *Secretary*

APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The 1977 Spring Meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held at the University of Maryland on April 30, 1977. There were 81 persons in attendance including 76 members.

The principal speaker was S. H. Schot of the American University. The title of Professor Schot's presentation was *Aberrancy: A Geometric Interpretation of the Third Derivative.*

The participants were graciously welcomed to the campus of the University of Maryland by the Mathematics Department. Several members of the department were available for assistance with the various sessions.

The Chairman of the Section, Professor Ronald Davis, presided over a brief business session. He announced that the Fall Meeting would be held at American University on November 19, 1977.

The contributed papers included:

1. *Reflections on mathematical talent, An exercise in the Verstehen approach*, by J. Fang, Old Dominion University.

2. *An interesting recurrence relation arising from a problem in optical physics*, by D. W. Lozier, National Bureau of Standards.
3. *A brief look at some generalizations of first countability*, by D. A. Schedler, Virginia Commonwealth University.
4. *Unstable polyhedral structures*, by Michael Goldberg, Washington, D.C.
5. *There are no unusual normed vector spaces in which bounded subsets are basically bounded*, by R. A. Herrmann, United States Naval Academy.
6. *Euclid's notion of numbers*, by S. S. Kutler, St. John's College.
7. *An approximation to Dawson's integral*, by T. S. Schreiber, Consultant, Fairfax, Virginia.
8. *Geometry without triangle congruence*, by C. J. Maloney, Department of Health, Education, and Welfare.
9. *An integral inequality through rearrangements*, by John Milcetich, University of the District of Columbia.
10. *Remarks on the binary Euclidean algorithm*, by G. W. Reitwiesner, Silver Spring, Maryland.
11. *Decoding of MCJ codes*, by H. M. Beck, Naval Research Laboratory.
12. *The Jacchia-Roberts atmospheric model*, by J. W. Kennedy and M. C. Fan, Computer Sciences Corporation.
13. *Approximating e via Cayley's formula*, by B. L.Golden, University of Maryland.

R. C. DRAKE, *Secretary*

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The thirty-sixth annual meeting of the Metropolitan New York section of the MAA was held at Sarah Lawrence College on April 24, 1977. One hundred twenty-five people attended.

Professor Gerald Freilich of Queens College, Chairperson of the Section, presided at the morning session which began with the business meeting. Professor William Zlot of New York University, Sectional Governor, reported on major items before the Board of Governors in the past year. Of particular interest, the proposed amendment to the Section By-Laws affecting the nomination and election of officers was approved by the Board of Governors. Then the committee chairpersons gave their reports as follows: Professor Helen Siner, the College of Staten Island, reported on the Section Committee for Two-Year Colleges; Professor Robert Bumcrot, Hofstra University, reported on the Mathematics Services Committee; Dr. Harry Ruderman, Hunter College High School, reported on the Greater Metropolitan New York Math Fair, and the MAA High School Contest.

The Section Award to the highest regional scorer in the Putman Competition was presented to Mr. Eric R. Jablow of Brooklyn College, CUNY.

Professor Israel Rose, Herbert H. Lehman College, presented the slate of nominees for Section Officers for 1977–79. There were no other nominees from the floor and the following were elected unanimously: Chairperson, Robert Bumcrot, Hofstra University; Vice-Chairperson for Colleges, Godfrey Isaacs, Lehman College; Vice-Chairperson for Two-Year Colleges, F. R. Buianouckas, Bronx Community College; Vice-Chairperson for High School, Alfred Kalfus, Babylon High School.

The main part of the morning session consisted of the following two lectures:

1. *A Little Analysis of Wind Instruments*, by H. E. Rauch, Graduate School and University Center, CUNY.
2. *How to Tell That a Simple Overhand Knot Is Really Knotted*, by E. E. Moise, Queens College, CUNY.

Professor Helen B. Siner of the College of Staten Island, Vice-Chairperson for Community Colleges, presided at the afternoon session. Professor Howard Kleiman, Queensborough Community College, reported on the Speakers' Bureau and gave the Treasurer's report.

The main part of the afternoon session was a panel discussion on *How I Teach Mathematics*. The panelists were: Professor John Impagliazzo, State University of New York at Farmingdale; Professor Melvin Hausner, Courant Institute, NYU; Mr. J. C. Hurley, Theodore Roosevelt High School; Dr. Harry Ruderman, Hunter College High School.

The afternoon session continued with the following twenty-one student and faculty papers given in five parallel sessions coordinated by Professor Samuel Graff, John Jay College of Criminal Justice:

1. *A new course: mathematics on the hand-held calculator*, by Leon Ablon, College of Staten Island.
2. *The Arkin–Hoggatt game*, by Joseph Arkin, New York Academy of Science.
3. *Extensions of the W. Mnich problem on rational sums and products*, by Haig Bohigian, John Jay College of Criminal Justice.
4. *Various schools of thought on probability*, by Allan Caesar, U. S. Merchant Marine Academy.
5. *Nonstandard analysis*, by Evan Cohn, Smithtown West High School.
6. *Extensions of meaning: what is a fractional derivative?* by Michael Ecker, Lehman College.
7. *Applicability criteria for the Monte Carlo solution of systems of linear equations*, by William Edelson, Riverside Research Institute, and Stanley Preiser, Polytechnic Institute of New York.
8. *Pythagoras' theorem for regular polytopes*, by Leon Gerber, St. John's University.

9. *Kepler's second law of planetary motion*, by Christopher Kingsley, Northport High School.
10. *Initial digit problem*, by David Laster, Bronx High School of Science.
11. *On the convergence of the series* $\Sigma_{n=1}^{\infty}((-1)^{[n\alpha]}/n)$, Leo Levine, Queensboro Community College.
12. *Asymptotic estimates in geometric number theory and extendible computer storage of planar arrays*, by John Lew and Arnold Rosenberg, Mathematical Sciences Department—I.B.M.
13. *An elementary proof of the law of anomalous numbers*, by James Peters, St. Bonaventure University.
14. *The existence of the infinitesimal*, by Keith Purcell.
15. *Primitive modulo prime*, by Jonathan Roberts, Bronx High School of Science.
16. *Use of the computer in undergraduate mathematics courses*, by F. Scalzo, Queensboro Community College, and A. Paullay, Bronx Community College.
17. *Asymptotic and formal solutions of ordinary differential equations*, by Arthur Schlissel, John Jay College of Criminal Justice.
18. *Space and time*, by Joel Schwartz, John F. Kennedy High School.
19. *The incremented totient summatory function*, by Paul Tartell, Stuyvesant High School.
20. *Some increasing functions of Baire class 1*, by Aaron Todd, Brooklyn College.
21. *Library search system*, by Robert Wilson, Syosset High School.

L. E. CHRIST, *Secretary*

APRIL MEETING OF THE NEBRASKA SECTION

The Annual Meeting of the Nebraska Section was held on Friday and Saturday, April 14 and 15, 1977, jointly with the Annual Meeting of the Nebraska Academy of Sciences, at Nebraska Wesleyan University, Lincoln.

Fifty persons were in attendance of whom thirty-five were members of the MAA.

Professor Stanley Luke, Chairman of the Section, reported on a program of visitation to two and four year colleges out of which a panel discussion was developed for the Annual Meeting. Professor R. C. Buck of the University of Wisconsin represented the MAA; he also presented two papers on the program. A report was given by Professor Mildred Gross, Sectional Governor.

Officers elected for 1977–1978 were: Chairman, P. A. Haeder, University of Nebraska at Omaha; Chairman-Elect, T. S. Shores, University of Nebraska-Lincoln; Chairman of High School Contest Committee, Stanley Luke, Nebraska Wesleyan University. H. M. Cox continues as Secretary-Treasurer for the second year of a three-year term.

The following papers were presented:

1. *Subgroups of the orthogonal group and their invariant surfaces: a geometric approach to representation theory of groups*, by S. A. Wiitala, Nebraska Wesleyan University.
2. *A definition for ordered pairs in class theory*, by H. H. Schneider, University of Nebraska-Lincoln.
3. *An extremal problem for a class of regular functions*, by P. D. Tuan, University of Nebraska at Omaha.
4. *Distributional supports of locally* 1′ *function*, by W. T. Franke, University of Nebraska-Lincoln.
5. *Cyclic quasi-Hadamard matrices*, by R. H. Kosloski, University of Nebraska at Omaha.
6. *Calculus with infinitesimals*, by G. H. Meisters, University of Nebraska-Lincoln.
7. *Functions of several variables: Hilbert's 13th and beyond*, by R. C. Buck, University of Wisconsin.
8. *Proportional reasoning as a predictor of success in science courses*, by M. A. Thornton, University of Nebraska-Lincoln.
9. *Existence theory for a quasilinear elliptic system*, by A. V Lair, University of South Dakota.
10. *A sufficiency test for constrained optimization*, by Stephen Montague, University of Nebraska at Omaha.
11. *Some intersection properties of cut sets of an irreducible connected subspace*, by Edwin Halfar, University of Nebraska-Lincoln.
12. *Results and statistical summary of the* 1977 *annual high school mathematics contest in Nebraska and South Dakota*, by L. J. Stephens, University of Nebraska at Omaha.
13. *Panel discussion: future curricula in collegiate mathematics*, by D. W. Behrens, J. S. Downing, J. A. Kaus, and J. F. Wampler.
14. *A role for speculation*, by R. C. Buck, University of Wisconsin.

An informal luncheon was held at the Nebraska Wesleyan Student Center.

H. M. COX, *Secretary*

MAY MEETING OF THE ILLINOIS SECTION

The Illinois Section of the MAA held its 56th annual meeting at Chicago Loop College on May 6–7, 1977.

Chairman John Christiano presided. Dr. Henry Alder, President of the Association, was our honored guest and keynote speaker at the annual banquet.

Invited addresses from Professor Robert Troyer of Lake Forest College on "Infinitesimal Calculus," and from Professor Vera Pless of University of Illinois at Chicago Circle on "Women in Mathematics" were presented. Other sessions and their topics included:

Ambiguity and indetermination for functions defined in a disk, by Peter Colwell, Iowa State University.

Instructional materials for intermediate algebra, by Al Otto, Illinois State University; John Bradburn, Elgin Community College; Ray Moehrlin, William Rainey Harper College.

Realizing an automaton, by Robert McFadden, Northern Illinois University.

The new student in mathematics classes, by William Drezdzon, Oakton Community College.

Professor Alder spoke on "Why we must and how we can improve the teaching of mathematics."

Committee reports were received and accepted at the annual business meeting, with Professor John Bradburn of Elgin Community College installed as Chairman for 77–78, Professor Gordon Mock of Western Illinois University named Chairman-elect (78–79), and Professor William Drezdzon of Oakton Community College elected 2nd Vice-Chairman for a two-year term.

H. C. SAAR, *Secretary*

MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held at Wabash College at Crawfordsville on Saturday, April 30, 1977, with approximately 90 persons in attendance. The Chairman of the Section, M.C. Gemignani of IU-PU-Indianapolis, presided over a program celebrating Gauss' 200th birthday.

The following papers were presented:

1. *Gauss: His life and times*, by P.T. Mielke, Wabash College.
2. *Infinite series from a contemporary point of view I*, by R.P. Boas, Northwestern University.
3. *Infinite series from a contemporary point of view II*, by R.P. Boas, Northwestern University.
4. *Who was the first non-Euclidean?*, by U. Dudley, DePauw University.
5. *Gauss and applied mathematics*, by E. Cumberbatch, Purdue University.

L.J. Cote, Purdue University, recognized James Keller, North Side High School, Fort Wayne, for solving problems appearing in the Indiana School Mathematics Journal.

At the business meeting Harold Hanes, Earlham College, as chairman of the Nominating Committee, presented the following slate of officers for 1977–78 (which was unanimously approved): Chairman, G.J. Sherman, Rose-Hulman Institute of Technology; Vice-Chairman, M. Jerison, Purdue University; Secretary-Treasurer, D.E. Wilson, Wabash College.

Memberships in the MAA were awarded to R.M. Priem and R.A. Strickland, both of Rose-Hulman Institute of Technology, in recognition of their performances on the Putnam examination.

D.E. WILSON, *Secretary-Treasurer*

MAY MEETING OF THE MICHIGAN SECTION

Approximately 100–120 persons attended the Annual Meeting of the Michigan Section on May 6 and 7, 1977, at Eastern Michigan University in Ypsilanti, Michigan.

An interesting session was a panel discussion of the mutual interests of the two-year and four-year colleges and universities in the State. Comments and responses from the discussants were enthusiastic, and a similar follow-up panel is hoped for a future meeting of the Section.

Dr. A.B. Willcox, Executive Director of the MAA, was an active participant of this year's meeting. He held a rap session with members during the Business Meeting to answer questions and comments. The following day he gave the closing address. Members enjoyed meeting our Executive Director and found his talks stimulating.

Professor J.S. Frame, who will be retiring from Michigan State University this year, was honored with the presentation of a certificate of resolution recognizing his numerous contributions to the Michigan Section and to the mathematical community at large.

In keeping with the Section practice of inviting speakers from out of state to our annual meetings, we are particularly pleased to report the excellent talks given by Dr. James Roseblade, Cambridge University, England, and Dr. Jack Hale, Brown University.

The following persons were elected to be officers of the Michigan Section for 1977–78: Chairperson: J.E.

Adney, Michigan State University; Vice-chairperson: D.G. Malm, Oakland University; Vice-chairperson: Donald Ross, Washtenaw Community College; Secretary-Treasurer: R.A. Chaffer, Central Michigan University.

Professor Yousef Alavi, Western Michigan University, was elected to serve a three-year term as governor of the Section beginning July 1, 1977. The balloting for the governor of the Section was conducted by mail by the Executive Director of the MAA.

The list of speakers at the Meeting and their topics follows. Invited speakers:

James Roseblade, Cambridge University, *Infinite groups.*

Jack Plotkin, Michigan State University, *Dare to err: Remarks on computational complexity.*

Jack Hale, Brown University, *Bifurcations and nonlinear oscillations.*

J.v. Iwaarden, Hope College, *Using the computer in teaching undergraduate mathematics courses.*

A.B. Willcox, Executive Director, MAA, *Some bridges to and from mathematics.*

Contributed papers:

M.B. Suryanarayana, Eastern Michigan University, *Generalized Fibonacci sequences and polynomials.*

J. Matti, Saginaw Valley State College, *Some number theoretic properties of the common calendar.*

John Whittle, Hope College, *Modularized approach to precalculus using a computerized test generator.*

Sister Mary Brechting, Aquinas College, *The effects of small group-discovery learning on student achievement and attitudes in calculus.*

John Kiltinen, Northern Michigan University, *What does* $(a+b)^m = a^m + b^m$ *imply about the characteristic of a ring?*

Student papers:

John Gimbel, Andrews University, *A mathematical model of a sewage treatment system.*

Peteris Graube, University of Michigan, *Pedagogical considerations in studying the equation* $y'' + y = 0$.

Bruce Herman, Hope College, *A user oriented linear regression package.*

Lee Kuivinen, Hope College, *Computer art using parametric equations.*

DELIA KOO, *Secretary-Treasurer*

MAY MEETING OF THE SEAWAY SECTION

The Spring Meeting of the Seaway Section of the MAA was held at State University of New York College at Buffalo on May 7, 1977, with a registered attendance of 81 people, including 65 members of the Association. Professor F.D. Parker, Chairman of the Section, presided.

At the morning session, Rudy Zimmer, Coordinator of the Mathematics Learning Centre, Fanshawe College, London, Ontario, gave a talk on his experiences with the Mathematics Learning Centre.

At the business meeting the following officers were elected: Chairman, P.T. Schaefer, State University College at Geneseo; First Vice-chairman, Violet Laraney, State University of New York at Albany; Second Vice-chairman, F.K. Harris, Alfred Ag. & Tech. It was announced that Dennis Martin, State University College at Brockport, had been appointed as the Section Chairman of the MAA High School Mathematics Contest.

It was announced that D.J. Wright, student at Cornell University, would receive a congratulatory check of $10 from the Section for scoring highest of anyone in the Section in this year's Putnam Mathematical Contest.

W.F. Lucas, Center for Applied Mathematics, Cornell University, presented the Harry M. Gehman Lecture, entitled "Mathematical Modeling: Examples and Course Materials"

The following contributed papers were presented during the afternoon:

Linear algebra in undergraduate statistics, by Harold Still, Queen's University.

Forms with 0-orthogonal Lie algebras, by Frank Servedio, McMaster University.

A computer lab in probability, by D.S. Martin, State University College at Brockport.

Coefficients of the cyclotomic polynomial $F_{pqr}(x)$, by Sister M. Beiter, Daemen College.

On the classification of critical points of real valued functions of several variables, by T.S. Bolis, State University College at Oneonta.

A structure theorem for partial isometries, by James Guyker, State University College at Buffalo.

Making meaning about making meaning: A dialog, by Larry Copes, Ithaca College, and David Pimm, Cornell University.

An elementary example of constructive versus non-constructive proofs, Northrup Fowler, Hamilton College.

EMMET STOPHER, *Secretary-Treasurer*

REPORT OF THE TREASURER FOR THE YEAR 1976

Herewith is a summary of the report of the Treasurer of the Association for the year 1976. In this summary, all entries have been rounded to the nearest dollar; therefore sums of entries may differ from the entered total. The full report has been approved by the Finance Committee and accepted by a vote of the Board of Governors. Any member of the Association who wishes to have a copy of the full report may obtain one by writing to the Washington Office of the Association.

ASSETS	*Dec. 31, 1976*
Cash	$ 73 511
Short-term investments	163 544
Securities (at cost), unrestricted	280 568
Securities (at cost), restricted	151 010
Accounts Receivable	136 637
Furniture and Equipment	19 470
Prepaid Expenses	21 537
Total Assets	$ 846 278
LIABILITIES	
Accounts Payable	$ 52 334
Unearned Income	
Dues and Subscriptions	324 566
Other	37 027
NSF Fund	(11 039)
High School Contest Fund	32 100
Total Liabilities	$ 434 988
Assets Minus Liabilities (Net Worth, including restricted funds)	$ 411 290

OPERATING INCOME	
Dues	$ 387 655
Publications	314 060
Dividends and Interest	29 222
Contributions	21 127
Registration Fees	14 710
Indirect Costs—Outside Agencies	28 130
Indirect Costs—High School Contest	6 000
Miscellaneous	8 008
Total Operating Income	$ 808 910

OPERATING EXPENDITURES	
Salaries	$ 259 579
Office Expenses	121 230
Publications (excl. salary & office exp.)	248 465
Travel and Meeting Expenses	58 871
Taxes and Fees	9 131
Dues and Contributions	21 448
Awards and Grants	12 954
Miscellaneous	6 568
Total Operating Expenditures	$ 738 246
Operating Income over (under) Operating Expenditures	$ 70 665

LEONARD GILLMAN, *Treasurer*

CALENDAR OF FUTURE MEETINGS

Sixty-first Annual Meeting, Atlanta, Georgia, January 6–8, 1978.

Fifty-eighth Summer Meeting, Brown University, Providence, August 8–10, 1978.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, University of Pittsburgh, Pennsylvania, April 14–15, 1978.

FLORIDA, St. Petersburg Junior College, Clearwater, March 3–4, 1978.

ILLINOIS, Western Illinois University, Macomb, May 5–6, 1978.

INDIANA, Earlham College, Richmond, April 22, 1978.

INTERMOUNTAIN

IOWA, University of Northern Iowa, Iowa Falls, April 22, 1978.

KANSAS, Wichita State University, Wichita, late March—early April 1978.

KENTUCKY, Northern Kentucky University, Highland Heights, April 7–8, 1978.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel-Motel, Biloxi, Mississippi, February 17–18, 1978.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Queensborough Community College, May 7, 1978.

MICHIGAN, Michigan State University, East Lansing, May 5–6, 1978.

MISSOURI, Central Missouri State University, Warrensburg, April 7–8, 1978.

NEBRASKA, University of Nebraska at Omaha, April 14–15, 1978.

NEW JERSEY, Steinhart High School, Trenton, April 28, 1978.

NORTH CENTRAL, College of St. Thomas, St. Paul, Minnesota, April 21–22, 1978.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, College of Notre Dame, Belmont, February 18, 1978.

OHIO, University of Akron, Akron, April 28–29, 1978.

OKLAHOMA-ARKANSAS, Henderson State University, Arkadelphia, Arkansas, March 31–April 1, 1978.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 16–17, 1978.

PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.

ROCKY MOUNTAIN, South Dakota School of Mines and Technology, Rapid City, April 28–29, 1978.

SEAWAY, Brock University, St. Catharines, Ontario, Canada, May 5–6, 1978.

SOUTHEASTERN, Clemson University, Clemson, South Carolina, March 31–April 1, 1978.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, New Mexico Institute of Mining and Technology, Socorro, Spring 1978.

TEXAS, Stephen F. Austin State University, Nacogdoches, March 31–April 1, 1978.

WISCONSIN, University of Wisconsin, Whitewater, late April 1978.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Washington, February 12–17, 1978.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, Atlanta, Georgia, January 4–7, 1978.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of British Columbia, Vancouver, June 19–22, 1978.

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC, Madison, Wisconsin, June 18–24, 1978.

ASSOCIATION FOR WOMEN IN MATHEMATICS, Atlanta, Georgia, January 4–8, 1978.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHEMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, San Diego, California, April 12–15, 1978.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, New York City, May 1–3, 1978 (Joint Meeting with the Institute of Management Sciences).

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SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

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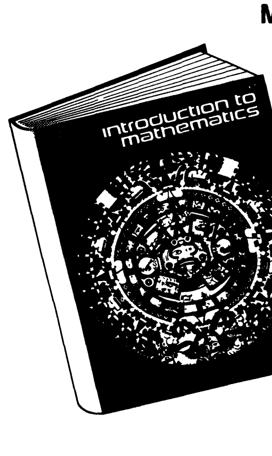

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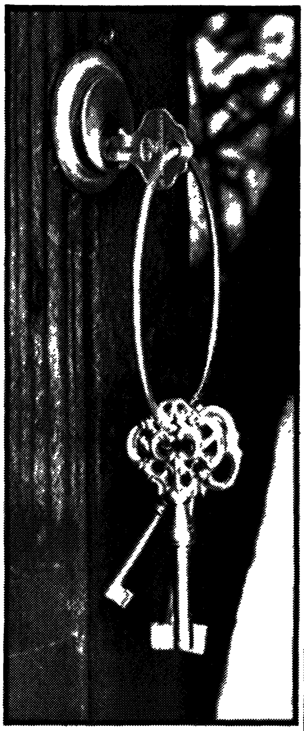

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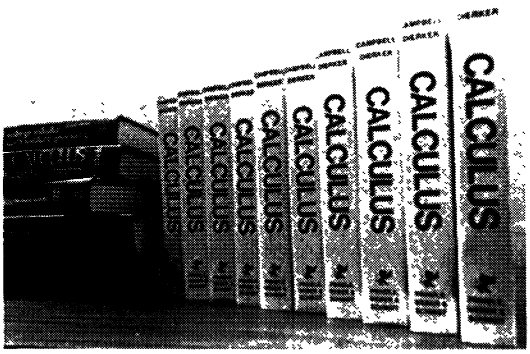

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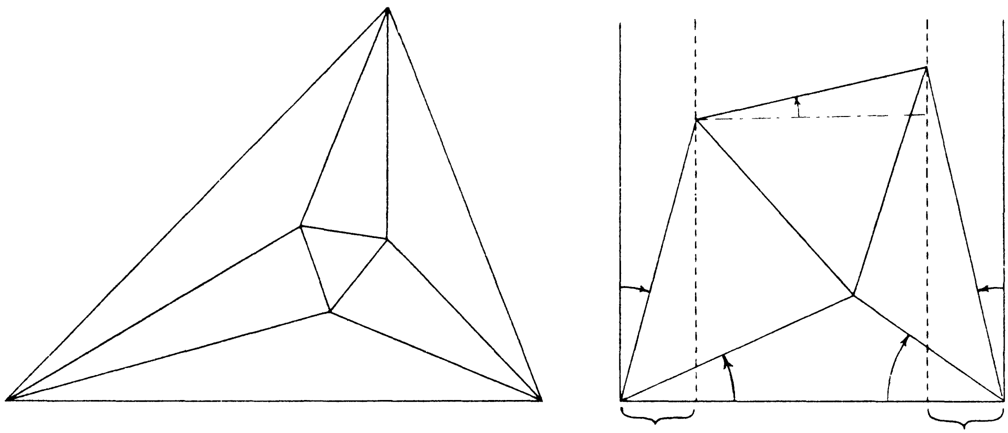

Morley's theorem — Generalized? (p. 100)

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AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR R. D. ANDERSON

Richard Davis Anderson has been one of the most active leaders of our time in promoting mathematics. He has worked through SMSG and CUPM to improve the mathematics curriculum at the secondary and collegiate levels. As an individual teacher and through committees he has promoted strong undergraduate and graduate programs in mathematics. His analyses of the job market have alerted us to the consequences of a heavy Ph.D. production. His emphasis of the long range demographic factors in academic employment has helped the mathematical community and many of its members adjust to the changed conditions of the seventies. His success at research, his interest in people, and his participation in worldwide conferences have made him an effective international spokesman for our national mathematical community.

He is known professionally as R. D. Anderson, but his friends call him Dick. Dick obtained his bachelor's degree from Minnesota in two years. At a meeting of the Association and the Society in the summer following his graduation he was recruited by Professor R. L. Moore to come to Texas. He arrived in Texas at the age of 19 to begin graduate work.

Anderson's work as a graduate student was interrupted by World War II. He served in the Navy as an officer on a submarine in the Atlantic and later as an engineering officer on two naval auxiliaries in the Pacific. With the war over in 1945, Dr. Moore immediately contacted his star student about returning to The University of Texas to resume his study.

We know Anderson as a fierce competitor whether it be at debate, tennis, chess, or mathematics. Competition had started early for Dick since he had a twin brother, Jack. However, Jack went into Chemistry and left Mathematics to Dick. Those doing graduate work in Topology at Texas during Anderson's time included Gail Young, R. H. Bing, Ed Moise, Ed Burgess and Mary Ellen Estill (later to become Mrs. Rudin)—not a slouch in the crowd. (Even as Anderson was completing his doctorate, Eldon Dyer, Mary Elizabeth Hamstrom and Billy Joe Ball were beginning to prove good theorems.)

Anderson was the only one of these who could nearly hold his own in heated debates with Ed Moise. I sometimes played chess with Anderson during our lunch hour at Texas, until Dr. Moore discouraged it. Dr. Moore asked me if I was not concerned that this recreation would rob Anderson of time that he might better spend on proving theorems. The thought had not occurred to me since my concern was in checkmating his king—which I was able to do only too infrequently.

After receiving his Ph.D., Anderson went through the ranks of Instructor, Assistant Professor and Associate Professor at Pennsylvania. During this period he spent two years at the Institute for Advanced Study. Eleven years after receiving his Ph.D. he had been named a Boyd Professor at LSU.

At LSU Anderson was very effective in setting up a program in Infinite Dimensional Topology. He and his students—including such stalwarts as Raymond Wong, Jim West and Tom Chapman—did fundamental work in this area which attracted many to attend LSU sponsored conferences.

Because of the depth of his mathematics and the charm of his personality Anderson is welcomed abroad. He has spent two sabbaticals in Holland and has made many visits to Eastern Europe. His rapport with foreign mathematicians is no accident. For example, in preparation for his first visit to Russia, he studied the Russian language for a considerable time. At last year's St. Louis meeting he wrote down a list from the top of his head of the names of about 25 Russian topologists (probably with the correct spelling) who should have copies of Chapman's book on Q-manifolds and arranged for them to be sent.

Anderson's influence has been boosted by the vigorous support of his wife Jeanette. They have made friends all over the world and are very effective ambassadors. Jeanette carries on correspondence with friends far and wide. Anderson's true spirit cannot be fully appreciated until one understands the concern of the Anderson family for others.

RICHARD DAVIS ANDERSON

Anderson has been on 25 Association committees dealing with such diverse topics as finances, employment, membership, developing colleges, visiting lecturers, consultants bureau, the undergraduate curriculum, and cooperation with other organizations. He led many of these committees and in particular chaired the important and very busy Committee on the Undergraduate Program in Mathematics (CUPM). Currently he is on the AMS–MAA–SIAM Joint Projects Committee in Mathematics.

Anderson does not restrict his contributions to MAA work. He was a vice-president of the Society and led its Committee on Employment and Educational Policy to study the changing job situation. Anderson is Chairman of the Board of Trustees of the Conference Board. For NSF he has been on committees giving advice on science education and minority programs. He earlier chaired the NSF Mathematics Advisory Panel. For GRE he is a member of the Panel on Graduate Record Examinations in Mathematics. For NRC he is a member of the Advisory Committee of the Office of Mathematical Sciences and was formerly on the Soviet and Eastern European Exchange Program Advisory Committee.

Anderson is a man who has served the mathematical community well. The Award for Distinguished Service in Mathematics is given for past performance, but in the case of a man as vigorous as Anderson we can look for additional contributions in the years that lie ahead.

R. H. Bing

AWARD OF THE CHAUVENET PRIZE TO PROFESSOR SHREERAM SHANKAR ABHYANKAR

The Board of Governors of the Mathematical Association of America voted to award the 1978 Chauvenet Prize to Professor Shreeram Shankar Abhyankar for his paper, *Historical Ramblings in Algebraic Geometry and Related Algebra*, which appeared in the American Mathematical Monthly, Vol. 83, pp. 409–448.

A certificate and monetary award in the amount of five hundred dollars was presented to Professor Abhyankar at the Business Meeting of the Association on January 7, 1978, in Atlanta, Georgia.

The Chauvenet Prize is awarded for a noteworthy paper of an expository or survey nature published in English, which comes within the range of profitable reading for members of the Association. The purpose of the Prize is to stimulate the writing of expository and survey articles. The 1978 Prize, awarded for a paper published in the three-year period 1974–76, is the twenty-sixth award of the Chauvenet Prize since its institution by the Association in 1925. For the list of names of the previous winners, see this Monthly, 71 (1964) 589; 72 (1965) 2–3; 74 (1967) 3; 75 (1968) 3–4; 77 (1970) 117–118; 78 (1971) 112–113; 79 (1972) 112–113; 80 (1973) 120; 81 (1974) 113–114; 82 (1975) 108–109; 83 (1976) 84–85, and 84 (1977) 417.

Professor Abhyankar was born on July 22, 1930, in Ujjain, India. He received the B.Sc. from Bombay University in 1951 and M.A. and Ph.D. degrees from Harvard University in 1952 and 1955, respectively. He has held regular faculty positions at Columbia, Cornell, Johns Hopkins, and Purdue and visiting faculty positions at Columbia, Princeton, Harvard, Münster, Erlangen, Matscience Madras, India, Yale, Tata Institute (Bombay), Leiden (Holland), and Kyoto (Japan). Currently he is a Distinguished Professor at Purdue University.

Professor Abhyankar has authored numerous papers in the area of algebraic geometry, commutative algebra, local algebra, theory of functions of several complex variables, and circuit theory. He has served as major professor for nine Ph.D. students.

Texts published by Professor Abhyankar are *Ramification Theoretic Methods in Algebraic Geometry* (Princeton University Press, 1959), *Local Analytic Geometry* (Academic Press, 1964),

Resolution of Singularities of Embedded Algebraic Surfaces (Academic Press, 1966), *A Glimpse of Algebraic Geometry* (University of Poona, 1971), *Algebraic Space Curves* (Les Presses de l'Université de Montréal, 1971), *Geometric Theory of Algebraic Space Curves* (with A.M. Sayhaye, Springer-Verlag, 1974), and *Expansion Techniques in Algebraic Geometry* (Tata Institute, 1977).

Professor Abhyankar has recently helped found BHASKARACHARYA PRATISHTHANA, a mathematical research institute at 39/5 Erandavane, Pune 4, India. Professor Abhyankar serves as Prime Member and Director of the Governing Body of BHASKARACHARYA PRATISHTHANA.

In his acceptance speech, Professor Abhyankar said, "I am very pleased and thankful for this honor. My attempt of writing readable mathematics was inspired by Bhaskaracharya's book on geometry called *Lilavati* which he composed in Sanskrit poetry around 1114 A.D. and through which my father initiated me to mathematics. With the aim of facilitating the cultivation and growth of mathematics in one of its original ancient homes, I have recently participated in founding the Bhaskaracharya Pratishthana, a mathematical institute in Pune in India, named after this great poet-mathematician. Hereby I appeal to all interested persons everywhere to help us in this venture by giving generous donations. Donations, payable to Bhaskaracharya Pratishthana, may be sent to me at: Mathematics Department, Purdue University, West Lafayette, IN 47907, U.S.A."

DAVID P. ROSELLE, *Secretary*

HOW TO GIVE AN EXPOSITION OF THE ČECH-ALEXANDER-SPANIER TYPE HOMOLOGY THEORY

W. S. MASSEY

1. Introduction. In spite of the fact that many books and papers speak glibly of "general homology theories," there are really only two types in existence: the *singular* type, and the *Čech-Alexander-Spanier* type.[1] These two types of theories give the same results on spaces that are "nice" locally, such as manifolds or simplicial polyhedra, but can give different results on more general spaces. There are several different ways of defining each.

Most graduate students in mathematics nowadays are exposed only to the singular type theory in their training, and with good reason: it is much easier to give a complete exposition of the singular type theory. Nevertheless, for some purposes, it is necessary to use the Čech-Alexander-Spanier type theory. We offer the following three examples to illustrate this point:

(a) The famous Alexander Duality Theorem asserts that for any compact subset X of Euclidean n-space, R^n, the (reduced) cohomology group $H^q(X)$ is naturally isomorphic to the (reduced) homology group, $H_{n-q-1}(R^n - X)$, of the complementary space. In this statement, $H_{n-q-1}(R^n - X)$ can refer to either the singular or Čech type homology group, because $R^n - X$ is a manifold. However, $H^q(X)$ *must* be chosen to be the Čech type cohomology group in order for the theorem to be true for arbitrary compact subsets X.

(b) There is an intimate relation between Čech-type homology or cohomology groups and dimension theory for topological spaces. For example, if X is any compact subset of Euclidean n-space, then the Čech-type homology and cohomology groups of X vanish in dimensions greater

1. In this paper, when we speak of "homology" or "cohomology", we are referring to an ordinary homology or cohomology theory defined on some category of topological spaces with an abelian group for coefficients. Thus we exclude from consideration homology or cohomology with local coefficients or with coefficients in a sheaf; we also exclude extraordinary homology and cohomology theories, such as K-theory.

than n. It is known that this statement is nowhere near true for the singular type homology and cohomology groups of X. There is an example, due to M. Barratt and J. Milnor, of a compact subset X of Euclidean 3-space such that the singular homology groups of X are non-zero in all dimensions greater than 1 (X is the union of a certain countable collection of 2-spheres).

(c) Let X denote a space which is locally nice, such as a manifold or simplicial polyhedron, let π denote an abelian group and n a positive integer. It is a well-known theorem that the homotopy classes of maps of X into the Eilenberg-Mac Lane space $K(\pi, n)$ are in natural 1-1 correspondence with the elements of the cohomology group $\check{H}^n(X, \pi)$. If we wish to generalize this theorem to more general spaces X, it is necessary to use the Čech-type cohomology theory; singular cohomology will not do.

In addition to being absolutely necessary for the correct statement of certain theorems, the Čech-type homology and cohomology theories offer certain technical advantages over the singular theory which make it easier to use in some situations. The most obvious of these advantages is the fact that the Čech-type theory satisfies a stronger version of the excision property than the singular type theory. For example, if X and Y are compact Hausdorff spaces, and A and B are closed subsets of X and Y respectively, then a continuous map $f: (X, A) \to (Y, B)$ which maps $X - A$ homeomorphically onto $Y - B$ induces isomorphisms $H_q(X, A) \approx H_q(Y, B)$ of the relative Čech homology and cohomology groups. This means that the relative Čech homology groups $H_q(X, A)$ depend only on the topological type of the complementary space $X - A$. It is easy to give examples to show that this need not be true for the singular homology and cohomology theory. This difference may also be expressed as follows: for the Čech type theory, *every* triad $(X; A, B)$ consisting of a compact Hausdorff space X and closed subsets A and B is *excisive*; for the singular theory this is not true.

It is the purpose of this paper to outline a new and improved method of exposition of the Čech-Alexander-Spanier type homology theory which makes it almost as easy to learn as the singular type. This new exposition depends crucially on a theorem proved by G. Nöbeling **[12]** in 1968.

2. Review of the usual treatment of Alexander–Spanier cohomology theory. If one is willing to restrict oneself to the use of *cohomology* theory (i.e., avoid any reference to homology theory), there is an easy method of exposition described in Spanier's thesis **[16]** and various textbooks. For future reference, we review the essential points of that development here.

Let X denote a topological space and G an additive abelian group, which will serve as coefficient group for our cohomology theory. Let $\Phi^p(X, G)$ denote the additive abelian group of all p-functions on X with values in G, $p \geqq 0$: an element $\phi \in \Phi^p(X, G)$ is an *arbitrary* mapping of the $(p+1)$-fold cartesian product, X^{p+1}, into G. Let Δ denote the diagonal subset of X^{p+1}, i.e., the set of points all of whose coordinates are equal. The notation $\Phi_0^p(X, G)$ denotes the subgroup consisting of all p-functions ϕ which vanish on some neighborhood of the diagonal Δ, and $\check{C}^p(X, G)$ denotes the quotient group $\Phi^p(X, G)/\Phi_0^p(X, G)$. The elements of $\check{C}^p(X, G)$ are called Alexander-Spanier p-cochains; they can be looked on as "germs" of p-functions defined on some neighborhood of the diagonal Δ. Next, one defines a coboundary operator $\delta': \Phi^p(X, G) \to \Phi^{p+1}(X, G)$ by the classical formula

$$(\delta'\phi)(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_p)$$

and proves that $\delta'\delta' = 0$. It is readily verified that δ' maps the subgroup $\Phi_0^p(X, G)$ into $\Phi_0^{p+1}(X, G)$, and hence it induces a homomorphism $\delta: \check{C}^p(X, G) \to \check{C}^{p+1}(X, G)$ of quotient groups. The sequence of groups $\check{C}^p(X, G)$, $p = 0, 1, 2, \ldots$, together with the coboundary operator δ constitutes a cochain complex, and the cohomology groups of this cochain complex, denoted by $\check{H}^p(X, G)$, $p = 0, 1, 2, \ldots$, are the Alexander-Spanier cohomology groups of X (with coefficient group G). A continuous map $f: X \to Y$ induces a homomorphism $f': \Phi^p(Y, G) \to \Phi^p(X, G)$ according to the rule $(f'\phi)(x_0, \ldots, x_p) = \phi(fx_0, \ldots, fx_p)$ for any $\phi \in \Phi^p(Y, G)$ and $x_0, \ldots, x_p \in X$; this homomorphism has the property that $f'[\Phi_0^p(Y, G)] \subset \Phi_0^p(X, G)$, and hence it induces a homomorphism $f^{\#}: \check{C}^p(Y, G) \to \check{C}^p(X, G)$. Note

also that f' commutes with the coboundary operator δ', and $f^{\#}$ commutes with δ, i.e., they are cochain maps. Thus there is induced a homomorphism $f^*: \check{H}^p(Y, G) \to \check{H}^p(X, G)$ with all the usual "functorial" properties.

For the sake of completeness, we shall also define the relative cohomology groups. Let A denote a subspace of X, and $i: A \to X$ the inclusion map. The induced homomorphism $i^{\#}: \check{C}^p(X, G) \to \check{C}^p(A, G)$ is readily seen to be an epimorphism; we define $\check{C}^p(X, A, G)$ to be the kernel of $i^{\#}$. Then the sequence of groups $\check{C}^p(X, A, G)$ for $p = 0, 1, 2, \ldots$ together with the coboundary operator δ restricted to these subgroups constitutes a sub-cochain-complex of $\check{C}^*(X, G)$; its cohomology groups, denoted by $\check{H}^p(X, A, G)$, are the relative Alexander-Spanier cohomology groups of the pair (X, A). Furthermore, for each integer p we have a short exact sequence

$$0 \to \check{C}^p(X, A, G) \to \check{C}^p(X, G) \xrightarrow{i^{\#}} \check{C}^p(A, G) \to 0.$$

This short exact sequence of cochain complexes gives rise to the exact cohomology sequence of the pair (X, A) as usual.

For the proof of the other main properties of Alexander-Spanier cohomology (e.g., the excision property, the homotopy property) we refer the reader to Spanier's paper **[16]**, or to his book **[17]**, chapter 6.

3. History of the Čech–Alexander–Spanier homology theory. The straightforward method of defining the Čech–Alexander–Spanier cohomology groups described in the preceding paragraphs has been known for almost thirty years now. Unfortunately, the development of the corresponding homology theory, which is dual to the cohomology theory just described, has lagged behind rather badly. In this section we shall briefly describe the various methods which have been used to define this homology theory in the past; the reader will note that these methods, although successful from a strictly logical point of view, have all been so complicated that they are only of interest to the expert or the ardent enthusiast.

First of all, it must be pointed out that the homology groups defined by Vietoris **[20]** in 1927 and Čech **[2]** in 1932 are unsatisfactory. The precise nature of the difficulty was indicated by Eilenberg and Steenrod when they developed their axiomatic framework for homology theory about 1945 (cf. **[3]**): the Čech-Vietoris homology sequence of a pair (X, A) need not be exact! There is a nice example to illustrate this point in Chapter X, §4, of Eilenberg and Steenrod's book **[4]**. Of course, for sufficiently nice spaces (e.g., simplicial polyhedra) or for special choices of the coefficient group G (e.g., G a divisible group) the Čech-Vietoris homology sequence of a pair is exact. This lack of exactness severely restricts the possible uses of the Čech-Vietoris homology theory. Unfortunately, Čech's original definition of homology groups for compact spaces has been repeated in textbooks in recent years even though there has been available in the literature a definition of a Čech-type homology theory for such spaces which *does* give rise to an exact homology sequence for any compact pair (X, A).

It is ironic that Eilenberg and Steenrod should have pointed out this difficulty with the Čech-Vietoris definitions of homology without realizing that Steenrod himself had defined several years earlier a Čech-type homology theory for compact metric spaces that satisfied all their axioms (see **[19]**). Perhaps the reason for this oversight was the fact that Steenrod was considering a rather special problem when he defined these homology groups just before World War II; then the War intervened and he never went back to them again after it ended. In any case, Steenrod's method of definition is rather complicated and one would hope for an easier procedure.

The fact that Steenrod's homology groups based on "regular cycles" had these nice properties was first pointed out by Milnor in an unpublished manuscript **[11]** in 1960. In this same manuscript, Milnor offered an alternative way of defining these homology groups. Since copies of Milnor's manuscript are not readily available, we will quote the essential parts of his definition *verbatim*:

"Let S^m denote the m-sphere with base point b. For any compact pair (X, A) let $F^m(X, A)$ denote the function space consisting of all maps

$$(X, A) \to (S^m, b).$$

The symbol b will also be used for the constant map, considered as a base point in $F^m(X, A)$.... Let S stand for the reduced suspension operation, and identify the sphere S^{m+1} with SS^m. Then there is a canonical embedding

$$SF^m(X, A) \subset F^{m+1}(X, A),$$

... observe that there is a canonical imbedding of the [integral] singular chain group $C_k(Y, b)$ into the singular chain group $C_{k+1}(SY, b)$ of the suspension. ... Hence there is a canonical imbedding

$$C_{m-q}(F^m X, b) \subseteq C_{m+1-q}(F^{m+1}X, b).$$

Define $\bar{C}^q(X) = \operatorname{dir\,lim} C_{m-q}(F^m X, b)$ to be the union of these groups. (Caution: q can be positive or negative.) It is clear that

(1) $\bar{C}^*X$ is a contravariant functor of X,

(2) Each $\bar{C}^q X$ is a free abelian group, and

(3) $H^q(\bar{C}^*X)$ is isomorphic to the Čech group $H^q(X)$. In particular this homology group is zero, for $q < 0$.

"This construction extends to a pair (X, A) as follows.

"LEMMA 4. *For a compact pair (X, A), the natural homomorphism $\bar{C}^q X \to \bar{C}^q A$ is onto. Defining $\bar{C}^q(X, A)$ as the kernel of this homomorphism, the groups $H^q\bar{C}^*(X, A)$ are canonically isomorphic to the Čech cohomology groups of (X, A). ...*

"DEFINITION. For any compact pair (X, A) define

$$\bar{C}_q(X, A; G) = \operatorname{Hom}(\bar{C}^q(X, A), G).$$

The homology groups of this chain complex will be denoted by $\bar{H}_q(X, A; G)$ and called the *Steenrod homology groups* of (X, A)."

After reading this definition, the astute reader may wonder why one could not simply define

$$\bar{C}_q(X, A; G) = \operatorname{Hom}(\check{C}^q(X, A; Z), G)$$

instead of using the complicated definition given by Milnor (here $\check{C}^q(X, A; Z)$ denotes our previously defined Alexander-Spanier integral cochain group). The answer is that if one applies the functor $\operatorname{Hom}(\ \ , G)$ to the short exact sequence

$$(E_q) \qquad 0 \to \check{C}^q(X, A, Z) \to \check{C}^q(X, Z) \to \check{C}^q(A, Z) \to 0,$$

there is no reason to expect the resulting sequence to be exact. Thus we would not get an exact homology sequence for the pair (X, A) by this procedure. On the other hand, if $\check{C}^q(A, Z)$ were a *free* abelian group for all q, then the sequence (E_q) would be a *split* exact sequence for each q, and our troubles would disappear. This explains why Milnor went to such great lengths to define a *free* cochain complex for any compact space such that the cohomology of this cochain complex is the Čech-Alexander-Spanier cohomology of the given space.

At about the same time that Milnor did this work, A. Borel and J. Moore [1] discussed this problem from the point of view of general sheaf theory. They seemed to be mainly interested in defining a Čech-type homology theory (for arbitrary locally compact Hausdorff spaces) with coefficients in a sheaf, so that they could state and prove the most general version possible of the Poincaré duality theorem for generalized manifolds.

In 1964 the present author **[10]** gave an exposition of the Steenrod homology theory based on the Alexander-Spanier cochain groups $\check{C}^q(X, A, Z)$ as defined above. Following a suggestion contained in the paper of Borel and Moore **[1]**, he got around the difficulty created by the fact that the sequence (E_q) is not a split exact sequence as follows. Let

$$0 \longrightarrow G \xrightarrow{\varepsilon} K^0 \xrightarrow{\delta_0} K^1 \xrightarrow{\delta_1} \cdots$$

denote an *injective* resolution of the given coefficient group G. We may consider the sequence $K^0, K^1, \ldots$ of injective modules together with the homomorphisms $\delta_0, \delta_1, \ldots$ as a cochain complex K^*. There is a natural way to make the sequence of groups

$$\check{C}_q(X, A; K^*) = \prod_n \operatorname{Hom}[C^n(X, A), K^{n-q}]$$

for $q = 0, 1, 2, \ldots$, into a chain complex; then define the Steenrod homology groups of (X, A) with coefficient group G to be the homology groups of this chain complex. Of course, one must prove that the resulting homology groups are independent of the choice of the injective resolution of G. This method works because the functor $\operatorname{Hom}[\ \ , K]$ is exact if K is injective. It may very well be simpler than the methods used by Steenrod, Milnor and Borel-Moore. However, most students find the algebraic complications resulting from the systematic use of injective resolutions unpleasant in comparison with the simple development of the singular homology and cohomology theory.

In 1969, E. Sklyarenko wrote a long survey paper **[15]** on this subject. Given any compact metric space X he showed how to construct a *free* cochain complex whose cohomology groups are the Alexander-Spanier cohomology groups of X. Thus one can apply the functor $\operatorname{Hom}(\ \ , G)$ to this cochain complex to obtain a chain complex which gives the Steenrod homology groups of X. Sklyarenko's construction is not canonical, it involves *choosing* an infinite sequence of closed coverings of X of a special kind whose "mesh" tends to zero. One has to prove that the end result is independent of all choices. Given a continuous map $f: X \to Y$, in order to define the induced homomorphism $f^*: H^q(Y) \to H^q(X)$, one must choose sequences of coverings of X and Y that are appropriately related to the given map f. Clearly, it would be preferable to have a more canonical type of definition.

4. A simple variant of the Alexander–Spanier cochain complex which is free. In a short paper **[8]** published in 1950, J. W. Keesee proposed the following variant of the usual Alexander-Spanier cochain complex. Let $\Phi^p_F(X, G)$ be the subgroup of $\Phi^p(X, G)$ consisting of those p-functions ϕ which take on only a *finite number of different values.* Then define

$$\Phi^p_{F0}(X, G) = \Phi^p_F(X, G) \cap \Phi^p_0(X, G) \quad \text{and} \quad C^p(X, G) = \Phi^p_F(X, G)/\Phi^p_{F0}(X, G).$$

It is clear that $C^p(X, G)$ is a subgroup of the usual Alexander-Spanier cochain group $\check{C}^p(X, G)$, and that the coboundary operator δ maps $C^p(X, G)$ into $C^{p+1}(X, G)$. Thus there is defined a subcomplex of the Alexander-Spanier cochain complex. Similarly, for any pair (X, A), one can define the sub-complex $C^*(X, A, G)$ of $\check{C}^*(X, A, G)$ based on finitely-valued cochains. Clearly, $C^*(X, A, G)$ is a functor of (X, A) and G.

In his paper, Keesee proved the following theorem: *For compact Hausdorff pairs, the inclusion map* $C^*(X, A, G) \to \check{C}^*(X, A, G)$ *induces an isomorphism of cohomology groups.* Thus we could equally well use the cochain complex $C^*(X, A, G)$ to define the Alexander-Spanier cohomology of the pair (X, A), provided it is compact Hausdorff.

Keesee gave absolutely no motivation for considering finitely-valued cochains in his paper. At the time he wrote it, he could not possibly have known the following surprising and important fact:

THEOREM. *The finitely-valued integral cochain group* $C^p(X, A, Z)$ *is a free abelian group.*

The proof depends on a theorem proved in 1968 by G. Nöbeling **[12]**, which we will now take up.

5. The theorems of Specker and Nöbeling. In 1950, E. Specker published a very interesting paper **[18]**; it was apparently motivated by the problem of describing the structure of the cohomology group $H^1(X; Z)$ if X is an *infinite* complex. This paper contained the following theorem, among others:

THEOREM. *Let I be a countable set, and let F denote the abelian group consisting of all bounded, integer-valued functions defined on I. Then F is a free abelian group.*

In the years following the publication of this paper, many mathematicians must have tried to prove (or disprove) this theorem with the countability hypothesis omitted. Probably many were thrown off the track by the fact that Specker used the continuum hypothesis in his proof. Finally, in 1968, G. Nöbeling **[12]** succeeded in proving the generalized Specker theorem without any countability hypothesis and without any reference to the continuum hypothesis. A very neat exposition of Nöbeling's theorem[2] was given by L. Fuchs in his textbook **[5]** published in 1973. In order to explain this theorem, we will follow Fuchs' exposition. Let I be a set of arbitrary cardinality, and let F denote the abelian group consisting of all *bounded*, integral-valued functions defined on I. Given any function $f \in F$, if $f \neq 0$, it can be written *uniquely* in the canonical form

$$(*) \qquad f = a_1 h_{X_1} + a_2 h_{X_2} + \cdots + a_k h_{X_k},$$

where $a_1, a_2, \ldots, a_k$ are distinct integers, $X_1, X_2, \ldots, X_k$ are disjoint non-empty subsets of I, and for any subset X of I, the symbol h_X denotes the characteristic function of the set X (i.e., for any $i \in I$, $h_X(i) = 1$ or 0, according as $i \in X$, or $i \notin X$).

DEFINITION. A subgroup S of F is called a *Specker group* if and only if for any $f \in S$, when f is expressed in the canonical form $(*)$, the functions $h_{X_1}, h_{X_2}, \ldots, h_{X_k}$ all belong to S.

DEFINITION. A subgroup S of F has a *characteristic basis* if S is a free abelian group with a basis consisting of characteristic functions.

NÖBELING'S THEOREM. *Let S and T be Specker subgroups of F such that $S \subset T$. Then there exists a free subgroup $R \subset T$ with a characteristic basis such that $T = R \oplus S$.*

Note that we may have $S = \{0\}$ in the above theorem; it follows that T is free and has a characteristic basis. Also, F itself is obviously a Specker group, and hence is free. Finally, the theorem implies that the quotient group T/S is free.

Naturally the proof of this theorem is highly non-constructive; it involves a transfinite induction. However, it does not require the continuum hypothesis.

Now we shall show how to use Nöbeling's theorem to prove that $C^p(X, A, Z)$ is free. First we shall prove that $C^p(X, Z)$ is free for any space X. Take $I = X^{p+1}$; then $\Phi^p_F(X, Z)$ is obviously a Specker group on I, and $\Phi^p_{F0}(X, Z)$ is a Specker subgroup. Hence the quotient group $C^p(X, Z)$ is free. Now consider the short exact sequence:

$$0 \to C^p(X, A, Z) \xrightarrow{j^{\#}} C^p(X, Z) \xrightarrow{i^{\#}} C^p(A, Z) \to 0.$$

We have already proved that $C^p(X, Z)$ and $C^p(A, Z)$ are free abelian; it follows that $C^p(X, A, Z)$ is free abelian, and the sequence splits. Q.E.D.

6. Suggested program for an exposition of Čech–Alexander–Spanier type homology and cohomology theory. Such an exposition could very well begin with the cohomology of compact Hausdorff pairs

2. Nöbeling's paper was first brought to the author's attention by Michael Keene several years ago. The author is grateful to Paul Eklof for pointing out that there is an account of Nöbeling's theorem in Fuchs' book.

(X, A) using the group of finitely-valued cochains $C^p(X, A, G)$ described above. There is no need to even mention the usual Alexander-Spanier cochain complex, $\check{C}^*(X, A, G)$. One can prove directly that the cohomology groups defined using these finitely-valued cochains satisfy all the Eilenberg-Steenrod axioms and the various other properties of Čech-Alexander-Spanier cohomology for compact pairs. The proofs would involve only slight modifications of those given in the literature based on the usual Alexander-Spanier cochain complex.

Next, one could use Nöbeling's theorem to prove that $C^p(X, A, Z)$ is a free abelian group, as outlined in the preceding section; then define a chain group $C_p(X, A, G)$ by

$$C_p(X, A, G) = \operatorname{Hom}[C^p(X, A, Z), G].$$

These chain groups have all the desired naturality properties, both with respect to continuous maps of compact pairs, and homomorphisms of coefficient groups. Also, any compact pair (X, A) gives rise to a short exact sequence

$$0 \to C_p(A, G) \to C_p(X, G) \to C_p(X, A, G) \to 0$$

of chain groups, and hence the usual exact homology sequence of the pair. The proofs of the basic properties of the homology theory thus defined follow from the corresponding properties already proved for cohomology theory; the techniques of proof are similar to those used to prove the basic properties of *singular* cohomology theory, assuming one has already developed singular homology theory.

At this step, one could derive the "universal coefficient theorem" short exact sequence

$$0 \to \operatorname{Ext}[H^{p+1}(X, A, Z), G] \to H_p(X, A, G) \to \operatorname{Hom}[H^p(X, A, Z), G] \to 0$$

by the methods given in books on homological algebra (e.g., MacLane **[9]**, p. 77 or Hilton-Stammbach, **[7]**, p. 179). It also follows easily from Fuchs **[5]** that one has a natural isomorphism of cochain complexes,

$$C^p(X, A, G) \approx C^p(X, A, Z) \otimes G$$

for any abelian group G (this relation would *not* be true for the usual Alexander-Spanier cochain complex). Hence by standard methods of homological algebra, one also has the following short exact sequence:

$$0 \to H^p(X, A, Z) \otimes G \to H^p(X, A, G) \to \operatorname{Tor}[H^{p+1}(X, A, Z), G] \to 0$$

(cf. MacLane, **[9]**, p. 171 or Hilton-Stammbach **[7]**, p. 176).

Other topics that could be treated at this stage include the Künneth theorem for the cohomology (*not* the homology) of the product of two compact spaces, and cup and cap products. Finally, a discussion could be given of the behavior of this homology and cohomology theory vis-à-vis inverse limits of compact spaces. It is well known that the Alexander-Spanier cohomology theory satisfies a so-called "continuity property": the cohomology groups of an inverse limit of compact spaces are naturally isomorphic to the direct limits of the corresponding cohomology groups of the spaces involved (cf. Spanier **[17]**, chapter 6, section 6). The analogous property does not hold for homology groups however. Instead, one has the following weaker statement: Let

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots,$$

be an inverse sequence of compact metric spaces with inverse limit X. Then there is a short exact sequence

$$0 \to \lim{}^1[H_{q+1}(X_i)] \to H_q(X) \to \operatorname{lim\,inv} H_q(X_i) \to 0$$

for each integer q, where the notation "lim^1" denotes the first derived functor of the inverse limit functor, "lim inv" (this theorem is due to Milnor, **[11]**). If we make use of the known fact that every compact metric space X can be represented as the limit of an inverse sequence of compact *polyhedra* $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$, then lim inv $[H_q(X_i)]$ is naturally isomorphic to the qth homology group of X as originally defined by Vietoris or Čech long ago; thus the term $\lim^1[H_{q+1}(X_i)]$ in the above short exact sequence measures the difference between the homology groups we have defined and those originally defined by Čech and Vietoris.

Inevitably some will want to consider the homology and cohomology of non-compact spaces. There are no real difficulties in doing this, but there are several different ways to proceed. As a result, the situation is more complicated than the compact case. For the sake of our more knowledgeable readers, we offer the following outlines of an exposition of the theory for non-compact spaces.

First we shall discuss the case of locally compact Hausdorff spaces. Here there is a cohomology theory based on Alexander-Spanier cochains with "compact supports." The dual homology theory is one based on chains with non-compact support, or "infinite chains." There are two different methods for giving an exposition of this type of homology and cohomology theory. The first method is to make use of the Alexandroff 1-point compactification to convert every locally compact (Hausdorff) pair into a compact pair, and then to apply the homology or cohomology theory we have already constructed to the resulting compact pair. This procedure is clearly described in Eilenberg and Steenrod **[4]**, chapter X, pp. 269–271. The success of this method depends fundamentally on the fact that the Čech-Alexander-Spanier type theory satisfies a strong form of the excision property, as was explained in the introduction to this paper. For this reason, this method would not work for the singular type theory. Note that it is also possible to formulate this homology or cohomology theory as a "single space theory" as outlined in **[4]**, chapter X, section 7. (Warning: In the list of axioms on p. 274 of **[4]**, one rather obvious axiom is omitted. Apparently this omitted axiom is needed for the proof of Theorem 7.4 on p. 276.) The second method of defining this cohomology theory for locally compact spaces is to use the subcomplex of the Alexander-Spanier cochain complex consisting of cochains with compact support; for a description of this method, see **[10]** or **[17]**, pp. 320–323. If one uses *finitely-valued* integral cochains with compact support, then by using Nöbeling's theorem, it can be proved that the resulting cochain complex is free. Hence to define chain groups, one can apply the functor $\mathrm{Hom}[\ \ ,G]$ to this cochain complex, as described above.[3]

Next, we shall discuss the case of arbitrary Hausdorff spaces (i.e., those which are not necessarily locally compact). To define cohomology groups for such spaces, the obvious thing to do is to use the full Alexander-Spanier cochain complex, as described in Spanier's paper **[16]**, or his book **[17]**, chapter 6. The best procedure for defining the corresponding homology theory for such spaces seems to be the following: for any Hausdorff space X, define $H_q(X, G)$ to be the direct limit of the groups $H_q(X_\alpha, G)$, where X_α ranges over all compact subsets of X. This procedure is described in Exercises C and D, chapter IX of **[4]**. Alternatively, one could first form the direct limit (or union) of the chain groups $C_q(X_\alpha)$, and then take the homology of the resulting chain complex. This homology theory was used by Sitnikov, **[13]** and **[14]**, to prove a far-reaching generalization of the Alexander duality theorem for subsets of Euclidean space which are neither open nor closed. Other than this, it does not seem to have occurred in the literature, and its properties have not been extensively investigated. It seems to be the principal candidate for a homology theory which is "dual" to the Alexander-Spanier cohomology on general spaces.

Now let us consider once again the possible Čech-type cohomology theories for locally compact Hausdorff spaces. We see that there are two different possibilities: that defined by the full Alexander-Spanier cochain complex, and that defined by the sub-complex of cochains with compact support. Similarly, there are two different homology theories, which are "dual" to the respective

3. This procedure will be described in detail in a forthcoming book by the author.

cohomology theories. In some situations, such as the discussion of the Poincaré duality theory for open manifolds, it seems necessary to consider both theories simultaneously, together with the various possible cup and cap products between them. In such a case, it is desirable that the cup product of cochain with compact support and an arbitrary cochain should be a cochain with compact support. Assuming that for cohomology with compact supports one uses finitely-valued cochains exclusively, this condition necessitates the use of *locally finitely valued* cochains for those which do not have compact support. Fortunately, the theorem needed here was proved by W. L. Gordon **[6]** in 1955: for fully normal Hausdorff spaces, the sub-complex of locally finitely valued cochains has the same cohomology groups as the full Alexander-Spanier cochain complex.

To summarize, it seems most appropriate to use only the following subcomplexes of the full Alexander-Spanier cochain complex:

(a) For compact spaces and for cohomology with compact support of locally compact spaces, use only finitely-valued cochains.

(b) For cohomology with non-compact supports of arbitrary spaces, use locally finitely-valued cochains.

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(NOTE: This bibliography makes no pretensions to completeness. For further information on this subject, see *Reviews of Papers in Algebraic and Differential Topology, etc.* (classified by N. Steenrod), vol. I, pp. 154–162.)

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EXTENDED MEAN VALUES

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The history of mean values is long and laden with detail. From a few antique means there evolved a continuum of classical means, sometimes called power means, and from these grew a profusion of generalizations. One finds the story, aside from its latest chapters, in Hardy-Littlewood-Pólya [**3**] and in Beckenbach-Bellman [**1**]. A survey of more recent developments is beyond the scope of this paper and of its authors. We are here concerned with a marked underdevelopment.

The Mean Value Theorem for derivatives is commonly extended in two directions. There is a generalized form given in Taylor's theorem and there is an extended form, involving a pair of functions, that one meets in proofs of L'Hospital's Rule. In [**4**], Leach gives a common generalization of the two. The extended form, applied to power functions, attests the existence of mean values whose study, aside from the classical subcase, has been widely neglected.

Clearly, to distinguish these means from others, it is suitable to call them extended means. A full definition of two-variable extended means, free of past restrictions, is found in Section 2 below. These particular means are special cases of means introduced by Tobey [**8**]. Apparently Stolarsky was first to isolate them and initiate their independent investigation. In [**7**] he discusses them briefly and then proceeds to a multivariable generalization.

We give below, for extended means, basic results concerning monotonicity, comparability, and other inequalities. Also, not unimportant for an area which is relatively new, destined to grow and become better known, we propose a cohesive selection of appropriate notations and word usage. An initial selection is given in Section 1. In terminology found there, this paper and its sequel survey the two-variable, unabstracted, unweighted extended mean. Discussion of various generalizations is planned for later papers.

1. Backround and initial notation. In the beginning there was the arithmetic mean or average, $A = A[x, y] = (x + y)/2$. It was joined by the geometric mean or mean proportional, $G = G[x, y] = (xy)^{1/2}$, and they multiplied. There came the harmonic mean $H = G^2/A$ and the root-mean-square $N = (G + A)/2$. From these rose the *classical means* $M_r = M_r[x, y] = ((x^r + y^r)/2)^{1/r}$. When $r \leqq 0$, $M_r[0, y]$ is defined as 0. The variables x and y are, in this paper, never negative.

Means M_r increase with r, $-\infty < r < \infty$. Special cases include $H = M_{-1}$, $N = M_{1/2}$, $A = M_1$, and $G = M_0$. The last occurs as a limiting case as do also the bounds $M_{-\infty} = \min[x, y]$ and $M_\infty = \max[x, y]$.

Further evolution led to multivariable means with $[x_1, x_2, \ldots, x_n]$ replacing $[x, y]$, to abstracted means $M_\varphi = \varphi^{-1}((\varphi(x) + \varphi(y))/2)$ which reduce to M_r when $\varphi(x) = x^r$, and to weighted means. Weighted means $A[x, y]$ and $G[x, y]$ are given by $(1-\alpha)x + \alpha y$ and $x^{1-\alpha}y^\alpha$, $0 \leqq \alpha \leqq 1$. There are mixtures of these types and there are mutations of sundry sorts. Natural evolution produces another type, the extended mean of Section 2.

Along with means M_r there are two extended means, L and I, of particular interest. We find in Pólya-Szegö [**6**] that the *logarithmic mean* L is no newcomer. Recent advances in its theory have been made by Carlson [e.g., **2**]. The full definition of L is three-part.

(1.1) For $xy \neq 0$ and for $x \neq y$, let

$$L = L[x, y] = (x - y)/(\ln x - \ln y).$$

Let $L[x, 0] = L[0, x] = 0$ and $L[x, x] = x$.

The *identric mean* I (our choice of name is consistent with others chosen in a sequel) was introduced by Stolarsky [**7**]. Its full definition is:

(1.2) For $xy \neq 0$ and for $x \neq y$, let

$$I = I[x, y] = e^{-1}(x^x y^{-y})^{1/(x-y)}.$$

Let $I[x,0] = I[0,x] = x/e$ and $I[x,x] = x$.

It is known that (it follows from (3.11) when $b = 1$)

$$G \leqq L < N < I < A \quad \text{unless } x = y. \tag{1.3}$$

It is not automatically known what an author has in mind when he refers to "The Theorem of the Means" or even to the annoyingly attenuated "Arithmetic-Mean Geometric-Mean Inequality." Does he mean "$G \leqq A$" for these means as first defined above? In their weighted multivariable form? Does the theorem cover the reverse inequality for weighted non-means, say $2x - y \leqq x^2y^{-1}$? In this paper we use the (1.4) version in a generally unknown symmetric form, the Trinomial Sign Test of (1.5).

(1.4) Let x and y be positive. For $0 < \alpha < 1$, $x^{1-\alpha}y^{\alpha} < (1-\alpha)x + \alpha y$ unless $x = y$. For $\alpha < 0$ or for $\alpha > 1$, the inequality reverses.

(1.5) Let $0 < \theta \neq 1$. If $a + b + c = a\alpha + b\beta + c\gamma = 0$, then

$$\operatorname{sgn}(a\theta^{\alpha} + b\theta^{\beta} + c\theta^{\gamma}) = -\operatorname{sgn}(abc).$$

Proof. The case $abc = 0$ gives no trouble. When $abc \neq 0$, two of these numbers have the same sign. We may assume that they are positive (replacing all three by their negatives if need be) and that $\operatorname{sgn} a = \operatorname{sgn} b$ (by symmetry). Then $abc < 0$ and $\theta^{\gamma} = (\theta^{\alpha})^{a/(a+b)} \cdot (\theta^{\beta})^{b/(a+b)}$. Hence

$$a\theta^{\alpha} + b\theta^{\beta} + c\theta^{\gamma} = (a+b)\left(\frac{a\theta^{\alpha} + b\theta^{\beta}}{a+b} - \theta^{\gamma}\right) > 0.$$

We note that, conversely, (1.4) follows from (1.5) with $\theta = e$ since $x = e^{\ln x}$ and $y = e^{\ln y}$.

As an application, we prove part of (2.22), a property which Stolarsky doesn't list.

(1.6) If E is defined as in (2.2), then $\partial E/\partial x > 0$.

Proof. Let $P = E^{s-r} \quad s/r = (x^s - y^s)/(x^r - y^r)$. Then $(\partial P/\partial x)(x^r - y^r)^2/y^{s+r-1}$ equals $(s-r)\theta^{s+r-1} - s\theta^{s-1} + r\theta^{r-1}$ where $\theta = x/y$. Since the conditions of (1.5) are met, $\operatorname{sgn} \partial P/\partial x = \operatorname{sgn}(sr(s-r))$. From

$$\operatorname{sgn}\frac{\partial}{\partial x}E^{s-r} = \operatorname{sgn}\left(\frac{r}{s}\frac{\partial P}{\partial x}\right) = \operatorname{sgn}(s-r),$$

it follows that E increases with x.

Before we continue discussing extended means, we pause and define them.

2. Definition and basic properties of *E*. The extended means here are binary operations applied to non-negative variables x and y. They are a two-parameter family involving real indices r and s. We write, shifting notation to suit the context,

$$E = E(r,s;x,y) = E(r,s) = E[x,y]. \tag{2.1}$$

We leave, on occasion, the context to indicate whether we are speaking of means or of mean values.

Since we are giving here a full definition, properties of E listed below only partially duplicate those listed by, say, Stolarsky. We save space by stating a given property in its final form. For example, from (2.2) we have $E > 0$. However, (2.3) has $E \geqq 0$ since later stages in the definition permit equality.

Suppose first that $x \neq y$, $r \neq s$, and $rs \neq 0$. Then either if $xy \neq 0$, or if $xy = 0$ with r and s positive,

$$\text{we define } E = \left(\frac{y^s - x^s}{y^r - x^r}\,\frac{r}{s}\right)^{1/(s-r)}. \tag{2.2}$$

All other values of E are found as limiting cases. For example, (2.6) is justified by (2.5).

$$E > 0 \text{ unless } xy = 0 \text{ and } \min(\operatorname{sgn} r, \operatorname{sgn} s) \leqq 0. \tag{2.3}$$

$$E[x,y] = E[y,x]. \tag{2.4}$$

$$\min[x,y] < E < \max[x,y] \quad \text{unless } x = y \text{ or } E = 0. \tag{2.5}$$

$$\text{We define } E[x,x] = x \text{ for all } (r,s). \tag{2.6}$$

Consider now the index plane. We define $E(r, s)$ and $E(s, r)$ as *twin means.* We find, in the sense of (2.9), that $E(r, s)$ and $E(-r, -s)$ are *inverse means.* Property (2.9) justifies definition (2.10). For $r = 0$, (2.7) is to be taken as a definition.

(2.7) $$E(r, -r) = G \text{ for all } r.$$

(2.8) $$E(r, s) = E(s, r).$$

(2.9) $$E(r, s)E(-r, -s) = G^2.$$

(2.10) For $xy = 0$, $r < 0$, and $s < 0$, we define $E(r, s) = 0$.

We call $E(r, s)$ and $E(r, -s)$ *conjugate means.* We call $E(r, s)$ and $E(-r, s)$ *co-conjugate means.* Property (2.11), though simple, is quite useful. It has as corollaries the conjugate relation (2.12) and the co-conjugate relation (2.13). They in turn justify (2.14). In (2.11)–(2.13), it is understood that cases where 0 is raised to a nonpositive power are excluded.

(2.11) $$E(r, t)^{r-t} = E(r, s)^{r-s}E(s, t)^{s-t}. \text{ (The triangle property.)}$$

(2.12) $$E(r, -s)^{r+s} = E(r, s)^{r-s}G^{2s}.$$

(2.13) $$E(-r, s)^{s+r} = E(r, s)^{s-r}G^{2r}.$$

(2.14) For $xy = 0$ and $rs \leqq 0$, we define $E(r, s) = 0$.

We denote $E(r, 0) = E(0, r)$ by L_r and call it a *logarithmic mean.* By definitions above, $L_r[x, y] = 0$ for $xy = 0$. Using (1.1) notation:

(2.15) We define $L_r[x, y] = L[x^r, y^r]^{1/r}, xy \neq 0$.

(2.16) For $xy = 0$, $r > 0$, and $s > 0$, we define

$$E(r, s; x, y) = \max[x, y] \exp(-1/L[r, s]).$$

We denote $E(r, r)$ by I_r and call it an *identric mean.* By definitions above, when $xy = 0$, $I_r = 0$ for $r \leqq 0$ and $I_r = \max[x, y]\exp(-1/r)$ for $r > 0$. Using (1.2) notation:

(2.17) We define $I_r[x, y] = I[x^r, y^r]^{1/r}, \quad xy \neq 0$.

Finally, we may define extended means for ideal points in the extended index plane. These means, $E_{-\infty}(m)$ and $E_{\infty}(m)$, are discussed in a sequel.

The definition of extended means $E(r, s)$ is now completed. That extended means generalize classical means follows from the equations $M_r = E(r, 2r) = E(2r, r)$.

If we denote $E(r, 1)$ by S_r we have $G = S_{-1}$, $L = S_0$, $N = S_{\frac{1}{2}}$, $I = S_1$, and $A = S_2$. As Stolarsky points out, means S_r are generated by the (non-extended) Mean Value Theorem. We may, by (2.11), express extended means E either in terms of means S_r or in terms of means L_r.

The next two properties are immediate.

(2.18) $$kE[x, y] = E[kx, ky], \ k \geqq 0.$$

(2.19) For $n > 0$, or for $n < 0$ when $xy \neq 0$,

$$E(r, s; x, y)^n = E(r/n, s/n; x^n, y^n).$$

In proving (2.20) and (2.21), we fix y and r. We let $\theta = x/y$ and $t = \theta^r$.

(2.20) LEMMA. *L_r increases as x increases.*

Proof. From $\ln L_r = r^{-1} \ln((t-1)/\ln t) + \ln y$, we obtain

$$\theta \frac{d}{d\theta}(\ln L_r) = \frac{t \ln t - t + 1}{(t-1)\ln t} > 0 \quad \text{for} \quad t \neq 1.$$

(2.21) LEMMA. *I_r increases as x increases.*

Proof. From

$$\ln I_r = r^{-1}\left(\frac{t\ln t}{t-1}-1\right)+\ln y, \text{ we obtain } \theta\frac{d}{d\theta}(\ln I_r)=\frac{t(t-1-\ln t)}{(t-1)^2}>0, \quad t\neq 1.$$

(2.22) THEOREM. *$E[x,y]$ increases with increase in either x or y.*

Proof. This follows from (1.6), (2.20), and (2.21).

Formulation of extended means as integrals is to be given in a paper on "extended means of sequences." For the present sequence $[x,y]$, if we let $z=(1-u)x+uy$, $t=(1-v)r+vs$, and $F(t)=(\partial/\partial t)\ln\int_0^1 z^{t-1}du$, we have

$$\begin{aligned}\ln E &= \ln\left(\int_x^y z^{s-1}dz\Big/\int_x^y z^{r-1}dz\right)/(s-r)\\ &= \ln\left(\int_0^1 z^{s-1}du\Big/\int_0^1 z^{r-1}du\right)/(s-r)\\ &= \int_r^s F(t)dt/(s-r)=\int_0^1 F(t)dv.\end{aligned}$$

Stolarsky, giving the penultimate form, notes that

$$F(t)=\frac{\partial}{\partial t}\ln\left|\frac{y^t-x^t}{t}\right| = \ln I_t. \tag{2.23}$$

Integral representations give E in a less fragmented form, thus making evident its continuity and, for positive variables and finite indices, its analyticity. Still other representations of E are given in the next section.

3. Normalization and applications. We have yet to exploit (2.18), homogeneity of E in x and y. Let $\theta=x/y$ and let v be an expression in x and y which is homogeneous of degree 1. Then $v^{-1}E[x,y]=E[x/v,y/v]$ is a function of θ. For many purposes we are able to assume that $v=1$ and thus *normalize E.* When $\theta=f(w)$, inequalities for means convert to inequalities involving f and functions related to f. The following table gives:

(3.1) Examples of normalization.

v	θ	G	L	A
y	θ	$\theta^{\frac{1}{2}}$	$(\theta-1)/\ln\theta$	$(\theta+1)/2$
A	θ	$\dfrac{2\theta^{\frac{1}{2}}}{\theta+1}$	$\dfrac{2(\theta-1)}{(\theta+1)\ln\theta}$	1
A	$\tan^2(w/2)$ $0<w<\pi$	ain w	$\dfrac{\cos w}{2\ln\tan(w/2)}$	1
A	e^{2w} $-\infty<w<\infty$	sech w	$\dfrac{\tanh w}{w}$	1
G	θ	1	$\dfrac{\theta^{\frac{1}{2}}-\theta^{-\frac{1}{2}}}{\ln\theta}$	$\dfrac{\theta^{\frac{1}{2}}+\theta^{-\frac{1}{2}}}{2}$

G	e^{2w} $-\infty < w < \infty$	1	$\frac{\sinh w}{w}$	$\cosh w$
$x-y$ >0	$\theta > 1$	$\frac{\theta^{\frac{1}{2}}}{\theta - 1}$	$\frac{1}{\ln\theta}$	$\frac{1}{2}\frac{\theta+1}{\theta-1}$
$x-y$ >0	$1+1/n$ $n>0$	$(n^2+n)^{\frac{1}{2}}$	$\frac{1}{\ln(1+1/n)}$	$\frac{2n+1}{2}$

From the inequality

(3.2) $$(1+1/n)^n < e < (1+1/n)^{n+1}, \; n>0,$$

we have $n < \log_b e < n+1$ where $b = 1+1/n$. Noting that

$$\log_b e = 1/\ln b = \frac{(n+1)-n}{\ln(n+1)-\ln n},$$

we "discover" the logarithmic mean as given in the last (3.1) example.

(3.3) $$(1+1/n)^{L[n,n+1]} = e.$$

Similarly, we have from (3.2) that $1 < e^{-1}(1+1/n)^{n+1} < 1+1/n$ or that $1 < e^{-1}b^{b/(b-1)} < b$ and we discover the identric mean $I[b,1]$. We also have $1 < e(1+1/n)^{-n} < 1+1/n$ or $1 < e b^{1/(1-b)} < b$. The resulting mean is the inverse mean of $I[b,1]$. In these cases the identric mean appears normalized as in the first (3.1) example. We adopt, in work given below, this normalization.

(3.4) *Convention.* We assume, when possible, that $x=\theta$ and $y=1$. We assume, unless stated otherwise, that $x \leqq y$, or $0 \leqq \theta \leqq 1$.

From $I = e^{-1}\theta^{\theta/(\theta-1)}$ we have

(3.5) $$\ln I = -1 + \theta/L \quad \text{and} \quad I^{1-\theta} = \theta^{L-\theta}.$$

Thus, bounds for L lead to bounds for I. In particular, from $\theta^{1/2} = G \leqq L \leqq N = (\theta^{1/2}+1)^2/4$, we obtain $\theta^a \leqq I \leqq \theta^b$ where $a = (1+3\sqrt{\theta})/(4+4\sqrt{\theta})$ and $b = \sqrt{\theta}/(1+\sqrt{\theta})$. This translates into

(3.6) $$x^{1-\alpha}y^{\alpha} \leqq I[x,y] \leqq x^{1-\beta}y^{\beta}$$

where $\alpha = (\sqrt{x}+3\sqrt{y})/(4\sqrt{x}+4\sqrt{y})$ and $\beta = \sqrt{y}/(\sqrt{x}+\sqrt{y})$.

In the rest of the paper, we let $t = \theta^r$. The function $\phi(t)$ in (3.7) is used in the proof of (3.8) and in the proof of Stolarsky's result, (3.9). It is easy to show that

$$\operatorname{sgn}((t-1)/\sqrt{t} - \ln t) = \operatorname{sgn}(t-1)$$

and we have, in consequence,

(3.7) $$\phi(t) = (t-1)^2 - t\ln^2 t > 0 \quad \text{unless} \quad t=1.$$

(3.8) LEMMA. *For $xy \neq 0$, L_r strictly increases with increase in r.*

Proof. Let $\psi(t) = (t\ln t/(t-1)) - 1 - \ln((t-1)/\ln t)$. From $\ln L_r = r^{-1}\ln(t-1)/\ln t$, we have $r^2(d/dr)\ln L_r = \psi(t)$ and, by further computation,

$$t(t-1)^2(\ln t)\psi'(t) = \phi(t).$$

From (3.7), $\psi(t) \geqq \lim_{t\to 1} \psi(t) = 0$ and hence $\ln L_r$ increases with r.

(3.9) LEMMA. *For $xy \neq 0$, I_r strictly increases with increase in r.*

Proof. From $\ln I_r = r^{-1}((t \ln t/(t-1)) - 1)$, we obtain $r^2(t-1)^2(d/dr)\ln I_r = \phi(t) \geqq 0$. The lemma follows.

It is well known that $(\theta^r - 1)/r$ increases with θ and that, for real r and positive θ, the function

$$\mu(r) = \begin{cases} (\theta^r - 1)/r & r \neq 0 \\ \ln \theta & r = 0 \end{cases}$$

increases with r. (For $\theta < 1$, from $-\infty$ to 0. For $\theta > 1$, from 0 to ∞.) Let $\lambda(r) = \ln|\mu(r)|$. One finds that $\lambda'(r) = \ln I_r$. As a corollary of (3.9), we have

(3.10) *$\lambda(r)$ is strictly convex in r.*

(3.11) LEMMA. *For $xy \neq 0$ and $b \neq 0$, $E(r, b)$ strictly increases with increase in r.*

Proof. Since it equals $\lambda(r) - \lambda(b)/(r-b)$, $\ln E(r, b)$ can be interpreted as the slope of a chord joining two points on a convex curve.

(3.12) THEOREM. *$E(r, s)$ increases with increase in either r or s.*

Proof. Lemmas above dispose of the case $xy \neq 0$. For $xy = 0$, $E = 0$ unless $r > 0$ and $s > 0$. For that case, if $0 < y$, we have $E(r, s) = y \exp(-1/L[r, s])$ and the proof is completed by appeal to (2.20).

The preceding theorem establishes comparability of certain means, cases where $E(r_1, s_1) \leqq E(r_2, s_2)$ for all x, y. What other cases exist? In particular, what cases involving $E(r, s)$ with (r, s) fixed at (r_1, s_1) and with (r_2, s_2) a nearby point in the index plane?

In discussing this and other questions, it is convenient to divide the plane into octants I-VIII defined by $0 < s < r$, $0 < r < s$, $0 < -r < s$, $0 < s < -r$, $0 < -s < -r$, $0 < -r < -s$, $0 < r < -s$, and $0 < -s < r$. For, say, (r, s) in Octant II, we find that $E(r, s)$ has an inverse in VI, a conjugate in VII, a co-conjugate in III, a twin in I, a twin inverse in V, a twin conjugate in IV, and a twin co-conjugate in VIII. Note that twin conjugate $E(r, s) =$ co-conjugate twin $E(r, s)$ and that twin co-conjugate $E(r, s) =$ conjugate twin $E(r, s)$. From (2.8), comparable means have comparable twins. Thus, in the search for comparability, we may assume that $s \geqq r$. From (2.9), comparable means have their inverse means comparable in the reverse order. Thus, in that search, we may also assume that $s \geqq -r$ and restrict our study to the upper quadrant $s \geqq |r|$, the union of Octants II and III. Even here, (2.13) might aid in simplifying the search.

Of similar potential aid is (2.19). The four following statements, for $h > 0$, are equivalent. (For $h < 0$, the last inequality is reversed.) They lead, for a "linear family" of means $E(r, mr + b)$, to (3.13). Let $[x, y]$ be $[\theta, 1]$, $0 < \theta \neq 1$.

$$E(r_1, s_1; \theta, 1) < E(r_2, s_2; \theta, 1), \quad \text{all} \quad \theta.$$

$$E(r_1, s_1; \theta^h, 1) < E(r_2, s_2; \theta^h, 1), \quad \text{all} \quad \theta.$$

$$E^h(hr_1, hs_1; \theta, 1) < E^h(hr_2, hs_2; \theta, 1), \quad \text{all} \quad \theta.$$

$$E(hr_1, hs_1; \theta, 1) < E(hr_2, hs_2; \theta, 1), \quad \text{all} \quad \theta.$$

(3.13) *Similitude Property.* For $h > 0$ [$h < 0$], the affine transformation which maps (r, s) onto (hr, hs) preserves [reverses] comparability of $E(r, s)$. In mapping a linear segment of comparable means on another, it preserves [reverses] monotone behavior. Moreover, since the transformation preserves midpoints, convexity or concavity of E on the one segment implies convexity or concavity on the other for suitable values of h.

A full discussion of comparability is reserved for a later paper. We state here a sample result,

(3.14), which has as a special case the inequality

$$L = E(0,1) \leqq E(1/3,2/3) = M_{1/3}$$

introduced by Lin **[5]**.

(3.14) THEOREM. *For $c > 0$ and $-\infty < r \leqq c$, $E(r, 2c-r)$ increases from G to I_c.*

Above properties lead to some terminal observations. We have, from (3.12), a 90° sector of directions away from (r, s) in which we find larger mean values. (Indeed, for a point $(r, -r)$, Theorem 3.12 and Property 2.7 present us with a sector of 180°.) When $r \neq s$, (3.12) and (3.14) used together give assurance of a 135° sector for neighboring points at which we find larger values. In the interior of the upper quadrant, and the lower one, the sector extends from $-\pi/4$ to $\pi/4$ radians. In the interior of the right hand and left hand quadrants, it extends from 0 to $3\pi/4$ radians.

There remains, at points in these quadrants, a sector of at most 45° (and its opposite sector) giving directions in which "neighboring means" are not comparable. Can we identify this sector? There is also a global problem: Given (r_0, s_0), describe the set of points (r, s) for which $E(r_0, s_0)$ and $E(r, s)$ are comparable. In work forthcoming, these problems are resolved.

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ERROR CORRECTING CODES: PRACTICAL ORIGINS AND MATHEMATICAL IMPLICATIONS

VERA PLESS

1. Introduction. Many important mathematical topics have had practical origins; the solutions of practical problems or engineering applications. I will just mention a familiar topic: calculus which is the study of motion. This applied origin in mathematics used to happen frequently; it is not so usual today. However, it is true of coding; this study was developed for practical reasons and these reasons are an interesting aspect of it. Indeed in **[6]** Levinson points out how coding theory refutes Hardy's notion that pure mathematics did not enter into any useful applied problems. Levinson does this by showing how finite fields and number theory play a central role in cyclic codes. In this paper we develop this theme further, though not in the context of cyclic codes, by pointing out important connections between coding theory, combinatorial designs, and finite groups. In **[13]** Sloane describes the use of invariant theory to prove significant coding theorems.

The initial problem was one in electrical engineering, namely the reliable communication of

digitally encoded information. Serious work in this area started only 28 years ago. We here think of a message as a sequence of zeroes and ones. When such a message is transmitted over a communications channel it can be distorted by noise. When this occurs, a zero is changed into a one or a one into a zero. The technique used to overcome this problem is to add redundancy to our message in an analytic manner so that the original message can be recovered if distorted. This consists of encoding, adding redundancy to the message, and decoding, recovering the original message if possible. We give an example of this process for the Hamming (7,4) code.

The 16 messages in this code are of length 7 and any 4-tuple can appear in the first four positions which are called information positions. The last three positions are the redundancy positions and are determined from the first four by the fact that this code is a vector space over GF(2) with the following four basis vectors:

1	2	3	4	5	6	7
1	0	0	0	0	1	1
0	1	0	0	1	0	1
0	0	1	0	1	1	0
0	0	0	1	1	1	1

For example, the code word whose information positions are 1 1 0 0 is the sum of the first two basis vectors and is 1 1 0 0 1 1 0. Since this code is a four dimensional subspace, the orthogonal space is three dimensional (in this case contained in the code itself) and has the following three vectors as a basis.

$$v_1 = 0\ 0\ 0\ 1\ 1\ 1\ 1$$

$$v_2 = 0\ 1\ 1\ 0\ 0\ 1\ 1$$

$$v_3 = 1\ 0\ 1\ 0\ 1\ 0\ 1$$

If $u = (u_1, u_2, \ldots, u_7)$ and $w = (w_1, w_2, \ldots, w_7)$ then $u \cdot w = \sum_{i=1}^{7} u_i w_i \pmod 2$. A vector u is in the code if and only if $u \cdot v_i = 0$, $i = 1, 2, 3$. If a single error occurs in transmission, then the received vector equals $u + e$ where e has a single one in the position in error. The matrix with rows v_1, v_2, v_3 has as columns the numbers 1 through 7 in binary so that the three dot products $u \cdot v_1, u \cdot v_2, u \cdot v_3$ give the position in error in binary. This is known as Hamming decoding and corrects any single error in a very efficient manner. We shall explain why the Hamming code is a single error-correcting code, but first we need some definitions.

2. Definitions and a theorem. If V is the set of all binary (ternary) n-tuples with $\oplus$ componentwise mod 2 (mod 3), then an (n, k) binary (ternary) code is a k dimensional subspace of V.

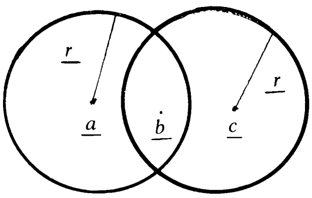
Another important concept in coding is the weight of a vector v, denoted $\mathrm{wt}(v)$. This is the number of non-zero components it has. Thus the first three vectors in the basis of the Hamming (7,4) code have weight 3 and the fourth has weight 4. An even more important concept is the minimum weight of a code. This is the weight of the non-zero vector in the code of smallest weight and is denoted by d. We shall see why this is important in the next theorem.

We let $[a]$ denote the largest integer less than or equal to a, for example $[3/2] = 1$.

THEOREM. *If d is the minimum weight of a code, the code can correct $[(d-1)/2]$ errors.*

Proof. We define the distance D between 2 vectors a and c to be the number of positions in which they differ. Then it follows that $D(a, c) = \mathrm{wt}(a + c)$. It is easy to prove that D is a metric. Hence the triangle inequality holds, i.e., $D(a, c) \leqq D(a, b) + D(b, c)$ for all a, b, c in the space. A sphere of radius r about a vector is defined to be the set of vectors in V at distance $\leqq r$ from the vector. The crucial part of the proof is the next statement. If the minimum weight of the code is d, the spheres of radius $r = [(d-1)/2]$ about the code-vectors are disjoint.

We demonstrate this by contradiction. Suppose not. Let a and c be in the code.



Then $D(a,c) = D(a,b) + D(b,c) = (d-1)/2 + (d-1)/2 = d-1$. But $D(a,c) = \text{wt}(a+c)$, the weight of a codeword. So this must be $= d$; a contradiction.

So if there are $[(d-1)/2]$ or fewer errors, the received vector is in one of these spheres; and we can decode it to the unique codeword in the center of the sphere.

From this theorem it follows that if we fix n and k, we want d as large as possible. This is a difficult thing to do.

Since d for the Hamming code is 3, the theorem tells us why this is a single error-correcting code, but does not explain the Hamming decoding algorithm which we described. One of the central and difficult problems in the practical use of codes is devising usable decoding algorithms.

THREE BINARY CODES

Longer codes are more practical because errors can be distributed in more ways. They are also more difficult to use and require greater analysis.

LENGTH	NUMBER OF INFORMATION SYMBOLS	NUMBER OF ERRORS CODE CAN CORRECT	NUMBER OF MESSAGES IN CODE	TOTAL NUMBER OF MESSAGES
7	4	1	16	128
23	12	3	4,096	8,388,608
47	24	5	16,777,216	140,737,488,355,328

FIG. 1

For many purposes longer codes are more practical because errors can be distributed in more ways. They are also more difficult to use and require greater analysis. Figure 1 illustrates these difficulties. The first code listed here is the single error-correcting Hamming code which has 16 messages. There are $128 = 2^7$ possible received messages in this situation and we have to decide for each of these which of the 16 messages was sent. As we saw, this is not too difficult. The next situation, however, is the (23, 12) triple-error correcting Golay code. Here there are $2^{23} = 8,388,608$ possible received messages which we must assign to the 4096 transmitted messages; clearly a much more difficult task. The numbers are even more astronomical for the 5-error correcting (47, 24) code.

3. Codes, groups, and designs. The theorem demonstrated that if d is the minimum weight of a code, the spheres of radius $[(d-1)/2]$ are disjoint. It is not necessarily true that the spheres cover the space. When they do, the code is called perfect.

The Hamming (7, 4) code and the Golay (23, 12) codes are perfect. To show you that the Hamming code is perfect I only have to demonstrate that each vector in the space of 7-tuples is in a sphere of radius 1 about a code word. There are 2^4 code words, 2^3 vectors in a sphere. $2^4 \cdot 2^3 = 2^7$. Q.E.D. An exactly analogous proof shows that the Golay (23, 12) code is perfect.

Perfect codes are useful in practical applications and they are also interesting mathematical objects, so it was an interesting, open problem: which codes are perfect?

In [**10**] it is shown that the binary Golay (23, 12) code is the unique perfect code with its parameters, and also that the ternary Golay (11, 6) code is the unique perfect code with its parameters. This problem was recently completed by van Lint and Tietäväinen with the result [**7**] that these 2 Golay codes and the family of single error-correct Hamming codes are the only non-trivial perfect codes of any length and over any field.

These 2 Golay codes have recently become interesting to finite group theorists. These 2 codes were constructed by an electrical engineer, Marcel Golay, for communication purposes, and his paper was published in the *Transactions of the I.E.E.E.* [**5**]. They are indeed very good for communications and are widely used for that. For the past one hundred years, from 1861 to 1966, all finite simple groups belonged to infinite families of groups, except for 5 exceptional ones, called sporadic groups, the Mathieu groups, discovered by Mathieu around 1861. No other sporadic groups were found for 100 years until Janko, a Yugoslav mathematician, found a new sporadic group in 1966. Since then two or three new ones have been discovered each year. A major problem in the simple sporadic groups has been whether there is anything unifying them. In 1969 John Conway [**2**] of England discovered a very large new, simple sporadic group CO_1, the automorphism group of the Leech lattice. The basis of the Leech lattice is either Golay code. This new group is not only interesting as a new sporadic group but also because it contains 12 of the 23 known sporadic groups as subgroups and might be a unifying principle for these groups.

Group theory intersects coding theory in other ways. One of the main problems in the practical use of codes is the decoding problem. Long codes are hard to decode and the group of the code is useful here, for more on this see MacWilliams [**8**]. The group of a code has also been very useful in classifying self-dual codes.

A code is called self-dual if it is equal to its orthogonal code. This is an important class of codes because it includes many of the best algebraic codes like the extended Golay (24, 12) code, the quadratic residue codes, and the symmetry codes, and also because more is known about the weight distributions of these codes via the Gleason polynomials [**4**]. So it was an interesting open problem to determine all binary self-dual codes of moderate lengths. A binary self-dual code is called doubly-even if the weights of all code words are divisible by 4. Both the extended Hamming (8, 4) code and the extended Golay (24, 12) codes are doubly even. The binary self-dual codes until length 20 were classified in [**9**], those of lengths 22 and 24 were classified in [**12**] and just recently the self-dual doubly even codes of length 32 were classified in [**3**] and from this latest classification all self-dual codes of length less than 32 can be identified. Two binary codes are called equivalent if one can be obtained from the other by a coordinate permutation. In all these classifications, a basis is given for one code from an equivalence class, the entire group of the code is given as is the number of equivalent codes and the code's weight distribution. The basic strategy for these classifications is as follows. The number N of self-dual $(n, n/2)$ codes is known, and if somehow we have a list $C_1, \ldots, C_k$ of self-dual codes and we know that g_i is the order of the group of C_i, then we have a complete listing of self-dual $(n, n/2)$ codes when $N = n/g_i$. This was done in all these cases although finding the list of $C_1, \ldots, C_k$ is very difficult, particularly for the (32, 16) codes. However, it can be seen from this that a crucial part of the problem is determining the entire group of a code. Since CO_1 is the group of a lattice derived from a code, groups of lattices arising from interesting codes could themselves be interesting. In the (32, 16) classification three new codes with minimum weight eight were discovered and their lattices might have interesting groups. This has not been determined yet.

Codes also play a part in the discovery of some very unusual and rare designs. First we will define t-designs. A $\lambda: t - r - n$ design consists of n points and a set of blocks where each block consists of r points, such that every t points lie on exactly λ blocks. Projective planes are 2-designs, there are many interesting 3-designs, 4-designs are somewhat unusual and interesting, but until recently only a handful of 5-designs were known. These were associated with the old Mathieu groups mentioned

SOME 5-DESIGNS FOUND IN SELF-DUAL CODES

Vectors of Weight		in	Hold Design
Over GF(2):	8	Golay (24, 12)	1: 5–8–24
Over GF(3):	6	Golay (12, 6)	1: 5 − 6 − 12
	9	Symm.(24, 12)	6: 5 − 9 − 24
	12	Symm.(24, 12)	576: 5–12–24
	15	Symm.(24, 12)	8,580: 5–15–24
	12	Symm.(36, 18)	45: 5–12–36
	15	Symm.(36, 18)	5,577: 5–15–36
	18	Symm.(36, 18)	209,685: 5–18–36
	15	Symm.(48, 24)	364: 5–15–48
	18	Symm.(48, 24)	50,456: 5–18–48
	18	Symm.(60, 30)	3,060: 5–18–60

FIG. 2

before. No 6 or higher designs are known, nor is it known if they can exist. Now many 5-designs have been found in the family of symmetry codes [**11**] (Figure 2) and in other codes [**1**].

4. Conclusion. In summary, much is known theoretically about codes, they are interesting mathematical topics in themselves, and they are even more interesting because of the great light they shed on other branches of mathematics, namely rare combinatorial configurations (5-designs) have been found in codes, and some codes are basic to the construction of some of the new, finite, simple groups. There is much mathematical interest in codes, many well-known mathematicians are working on them. I believe this demonstrates again that starting with some very practical problems, it is possible to develop some very interesting and exciting mathematics.

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PROGRESS REPORTS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

ARITHMETIC PROGRESSIONS

P. R. HALMOS AND C. RYAVEC

A set of positive integers with some algebraic structure, however rudimentary, is better to work with than one without any — it is likely to give more number-theoretic information. From this point of view arithmetic progressions are good sets, and a set that contains many arithmetic progressions is better than one that does not. It is reasonable to conjecture that "large" sets contain many arithmetic progressions.

A famous theorem of van der Waerden asserts that if the set of positive integers is partitioned into two subsets, then at least one of them must be large, in the sense that it contains arbitrarily long arithmetic progressions. (Caution: a set that contains arbitrarily long arithmetic progressions may fail to contain an infinitely long one. Example: the sequence

$$11, 101, 102, 1001, 1002, 1003, 10001, 10002, 10003, 10004, \ldots,$$

made up of the blocks $10^i + j$, with $i = 1, 2, 3, \ldots$ and $j = 1, \ldots, i$.)

Van der Waerden's theorem is implied by the following assertion: to each positive integer k there corresponds a positive integer $n(= n(k))$ such that if the set $\{1, \ldots, n\}$ is partitioned into two subsets, then at least one of them contains an arithmetic progression of k terms. (The first non-trivial example is easily obtained just by listing all possibilities: if $k = 3$, then $n = 9$.)

Many results in number theory concern the connection between primes and arithmetic progressions. It is good to know the extent to which primes are regularly distributed in arithmetic progressions, and it would be good to know whether the set of primes contains arbitrarily long arithmetic progressions — but the latter is a long-standing unsolved problem.

Erdös and Turán (1936) had once hoped to attack this problem (and others) by showing that if a sequence is sufficiently dense, then it must contain arbitrarily long arithmetic progressions. To make this precise, fix k and n and ask: how many numbers between 1 and n (inclusive) are needed to guarantee that they will contain an arithmetic progression with k terms? (If $k = 3$ and $n = 9$, the

answer is 5.) Equivalently: what is the largest number $r(=r_k(n))$ such that some set of r numbers between 1 and n does *not* contain an arithmetic progression with k terms? (The value of $r_3(9)$ is 4.)

The problem of long arithmetic progressions of primes would be solved by a proof that $r_k(n) < \pi(n)$, where $\pi(n)$ is the number of primes less than or equal to n. It would be good enough, of course, to show that for each k the inequality holds for all sufficiently large n.

Erdös and Turán observed that

$$r_k(m+n) \leqq r_k(m) + r_k(n);$$

they inferred that, for each k, the sequence $\{(1/n)r_k(n)\}$ has a limit c_k; and, finally, they conjectured that $c_k = 0$ for each k, i.e., that

$$\lim_n \frac{1}{n} r_k(n) = 0.$$

The conjecture is simple and elegant and would have pleasant consequences, but it turned out to be extraordinarily difficult to settle. Sample consequence: every set of positive density contains arbitrarily long arithmetic progressions. In detail: if E is a set of positive integers, if a_n is the number of elements of E between 1 and n, and if $\lim_n (1/n)a_n > 0$, then E contains arbitrarily long arithmetic progressions. Reason: the truth of the conjecture implies that, for each k, the inequality $r_k(n) < a_n$ holds for all sufficiently large n.

For $k=3$ the Erdös–Turán conjecture was proved in 1954 by K. F. Roth; for $k=4$, using van der Waerden's theorem, E. Szemerédi proved it in 1967. The proof was so intricate (some called it the apotheosis of the elementary method in number theory and combinatorics) that there was reluctance (to put it mildly) to attempt the case $k=5$ till substantial simplifications could be made in the case $k=4$. The next step was taken by Roth (1970), who put the proof on an analytic basis and, incidentally, removed the use of van der Waerden's theorem, but none of that seemed to help.

In 1972 Szemerédi announced that he had solved the problem in full generality. Erdös had offered a prize of \$1000.00 for a solution, but to collect it Szemerédi needed to solve another formidable problem, namely the problem of writing the proof down so as to make it comprehensible to others. A preliminary exposition of Szemerédi's proof was written by A. Hajnal (who became sufficiently convinced that he told Erdös that he was willing to buy the proof from Szemerédi for \$500.00). An abstract of Szemerédi's result appeared in 1974 and the full version a year later.

The leading idea of Szemerédi's proof can probably not be said better in a short paragraph than the way Szemerédi himself put it. "The basic objects with which the proof of the main theorem deals are not just arithmetic progressions themselves but rather generalizations of arithmetic progressions called *m-configurations*. Roughly speaking, a 1-configuration is just an arithmetic progression; an m-configuration is an "arithmetic progression" of $(m-1)$-configurations. In a nutshell, one can show that for any given set of integers R of positive upper density, a very long m-configuration which intersects R in a moderately regular way must always contain a shorter (but still quite long) $(m-1)$-configuration which intersects R in an even more regular way. In this way we eventually conclude that R must contain arbitrarily long 1-configurations, i.e., arithmetic progressions, and we are done."

Reference

E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, Proc. Int. Congress Math., Vancouver, 2 (1974) 503–505; Acta Arithmetica, 27 (1975) 199–245 (MR 51 # 5547).

MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

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THE CONGRUENCE $a^{r+s} \equiv a^r \pmod{m}$

A. E. LIVINGSTON AND M. L. LIVINGSTON

1. Introduction. As is well known, the sequence of power residues of the integer a modulo m is eventually periodic; that is, for each a there exist positive integers $r = r(a, m)$ and $s = s(a, m)$ such that

$$a^{r+s} \equiv a^r \pmod{m}. \tag{1.1}$$

If $(a, m) = 1$ then the Euler–Fermat theorem gives (1.1) with $s = \phi(m)$ and all integers r. Without the restriction $(a, m) = 1$ there is the following result attributed to L. Redei

$$a^m \equiv a^{m-\phi(m)} \pmod{m}$$

and, more recently, the result, [2],

$$a^{\phi(m)+1} \equiv a^{\phi(m)+1-\phi(Q)} \pmod{m}, \tag{1.2}$$

where Q is the largest factor of m which is relatively prime to a. In [2], R. Osborn formulated what he termed 'the ideal generalization of the Euler–Fermat Theorem,' namely:

If a is an integer and m is an integer >2, then

$$a^{f(m,a)} \equiv a^{g(m,a)} \pmod{m}, \tag{1.3}$$

where $f(m, a)$ and $g(m, a)$ are the least positive integers, such that (1.3) holds and $g(m, a) < f(m, a)$.

The purpose of the present paper is to prove just such an ideal generalization (Theorem 3.1) and to establish, as a consequence, the following result uniform in a (Corollary 3.1):

A GENERALIZED EULER–FERMAT THEOREM. *If m is a positive integer and r is a non-negative integer then*

$$a^{r+\phi(m)} \equiv a^r \pmod{m} \tag{1.4}$$

for all integers a if and only if m is $(r+1)$-st power-free.

2. Preliminary lemmas. Suppose a and m are integers with $m > 0$. Let $\xi_m(a)$ be the smallest positive integer r such that (1.1) is true for some positive integer s and let $\rho_m(a)$ denote the smallest positive integer s such that

$$a^{\xi_m(a)+s} \equiv a^{\xi_m(a)} \pmod{m}.$$

LEMMA 2.1. *If r and s are distinct positive integers then* (1.1) *is true if and only if $r \geqq \xi_m(a)$ and $\rho_m(a) \mid s$.*

Proof: We will write ρ and ξ for $\rho_m(a)$ and $\xi_m(a)$ respectively.

Since $a^{\xi+\rho} \equiv a^{\xi} \pmod{m}$ it follows that $a^{r+\rho} \equiv a^r \pmod{m}$ for all integers $r \geqq \xi$ and hence

$$a^{r+n\rho} \equiv a^r \pmod{m}$$

for all positive integers n and r with $r \geqq \xi$.

Conversely, suppose that (1.1) holds for the distinct integers r and s. Then we must have $r \geqq \xi$ by

the definition of ξ. Let A and B be the integers determined by $s = A\rho + B$ and $0 \leqq B < \rho$ and let k be any non-negative integer for which $r < \xi + k\rho$. Then by (1.1) and the definition of ξ and ρ,

$$\begin{aligned} a^{\xi} &\equiv a^{\xi+k\rho} && (\operatorname{mod} m) \\ &\equiv a^{\xi+k\rho-r} \cdot a^{r+s} && (\operatorname{mod} m) \\ &\equiv a^{\xi+(k+A)\rho+B} && (\operatorname{mod} m) \\ &\equiv a^{\xi+B} && (\operatorname{mod} m). \end{aligned}$$

It now follows from the definition of ρ that $B = 0$, so that $\rho \mid s$ as claimed.

Before we determine $\rho_m(a)$ and $\xi_m(a)$ explicitly we shall establish the following technical lemma.

LEMMA 2.2. *Let $d_0 = (a, m)$ and $d_{k+1} = (d_k, m/\Pi_{i=0}^{k} d_i)$ for $k = 0, 1, 2, \ldots$. If r is a non-negative integer then*

$$a^{r+s} \equiv a^{r} \pmod m \tag{2.1}$$

for some positive integer s if and only if $d_r = 1$.

Proof. Let $d_{-1} = a$, $m_{-1} = m$, $d_{k-1} = d_k D_k$ and $m_k = m/\Pi_{i=0}^{k} d_i$ for $k = 0, 1, 2, \ldots$. Denote by S_k the statement

$$d_{k-1}^{r-k+s}\left(\prod_{i=0}^{k-1} D_i\right)^{s} \equiv d_{k-1}^{r-k} \pmod{m_{k-1}}$$

for $k = 0, 1, 2, \ldots, r$. S_0 is the congruence (2.1) while S_r is the statement

$$d_{r-1}^{s}\left(\prod_{i=0}^{r-1} D_i\right)^{s} \equiv 1 \pmod{m_{r-1}}. \tag{2.2}$$

If $d_r \neq 1$ then (2.2) cannot hold for any positive integer s while if $d_r = 1$ and we choose $s = \phi(m)$ then (2.2) holds by the Euler–Fermat theorem and the fact that $\phi(m_{r-1}) \mid \phi(m)$. Thus, if we prove the equivalence of S_k and S_{k+1} for $k = 0, 1, 2, \ldots, r-1$ and $r \geqq 1$, the lemma will follow.

Since $d_k = (d_{k-1}, m_{k-1})$ and $d_{k-1} = d_k D_k$ we have $(m_k, D_k) = 1$. S_k may be rewritten as

$$d_k^{r-k+s} D_k^{r-k+s}\left(\prod_{i=0}^{k-1} D_i\right)^{s} \equiv d_k^{r-k} D_k^{r-k} \pmod{d_k m_k}$$

which is equivalent to

$$d_k^{r-k-1+s} D_k^{r-k+s}\left(\prod_{i=0}^{k-1} D_i\right)^{s} \equiv d_k^{r-k-1} D_k^{r-k} \pmod{m_k}$$

and this congruence, in turn, is equivalent to S_{k+1}.

In the following, $[x]$ denotes the greatest integer $\leqq x$, $\langle a, b\rangle$ denotes the ordered pair with first element a and second element b, p denotes a prime and $p^{\gamma} \| n$ means $p^{\gamma} \mid n$ while $p^{\gamma+1} \nmid n$.

LEMMA 2.3. *Let $S = \{\langle \alpha, \beta\rangle \mid 0 < \beta \leqq \alpha,\ p \text{ prime},\ p^{\beta} \| a, p^{\alpha} \| m\}$ and let $K = \max\{1\} \cup \{[\alpha/\beta] \mid \langle \alpha, \beta\rangle \in S\}$. Then*

$$\xi_m(a) = \begin{cases} K+1 & \text{if } \exists \langle \alpha, \beta\rangle \in S \ \ni K < \alpha/\beta < K+1 \\ K & \text{otherwise.} \end{cases} \tag{2.3}$$

Proof. S is empty if and only if $(a, m) = 1$ or $(a, m) > 1$ and $\beta > \alpha$ whenever $p^{\beta} \| a, p^{\alpha} \| m$ and $\alpha > 0$. In either case $\xi_m(a) = K = 1$.

Suppose S is not empty. We will show that, in the notation of Lemma 2.2, the least positive integer

r such that $d_r = 1$ is given by the right-hand side of (2.3). The lemma will then follow from the definition of $\xi_m(a)$ and Lemma 2.2.

If $p^\beta \| a$, $p^\alpha \| m$ and $\langle \alpha, \beta \rangle \notin S$ then $p^\alpha \| d_0$ but $p \nmid d_k$ for $k \geqq 1$. On the other hand, if $\langle \alpha, \beta \rangle \in S$ we now prove that

(i) $p^\beta \| d_k$ for $k < [\alpha/\beta]$
(ii) $p^\lambda \| d_k$ where $\lambda = \alpha - [\alpha/\beta]\beta$ and $k = [\alpha/\beta]$, and
(iii) $p \nmid d_k$ for $k > [\alpha/\beta]$.

We establish (i) by induction. Clearly $p^\beta \| (a, m) = d_0$, so suppose, for some positive integer $k < [\alpha/\beta]$, that $p^\beta \| d_j$ for all $0 \leqq j < k$. Then, since $k + 1 \leqq \alpha/\beta$, we have $p^\beta \mid m/\Pi_{i=0}^{k-1} d_i$ and, therefore, $p^\beta \| d_k = (d_{k-1}, m/\Pi_{i=0}^{k-1} d_i)$. To prove (ii) we need merely observe that if $k = [\alpha/\beta]$ and $p^\lambda \| d_k$ then $\lambda = \min\{\beta, \alpha - [\alpha/\beta]\beta\} = \alpha - [\alpha/\beta]\beta$. Finally, (iii) follows from the fact that $p \nmid m/(\Pi_{i=0}^{k} d_i)$ where $k = [\alpha/\beta]$.

The above results imply, when S is non-empty, that $d_k \neq 1$ if $k < K$, $d_k = 1$ if $k > K$ and that $d_K \neq 1$ if and only if there exists $\langle \alpha, \beta \rangle \in S$ such that $\alpha - [\alpha/\beta]\beta > 0$. Thus the lemma is proved.

Let m, x be integers for which $(m, x) = 1$ and let $O(m, x)$ denote the least positive integer s such that $x^s \equiv 1 \pmod{m}$.

LEMMA 2.4. *Let $m_a = \Pi p^\alpha$ where $\alpha = \alpha(p)$ and the product is over primes p for which $p^\alpha \| m$ and $p \nmid a$. Then*

$$\rho_m(a) = O(m_a, a).$$

Proof: Let $A = a/\Pi p^\beta$ where $\beta = \beta(p)$ and the product is over primes p for which $p^\beta \| a$ and $p \mid (a, m)$. If r is any integer $\geqq \xi_m(a)$ then $\beta r \geqq \alpha$ by Lemma 2.3 and the following congruences are equivalent:

$$a^{r+s} \equiv a^r \pmod{m}, \tag{2.4}$$

$$A^{r+s} \prod_{p \mid (a,m)} p^{\beta(r+s)} \equiv A^r \prod_{p \mid (a,m)} p^{\beta r} \pmod{m},$$

$$A^{r+s} \prod_{p \mid (a,m)} p^{\beta(r+s)-\alpha} \equiv A^r \prod_{p \mid (a,m)} p^{\beta r-\alpha} \pmod{m_a},$$

$$A^s \prod_{p \mid (a,m)} p^{\beta s} \equiv 1 \pmod{m_a},$$

$$a^s \equiv 1 \pmod{m_a}. \tag{2.5}$$

Setting $s = \rho_m(a)$ and $r = \xi_m(a)$ we have that (2.4) holds and it follows then from (2.5) that $O(m_a, a) \mid \rho_m(a)$. On the other hand, choosing $s = O(m_a, a)$ and $r = \xi_m(a)$ we find that (2.5) and, hence, (2.4) holds. It then follows from the definition of $\rho_m(a)$ that $\rho_m(a) \leqq O(m_a, a)$. Thus $\rho_m(a) = O(m_a, a)$.

3. Main results. We come now to the 'ideal generalization of the Euler–Fermat theorem' and some of its consequences.

THEOREM 3.1. *If m, r, and s are positive integers and a is an integer then*

$$a^{r+s} \equiv a^r \pmod{m}$$

if and only if $r \geqq \xi_m(a)$ and $\rho_m(a) \mid s$ where $\xi_m(a)$ is given by Lemma 2.3 and $\rho_m(a)$ is given by Lemma 2.4.

Proof: The theorem follows directly from Lemmas 2.1, 2.3 and 2.4.

Now, let $\lambda(m)$ denote the 'universal exponent' of m, [**1**; p. 53], that is

$$\lambda(m)=\begin{cases} 1 & \text{if } m=1 \\ \frac{1}{2}\phi(m) & \text{if } m=2^\alpha \text{ and } \alpha>2 \\ \phi(m) & \text{if } m=2,4,p^\alpha \text{ with } p \text{ an odd prime} \\ \text{l.c.m.} & \{\lambda(p^\alpha)\mid p^\alpha \,\|\, m\} \quad \text{otherwise} \end{cases}$$

and let $\varepsilon(m)=\max\{1\}\cup\{\alpha \mid p^\alpha \,\|\, m\}$. In the next theorem we give necessary and sufficient conditions, uniform in a, for which (1.1) holds.

THEOREM 3.2. *If r, s, and m are positive integers, then*

$$a^{r+s}\equiv a^r \pmod m \tag{3.1}$$

for all integers a if and only if $r \geqq \varepsilon(m)$ and $\lambda(m)\mid s$.

Proof: By Theorem (3.1), congruence (3.1) holds for each integer a if and only if $r \geqq \xi_m(a)$ and $\rho_m(a)\mid s$ for each a. It follows from Lemma 2.3 that $r \geqq \xi_m(a)$ for all a if and only if $r \geqq \varepsilon(m)$. In addition, $\rho_m(a)\mid s$ for all a if and only if $\rho_m = \text{l.c.m.}\{\rho_m(a)\mid a \text{ an integer}\}$ divides s. All that remains now is to show that $\rho_m=\lambda(m)$.

Let n be a positive integer, then $a^{\lambda(n)}\equiv 1 \pmod n$ holds for each integer a such that $(a,n)=1$, [**1**, Theorem 4–8]. This result with

$$n=\prod_{p\nmid a,\, p^\alpha\|m} p^\alpha \text{ and Lemma 2.4 show that } \rho_m(a)\mid\lambda\Big(\prod_{p\nmid a,\, p^\alpha\|m} p^\alpha\Big)$$

for each integer a. Furthermore, from the definition of $\lambda(m)$ we see that $\lambda(d)\mid\lambda(m)$ if $d\mid m$. Consequently, $\rho_m\mid\lambda(m)$. On the other hand, since there exists an integer a' such that $(a',m)=1$ and $O(m,a')=\lambda(m)$, [**1**, Theorem 4–9], it follows that we also have $\lambda(m)\mid\rho_m$ and hence $\lambda(m)=\rho(m)$.

COROLLARY 3.1. (*Generalized Euler–Fermat Theorem*).

Proof: From Theorem 3.2 we see that if m and r are positive integers then congruence (1.4) can hold if and only if $r \geqq \varepsilon(m)$.

Added in proof. A variation of Corollary 3.1 in which $\phi(m)$ is replaced by $\lambda(m)$ has appeared as Theorem 1 in the article *A maximal generalization of Fermat's theorem* by D. Singmaster (Math. Mag., 39 (1966) 103–107). This theorem can also be obtained as corollary of our Theorem 3.2.

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MORLEY'S THEOREM AND A CONVERSE

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A very elegant and rather classical type of theorem on triangles was not discovered until the beginning of this century by the mathematician Frank Morley (1860–1937), who incidentally, was the father of novelist Christopher Morley. The theorem is that the three intersection points of adjacent trisectors of the angles in any triangle are the vertices of an equilateral triangle. See Fig. 1. Morley

discovered this sometime around 1900 as an interesting special result from his geometric studies. He told his friends of the result, and the news spread through the mathematical world in the form of gossip. Morley, however, did not publish a proof of this theorem until 1924 when it appeared in a Japanese journal for secondary mathematics [**5**]. His proof also appeared in 1929 as a special result in the middle of a paper in the *American Journal of Mathematics* [**6**]. It would, however, take a reader some time to be able to follow this paper. The first published proof came in 1909 in response to the theorem being presented as a problem in the *Educational Times* [**7**]. This proof is given by Coxeter in [**2**, p. 47–50].

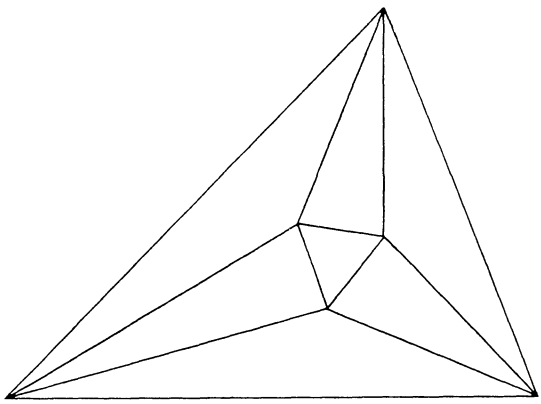

FIG. 1

Many proofs of the theorem have been published, some using trigonometric methods, and others using purely elementary techniques. A very clean and elementary proof is given by Coxeter in [**1**, p. 23–25]. The method of this proof, adapted from one given by K. Venkatachaliengar, [**8**], is to construct an arbitrary triangle in the desired position about an equilateral triangle. A nice trigonometric proof is given by David Kay in [**3**, p. 15–17].

A surprising fact about the Morley triangle is its relation to the Steiner hypocycloid. This is a hypocycloid of three cusps which is the envelope of the Simson lines (lines through the three collinear feet of perpendiculars to the sides of a triangle drawn from a point on the circumcircle) of a triangle. The Morley triangle is oriented so that its altitudes are parallel to the tangents at the cusps of the hypocycloid. This relation is shown in [**4**, p. 72–79] and also [**3**, p. 248–253].

Such a theorem as Morley's invites variations. People have looked at the resulting n-gon that one gets by taking the intersection points of adjacent trisectors of the angles in an n-gon. Examples in [**9**] show that the Morley n-gon may be regular, even when the given n-gon is irregular. However, only in the case of the triangle is it always regular.

The elegance and relatively recent discovery of Morley's Theorem lead one to wonder if there are not other similar relations lurking about unnoticed. One especially may wonder, since the bisectors of the angles of a triangle are concurrent, and the trisectors give rise to an equilateral triangle; are we in for more surprises if we take one-fourth (quadrisect?) the angles or divide them in other ways? The following theorem shows that we cannot get another theorem like Morley's by dividing the angles in some other way.

THEOREM. *Take k_1, k_2, and k_3 between* 0 *and* 1/2. *Starting with any triangle OPQ, at each vertex form angles on each of the two sides equal to k_i (for the i-th vertex) times the angle of the triangle at the vertex and falling inside the triangle. The interior sides of two of these angles having a side of the triangle in common will intersect inside the triangle. Call these three intersection points A, B, and C. The triangle ABC will be equilateral for any starting triangle OPQ if and only if $k_1 = k_2 = k_3 = 1/3$.*

This is illustrated in Fig. 2, where α = angle OQP, β = angle POQ, and γ = angle QPO.

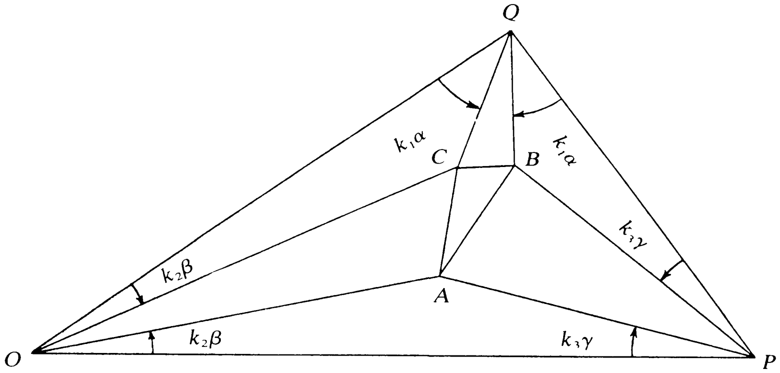

FIG. 2

In the proof, we look at a special triangle that will allow convenient calculations. Actually, it is a "triangle" with two right angles which is obtained as the limit of isosceles triangles. From it we first conclude that the k_i's are equal; secondly, we get an equation which the common values must satisfy. Finally, a little analysis shows that the equation has only the desired solution.

Proof. The "if" part is Morley's Theorem.

Now assume triangle ABC to be equilateral. If the numbers, k_1, k_2, k_3, are not all equal, we assume $k_2 \neq k_3$. We exclude the known case of $k_1 = k_2 = k_3 = 1/2$. Choose OPQ to be isosceles with base $OP = 2$. Then $\beta = \gamma$ and $\alpha = \pi - 2\beta$. Install a cartesian coordinate system with center O, and P on the positive x-axis.

The coordinates of Q are $(1, \tan\beta)$. Line OC makes an angle $\beta - k_2\beta$ with the x-axis, and CQ an angle $\beta + k_1\alpha$. Hence, if x and y are the coordinates of C, they satisfy the equations

$$y = \tan(\beta - k_2\beta)x$$

$$y - \tan\beta = \tan(\beta + k_1\alpha)(x - 1).$$

Solving these for x gives

$$x = \frac{\tan\beta - \tan(\beta + k_1\alpha)}{\tan(\beta - k_2\beta) - \tan(\beta + k_1\alpha)}.$$

Multiply numerator and denominator by $\cos(\beta + k_1\alpha)$ to get

$$x = \frac{(\cos(\beta + k_1\alpha)/\cos\beta)\sin\beta - \sin(\beta + k_1\alpha)}{\cos(\beta + k_1\alpha)\tan(\beta - k_2\beta) - \sin(\beta + k_1\alpha)}.$$

$$\frac{d}{d\beta}\left(\beta + k_1\alpha\right) = 1 - 2k_1, \quad \text{since} \quad \beta + k_1\alpha = \beta + k_1(\pi - 2\beta) = (1 - 2k_1)\beta + k_1\pi.$$

Hence it follows from l'Hôpital's Rule that

$$\lim_{\beta \to \pi/2} \frac{\cos(\beta + k_1\alpha)}{\cos\beta} = 1 - 2k_1.$$

Finally

$$\lim_{\beta \to \pi/2} x = \frac{(1 - 2k_1)\sin(\pi/2) - \sin(\pi/2)}{\cos(\pi/2)\tan(\pi/2 - k_2\pi/2) - \sin(\pi/2)} = 2k_1.$$

Since we are assuming the triangles ABC to be equilateral, the triangle we get in the limit as

$\beta \to \pi/2$ must also be equilateral. We continue using the letters A, B, and C, now designating the limit points.

From our work above, we know that C has coordinates $(2k_1, 2k_1 \tan(\pi/2 - k_2\pi/2))$ or $(2k_1, 2k_1 \cot(k_2\pi/2))$. Similarly, the difference in x-coordinates of B and P must be $2k_1$.

Now let $x = \cot(k_2\pi/2)$, $y = \cot(k_3\pi/2)$, z = one-half the length of CB, and ω = the angle CB makes with the positive x-direction.

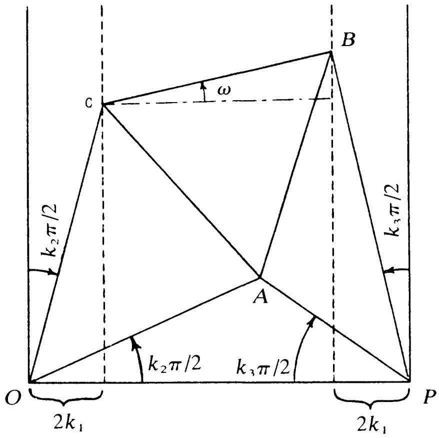

FIG. 3

Using these, C has coordinates $(2k_1, 2k_1x)$, and B has $(2-2k_1, 2k_1y)$. We can now write

$$\sin\omega = (2k_1y - 2k_1x)/(2z) = k_1(y-x)/z \tag{1}$$

$$\cos\omega = (2-4k_1)/(2z) = (1-2k_1)/z. \tag{2}$$

CA makes an angle of $\omega - \pi/3$ with the x-direction. Hence the x and y-coordinates of A are given respectively by $2k_1 + 2z\cos(\omega - \pi/3)$ and $2k_1x + 2z\sin(\omega - \pi/3)$. These can be used to write the cotangents of angles AOP and APO, which are x and y respectively since angle AOP is $k_2\pi/2$ and APO is $k_3\pi/2$. This gives

$$x = \frac{2k_1 + 2z\cos(\omega - \pi/3)}{2k_1x + 2z\sin(\omega - \pi/3)}$$

$$y = \frac{2 - 2k_1 - 2z\cos(\omega - \pi/3)}{2k_1x + 2z\sin(\omega - \pi/3)}.$$

Now rewrite the trigonometric functions using the addition law and equations (1) and (2):

$$2z\cos(\omega - \pi/3) = 1 - 2k_1 + \sqrt{3}k_1(y-x)$$

$$2z\sin(\omega - \pi/3) = k_1(y-x) - \sqrt{3}(1-2k_1).$$

Substitute these in the two preceding equations and simplify to get

$$k_1x^2 + k_1xy + \sqrt{3}(3k_1 - 1)x - \sqrt{3}k_1y - 1 = 0 \tag{3}$$

$$k_1xy + k_1y^2 + \sqrt{3}(3k_1 - 1)y - \sqrt{3}k_1x - 1 = 0. \tag{4}$$

As expected from symmetry, the equations transform to each other under interchange of x and y. If $x \neq y$, subtract equation (4) from (3) and divide by $x - y$. This gives

$$k_1x + k_1y + \sqrt{3}(3k_1 - 1) + \sqrt{3}k_1 = 0. \tag{5}$$

Using this to replace y in equation (3), the terms involving x cancel, and we conclude that $k_1 = 1/3$. But then (5) becomes $x + y + \sqrt{3} = 0$, which is impossible since x and y are at least 1. Therefore, it must be that $x = y$, and (3) and (4) reduce to one equation, which can be written $2k_1x^2 + \sqrt{3}(2k_1 - 1)x - 1 = 0$. Solving this for k_1 gives

$$k_1 = \frac{\sqrt{3}x + 1}{2x(x + \sqrt{3})}.$$

Since $x = y$, we have $k_2 = k_3$. Also, because we assumed at the beginning that k_2 and k_3 were unequal if the three were not all the same, we have $k_1 = k_2 = k_3$. Hence $x = \cot(k_2\pi/2) = \cot(k_1\pi/2)$, and $k_1 = (2/\pi)$ arccot x. Setting the two expressions for k_1 equal gives

$$\frac{2}{\pi}\operatorname{arccot} x = \frac{\sqrt{3}x + 1}{2x(x + \sqrt{3})}. \tag{6}$$

We form the function

$$D(t) = \frac{\sqrt{3}t + 1}{2t(t + \sqrt{3})} - \frac{2}{\pi}\operatorname{arccot} t$$

and show that the only positive zeros it has are $t = 1$ and $t = \sqrt{3}$, which are easily verified to be zeros. Consider

$$\begin{aligned} D'(t) &= -\frac{\sqrt{3}t^2 + 2t + \sqrt{3}}{2t^2(t + \sqrt{3})^2} + \frac{2}{\pi(1 + t^2)} \\ &= \frac{4t^2(t + \sqrt{3})^2 - \pi(1 + t^2)(\sqrt{3}t^2 + 2t + \sqrt{3})}{2\pi t^2(t + \sqrt{3})^2(1 + t^2)}. \end{aligned}$$

Then the numerator can be written

$$-[(\pi\sqrt{3} - 4)t^4 - (8\sqrt{3} - 2\pi)t^3 - (12 - 2\pi\sqrt{3})t^2 + 2\pi t + \pi\sqrt{3}].$$

By Descartes' Rule of Signs (the number of positive zeros of a polynomial is no more than the number of changes in sign of the coefficients), this can have no more than two positive zeros.

If T is the largest zero of D', then I claim it is greater than the zeros of D. Clearly D is decreasing on $[T, \infty)$. If $r > T$ is a zero of D, then D is negative on (r, ∞). But

$$tD(t) = \frac{\sqrt{3} + 1/t}{2(1 + \sqrt{3}/t)} - \frac{2}{\pi}t \operatorname{arccot} t \to \frac{\sqrt{3}}{2} - \frac{2}{\pi} > 0$$

as $t \to \infty$. So $D(t)$ is positive for large t.

If there were three or more positive zeros of D, then by Rolle's theorem, there would be two or more zeros of D' between the smallest and largest of them. Since there is a zero of D' greater than they are, there would be three or more positive zeros of D'. But D' has no more than two positive zeros, so there can be only the two positive zeros of D. Since x is a zero of D and $x > 1$, we have $x = \sqrt{3}$ and $k_1 = k_2 = k_3 = 2/\pi$ arccot $\sqrt{3} = 1/3$.

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THE TRIANGLE INEQUALITY

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Many extensions, generalizations and modifications of the triangle inequality have been given. We believe the version below has a certain appeal in light of its completeness.

THEOREM 1. *Let $x_1, x_2, \ldots, x_n$ and z be complex numbers. Then z satisfies*

$$\max\left(2\max_{1\le j\le n}(|x_j|)-\sum_1^n |x_j|,\ 0\right)\leqq |z|\leqq \sum_1^n |x_j| \tag{1}$$

if and only if $z=\sum_1^n c_jx_j$ where $|c_j|=1 (1\leqq j\leqq n)$.

In case $n=2$, the left side of (1) reduces to $\big||x_1|-|x_2|\big|$ and we obtain the following generalization of the triangle inequality.

COROLLARY. *Let x_1, x_2 and z be complex numbers. Then z satisfies*

$$\big||x_1|-|x_2|\big|\leqq |z|\leqq |x_1|+|x_2| \tag{2}$$

if and only if $z=c_1x_1+c_2x_2$ where $|c_j|=1$ $(j=1,2)$.

Proof. If $z=\sum_1^n c_jx_j$ where $|c_j|=1$, then for each fixed $1\leqq k\leqq n$ we have

$$|z|\geqq |x_k|-\left|\sum_{j\ne k}^n c_jx_j\right|\geqq 2|x_k|-\sum_1^n |x_j|$$

and the left side of (1) follows. The right side of (1) clearly holds.

The converse is immediate if $n=1$. To establish the converse for $n\geqq 2$ we consider two cases.

Case 1. $(n=2)$. Assume that x_1, x_2 and z satisfy (2). A moment's reflection shows that $z=c_1x_1+c_2x_2$ where $|c_j|=1$ if any of x_1, x_2 or z is zero. Thus, we may suppose that none of these numbers equals zero. Then from (2)

$$\left|\frac{|x_1|^2-|x_2|^2+|z|^2}{2|x_1||z|}\right|\leqq 1 \text{ and } \left|\frac{|x_2|^2-|x_1|^2+|z|^2}{2|x_2||z|}\right|\leqq 1.$$

Thus, there exist $0\leqq \alpha,\ \beta\leqq \pi$, such that

$$\cos\alpha=\frac{|x_1|^2-|x_2|^2+|z|^2}{2|x_1||z|};\quad \cos\beta=\frac{|x_2|^2-|x_1|^2+|z|^2}{2|x_2||z|}.$$

From the identity $\sin^2 x+\cos^2 x=1$, we obtain

$$\sin\alpha=\frac{\sqrt{a}}{2|x_1||z|};\quad \sin\beta=\frac{\sqrt{a}}{2|x_2||z|},$$

where (again by (2))

$$a=[(|x_1|+|x_2|)^2-|z|^2]\,[|z|^2-(|x_1|-|x_2|)^2]\geqq 0.$$

Hence,

$$e^{i\alpha}|x_1|+e^{-i\beta}|x_2| = \left[\frac{|x_1|^2-|x_2|^2+|z|^2}{2|x_1||z|}+i\frac{a}{2|x_1||z|}\right]|x_1| + \left[\frac{|x_2|^2-|x_1|^2+|z|^2}{2|x_2||z|}-i\frac{a}{2|x_2||z|}\right]|x_2|$$

$$= |z|.$$

Writing $x_1 = d_1|x_1|$, $x_2 = d_2|x_2|$, and $z = d_3|z|$ where $|d_j| = 1$, we have $z = e^{i\alpha}d_3\bar{d}_1x_1 + e^{-i\beta}d_3\bar{d}_2x_2 = c_1x_1 + c_2x_2$ where $|c_j| = 1$.

Case 2. ($n \geqq 3$). By reindexing if necessary, we may suppose that $|x_1| \geqq |x_2| \geqq \cdots \geqq |x_n|$. Then (1) becomes

$$\max\left(|x_1| - \sum_2^n |x_j|, 0\right) \leqq |z| \leqq \sum_1^n |x_j|. \tag{3}$$

Let k denote the smallest integer satisfying $1 \leqq k \leqq [(n+1)/2]$ for which $0 \leqq \Sigma_1^k|x_j| - \Sigma_{k+1}^n|x_j|$. Such an integer exists since the numbers $(|x_j|)_1^n$ are decreasing. If $\Sigma_1^k|x_j| - \Sigma_{k+1}^n|x_j| \leqq |z|$, then

$$\left|\sum_1^k |x_j| - \sum_{k+1}^n |x_j|\right| \leqq |z| \leqq \sum_1^k |x_j| + \sum_{k+1}^n |x_j|,$$

and by Case 1 there exist c_1', c_2' with $|c_j'| = 1$ such that

$$z = c_1'\sum_1^k |x_j| + c_2'\sum_{k+1}^n |x_j|.$$

It follows that $z = \Sigma_1^n c_jx_j$ where $|c_j| = 1$. On the other hand, if $|z| < \Sigma_1^k|x_j| - \Sigma_{k+1}^n|x_j|$, then the hypothesis in (3) forces $k \geqq 2$. This, together with the definition of k, implies that

$$\sum_1^{k-1}|x_j| - \sum_k^n |x_j| < 0 \leqq |z| < \sum_1^k |x_j| - \sum_{k+1}^n |x_j|.$$

Thus,

$$\sum_1^{k-1}|x_j| - |x_k| < |z| + \sum_{k+1}^n |x_j| < \sum_1^k |x_j|.$$

The left side of this last inequality is nonnegative so that again by Case 1 there exist c_1', c_2' with $|c_j'| = 1$ such that

$$|z| + \sum_{k+1}^n |x_j| = c_1'\sum_1^{k-1}|x_j| + c_2'|x_k|.$$

It follows that $z = \Sigma_1^n c_jx_j$ where $|c_j| = 1$ and the proof is complete.

Finally, it seems worth mentioning that Theorem 1 may be extended to absolutely convergent series by a diagonal argument:

THEOREM 2. *Let $\Sigma_1^\infty x_j$ be absolutely convergent. A complex number z satisfies*

$$\max\left(2\max_j(|x_j|) - \sum_1^\infty |x_j|, 0\right) \leqq |z| \leqq \sum_1^\infty |x_j|$$

if and only if $z = \Sigma_1^\infty c_jx_j$ where $|c_j| = 1$ ($j \geqq 1$).

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

IS A SEQUENCE OF POLYNOMIALS COMPLETE?

J. S. HUANG

Let m be a positive integer, $\{k_n\}$ be a sequence of positive integers with $1 \leqq k_n \leqq n$. We ask: Is the sequence of polynomials of degree $n-1 \geqq m-1$:

$$\{x^{k_n-1}(1-x)^{n-k_n}: n = m, m+1, m+2, \dots\} \tag{1}$$

L_1-complete? In other words, if $g \in L_1(0,1)$, does

$$\int_0^1 g(x)x^{k_n-1}(1-x)^{n-k_n}dx = 0, \quad n = m, m+1, \dots \tag{2}$$

imply $g = 0$ a.e. $(0,1)$?

The problem originated in a moment problem of order statistics as follows. Let $X_1, \dots, X_n$ be independent random variables with a common distribution function F and let the order statistics be denoted by $X_{1,n} \leqq \cdots \leqq X_{n,n}$. It is known **[6]** that F is uniquely determined by the triangular array of numbers

$$\{E(X_{k,n}): k = 1, \dots, n; \quad n = 1, 2, \dots\}, \tag{3}$$

whenever $E|X_{1,1}| < \infty$. It is of interest (see **[2]**) to ask if an arbitrary infinite subset of (3) characterizes F. And what about a subset of form

$$\{E(X_{k_n,n}): n = m, m+1, \dots\}? \tag{4}$$

The last question is equivalent to the analytical question of whether (2) implies $g = 0$ a.e. $(0,1)$ or not. The problem is solved **[4, 8, 5, 1]** when either k_n or $n - k_n$ is independent of n. For $m = 1$ the condition (2) is equivalent to

$$\int_0^1 g(x)x^n dx = 0, \quad n = 0, 1, 2, \dots, \tag{5}$$

and it is clear **[3**, p. 235**]** that $g = 0$ a.e. $(0,1)$. The most general result known to date is **[7]**, where it is shown that for $m = 2$, the condition (2) does imply $g = 0$ a.e. $(0,1)$. This result can be extended to $m \geqq 3$ with the additional assumption that each pair of the $m-1$ numbers

unsolved.

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MISCELLANEA

STEENROD SQUARE CANTICLE*

G. B. FOLLAND

Are you going to Steenrod Square?**
(Riemann, Gauss, von Neumann, and Klein)
Remember me to one who lives there:
She once was a true love of mine.

Tell her to find me a cardinal, D,
(Riemann, Gauss, von Neumann, and Klein)
That lies in between aleph-null and c:
Then she'll be a true love of mine.

Let her find me an integer n greater than 2,
And positive integers x, y, and z,
Such that x^n plus y^n equal z^n is true:
Then a true love of mine she will be.

Tell her to bring me a manifold here,
Closed, simply connected, of dimension 3,
That is not a homeomorph of the sphere:
Then a true love of mine she will be.

Tell her to find me a proof that it's true
That a non-real zero of zeta of z
Must have real part equal 1 over 2:
Then a true love of mine she will be.

And when she's solved all these problems for me,
(Riemann, Gauss, von Neumann, and Klein)
She'll win a Fields medal as sure as can be:
And she'll be a true love of mine.

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* For technical reasons it is not possible to include the music. The tune is "Scarborough Fair," also known as "The Lover's Tasks" or "The Elfin Knight"; the version preferred by the author is by Simon and Garfunkel and was published by Charing Cross Music, 1967. — *Editors.*

** This song was written in Princeton, and the original second line was "Nelson, Moore, Shimura, and Stein." Performing it later at the Courant Institute, I used "Friedrichs, Lax, Shapiro, and Kline."

EDITED BY RICHARD A. BRUALDI

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AN EQUIVALENT VIEW OF MEASURE-PRESERVING TRANSFORMATIONS

LOUIS H. BLAKE

Introduction. Fundamental to both ergodic and information theory is the concept of a measure-preserving transformation with respect to a probability space. Both Birkhoff's ergodic theorem, 1931, [see **2**] and what has become known as the Shannon–McMillan–Breiman theorem of information theory [see **1**] are results which depend on the measure-preserving nature of the transformations involved.

With regard to information theory, it is clear that measure-preserving transformations are important because they preserve the information of partitions. One might well ask: why not talk directly of an "information-preserving" transformation? What will be proved below is that information-preserving transformations are—except in the simplest cases—actually equivalent to measure-preserving transformations.

The main results. Throughout this paper let $(\Omega, \mathfrak{A}, P)$ be a probability space and T a measurable transformation. By $\mathscr{A}$ denote a finite partition of Ω. $H(\mathscr{A}) \equiv -\Sigma_{A \in \mathscr{A}} P(A) \log P(A)$ is the entropy of the partition $\mathscr{A}$. By $T^{-1}(\mathscr{A})$ denote the transformation of the partition $\mathscr{A}$.

DEFINITION: The transformation T will be called information-preserving if for every partition $\mathscr{A}, H(\mathscr{A}) = H(T^{-1}(\mathscr{A}))$.

LEMMA 1. *Let* $\mathscr{A} = \{A, A^c\}$. *If* $H(\mathscr{A}) = H(T^{-1}(\mathscr{A}))$, *then* $P(T^{-1}(A)) = P(A)$ *or* $P(T^{-1}(A)) = P(A^c)$.

Proof. If $P(A) = 0$ or $P(A) = 1$, the result is immediate. Thus consider $0 < P(A) < 1$ and $P(A)\log P(A) + P(A^c)\log P(A^c) = k \neq 0$. Since T preserves information, it is obvious that $0 < P(T^{-1}(A)) < 1$. Next, examine $f(x) \equiv x \log x + (1-x)\log(1-x) - k$ for $0 < x < 1$. It is obvious that $f(x) = 0$ for $x = P(A)$ and for $x = 1 - P(A)$. Since $f'(x) = \log x/(1-x)$ and $f'(x) = 0$ only when $x = 1/2$, then $f(x)$ has only one critical point. Since $f(x)$ is not constant, then $f(x)$ cannot have a zero other than at $x = P(A)$ and $x = 1 - P(A)$.

THEOREM 1. *If* $\mathscr{A}$ *consists of at least three sets each of positive measure and* T *is information-preserving, then* T *is measure-preserving.*

Proof. Let $D \in \mathfrak{A}$ with $0 < P(D) < 1$. Then there exists a partition of $\Omega, \mathscr{P} = \{A, B, C\}$, each set with positive measure, such that either D equals some set of $\mathscr{P}$ or D is the union of two sets in $\mathscr{P}$. It will be established that $P(T^{-1}(A)) = P(A), P(T^{-1}(B)) = P(B)$ and $P(T^{-1}(C)) = P(C)$. This will be sufficient to establish that T preserves the measure of (an arbitrary set) D and thus T is a measure-preserving transformation.

Consider three partitions:

$$\mathscr{P}_1 = \{A \cup B, C\}$$
$$\mathscr{P}_2 = \{A \cup C, B\}$$
$$\mathscr{P}_3 = \{A, B \cup C\}.$$

From Lemma 1 it follows that with reference to $\mathcal{P}_1$:

(i) $$P(T^{-1}(A \cup B)) = P(A \cup B) \text{ or}$$

(ii) $$P(T^{-1}(A \cup B)) = P(C),$$

with reference to $\mathcal{P}_2$:

(i) $$P(T^{-1}(A \cup C)) = P(A \cup C) \text{ or}$$

(ii) $$P(T^{-1}(A \cup C)) = P(B),$$

and with reference to $\mathcal{P}_3$:

(i) $$P(T^{-1}(B \cup C)) = P(B \cup C) \text{ or}$$

(ii) $$P(T^{-1}(B \cup C)) = P(A).$$

The previous situations will be examined.

Assume $\mathcal{P}_1$-(i). Then $P(T^{-1}(C)) = P(C)$. $\mathcal{P}_1$-(i) and $\mathcal{P}_2$-(i), or $\mathcal{P}_1$-(i) and $\mathcal{P}_3$-(i) give $P(T^{-1}(B)) = P(B)$ or $P(T^{-1}(A)) = P(A)$ respectively—thus T preserves the measure of A, B and C.

$\mathcal{P}_1$-(i) and $\mathcal{P}_2$-(ii) and $\mathcal{P}_3$-(ii) gives

$$P(T^{-1}(B)) + P(C) = P(A),$$

$$P(T^{-1}(A)) + P(T^{-1}(B)) = P(A) - P(C) + P(B) - P(C)$$

$$P(A) + P(B) = P(A \cup B) = P(T^{-1}(A \cup B)) = P(A) + P(B) - 2P(C)$$

and so $P(C) = 0$—a contradiction. Thus $\mathcal{P}_1$-(i) and any other condition which does not force a logical contradiction imply T preserves the measure of A, B and C.

From the previous paragraph and the symmetric nature of the argument, it follows that only $\mathcal{P}_1$-(ii), $\mathcal{P}_2$-(ii) and $\mathcal{P}_3$-(ii) are left to be considered. These three conditions taken simultaneously give

$$P(T^{-1}(A \cup B)) + P(T^{-1}(A \cup C)) + P(T^{-1}(B \cup C)) = P(A) + P(B) + P(C) = 1.$$

Thus, $2P(T^{-1}(A)) + 2P(T^{-1}(B)) + 2P(T^{-1}(C)) = 1$ or $2 = 1$—a contradiction.

It therefore follows that T preserves the measure of A, B and C.

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NONEXISTENCE PROOFS FOR PROJECTIVE DESIGNS

E. F. ASSMUS, JR. AND DAVID P. MAHER

We wish to present elementary nonexistence proofs for projective designs, otherwise known as symmetric 2-designs or (v, k, λ) designs. Such a design is an arrangement of v points into v subsets called blocks, each containing k points in such a way that any pair of distinct points is contained in exactly λ blocks. It is easy to show [1, Chapter 8] that every pair of distinct blocks meets in exactly λ points and that these parameters satisfy

(*) $$k(k-1) = (v-1)\lambda.$$

The *order* of a projective design is defined to be $N = k - \lambda$. A projective plane of order N is an

Edited by Paul T. Mielke and W. E. Mastrocola

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A LABORATORY FOR AN ELEMENTARY STATISTICS COURSE

A. Sterrett and Z. A. Karian

I. Introduction. In spite of a fairly large range of specific objectives that an introductory non-calculus statistics course may have, almost all such courses have a few fundamental common goals such as the computation and interpretation of various sample statistics and the ability to deal with large amounts of data. A more elusive goal is the development of an understanding of basic statistical concepts. The difficulty in this regard is that the concepts involved are abstract and their definitions often are beyond the scope of students in elementary courses. Computers can be used to advantage in trying to achieve all these goals, and in recent years there have been attempts by many to include computational techniques in introductory statistics courses. Frequently this has consisted of exercises which require students to use pre-programmed statistical packages. Experience in using a general purpose statistical package will be useful to any student interested in applications, since many applications require computational data analysis. As beneficial as exercises which require the use of pre-programmed statistical packages are, they generally do little to enhance the understanding of statistical concepts.

We have attempted to produce a better understanding of statistical concepts through the use of a statistical laboratory. The exercises in our laboratory make extensive use of individually generated data, sometimes generated by students with tables of random digits but more frequently generated by a computer for individual students. Examples of each type of laboratory exercise are illustrated below.

II. Exercises using data from tables of random digits.

Experiment 1.

Instructors frequently ask a student to repeat an experiment many times in order to illustrate the long-run relative frequency interpretation of probability, but these experiments lose much of their effectiveness if the probability of the event of interest is too obvious. Students tend not to be overly impressed to learn that the ratio of the number of heads tossed to the number of tosses of a coin appears to be getting closer and closer to 1/2 — the answer is too obvious. Consequently they are not likely to give much thought to the sequence of relative frequencies. We suggest that an experiment for which the probability of success is not so obvious be used to illustrate the long-run relative frequency interpretation of probability.

Arnold and Dick agree to play a series of games in which each player has an equal chance to win a game. The player who first wins 5 games wins a prize. With Dick leading 3 games to 2, play is interrupted, and they cannot finish the series as originally intended.

Use the table of random digits to simulate the finish of the series and record the name of the winner. Repeat the simulation many times, recording the relative frequency with which Dick wins the series (i.e., $0/1, 0/2, 1/3, 2/4, \ldots$).

Plot the relative frequencies after 5, 10, 15, 20, 25, ... trials. Do the relative frequencies appear to be stabilizing around any particular number? If so, what is that number? If such a number exists, we will define it to be the probability that Dick would win the prize, given that he has a 3 to 2 advantage.

As well as illustrating the long-run relative frequency definition of probability, Experiment 1 permits the instructor to introduce a bit of history. The historical role of this so-called problem of points is described in [2].

EXPERIMENT 2.

From the table of random digits select six two-digit numbers. Let the second lowest, L(2), and second highest, H(2), numbers among the sample of six determine the interval [L(2), H(2)]. Repeat the experiment many times, associating an interval with each sample. For each interval note whether it contains the median, 49.5, of the population. Record your data as follows, letting 1 designate that 49.5 belongs to an interval and 0 designate that it does not. For example,

Trial #	1	2	3	4
Interval	[46, 53]	[51, 59]	[21, 62]	[27, 43]
Contains 49.5	1	0	1	0
Cum. # 1's	1	1	2	2
Ratio, # 1's/n	1	.50	.67	.50

Graph the intervals against the number of trials for the first 10 intervals obtained.

Graph # 1's/n against n at intervals of n equal to 5. Exchange data with others, listing the names of those whose data you use. Does the ratio of # 1's/n appear to be stabilizing around any number? If so, what is its value? What is its interpretation?

Students ordinarily are quite successful in making the necessary substitutions in order to find a confidence interval for a given parameter. The interpretation of the result is, however, more difficult for the average student. We have found that this experiment is very helpful in discussing the interpretation of confidence intervals. Because the value of the population median is known, students will see that some of their intervals contain the value of the median and that some do not. After finding many intervals, they will understand that the procedure itself has associated with it a probability (confidence) that any interval which might be obtained by the same procedure will surround the value of the population median. After the assignment has been turned in, the instructor can help the class find the exact probability (as a review of the binomial model) with which such intervals contain the population median. The class should be impressed by the accuracy of the approximate probability which their data produced.

This nonparametric approach to confidence intervals is discussed in [**1**].

III. Use of the computer to generate data. As indicated earlier, many of our experiments make use of computer generated data. Each student in our class is given a separate computer account and a unique data number. Through the use of pre-stored programs, the student obtains his own data and is encouraged to discuss the exercises with other students to increase understanding of the concepts involved. Eventually each student must make the analysis using his own data.

In order to grade the exercises, we also have programs which provide answers to these problems for each data number (i.e., student). Certain exercises are graded on the computer.

Our computing facilities consist of a PDP 11/45 computer with 25 terminals, 16 of which are dedicated to instructional use for a student body of 2150. These terminals are available 24 hours a day. The authors will be glad to supply the BASIC-PLUS programs developed for these experiments.

EXPERIMENT 3.

The purpose of this experiment is to investigate which of several estimators are unbiased estimators of the mean and median of a particular finite population.

Run the program SAMPLE to get your data. You will be given a population of six numbers and a list of all (20) samples of size 3 which can be selected without replacement from your population. Assume that the numbers in the population are equally likely to occur and find the population mean, μ, and the population median ξ.

Suppose that each of the 20 samples listed has an equal chance to be drawn from the population.

(i) Is $\bar{x}$ an unbiased estimator of μ?

(ii) Is $\bar{x}$ an unbiased estimator of ξ?

(iii) Is the sample median an unbiased estimator of μ?

(iv) Is the sample median an unbiased estimator of ξ?

Justify your conclusion for each of your four responses. NOTE: Any "no's" stated above enable you to draw general conclusions about estimators; "yes's" provide a clue that an estimator might always be an unbiased estimator of a parameter, but one example does not constitute proof of a general principle.

The expected value of a random variable is a concept which cannot be easily discussed in a mathematical context in an elementary course but empirical evidence such as that provided by this exercise helps clarify the concept.

Recall that each student has different data and the computer provides answers for each data set.

EXPERIMENT 4

This experiment will provide empirical evidence regarding the sampling distribution of sample means, including relationships:

$$\mu_{\bar{x}} = \mu \qquad \text{and} \qquad \sigma_{\bar{x}}^2 = \sigma^2/n.$$

Run the program DGEND and request variable MEAN 4. You will be given 100 sample means calculated from samples of size 4 taken from a normal population with mean 100 and standard deviation 16.

(i) Make a histogram using eight classes of equal width.

(ii) Use your histogram as a guide and draw a smooth curve which you think represents the sampling distribution of means of samples of size 4 taken from the given normal population.

(iii) Use the mean of 100 sample means to estimate $\mu_{\bar{x}}$. What should be the approximate value of your estimate? Why? (We have told our students that $\mu_{\bar{x}} = \mu$ but have said that we could not prove it for them.)

(iv) Use the standard deviation of the 100 sample means to estimate $\sigma_{\bar{x}}$. What is the exact value of $\sigma_{\bar{x}}$? Why?

Because students choose their own intervals for classifying sample means, answers for this problem are not available.

EXPERIMENT 5

Run the program POKER. You will receive 50 poker hands (five cards selected from an ordinary deck without replacement). Find a 90% confidence interval for p, the probability that a poker hand contains a pair (this should include hands with 2 pairs, 3 of a kind, full house, 4 of a kind)

(i) using your sample data to estimate p in $\sigma_{x/n}^2 = pq/n$, and

(ii) replacing pq by 1/4.

Each trial of a binomial experiment results in a success or a failure. By asking students to classify outcomes, this property of the binomial model will become more meaningful. The accuracy provided by replacing pq by 1/4 when p is close to 1/2 is also illustrated.

EXPERIMENT 6

Run the program DGEND and ask for variable NORMAL. The nine observations given to you will be taken from a normal population. At a 20% level of significance test the null hypothesis that $\mu = 100$ against the alternative that $\mu \neq 100$.

Report your decision rule, the value of the statistic on which your decision was based, and whether you accept or reject the hypothesis.

A large value for the significance level is selected for this exercise so that a fairly high proportion of the students will reject the null hypothesis even though it is true. The meaning of significance level can be clarified by tabulating the percentage of students in a class who have rejected the null hypothesis and comparing the percentage to the given level of significance.

EXPERIMENT 7

Use your data NORMAL (EXPERIMENT 6) to find an 80% confidence interval for the mean of the population from which the sample was taken.

This experiment is not especially useful in helping students understand a concept, but we include it here to remind the reader how simple it is to give members of a class individual data sets. Once again we use the computer to check individual answers.

IV. Conclusions. We assign from 20 to 25 laboratory exercises during a one-semester course, and the laboratory scores count as one-sixth of the total grade. There are also three hourly examinations and a final examination which is equivalent to two hourly examinations.

We are convinced that many of these activities enhance our students' ability to understand the variable nature of experimental results and thereby better understand statistical reasoning.

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PROBLEMS AND SOLUTIONS

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All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before May 31, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2695. *Proposed by Eliyahu Beller, Bar-Ilan University, Israel*

Prove or disprove the following conjecture: For $a>1$ and $x>0$, show that $-\log(1-(1-e^{-x})^a)<x^a$.

E 2696. *Proposed by William P. Wardlaw, U.S. Naval Academy*

(a) If numbers are drawn randomly (using uniform distribution with replacements) from the set $\{1,2,\ldots,n\}$ until their sum first exceeds n, what is the expected number of draws?

(b) The same problem for numbers selected from $\{0,1,\ldots,n-1\}$ until their sum exceeds $n-1$.

E 2697. *Proposed by William Anderson and William Simons, West Virginia University*

Is there a dense subset S of the unit circle such that each point in S has rational coordinates and the (Euclidean) distance between any pair of points in S is also rational?

E 2698. *Proposed by Paul Monsky, Brandeis University*

Let A_n be an $n\times n$ chessboard. The n queens problem (placing n counters on A_n so that no two lie in any row, column, or diagonal) admits solutions for all $n\neq 2$ or 3.

Let B_n be the "chessboard" obtained from A_n by identifying opposite sides so that the resulting surface is a torus. (Now, every diagonal of B_n consists of n squares.)

(1) For which values of n does there exist a solution of the n queens problem on B_n?

(2)* If n satisfies (1) then a solution of (1) gives, by cyclic permutation, n superimposable solutions to the n queens problem on A_n. Do there exist n superimposable solutions (for A_n) for other values of n?

E 2699. *Proposed by Emile Haddad and Peter Johnson, American University of Beirut*

Suppose that $1=\theta_0>\theta_1>\cdots>\theta_k>0$, and that

$$\sum_{i=0}^{k} a_i \cos n\theta_i\pi \to 0$$

as $n\to\infty$ through the integers.

Does it follow that $a_i=0$ for all i?

E 2700. *Proposed by Richard Stanley, Massachusetts Institute of Technology*

Let L be a finite lattice with minimum element 0 and maximum element 1. Suppose that for all $x\neq 0$ in L, the interval $[0,x]$ contains an even number of elements. Show that L is complemented, i.e., for all x in L there is a y in L such that $x\wedge y=0$ and $x\vee y=1$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Symmetrical Networks with One-Ohm Resistors

E 2620 [1976, 740]. *Proposed by Albert Mullin, Ft. Hood, Texas, and Derek Zave, Sperry UNIVAC (independently).*

Let Γ be the graph consisting of the vertices and edges of one of the five regular polyhedra. Suppose all edges of Γ are one-ohm resistors. Compute the resistance between any two of the most remote vertices of Γ.

Answer the same question when Γ is the graph of the n-dimensional cube.

Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands. Let d be the diameter of Γ and choose two vertices x and y of Γ at distance d. Denote by $\Gamma^j(x)$ the set of vertices at distance j from x. Let γ_j be the number of edges between $\Gamma^{j-1}(x)$ and $\Gamma^j(x)$.

THEOREM. *If* $|\Gamma^d(x)|=1$, *then the resistance* $R(\Gamma)$ *of* Γ *between* x *and* y *is given by*

$$R(\Gamma)=\sum_{j=1}^{d}\gamma_j^{-1}.$$

Proof. Connect a unit current generator from x to y. Then for symmetry reasons all vertices in $\Gamma^j(x)$ have the same potential and therefore may be identified without influencing the potential distribution over Γ. Thus by Ohm's law

$$R(\Gamma)=R(\Gamma')=\sum_{j=1}^{d}\gamma_j^{-1},$$

where Γ' is the graph on $d+1$ vertices obtained from Γ by identifying all vertices in $\Gamma^j(x)$ for $j=1,\ldots,d-1$.

We list the results in the accompanying table. (The tetrahedron is easily dealt with separately.)

Γ	d	γ_j	$R(\Gamma)$
Tetrahedron	1	3	1/2
Cube	3	3, 6, 3	5/6
Octahedron	2	4, 4	1/2
Dodecahedron	5	3, 6, 6, 6, 3	7/6
Icosahedron	3	5, 10, 5	1/2
n-cube	n	$j\binom{n}{j}$	$\sum_{j=1}^{n}\left[j\binom{n}{j}\right]^{-1} = 2^{-n}\sum_{j=1}^{n}\frac{2^j}{j}.$

Also solved by Anders Bager (Denmark), Anthony Barkauskas, T. S. Bolis & A. D. Brackett, Oscar Box, Landy Godbold, Michael Goldberg, Mark Meyerson, Stephen Noltie, Eric Rosenthal, Temple University Problem Group, and University of South Alabama Problem Group.

The n-dimensional cube part of the problem was solved also by Tom Morley and by the proposer, Derek Zave.

Editor's Comment. Rosenthal notes that the solution to this problem for the case of the cube first appeared in H. S. M. Coxeter, *Regular Polytopes*, Methuen & Co., London, 1948 and that the solution for the other regular polyhedra appears in Martin Gardner, *The Second Scientific American Book of Mathematical Puzzles and Diversions*, Simon and Schuster, 1961, pp. 17 and 21.

Meyerson shows that for the n-simplex, the total resistance is $2/(n+1)$ and for the dual n-cube, the total resistance is $1/(n-1)$.

No Solutions in Positive Integers

E 2621 [1976, 741]. *Proposed by Barry Powell, Kirkland, Washington*

Prove that $x^n + 1 = y^{n+1}$ has no solutions in positive integers x, y, n $(n \geqq 2)$ with $(x, n+1) = 1$.

I. *Solution by University of South Alabama Number Theory Class.* Suppose that there exists such a solution. Every prime divisor p of $y-1$ divides x and since $(x, n+1) = 1$ we have $p \nmid (n+1)$. Since

$$1 + y + \cdots + y^n \equiv n + 1 \pmod{(y-1)}$$

it follows that $y - 1$ and $1 + y + \cdots + y^n$ are relatively prime. Then

$$x^n = (y-1)(1 + y + \cdots + y^n)$$

implies that $1 + y + \cdots + y^n$ is an nth power. This is a contradiction because $y^n < 1 + y + \cdots + y^n < (y+1)^n$.

II. *Solution by E. W. Trost, Zurich, Switzerland.* We show that the assertion remains true even when we omit the condition $(x, n+1) = 1$. The following three cases are exhaustive:

(1) $n = 2m$ $(m \geqq 1)$, $n + 1 = pu$, p an odd prime. Then our equation is

$$(x^m)^2 + 1 = (y^u)^p.$$

In 1850 V. A. Lebesgue proved that $u^2 + 1 = v^p$ is impossible for $u \geqq 1$.

(2) $n = 3^k$ $(k \geqq 1)$, $n + 1 = 2s$ $(s \geqq 2)$. Then our equation is $(x^{n/3})^3 + 1 = (y^s)^2$.

In 1770 Euler proved that $u^3 + 1 = v^2$ and $u \geqq 1$ imply that $v = 3$. But $3 = y^s$ is impossible $(s \geqq 2)$.

(3) $n = 2m + 1 = pt$, p $(\geqq 5)$ an odd prime. Our equation is

$$(x^t)^p + 1 = (y^{m+1})^2.$$

In 1964 Chao Ko proved that $u^p + 1 = v^2$ has no solution in positive integers.

For the results quoted above see L. J. Mordell, *Diophantine Equations*, p. 301–302.

Also solved by Max Agoston, W. J. Blundon (Canada), Robert Breusch, Charles Delzell (Switzerland), Ron Evans, Lorraine Foster, Irving Gerst, M. G. Greening (Australia), K. Inkeri (Finland), Mark Kleiman, Andrzej Makowsky (Poland), David Penney, Blair Spearman & Lenny Jones, J. M. Stark, Charles Wexler, and the proposer.

An Upper Bound for an Integral

E 2622 [1976, 741]. *Proposed by S. Zaidman, University of Montreal*

Let $f:[a,b]\to\mathbb{R}$ be a continuous function which is twice differentiable in (a,b) and satisfies $f(a)=f(b)=0$. Prove that

$$\int_a^b |f(x)|\,dx \leqq M(b-a)^3/12,$$

where $M=\sup|f''(x)|$ for $x\in(a,b)$.

Solution by Robert K. Meany, Iowa State University. Let

$$g(t)=(t-a)(t-b)f(x)-(x-a)(x-b)f(t), \qquad a<x<b.$$

Since $g(a)=g(x)=g(b)=0$, repeated application of Rolle's theorem shows that $g''(c)=2f(x)-(x-a)(x-b)f''(c)=0$ for some $c\in(a,b)$. This gives the estimate $|f(x)|\leqq M(x-a)(b-x)/2$, and integration completes the proof.

Also solved by 48 other contributors. Several solvers noticed that the above estimate follows easily from the Cauchy remainder formula for linear interpolation, see P. J. Davis, *Interpolation and Approximation*, Blaisdell 1963, p. 57, or S. D. Conte and Carl de Boor, *Elementary Numerical Analysis*, McGraw Hill 1972, p. 211. It also follows from Theorem 15.13 of Apostol, *Calculus*, vol. II, or Lemma 4, p. 390 of L. H. Loomis, *Calculus*, Addison–Wesley 1974.

Wilfred Herget (W. Germany) remarks that if the conditions $f(a)=f(b)=0$ be replaced by $f((a+b)/2)=0$, then

$$\left|\int_a^b f(x)dx\right| \leqq \frac{1}{24}M(b-a)^3.$$

Integrality of Some Fractions

E 2623 [1976, 812]. *Proposed by Ivan Niven, University of Oregon*

It is well known that if m and n are positive integers and $(m,n)=1$ then $(m+n-1)!/m!\,n!$ is an integer. For which positive integers k is it true that there are infinitely many pairs of positive integers m,n such that $(m+n-k)!/m!\,n!$ is an integer?

Solution by L. Kuipers, Mollens, Switzerland. This is true for every positive integer k. Let m be any positive integer $\geqq k+1$ and choose $n=m!-1$. Then

$$(m+n-k)!/m!\,n!=(m+n-k)\cdots(n+1)/m!=(m+n-k)\cdots(n+2),$$

which is an integer.

Also solved by Wayne Boucher, Peter de Buda, José Luis de Miguel (Spain), Robert Dressler, Irving Gerst, Marguerite Gerstell, Alan Hartman (Australia), Eli Isaacson, Elgin Johnson, Jordan Levy, S. C. Locke (Canada), O. P. Lossers, Jr. (Netherlands), Jerry Metzger, Aaron Meyerowitz (Israel), David Penney, Reinhard Razen (Australia), Harry Ruderman, Gillian Valk, Edward Wang (Canada), University of South Alabama Number Theory Class, and the proposer.

Chinese Remainder Theorem Applied

E 2624 [1976, 812]. *Proposed by Robert M. Hashway, West Warwick, Rhode Island*

James T. Rogers, *The Calculating Book*, Random House (1974) states that the number 3025 has a "remarkable quirk." If it is "split in two parts," 30 and 25, the sum of those parts is 55, and its square $55^2=3025$, the original number. Such numbers can be obtained by solving the Diophantine equation

$$a+b\cdot 10^k=(a+b)^2$$

where $0<a,\ b<10^k$. Find all solutions when $k\leqq 5$.

Solution by O. P. Lossers, Jr., Technological University, Eindhoven, Netherlands. With $x = a + b$ the given equation gives

$$x(x-1) = b(10^k - 1). \tag{1}$$

For a given k, x determines uniquely an ordered pair (u, v) of positive relatively prime integers such that

$$10^k - 1 = uv \tag{2}$$

and $u \mid x$, $v \mid (x-1)$.

Conversely, such a pair (u, v) satisfying (2) determines a unique x, $1 \leqq x \leqq 10^k - 1$ such that $x \equiv 0 \pmod u$ and $x \equiv 1 \pmod v$, by the Chinese Remainder Theorem.

It follows that the number of solutions of the given equation for a fixed k equals $2^i - 1$ where i is the number of prime divisors of $10^k - 1$. We have

k	1	2	3	4	5
i	1	2	2	3	3
$2^i - 1$	1	3	3	7	7

The solutions are found in two steps: first find all factorizations (2) of $10^k - 1$ and then find x by solving the two simultaneous congruences. The solutions $a + 10^k \cdot b$ are tabulated below:

$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
81	2025	088209	04941729	0023804641
	3025	494209	07441984	0300814336
	9801	998001	24502500	0493817284
			25502500	6049417284
			52881984	6832014336
			60481729	9048004641
			99980001	9999800001

Also solved by E. C. Buissant des Amorie (Netherlands), John Beidler, Robert Breusch, Peter de Buda, José Luis de Miguel (Spain), D. R. Deuerman, Thomas Elsner, Lorraine Foster, Irving Gerst, M. G. Greening (Australia), Louise Grinstein, Cornelius Groenewoud, Anita Harnadek, C. T. Haskell, G. A. Heuer, Eli Isaacson, Allan Johnson, Jr., R. J. Koch, L. Kuipers (Switzerland), John Lew, Deborah Lockhart, Graham Lord (Canada), Carolyn MacDonald, William Markel, Bernard Martin, Ronald McCuiston, Roger Nelsen, David Penney, Les Reid, Harry Ruderman, C. L. Sabharwal, August Sardinas, Michael Skalsky, H. R. van der Vaart, Gillian Valk, Michael Vowe (Switzerland), Charles Wexler, Wayne Wild, and Y. H. Yiu (Hong Kong).

Comments. The leading zeros are included in the table above (at the suggestion of Elsner) so that the "split" remains central. He also observes that these "false" cases with leading zeros have "partners" with the same end part (last k digits) occurring later.

Grinstein notes that this problem is discussed in detail by F. A. Halliday in *Mathematical Questions and Solutions*, 3 (1917), 70–73.

Groenewoud finds the following related examples

$$(1+29+50+29)^3 = 1295029,$$

$$(2+92+42+07)^3 = 2924207,$$

$$(3+65+22+64)^3 = 3652264,$$

$$(3+58+15+77)^3 = 3581577.$$

He also gives three consecutive cubes:

$$197^3 = 7645373 = (7+64+53+73)^3,$$

$$198^3 = 7762392 = (7 + 76 + 23 + 92)^3,$$

$$199^3 = 7880599 = (7 + 88 + 05 + 99)^3.$$

A Property of Conics

E 2625 [1976, 812]. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let A_i ($i \equiv 0, 1, 2, 3 \bmod 4$) be four points on a circle Γ. Let t_i be the tangent of Γ at A_i and let p_i and q_i be the lines parallel to t_i passing through the points A_{i-1} and A_{i+1} respectively. If $B_i = t_i \cap t_{i+1}$, $C_i = p_i \cap q_{i+1}$ show that the four lines B_iC_i have a point in common.

Solution by Jordi Dou, Barcelona, Spain. We shall prove a more general result.

THEOREM. *Let K be a non-degenerate conic in a real projective plane, A_i $(0 \leqq i \leqq 3)$ be four distinct points on K and r be a line such that $A_i \notin r$. Let t_i be the tangent of K at A_i, $B_i = t_i \cap t_{i+1}$, $T_i = t_i \cap r$, $p_i = T_iA_{i-1}$, $q_i = T_iA_{i+1}$ and $C_i = p_i \cap q_{i+1}$. Then the four lines B_iC_i are concurrent.*

Proof. Let π be the polarity with respect to K and $S = A_0A_2 \cap A_1A_3$. Put $s = \pi(S)$ and $R = \pi(r)$. Let σ be the harmonic homology with center S and axis s. Thus we have $\sigma^2 = 1$ and $\sigma(A_i) = A_{i+2}$. We claim that the point $Q = \sigma(R)$ lies on each of the lines B_iC_i.

Note that π interchanges S and s and consequently σ and π commute. Therefore, $\tau = \sigma\pi = \pi\sigma$ is also a polarity. We have

$$\tau(t_i) = \sigma\pi(t_i) = \sigma(A_i) = A_{i+2},$$

$$\tau(t_{i+1}) = A_{i+3} = A_{i-1},$$

$$\tau(r) = \sigma\pi(r) = \sigma(R) = Q$$

and consequently the two triangles $T_{i+1}T_iB_i$ and $A_{i+2}A_{i-1}Q$ are polar to each other with respect to K. By Chasles' theorem (see H. S. M. Coxeter, *The Real Projective Plane*, Cambridge University Press, 1961, p. 71) this is a pair of Desargues' triangles. Hence the lines $T_{i+1}A_{i+2} = q_{i+1}$, $T_iA_{i-1} = p_i$ and B_iQ are concurrent at $C_i = p_i \cap q_{i+1}$. Therefore we see that Q lies on the lines B_iC_i as claimed.

The statement of the problem is obtained by choosing $K = \Gamma$ and $r =$ line at infinity.

Also solved by L. Kuipers (Switzerland), and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before May 31, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6192*. *Proposed by Harry D. Ruderman, Hunter College Campus School*

Let R be a rectangular array of lattice points having at least 2 rows and 2 columns. Let each lattice point of R be labeled by one of the numbers: 1, 2, 3, or 4. Suppose that the boundary points of R contain at least one of each of the four numbers and the boundary is oriented, say counterclockwise, with repetitions permitted, and with possibly more than one cycle (1 is allowed to follow 4). Call two lattice points adjacent if they are vertices of a common small square. Call two lattice points opposite if they are labeled either 1 and 3, or 2 and 4. Prove that for every such R there is a square containing two lattice points that are both opposite and adjacent.

6193. *Proposed by Robert E. Shafer, Berkeley, California*

Given that n is such that $2\phi(n) = n - 1$ (ϕ is the Euler totient function), prove (1) $3 \nmid n$; (2) If p

and q are distinct prime divisors of n, then $p \not\equiv 1 \pmod{2q}$; (3) n has at least 11 distinct prime divisors. (*Note.* In Beiler, *Recreations in the Theory of Numbers*, Dover (1966), pp. 92–93, it is shown that n must have at least 7 distinct prime divisors.)

6194. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College of CUNY, New York*

Let N be an arbitrary integer > 6 and let $\{a_i\}$, $i = 1, 2, \ldots, m$, denote the set of positive composite integers less than N which are not powers of primes. Prove that

$$\sum_{i=1}^{m} \frac{1}{a_i} \quad \text{is not an integer.}$$

6195*. *Proposed by Andreas N. Philippou, University of Patras, Greece*

For $j = 1, 2, \ldots$ and $n \geqq j$, let X_{nj} and X_j be random variables defined on a probability space (Ω, A, P). Assume that $\sup_j \mathscr{E}|X_j|^r < \infty$ $(r > 0)$, where $\mathscr{E}$ denotes expectation under P. Then $\max[\mathscr{E}|X_{nj} - X_j|^r, 1 \leqq j \leqq n] \to 0$, if and only if

$$\max\{P[|X_{nj} - X_j| > \varepsilon],\ 1 \leqq j \leqq n\} \to 0 \quad \text{and}$$
$$\max[\mathscr{E}|X_{nj}|^r - \mathscr{E}|X_j|^r,\ 1 \leqq j \leqq n] \to 0.$$

6196*. *Proposed by Daniel Shanks, National Bureau of Standards.*

(A) Let $-5 < x_0 < 0$ and let

$$x_n = \begin{cases} \sqrt{x_{n-1} + 5} & \text{if } \; n \not\equiv 0 \pmod 4) \\ \\ -\sqrt{x_{n-1} + 5} & \text{if } \; n \equiv 0 \pmod 4). \end{cases}$$

Identify the numbers toward which x_{4m}, x_{4m+1}, x_{4m+2}, and x_{4m+3} converge as $m \to \infty$.

(B) Let p be a prime for which $(5|p) = +1$, so that $\sqrt{5}$ exists, modulo p. Then

$$(15 \pm 6\sqrt{5}|p) = +1, \quad \text{or} \quad -1,$$

according as $p \equiv \pm 1 \pmod{15}$, or $p \equiv \pm 4 \pmod{15}$, respectively.

(C) What is the relationship between problems A and B?

6197*. *Proposed by Manuel Scarowsky, McGill University, Canada*

Let p be a prime; a and b positive integers; and let (x_0, y_0) be a solution of $ax + by = p$ in positive integers with x_0 minimal, if such exists (otherwise take $x_0 = 0$). Find an estimate for $\Sigma_{a,b} x_0$.

SOLUTIONS OF ADVANCED PROBLEMS

ϕ-Convergence

6090 [1976, 385]. *Proposed by T. Šalát and O. Strauch, Bratislava, Czechoslovakia*

I. J. Schoenberg has defined (this MONTHLY, 66 (1959), 365) φ-convergence of a sequence $\{\gamma_n\}$ of real numbers in this way: φ-limit $\gamma_n = \lambda$ if and only if $\lim s_n = \lambda$ where $s_n = n^{-1}\Sigma_{d|n}\varphi(d)\gamma_d$, ($\varphi$ denotes Euler's function.) Find a sequence which is φ-convergent and is not convergent.

Solution by Paul Erdös, University of Washington. Let

$$x_n = \begin{cases} 1, & \text{if } \; n = 2 \cdot 3 \cdot 5 \cdots p_k, \\ \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

$\{x_n\}$ is not convergent. We show that

$$y_m = \frac{1}{m}\sum_{d\mid m}\varphi(d)x_d \to 0 \quad \text{as} \quad m\to\infty.$$

Let the largest divisor of m, which is a product of consecutive primes starting with 2, be $2\cdot 3\cdot 5\cdots p_k$, and set $m=(2\cdot 3\cdots p_k)d_m$ ($p_{k+1}\nmid d_m$). Then

$$y_m = \frac{1}{m}\sum_{d\mid 2\cdot 3\cdots p_k}\phi(d)x_d = \frac{1}{d_m}\cdot\frac{1}{2\cdot 3\cdots p_k}\sum_{d\mid 2\cdot 3\cdots p_k}\varphi(d)x_d.$$

As $m\to\infty$, $d_m+p_k\to\infty$. It is therefore sufficient to show that

$$\frac{1}{2\cdot 3\cdots p_k}\sum_{d\mid 2\cdot 3\cdots p_k}\varphi(d)x_d\to 0 \quad \text{as} \quad k\to\infty. \tag{2}$$

This will follow from

$$\begin{aligned}\sum_{d\mid 2\cdot 3\cdots p_k}\varphi(d)x_d &= \{\varphi(2)+\varphi(2\cdot 3)+\cdots+\varphi(2\cdot 3\cdots p_{k-1})+\varphi(2\cdot 3\cdots p_k)\}\\ &\leqq \sum_{d\mid 2\cdot 3\cdots p_{k-1}}\varphi(d)+\varphi(2\cdot 3\cdots p_k) = 2\cdot 3\cdots p_{k-1}+\varphi(2\cdot 3\cdots p_k)\\ &= 2\cdot 3\cdots p_k\left\{\frac{1}{p_k}+\prod_{\nu=1}^{k}\left(1-\frac{1}{p_\nu}\right)\right\} = o(2\cdot 3\cdots p_k).\end{aligned}$$

Also solved by Gustaf Gripenberg (Finland), L. E. Mattics, I. J. Schoenberg, Joel Spencer, and the proposers.

Addition of 'Student' Random Variables

6092 [1976, 385]. *Proposed by Ignacy I. Kotlarski, Oklahoma State University*

Let X_1 and X_2 be two independent Student distributed random variables with 1 and 3 degrees of freedom respectively. Denote

$$Y=\tfrac{1}{2}X_1\sqrt{3}+\tfrac{1}{2}X_2. \tag{1}$$

Show that the probability density functions of X_1, X_2, Y satisfy the relation

$$f_Y(y)=\tfrac{1}{2}f_{X_1\sqrt{3}}(y)+\tfrac{1}{2}f_{X_2}(y) \tag{2}$$

almost everywhere on R.

QUERY. Can this analogy be generalized to

$$Y=a_1X_1+a_2X_2+\cdots+a_nX_n, \tag{1'}$$

where $X_1,\cdots,X_n$ are independent Student distributed random variables with $1,3,\ldots,2n-1$ degrees of freedom and $a_1, a_2,\ldots,a_n$ are constants?

Solution by Michael Skalsky, Southern Illinois University. Denoting by U a random variable with the probability density function

$$\tfrac{1}{2}f_{X_1\sqrt{3}}(u)+\tfrac{1}{2}f_{X_2}(u),$$

and by $\phi_Y(t)$ and $\phi_U(t)$ the characteristic functions of Y and U respectively, we have

$$\varphi_Y(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{\frac{1}{2}\sqrt{3}xit}}{1+x^2}\,dx\cdot\frac{2}{\sqrt{3}\pi}\int_{-\infty}^{\infty}\frac{e^{\frac{1}{2}xit}}{((x^2/3)+1)^2}\,dx,$$

$$\varphi_U(t)=\frac{1}{2\pi\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{ity}}{1+y^2/3}\,dy+\frac{1}{\pi\sqrt{3}}\int_{-\infty}^{\infty}\frac{e^{ity}}{(1+y^2/3)^2}\,dy.$$

Applying the residue theorem, we find

$$\varphi_Y(t)=\varphi_U(t)=e^{-|t|\sqrt{3}}\left(\frac{|t|\sqrt{3}}{2}+1\right),$$

and since the characteristic function uniquely determines the distribution, we conclude that the random variables U and Y are identically distributed.

To see that the analogy cannot be extended to $Y=a_1X_1+\cdots+a_nX_n$, $n>2$, it is sufficient to notice that the characteristic function of Y will be of the form $P(|t|)e^{-c|t|}$ where c is a constant and $P(t)$ is a polynomial of degree $1+2+\cdots+n-1=n(n-1)/2$, whereas the characteristic function of U will be of the form

$$\sum_{i=1}^{n} Q_i(t)e^{-c_i|t|},$$

where c_i, $i=1,2,\cdots,n$, is a constant, and $Q_i(t)$, $i=1,2,\ldots,n$, is a polynomial of degree $i-1$, and thus the degree of the polynomial $Q_n(t)$, which has the highest degree, will be only $n-1$.

Also solved by the proposer.

Connected Cells of a Chessboard

6096 [1976, 489]. *Proposed by Jan Mycielski, University of Colorado*

A set of cells of a chessboard is called connected if a rook can visit the whole set without moving over cells which are not in the set. Put $s=a_nn^2$ and let 2^s be the number of connected subsets for a chessboard of size $n\times n$. Prove that the sequence $a_1,a_2,\ldots,$ converges and estimate its limit.

Solution by the proposer. (A) $\lim a_n$ exists (Proof of A. Ehrenfeucht). Let B_m be the chessboard of size $m\times m$, and $C_m=2^s$ with $s=a_mm^2$. Let C'_m be the number of connected sets in B_m including the top row of cells and one complete column of cells. Every connected set in B_m can be obtained from such a set by deleting some cells from this row and this column. Then

(1) $$C'_m\geqq\frac{C_m}{2^{2m-1}m}$$

since there are 2^{2m-1} subsets of a set consisting of one row and one column, and m possible positions of a column.

Let $n\geqq m$ and consider a subboard B_k of B_n, of size $k\times k$, where $k=m[n/m]$, and $[x]$ denotes the largest integer not exceeding x. Thus B_k splits into disjoint subboards $B^1_m,\ldots,B^r_m$ each of size $m\times m$, where $r=[n/m]^2$. Let X_i be a connected set of cells of B^i_m which contains the top row and some column. Then $\bigcup_{i=1}^r X_i$ is a connected set. Hence

(2) $$C_n\geqq(C'_m)^r$$

By (1) and (2) we get

$$C_n\geqq\left(\frac{C_m}{2^{2m-1}m}\right)^r.$$

Hence

(3) $$a_n>a_mT^n_m-\frac{2+\log_2 m}{m},$$

where $T^n_m=(m^2/n^2)[n/m]^2$. We have $a_n\leqq1$ for all n. Let $a=\limsup a_m$. Thus for every $\varepsilon>0$ there is an m_0 such that $a_{m_0}>a-\varepsilon$ and $(2+\log_2 m_0)/m_0<\varepsilon$. Since $\lim_{n\to\infty}T^n_m=1$ there is an $N\geqq m_0$ such

that for all $n > N$ we have $T^n_{m_0} > 1 - \varepsilon$ and, by (3),

$$a_n > (a - \varepsilon)(1 - \varepsilon) - \varepsilon.$$

Hence $\lim a_n = a$.

(B) *Estimates of* $\lim a_n$ *from above.* We can find in B_n at least $(n-4)^2/5$ disjoint crosses each consisting of 5 cells of B_n (see Figure 1). If the center cell of a cross is in a set S of cells and the four adjacent cells of the cross are not in S then S is connected only if it contains no other cells at all. Since there are 32 subsets of a cross then there are no more than $31^{n^2/5} \cdot 2^{8n} + n^2/5$ connected sets in B_n. It follows that $\lim a_n \leqq (\log_2 31)/5$.

Better estimates from above are possible if we consider tessellations of B_n into other shapes.

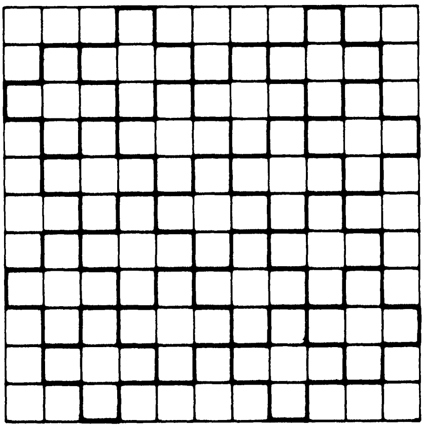

FIG. 1

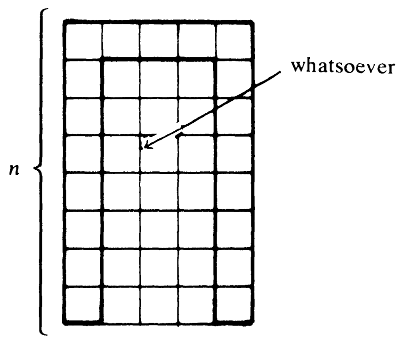

FIG. 2

(C) *Estimates of* $\lim a_n$ *from below.* Let us consider all sets of cells of B_n including the top row and each third column beginning with the first, and, in the case when 3 divides n, also the last column. It is clear that all such sets are connected. There are at least $2^{2(n-1)^2/3}$ of them. Hence $\lim a_n \geqq 2/3$.

P. Erdös gives a better estimate. Let E be the class of connected sets in B_n containing the top row and each fourth column. The number of connected sets of the form shown in Figure 2 is about $2^{2n}2^{n(1-1/4)}$. Hence the total number of connected sets in E is about $(2^{n(3-1/4)})^{n/4} = 2^{11n^2/16}$. It follows that

$$\lim a_n \geqq 11/16.$$

With the help of Markov chains, better estimates are available.

Maximally Symmetric Convex Bodies

6098 [1976, 489]. *Proposed by Peter L. Renz, San Francisco, California*

Let A be the group of affine bijections from $\mathbb{R}^n$ to $\mathbb{R}^n$. For any subset S of $\mathbb{R}^n$ define $A(S)$ to be the subgroup of A which takes S into itself. A convex body is a compact convex set with non-empty interior. We say a convex body K in $\mathbb{R}^n$ is maximally symmetric if $A(K)$ is not properly contained in $A(L)$ for any convex body L in $\mathbb{R}^n$. Characterize the maximally symmetric convex bodies in $\mathbb{R}^n$.

Solution by Detlef Laugwitz, Technische Hochschule, Darmstadt, Germany. Let $E \supseteq K$ be Loewner's ellipsoid of K which is the uniquely determined ellipsoid of minimal volume containing K. (See Herbert Busemann, *The Geometry of Geodesics.*) For $T \in A(K)$, $TK = K$ and T must be volume preserving, so $TE = E$ and $A(E) \supseteq A(K)$. If K is maximally symmetric then $A(E) = A(K)$. Now E and K have at least one boundary point in common. Since $A(E)$ operates transitively on the

boundary of E the boundaries of K and E coincide. So K must be an ellipsoid, this condition being sufficient.

The reasoning leads to the generalization for $\mathbb{R}^n$. Let $S \subseteq \mathbb{R}^n$ be a bounded set with non-empty interior of its convex hull. Then S is maximally symmetric if and only if it is the union of concentric homothetic ellipsoidal surfaces. For related results, refer to L. Danzer, D. Laugwitz and H. Lenz, *Über das Löwnersche Ellipsoid*, Archiv. Math. 8 (1957), 214–219.

Also solved by the proposer.

Sums of Squares in Fields with Rolle's Theorem

6101 [1976, 490]. *Proposed by Michael Slater, University of Bristol, England*

Suppose F is an ordered field in which Rolle's theorem holds for polynomials. (That is, if $f(x) \in F[x]$, $f(a) = 0 = f(b)$ with $a < b$, then there exists c with $a < c < b$ such that $f'(c) = 0$, where f' is the formal derivative.) Show that any sum of squares in F is a square in F.

Solution by Einar Andresen, Nordland Regional College, Norway. We shall weaken the hypothesis: F need not be ordered, but need only have the property: if $f \in F[x]$ with two different roots, then f' has a root. Notice that this excludes fields with characteristic $p \neq 0$. The polynomial $x^p - x$ has formal derivative -1 in such a field.

First we claim that $\sqrt{3} \in F$. Indeed, $g(x) = x^3 - 9x$ has roots 0, ± 3, while g' has roots $\pm\sqrt{3}$.

Let $a, b \in F$, and let $f(x) = x^3 - 3(a^2 + b^2)x + 2a^3 - 6ab^2$. Then f has roots $-2a$, $a \pm \sqrt{3}b$; f' has roots $\pm\sqrt{a^2 + b^2}$. Thus a sum of two squares (and hence any sum of squares) is a square.

QUESTION: What is the smallest subfield of the real algebraic numbers in which Rolle's theorem holds for polynomials? (It must contain $k^{1/m}$ when m, k are positive integers: Consider the polynomial $x^{km} - x^m$.)

Also solved by George Crofts, Gerald A. Edgar, Hugh M. Edgar, Thomas Foregger, Robert Gilmer, Ellen Hertz, Michael Josephy (Costa Rica), Ilias Kastanas, Carl Kohls, David Leep, Alvin Martin, L. E. Mattics, J. G. Mauldon, James T. Smith, Walter Taylor, Adrian Wadsworth, W. C. Waterhouse, William Watkins, Paul Woods, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

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Courant in Göttingen and New York, The Story of an Improbable Mathematician. By Constance Reid. Springer Verlag, New York, 1976. 314 pp. $12.80. (Telegraphic Review, February 1977.)

Mathematics is created in more than one sense. There is, of course, the creation of theorems and proofs by individual mathematicians. But there is also the creation of an environment, an atmosphere, physical facilities, directions of research and financial support that make possible the creative work of individuals.

In recent history Richard Courant was certainly the greatest of creators in the latter sense. His own research was by no means minor and his books have been highly influential. But it was his promotion of mathematical activity that entitles him most to a place in the annals of mathematics.

This biography details Courant's almost superhuman efforts and successes in building two world centers of mathematics. The first, Göttingen University, which he entered in 1907, was even then by no means a minor institution. In its past were great professors such as Gauss, Dirichlet, and Riemann; when Courant began his doctoral work there Felix Klein and Hilbert were in their prime. Nevertheless, until Klein began to devote himself to administration, each professor was autonomous and pursued his own interests exclusively. Professors were no more concerned with their students than they are today at the major universities. Klein envisioned better physical facilities, more concern for the education of all students and especially of school teachers, and above all, the improvement of connections between mathematics, the sciences, and even industry. Klein had noted the growing isolation of mathematics and wished to counter it.

Courant took over the leadership in 1920 and proceeded to carry out Klein's program. Despite the unfavorable conditions in post-World War I Germany, Courant succeeded in making Göttingen the best center for a broad spectrum of mathematical activities, and until Hitler came to power Göttingen was the mecca of mathematicians. The account of life in Göttingen during Courant's years there is informative and absorbing.

In 1933, Courant, a Jew, was forced to leave and in 1934 he came to New York University, which at that time certainly was not a major center for mathematics. His efforts, well described by Reid, to build what is now called the Courant Institute for Mathematical Sciences, show Courant's wisdom, perseverance, and administrative genius. Courant's greatness lay especially in his ability to judge men and ideas. Most administrators are content to count pages of publication.

Courant had to build from scratch in a country that had barely begun to understand and support research. He had to discover and find the means of supporting promising young men and women and to provide adequate physical facilities and a library; he had also to contend with the predominance of pure mathematics. He succeeded in establishing a first-class center where what are commonly called pure and applied mathematics are treated as a unity and wherein physicists mingle freely with mathematicians.

Research mathematicians like to believe that they are born with a special talent but in a favorable atmosphere they spring up like weeds. The rarity of excellent administrators, on the other hand, would seem to point to the fact that if there is a special talent it is administrators such as Courant who possess it.

MORRIS KLINE, Courant Institute

MISCELLANEA

6. Important though the general concepts and propositions may be with which the modern industrious passion for axiomatizing and generalizing has presented us,... nevertheless I am convinced that the special problems in all their complexity constitute the stock and core of mathematics; and to master their difficulties requires on the whole the harder labor.

Hermann Weyl, *The Classical Groups*, 1939, pp. xi–xii. (Copyright © by Princeton University Press and reprinted by permission of the Press.)

TELEGRAPHIC REVIEWS

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S = supplementary reading L = undergraduate library purchase
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1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, L*, *Use of Mathematical Literature*. Ed: A.R. Dorling. Butterworths, 1977, xii + 260 pp, $24.95. [ISBN: 0-408-70913-8] A concise yet rich resource for anyone concerned about mathematics library collections, as well as a useful guide for someone seeking guidance outside his own fields of expertise. Three chapters on general reference sources (spanning nations and decades) are followed by 14 topical bibliographical essays by specialists providing guidance to major references. Altogether the volume includes comments on several thousand mathematics references, quite a number of which are likely to be unfamiliar to the average mathematician. Concludes with author and subject indices. LAS

GENERAL, P**, L**, *Encyclopedic Dictionary of Mathematics*. Ed: Shôkichi Iyanaga, Yukiyosi Kawada. MIT Pr, 1977. *V. I:* xv + 883 pp; *V. II:* 864 pp, $125 (set). [ISBN: 0-262-09016-3] Translation of the Japanese encyclopedia *Iwanami Sūgaku Ziten*, itself prepared in three editions from 1947-1968, which took ten years (1967-1977) to prepare. (The translation was sponsored by the AMS, organized by Edwin Hewitt, and reviewed by K.O. May.) Contains 436 extensively referenced and cross-referenced moderate length articles (and biographies) on topics similar to the major subheadings of the AMS subject classification, but arranged alphabetically. Half of the second volume is devoted to very useful appendices and reference tables: formulas (from e.g., algebra, trigonometry, Lie algebras, knot theory, differential equations, algebraic topology), numbers (e.g., prime numbers, class numbers, group characters), notation, subject-oriented lists of articles (in English, French, German, Russian), lists of contributors, translators, journals and publishers, name index (≈3000 names) and 25,000 item subject index. An invaluable, up-to-date, sophisticated reference, it is without precedent or peer in the mathematical literature. LAS

GENERAL, S?, *The Theory and Applications of ...π, e^x, i, ok, c, G,... Constants Analysis*. T.S. Davis. Constant Society, 1977, iii + 236 pp, $10 (P). Motley collection of facts and formulas, most involving the mathematical constants π and e, with essentially no textual material. (The ok in the title appears only once, as the symbol for absolute zero.) Almost all pages are photo reproduced from nearly illegible handwritten copy. RSK

GENERAL, S(13-15), L, *Patterns of Symmetry*. Ed: Marjorie Senechal, George Fleck. U of Mass Pr, 1977, viii + 152 pp, $12. [ISBN: 0-87023-232-0] A diverse collection of brief essays, illustrations, poems, and dialogue based on a Symmetry Festival held at Smith College in February, 1973. Touches on music, art, science, mathematics and much more; a substantial interdisciplinary bibliography provides guidance for those wishing a deeper study. LAS

GENERAL, S?(13), *Numbers: Shortcuts & Pastimes*. Jack Gilbert. TAB Books, 1976, 336 pp, $9.95. [ISBN: 0-8306-6675-3] The first half of this book deals with techniques for calculating easily and accurately. The second half looks at some well-known patterns in numbers such as magic squares and divisibility tests. Of little interest to anyone with a minimal background in mathematics. CEC

GENERAL, P*, L**, *Index of the American Mathematical Monthly, V. 1-80 (1894-1973)*. Kenneth O. May. MAA, 1977, vi + 269 pp, $16. [ISBN: 0-88385-426-0] In three parts: a contents register of each volume (excluding book reviews, meeting reports and official announcements), an author index referenced to the contents register by volume and initial page number, and a dictionary-type index of major terms, also referenced to the contents register. An indispensible reference for college mathematics teachers and libraries. LAS

GENERAL, L*, *Index of Mathematical Papers, V. 7*. AMS, 1977. *Part 1*, 860 pp; *Part 2*, 630 pp, $95 (P). [ISBN: 0-8218-4010-X] Author (Part 1) and subject (Part 2) indices to *Mathematical Reviews*, Vols. 49 and 50 (1975). This year the index includes every item reviewed by MR, not just major journal articles. LAS

GENERAL, *Lecture Notes in Mathematics-501-600: An Index and Other Useful Information*. Ed: A. Dold, B. Eckmann. Springer-Verlag, 1977, 30 pp, (P). Contents summaries of each volume, author and (very brief) subject indices, and instructions for preparation of LNM manuscripts. LAS

PRECALCULUS, P, L**, *Selected Papers on Precalculus*. Ed: Tom M. Apostol, *et al*. Brink Selected Papers, V. 1. MAA, 1977, xvii + 469 pp, $15. [ISBN: 0-88385-202-0] First in a new series of papers selected from the *Monthly* and the *Magazine*, named in honor of Raymond W. Brink. Patterned after the successful *Selected Papers in Calculus* (which shall become V. 2 in the new series), *Precalculus* contains papers and bibliographic entries on pedagogy, numbers, induction, trigonometry, algebra, solutions of equations, synthetic geometry, conic sections, analytic geometry, area and volume. An extraordinary resource for instructional ideas and enrichment topics. LAS

PRECALCULUS, T(13: 1, 2), *Elementary Functions*. Dennis T. Christy. Har-Row, 1978, x + 454 pp, $11.95. [ISBN: 0-06-041297-6] Standard precalculus curriculum, as recommended by CUPM. Includes chapters on functions and graphs, polynomials, exponential, logarithmic and circular functions, sequences and series. Concept of function serves to unify the material. Not enough non-routine exercises. JRG

EDUCATION, T*(15: 1), S, P*, L. *Mathematics Teaching.* Kenneth J. Travers, *et al.* Har-Row, 1977, x + 591 pp, $13.95. (ISBN: 0-06-045233-1] An outstanding new text which is appropriate for a secondary school methods course. This well-written book includes the standard topics along with sections on computer usage and individualization. It does a good job of cataloging current research and literature in math education. CEC

EDUCATION, P. *Capes Mathêmatique, Prêparation a l'Oral.* Denis Richard, Jean-Marc Braemer. Hermann, 1977, 305 pp, 58F (P). [ISBN: 2-7056-1387-0] Guide for teachers in classroom treatment of various subjects, e.g., continuity, isometries of the affine plane, Bezout's theorem. Each "lesson" gives pertinent definitions and proofs of major theorems. JRG

EDUCATION, P, L. *Conference on Graduate Training of Mathematics Teachers.* N.S. Mendelsohn. Canadian Math Congress, 1972, v + 110 pp, (P). [ISBN: 0-919558-02-X] Papers from a 1969 conference at Sir George Williams University on aspects of mathematics teaching. Participants include K.O. May, W.W. Sawyer, and G. Pôlya. LAS

HISTORY, P, L*. *Number Words and Number Symbols, A Cultural History of Numbers.* Karl Menninger. Trans: Paul Broneer. MIT Pr, 1977, xiii + 480 pp, $9.95 (P). [ISBN: 0-262-63061-3] Paperback edition of the 1969 hardcover translation (TR, January 1970) of the 1958 German original. A simple, direct fascinating mixture of linguistics and cultural anthropology that sheds light on both etymology and the history of arithmetic. LAS

HISTORY, S, P, L*. *The Muslim Contribution to Mathematics.* Ali Abdullah Al-Daffa'. Humanities Pr, 1977, 121 pp, $11. [ISBN: 0-391-00714-9] A concise well-referenced survey of 7th-13th century Muslim contributions to arithmetic (formalization of decimal notation and standard algorithms), algebra (linear and quadratic equations), trigonometry (theory of sin, cos, tan) and geometry (translation and interpretation) set in the context of Muslim history and religious philosophy. Concludes with an extensive bibliography of several hundred items, including many Arabic manuscripts in British museums. LAS

FOUNDATIONS, T(16-17), S, L*. *Complementarity in Mathematics.* Willem Kuyk. Math. and Its Appl., V. 1. Reidel, 1977, 186 pp, $15.95. [ISBN: 90-277-0814-2] A three-part inquiry into the philosophy of mathematics based on a series of lectures to mixed groups of mathematics and philosophy students. The first section treats the Gödel completeness and incompleteness theorems in moderate formal detail. The second section is a detailed historical essay on the epistemology of mathematics, from ancient times through Bourbaki. The final briefer section makes a case for a "principle of complementarity" (e.g., discrete vs. continuous, algebra vs. geometry) as a basis for a philosophy of mathematics. Well-suited to its intended audience, it would make an interesting choice for a seminar on the philosophy of mathematics. LAS

NUMBER THEORY, P. *Lecture Notes in Mathematics-601: Modular Functions of One Variable V.* Ed: J.-P. Serre, D.B. Zagier. Springer-Verlag, 1977, 294 pp, $11.40 (P). [ISBN: 0-387-08348-0; 3-540-08348-0] Proceedings of the International Conference of July 2-14, 1976, held at the University of Bonn. JAS

ALGEBRA, T(16-17: 1), S*, L. *Field Theory.* Masayoshi Nagata. Pure and Appl. Math., V. 40. Dekker, 1977, vii + 268 pp, $23.50. [ISBN: 0-8247-6466-8] Preliminary chapters on groups and rings followed by comprehensive treatment of algebraic and transcendental extensions, valuation theory, ordered fields and Galois theory. Not too many exercises, but answers and comments are provided. Standard definition-theorem-proof style. JRG

ALGEBRA, P, L**. *Selected Papers on Algebra.* Ed: S. Robert Gordon, *et al.* Brink Selected Papers, V. 3. MAA, 1977, xv + 537 pp, $15. [ISBN: 0-88385-203-9] Third volume in the new MAA series of reprints from the *Monthly* and the *Magazine*. Includes 108 articles, plus additional bibliographic entries, on group theory, ring theory, field theory, algebraic number theory, linear algebra, history.LAS

ALGEBRA, T(16-17: 2). *Module Theory, An Approach to Linear Algebra.* T.S. Blyth. Clarendon Pr, 1977, viii + 400 pp, $21. [ISBN: 0-19-853162-1] Development of the basic ideas of module theory and vector spaces, with special emphasis on the importance of module theory in linear algebra. Tensor and exterior algebras are discussed. Good supply of exercises. JRG

ALGEBRA, P. *Theory of Modules (An Introduction to the Theory of Module Categories).* Alexandru Solian. Trans: Mioara Buiculescu. Wiley, 1977, x + 420 pp, $26.50. [ISBN: 0-471-99462-6] Reference work giving parallel treatments of module theory and category theory with particular attention to abelian categories. Good basic bibliography. JRG

CALCULUS, S(13-14). *FORTRAN Programming: A Supplement for Calculus Courses.* William R. Fuller. Springer-Verlag, 1977, xii + 145 pp, $6.80 (P). [ISBN: 0-387-90283-X; 3-540-90283-X] Presents the basics of Fortran necessary for computer solutions of many calculus problems. Chapter presentations on numerical evaluation of integrals, functions of two variables, infinite series, and differential equations. Many exercises which include all types of calculus concepts. Standard exercises; exercises where closed form solutions are not possible; exercises where exact solutions can be compared to approximate solutions. Can be used with a conventional calculus text or in a mini-course independent of the regular class. Exercises and solutions. List of computer problems by topics. Course outline. Index. RJA

CALCULUS, T(13: 1). *Calculus for Business and Economics, Second Edition.* Robert L. Childress. P-H, 1978, xiii + 383 pp, $11.95. [ISBN: 0-13-111534-0] Greatly expanded second edition includes nine new chapters. More rigorous treatment of limit, derivative, integral; more material on extreme values. Detailed solutions of examples. (First edition, TR May 1972.) JRG

DIFFERENTIAL EQUATIONS, P. *Nonlinear Diffusion Problems.* Ed: O. Diekmann, N.M. Temme. Math Centrum, 1976, viii + 247 pp, Dfl. 30 (P). [ISBN: 90-6196126-2]

DIFFERENTIAL EQUATIONS, P. *Pseudo-Differential Operators and Applications: An Introduction.* A. Unterberger. Lect. Notes Ser., No. 46. Aarhus U, 1976, 114 pp, $2.50 (P).

DIFFERENTIAL EQUATIONS, T(15-18: 1, 2), S, L. *Numerical Methods for Partial Differential Equations, Second Edition.* William F. Ames. Comp. Sci. and Appl. Math. Acad Pr, 1977, xiv + 365 pp, $16.50. [ISBN: 0-12-056760-1; 0-17-771086-1] New material included in this second edition includes numerical fluid mechanics, explicit-implicit methods, Monte Carlo techniques, lines, the fast Fourier transform, and fractional steps methods. (First edition, TR March 1970; ER November 1970.) Chapter six on weighted residuals and finite elements has been added. Problems. Numerous references at the ends of chapters. Author and subject indexes. RJA

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-599: Complex Analysis.* Ed: J.D. Buckholtz, T.J. Suffridge. Springer-Verlag, 1977, x + 159 pp, $8.30 (P). [ISBN: 0-387-08343-X; 3-540-08343-X] Proceedings of the conference held in honor of Professor S.M. Shah at the University of Kentucky during May 18-22, 1976. JAS

NUMERICAL ANALYSIS, S**(13-18), P, L. *Scientific Analysis on the Pocket Calculator, Second Edition.* Jon M. Smith. Wiley, 1977, xii + 445 pp, $13.75. [ISBN: 0-471-03071-6] Major portion of text is devoted to the explanation of numerical analysis methods usable on pocket calculators--function evaluation, Fourier analysis, numerical integration, linear systems, polynomial approximations, roots of a function, statistical and probabilistic techniques. Includes discussion of the differences in the designs of the leading calculators, how to best program on a given calculator, and indications of trends in designs and uses for the future. Contains a special section on concepts and calculations of financial analysis. Four appendices. Index. Chapter references. (First edition, TR May 1976.) RJA

NUMERICAL ANALYSIS, T(15-18: 1), S, L. *Matrix Computation for Engineers and Scientists.* Alan Jennings. Wiley, 1977, xv + 330 pp, $25.50. [ISBN: 0-471-99421-9] Matrix algebra, selected applied examples, computer programming techniques (especially storage allocation), numerical techniques for linear equations, and eigenvalue problems. Detailed discussion of how to design numerical algorithms in a fast and storage-efficient manner. Small program segments in Algol and Fortran. Appendices on program layout, preparation, and verification. Chapter bibliographies. Index. RJA

NUMERICAL ANALYSIS, T(14-16: 1), S, L. *Computer Methods for Mathematical Computations.* George E. Forsythe, Michael A. Malcolm, Cleve B. Moler. P-H, 1977, xi + 259 pp, $15.95. [ISBN: 0-13-165332-6] Text organized around ten very well written and very useful numerical methods subroutines, each of which is written in Fortran. The mathematical background surrounding the methods used in the subroutines is carefully developed. Many appropriate comments in the code itself. Prerequisites are calculus and linear algebra. Problems. References. Index. RJA

NUMERICAL ANALYSIS, S(13), P, L. *Computational Analysis with the HP25 Pocket Calculator.* Peter Henrici. Wiley, 1977, xi + 280 pp, $11.50 (P). [ISBN: 0-471-02938-6] This book contains 35 programs written for the HP25 which implement algorithms in number theory, equation solving, algebraic stability theory, numerical integration and the evaluation of some special functions. Flow diagrams and detailed descriptions are provided which enable one to adapt the programs to calculators of comparable capacity. CEC

NUMERICAL ANALYSIS, S(15-16). *Numerical Methods for Engineers and Scientists: A Students' Course Book.* A.C. Bajpai, I.M. Calus, J.A. Fairley. Wiley, 1977, xii + 380 pp, $10.95 (P). [ISBN: 0-471-99542-8] A programmed workbook, reprinted by Wiley from the 1975 original Taylor & Francis edition (TR, May 1976). Standard topics: equations, matrices, finite differences, differential equations. Examples designed to be worked with a hand calculator. LAS

OPTIMIZATION, T(15-17: 1, 2), S, L. *Linear Programming and Network Flows.* Mokhtar S. Bazaraa, John J. Jarvis. Wiley, 1977, x + 565 pp, $21. [ISBN: 0-471-06015-1] The simplex method provides the underlying and unifying concept throughout. Linear programming: the simplex method and its initiation, optimality criteria, dual problem, decomposition principle, and large-scale programming. Network flows: transportation, assignment, minimal cost network flow, maximal flow, shortest path, and multicommodity minimal cost flow problems plus the out-of-kilter algorithm. Exercises. References. Appendix. Index. RJA

OPTIMIZATION, P. *Numerische Methoden bei Optimierungsaufgaben, Band 3.* L. Collatz, G. Meinardus, W. Wetterling. Int. Ser. Num. Math., V. 36. Birkhäuser, 1977, 216 pp, sFr. 42 (P). Proceedings of the February 22-28, 1976 conference "Optimierung bei graphentheoretischen und ganzzahligen Problemen" at Oberwolfach. JAS

OPTIMIZATION, T*(16-17: 2), S. *Mathematical Programming for Economics and Business.* Roger C. Pfaffenberger, David A. Walker. Iowa St U Pr, 1976, xii + 462 pp, $18. Gives theoretical bases for algorithms as well as methods. Adequate examples but small number of exercises. Linear, nonlinear, dynamic and stochastic models. Assumes only calculus and matrix algebra. LH

OPTIMIZATION, T(18: 1, 2), S, P. *Optimization: A Theory of Necessary Conditions.* Lucien W. Neustadt. Princeton U Pr, 1976, xiii + 424 pp, $22.50. Based upon a year long advanced seminar at Brown. Prerequisites: real analysis, integration theory. Includes some mathematical preliminaries. Establishes necessary conditions in several settings including optimization with operator equation restrictions, optimal control with ordinary differential equation constraints and with parameters. Bibliography. RWN

ANALYSIS, P. *Brownian Motion, Hardy Spaces and Bounded Mean Oscillation.* K.E. Petersen. London Math. Soc. Lect. Notes, No. 28. Cambridge U Pr, 1977, 104 pp, $6.95 (P). Investigation of the connection between Hardy spaces and Martingale theory. JEG

ANALYSIS, S(17-18), P. *Grundstrukturen der Analysis I.* Werner Gähler. Math. Reihe, B. 58. Birkhäuser, 1977, viii + 412 pp, sFr. 63. [ISBN: 3-7643-0901-6] The first of two volumes providing a systematic development of analysis in abstract terms (e.g., filter, convergence space). The flavor and many specifics of the approach are reflected in the use of Gödel-Bernays theory of types and of categorical language and constructions. JD-B

ANALYSIS, P. *Two Papers on Special Functions.* Ja. L. Geronimus, Gábor Szegö. Amer. Math. Soc. Trans., Ser. 2, V. 108. AMS, 1977, 130 pp, $19.60. Translations (into English) of Szegö's 1918 paper "Hankel forms," together with the appendix by Geronimus to the 1961 Russian edition of Szegö's 1959 revision of his 1939 monograph *Orthogonal Polynomials.* LAS

ANALYSIS, P*, L. *Generalized Functions.* I.M. Gel'fand, *et al.* Acad Pr. *V. 1, Properties and Operations,* Trans: Eugene Saletan, 1964, xviii + 423 pp, $9.50 (P); *V. 2, Spaces of Fundamental and Generalized Functions,* Trans: Morris D. Friedman, Amiel Feinstein, Christian P. Peltzer, 1968, x + 261 pp, $9 (P); *V. 3, Theory of Differential Equations,* Trans: Meinhard E. Mayer, 1967, x + 222 pp, $9 (P); *V. 4, Applications of Harmonic Analysis,* Trans: Amiel Feinstein, 1964, xiv + 384 pp, $9.25 (P); *V. 5, Integral Geometry and Representation Theory,* Trans: Eugene Saletan, 1966, xvii + 449 pp, $9.50 (P). Set $39.50. Paperback edition of the definitive treatise on the theory and applications of distributions. LAS

ANALYSIS, P. *Lecture Notes in Mathematics-587: Non-Commutative Harmonic Analysis.* Ed: J. Carmona, M. Vergne. Springer-Verlag, 1977, 240 pp, $11 (P). 11 papers from the second colloquium on noncommutative harmonic analysis held at Marseille-Luminy in July 1976. Proceedings of the first colloquium (July 1974) were published as LNM-466 (TR, April 1976). LAS

ANALYSIS, L*. *A Problem Book in Mathematical Analysis.* G.N. Berman. Trans: Leonid Levant. MIR, 1977, 462 pp. A valuable collection of almost 4500 problems beginning with basic properties of functions and moving through derivatives, integrals, series, functions of several variables, line and surface integrals, differential equations, and trigonometric series. Many of the problems are routine, some are difficult. If nothing else, this book will make the writing of calculus examinations an easier chore. SG

ANALYSIS, P. *Lecture Notes in Mathematics-580: Convex Analysis and Measurable Multifunctions.* C. Castaing, M. Valadier. Springer-Verlag, 1977, vii + 278 pp, $11 (P).

ANALYSIS, T(13-16), S. *Mathematik für Ingenieure, des Maschinenbaus und der Elektrotechnik.* Wolfgang Brauch, Hans-Joachim Dreyer, Wolfhart Haacke. Teubner, Stuttgart, 1977, 767 pp, (P). Fifth edition of a book on mathematics for engineers. Chapters on linear algebra and probability and statistics, as well as on more usual topics such as calculus, differential equations and Fourier series. JD-B

GEOMETRY, S**. *M.C. Escher Kaleidocycles.* Doris Schattschneider, Wallace Walker. Ballantine, 1977, 43 pp + 17 models, $8.95. [ISBN: 0-345-25686-7] An exciting extension of Escher's well-known periodic drawings into the third dimension: 17 die-cut paper models for constructing geometric solids decorated with vividly colored Escher prints, accompanied by a brief manual explaining how Escher's patterns are adapted to three-dimensional figures. Among the models are standard Platonic solids, and the "flexible" kaleidocycles formed from a chain of tetrahedra that can be continuously deformed into different configurations, each revealing a new Escher pattern. LAS

GEOMETRY, T*(15-16), L. *Elements of Differential Geometry.* Richard S. Millman, George D. Parker. P-H, 1977, xiv + 265 pp, $16.95. A well-written undergraduate level introduction suitable for a one semester course. Topics include local and global curve theory (including convex curves, the isoperimetric inequality, the 4-vertex theorem, and the Fary-Milnor theorem), local and global surface theory (various curvatures and fundamental forms, geodesics, *theorema egregium,* Gauss-Bonnet), and manifolds. No differential forms, rather a reliance on partial derivatives (à la Stoker) and linear algebra. Plenty of exercises, historical notes. SG

GEOMETRY, T*(16-17), L. *Linear Geometry, Second Edition.* K.W. Gruenberg, A.J. Weir. Springer-Verlag, 1977, x + 198 pp, $12.80. A reprinting with only minor changes (a couple of alternative proofs and more diagrams) of the 1967 edition. Still an excellent introduction to affine, projective, and metric geometry. SG

GEOMETRY, S(10-15), L. *Creating Escher-Type Drawings.* E.R. Ranucci, J.L. Teeters. Creative Pub, 1977, vii + 195 pp, $6.25 (P). [ISBN: 0-88488-087-7] An elementary explanation of translation, rotation, and reflection devices for creating tessellations of the plane based on curvilinear figures. Lavishly illustrated, full of potential for fun and learning. LAS

GEOMETRY, S, L*. *How to Draw a Straight Line.* A.B. Kempe. NCTM, 1977, vi + 55 pp, $6 (P). Another in NCTM's series "Classics in Mathematics Education," this is an unaltered republication of Kempe's 1877 monograph describing various mechanical devices which will draw a straight line the way a compass draws a circle. Kempe (famous for being the first to "solve" the four color problem) is concerned with constructing a straight line without being given one in the form of a straightedge. LAS

TOPOLOGY, S(16-17). *Introduction to Differentiable Manifolds.* Louis Auslander, Robert E. Mackenzie. Dover, 1977, v + 218 pp, $4 (P). Reprint of 1963 McGraw-Hill edition. Good undergraduate introduction. Special topics include projective spaces, Whitney imbedding theorem, Lie groups, and fibre bundles. Best features are numerous good examples and problems which illustrate ideas with concrete examples. Probably best used to supplement a more modern text. TLS

TOPOLOGY, T(15-18: 1, 2), S, L. *Geometric Topology in Dimensions 2 and 3.* Edwin E. Moise. Grad. Texts in Math., V. 47. Springer-Verlag, 1977, x + 262 pp, $19.80. [ISBN: 0-387-90220-1; 3-540-90220-1] Modern unified treatment of the development of geometric topology up through the

triangulation theorem and the Hauptvermutung, whose proofs follow Shalen's ideas. Text proceeds from a careful treatment of planar topology to analogous concerns in three dimensions. An appropriate amount of attention is paid to mathematical details. Chapter problem sets used to illustrate and expand on text material. Bibliography. Index. RJA

TOPOLOGY, T(16-17: 1), S, L. *Introduction to Knot Theory*. Richard H. Crowell, Ralph H. Fox. Springer-Verlag, 1977, x + 182 pp, $12.80. [ISBN: 0-387-90272-4; 3-540-90272-4] Slightly altered reprint of the 1963 original edition, itself based on the 1956 Philips lectures at Haverford College. A strictly elementary interpretation, quite suitable for senior seminars or independent study. LAS

TOPOLOGY, P**, L. *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*. Sidney A. Morris. London Math. Soc. Lect. Notes, No. 29. Cambridge U Pr, 1977, viii + 128 pp, $8.50 (P). It is well known that a finitely generated abelian group is the direct product of cyclic groups. There is a similar characterization of locally compact abelian topological groups. Well written, many exercises and examples and almost completely self-contained. JEG

PROBABILITY, T*(13-16: 1), S*, L*. *Basic Probability and Applications*. Miloslav Nosal. Saunders, 1977, x + 370 pp, $13.50. [ISBN: 0-7216-6865-8] A mathematical text, concerned with proofs but readable with little background. The first 16 chapters assume finite sample spaces so no calculus is required there. The final two chapters consider countable spaces and the central limit theorem. Includes considerable history, many applications, and problems oriented to pocket calculators. FLW

PROBABILITY, P. *Probability*. Ed: Joseph L. Doob. Proc. of Symp. in Pure Math., V. XXXI. AMS, 1977, 169 pp, $19.20. 16 research papers from a March 1976 conference held at Urbana, Illinois. LAS

PROBABILITY, S(18), P. *Lecture Notes in Mathematics-595: Stetige Faltungshalbgruppen von Wahrscheinlichkeitsmassen und erzeugende Distributionen*. Wilfried Hazod. Springer-Verlag, 1977, xiii + 157 pp, $8 (P). [ISBN: 0-387-08259-X; 3-540-08259-X]

PROBABILITY, T(17: 1), P. *Risk Theory: The Stochastic Basis of Insurance, Second Edition*. R.E. Beard, T. Pentikäinen, E. Pesonen. Chapman and Hall, 1977, xvi + 195 pp, $12.95. [ISBN: 0-470-99119-4] Revision of the 1969 Methuen Monograph (TR, April 1970), and now one of the Halsted Press Monographs on Applied Probability and Statistics. Covers the elementary theory (using stochastic processes) and its applications, with an updated final chapter on application of risk theory to business planning. RSK

STATISTICS, S(13-14), L. *Computational Handbook of Statistics, Second Edition*. James L. Bruning, B.L. Kintz. Scott F, 1977, 308 pp, $8.95 (P). Step-by-step arithmetic shown for dozens of statistical tests. No theoretical discussion whatever. Authors claim the most inexperienced assistant can do complex analysis without supervision. LH

STATISTICS, T?(16-17: 1). *An Introduction to Multivariate Techniques for Social and Behavioural Sciences*. Spencer Bennett, David Bowers. Halsted Pr, 1976, xii + 156 pp, $14.95. [ISBN: 0-470-09280-7] Primarily concerned with factor analysis, but also includes multiple group analysis, multidimensional scaling, discriminant analysis, and the analysis of qualitative data. Written at the intuitive level, with a bare minimum of matrix algebra. Numerical examples are presented in detail--unfortunately the first major example is completely botched. No exercises. RSK

STATISTICS, T(14-17: 2), P, L. *Experimental Design: Procedures for the Behavioral Sciences*. Roger E. Kirk. Brooks/Cole, 1968, xii + 577 pp, $17.95. Well-organized presentation of the principal designs and techniques used in behavioral research, presuming only college algebra and basic statistical inference. Assumptions are clearly stated, advantages and disadvantages of each design are given, and for most designs references are made to the literature where these designs have been applied. Its use as a text is diminished by its lack of exercises. Also, it (deliberately) uses only one (artificial) data set for all illustrations. RSK

STATISTICS, S*, P, L*. *The Theory of Gambling and Statistical Logic, Revised Edition*. Richard A. Epstein. Acad Pr, 1977, xv + 450 pp, $19.50. [ISBN: 0-12-240760-1] Major changes from the 1967 First Edition (TR, February 1968; ER, March 1969) are that proofs of the basic theorems have been omitted, and recent developments in the analysis of security prices and in the game of blackjack have been incorporated. A fascinating book for anyone interested in games of chance, requiring only a modest mathematical background. RSK

STATISTICS, S. *Statistical Tables for Science, Engineering, Management and Business Studies, Second Edition, Revised and Expanded*. J. Murdoch, J.A. Barnes. Halsted Pr, 1977, 46 pp, $1.95 (P). [ISBN: 0-470-62516-3] Expanded from the 1970 *Second Edition* to include three accounting tables of limited usefulness (only integer interest rates are tabled). In addition to the basic statistical tables and some obsolete mathematical tables, it contains some statistical quality control tables, attribute sampling tables, and significance tables for runs. RSK

STATISTICS, P, L. *Inference and Decision*. Günter Menges, *et al*. Selecta Statistica Canadiana, V. 1. U Pr of Canada, 1973, vi + 82 pp, (P). Seven invited brief survey papers from various Canadian university seminars during 1970-71. LAS

STATISTICS, T(15-17: 1), P. *The Analysis of Contingency Tables*. B.S. Everitt. Chapman and Hall, 1977, 128 pp, $8.50. [ISBN: 0-470-99144-5] In the Halsted Press (formerly Methuen's) Monographs on Applied Probability and Statistics series. First half is essentially a revision of A.E. Maxwell's work *Analysing Qualitative Data*, which deals with two dimensional tables. Last half provides an introduction to the analysis of multidimensional tables, principally through the use of log-linear models. No exercises. RSK

STATISTICS, S(15-17), P, L. *Classification and Clustering*. Ed: J. Van Ryzin. Acad Pr, 1977, x + 467 pp, $17. [ISBN: 0-12-714250-9] Proceedings of a seminar held at the University of Wisconsin in May, 1976. Several of the articles would be of interest and accessible to a good undergraduate. FLW

COMPUTER SCIENCE, S(15-18), P. *Data Structures, Computer Graphics, and Pattern Recognition.* Ed: A. Klinger, K.S. Fu, T.L. Kunii. Acad Pr, 1977, xiv + 498 pp, $23. [ISBN: 0-12-415050-0] Fifteen papers organized around computer graphics and pattern recognition applications of data structures methodology. References. Appendix. Index. RJA

COMPUTER SCIENCE, S(17-18), P. *Lecture Notes in Computer Science-53: Mathematical Foundations of Computer Science 1977.* Ed: J. Gruska. Springer-Verlag, 1977, xi + 595 pp, $18.40 (P). [ISBN: 0-387-08353-7; 3-540-08353-7] Contains 15 invited papers and 46 short communications contributed for presentation at the 6th Symposium on Mathematical Foundations of Computer Science--MFCS '77, held at Tatranská Lomnica, Czechoslovakia, September 5-9, 1977. Principal areas of conference interest: automata and formal languages, computability theory, analysis and complexity of algorithms, theoretical aspects of programming and of programming languages, theoretical aspects of operating systems and mathematical approaches to artificial intelligence. Appendix. RJA

COMPUTER SCIENCE, T(13-18: 1, 2), S, P, L. *Microprocessors and Microcomputers.* Branko Souček. Wiley, 1976, xv + 607 pp, $23. [ISBN: 0-471-81391-5] Begins with general programming and interfacing techniques common to all microprocessors. Second part presents detailed descriptions of representative microprocessor families: 4004/4040, 8008/8080, M6800, PPS-4, PPS-8, IMP 4/8/16, PACE. Last part presents new microprocessors and special purpose microsystems: LSI-11, F8, SMS, 3000 system, IM6100. Three appendices. Examples, illustrations, diagrams, tables throughout. Problems. Chapter references. Index. RJA

COMPUTER SCIENCE, T(15-16: 1), S, L. *Introduction to the Design and Analysis of Algorithms.* S.E. Goodman, S.T. Hedetniemi. McGraw, 1977, xi + 371 pp, $17.95. [ISBN: 0-07-023753-0] First presents several tools useful in the design and analysis of algorithms--structured programming, data structures, elementary probability and statistics. Then algorithm design methods follow--subgoals, hill climbing, heuristics, backtracking, branch and bound, recursion, and simulation. The last part contains illustrative examples and applications. Many exercises. A chapter devoted to topical references. Programs given in Fortran, Algol, and PL/I. Appendices. Index. RJA

COMPUTER SCIENCE, T(15: 1), S, L. *Analysis and Design of Digital Circuits and Computer Systems.* Paul M. Chirlian. Matrix Pub, 1976, xv + 606 pp, $22.50. [ISBN: 0-916460-03-7] Number systems, binary arithmetic, gates; switching circuits and minimization techniques; logic, sequential and digital circuits; memories; computer arithmetic. Presumes a bit of circuit analysis. Contains an appendix on semi-conductors. RWN

COMPUTER SCIENCE, T(13-18: 1), S. *Elements of Cobol Programming.* Wilson T. Price, Jack L. Olson. Dryden Pr, 1977, 375 pp, $10.95 (P). [ISBN: 0-03-018371-5] The essentials of Cobol are presented; then follows an elementary discussion of computing, various facets of hardware and software, and error detection. Advanced aspects of Cobol comprise most of second half. A chapter on structured programming is included. Two appendices: one on reserved words and another on statement formats. Features from both the 1968 and 1974 ANS Cobol standards are used plus some of their common extensions. In-chapter exercises and answers. Chapter problems. Index. RJA

COMPUTER SCIENCE, T*(13-18: 1), S, L. *Problem Solving and Structured Programming in Fortran.* Frank L. Friedman, Elliot B. Koffman. A-W, 1977, xvi + 472 pp, $10.50 (P). [ISBN: 0-201-01967-1] For solving any given problem three phases are emphasized: top-down analysis of the problem, stepwise specification of an algorithm, language implementation of the algorithm. Text uses an extended Fortran with format-free input and output, character variables (CHARACTER declaration), IF-THEN-ELSE (block-IF) decision structure, WHILE loop structure, FOR loop structure (generalized indexed-DO), remote block (parameterless subroutine), and loop escape and next iteration statements. Possible to use text in an environment where none or only some of these extended features are available. Exercises. Appendix. Glossary. Answers to selected exercises. Index. RJA

COMPUTER SCIENCE, T*(15-18: 2), S, L. *Artificial Intelligence.* Patrick Henry Winston. A-W, 1977, xvi + 444 pp, $14.95. First part presents basic terminology and surveys the key problems in artificial intelligence together with attempts at their solutions. This part is programming-independent. Second part introduces Lisp and its many uses in artificial intelligence research. The book taken as a whole provides one with an integrated view of the field with special emphasis on the approach researchers have actually taken to solve problems. Problems. Appendix on Lisp. Index. Chapter references. RJA

COMPUTER SCIENCE, T(13: 1), S. *Computers, Their Impact and Use: Structured Programming in Fortran.* Robert E. Lynch, John R. Rice. HR&W, 1977, ix + 453 pp, $10 (P). Begins with a discussion of what a computer is and how it processes information. Continues with the computer's historical development and its impact on society. Then chapters on writing programs in Fortran are interwoven with material on computer applications, algorithms, and computer languages and systems. Similar to the authors' earlier text on Basic. References. Problems. Appendices. Glossary. Index. RJA

SYSTEMS THEORY, T(17-18: 1, 2), S, P. *Decomposability, Queueing and Computer System Applications.* P.J. Courtois. Acad Pr, 1977, xiii + 201 pp, $17.50. [ISBN: 0-12-193750-X] Text is divided into three parts: (1) theory of nearly completely decomposable stochastic matrices; (2) analysis of stochastic queueing networks; (3) computer system performance evaluation. Rigorous mathematical treatment throughout. Four appendices. References. Index. RJA

SYSTEMS THEORY, T(18: 2), S, P. *Self-Organizing Control of Stochastic Systems.* George N. Saridis. Dekker, 1977, xxi + 488 pp, sFr. 128. Attempts to organize and classify methods for systems with uncertain dynamics. Very large bibliography. No problems. Estimation theory, stochastic and dual optimal control. Parameter-adaptive and performance-adaptive self-organizing methods are compared and applied to problems. LH

SYSTEMS THEORY, P. *Dynamical Systems.* Ed: A.R. Bednarek, L. Cesari. Acad Pr, 1977, xv + 516 pp, $19.50. 24 invited lectures and 31 contributed papers from a symposium held at the University of Florida in March, 1976. Printed from typescript. LAS

APPLICATIONS, S(14-16), P, L*. *Models for Public Systems Analysis.* Edward J. Beltrami. Acad Pr, 1977, xv + 218 pp, $14.50. Mathematical models extracted from analyses of municipal services (garbage, energy, firefighting) in New York City. Effective solutions are "less a matter of optimization than of accommodation among feasible alternatives." Mathematical appendices cover necessary elementary techniques of linear, integer and nonlinear programming, random processes and graph theory. LAS

APPLICATIONS, P. *Interdisciplinary Mathematics, V. XI: Geometric Structure of Systems-Control Theory and Physics, Part B.* Robert Hermann. Math Sci Pr, 1976, xviii + 484 pp, $35 (P). Continues the author's program of presenting the geometric foundations of engineering and physics. Examines the role of the geometry of manifolds in classical and quantum mechanics, optimal control theory, and (very briefly) economics. Exercises, many open-ended. Historical remarks. Printed from typescript; note price. DFA

APPLICATIONS, P. *Interdisciplinary Mathematics, V. XII: The Geometry of Non-Linear Differential Equations, Bäcklund Transformations and Solutions, Part A.* Robert Hermann. Math Sci Pr, 1976, xii + 313 pp, (P). A "geometric" study of nonlinear partial differential equations, whose aim is to expound the mathematical tools needed to right the "wrong headed and even perverse" directions into linearity taken by physicists probing the nature of elementary particles. In addition to the emphasis on the differential geometry of nonlinear equations, the analogy between the study of algebraic equations (Galois theory, algebraic geometry) and that of differential equations is stressed. While the presentation is often more suggestive than rigorous, it is certainly provocative, engaging, and interesting. SG

APPLICATIONS, P. *Interdisciplinary Mathematics, V. XIII: Algebro-Geometric and Lie-Theoretic Techniques in Systems Theory, Part A.* Robert Hermann, Clyde Martin. Math Sci Pr, 1977, vii + 256 pp, (P). Matrix algebra has been at the forefront of the mathematics of systems theory and electrical engineering for many years. The authors are convinced that certain techniques of algebraic geometry are the natural successors to these 19th century methods; their goal is to isolate and develop them. These notes extend earlier material on this topic in this series (especially *Vol. VIII, Linear Systems Theory and Introductory Algebraic Geometry,* TR, March 1976) and explore the problem of pole placement via state feedback in which "old-fashioned" techniques involving "canonical forms" are replaced by more powerful technique from algebraic geometry. The authors' antipathy for certain "modern" trends in mathematics ("mountains of gobbledygook") is made explicit. LCL

APPLICATIONS, P, L**. *Catastrophe Theory, Selected Papers, 1972-1977.* E.C. Zeeman. A-W (Adv. Bk. Prog.), 1977, x + 675 pp, $14.50 (P); $26.50. [ISBN: 0-201-09015-5; 0-201-09014-7] A useful collection of key papers, some formerly rather inaccessible, tracing the author's pathbreaking research into the uses and potential of René Thom's theory of elementary catastrophes. Concludes with a dialogue "Afterthought" containing a mild response to questions similar to those raised by the critics of catastrophe theory. The volume is handsomely designed, creatively overcoming the typically grotesque contrasts of reprints from diverse type styles and page dimensions. LAS

APPLICATIONS, P, L. *Modern Modeling of Continuum Phenomena.* Ed: Richard C. DiPrima. Lect. in Appl. Math., V. 16. AMS, 1977, x + 251 pp, $30.80. [ISBN: 0-8218-1116-9] Lectures from the ninth AMS-SIAM summer seminar on applied mathematics held at RPI in July, 1975. Very diverse topics: perturbation theory, stochastic equations, population dynamics, amoeboid motions, earthquake sources. A bicentennial monument to Euler's 1776 laws of continuum mechanics. LAS

APPLICATIONS (APPROXIMATION THEORY), P. *Optimal Estimation in Approximation Theory.* Charles A. Micchelli, Theodore J. Rivlin. Plenum Pr, 1977, ix + 300 pp, $29.50. [ISBN: 0-306-31049-X] IBM Research Symposium Series. Contains 14 papers presented at an international symposium held in Freudenstadt, West Germany, September 27-29, 1976. Includes application areas of crystallography, data transmission systems, cartography, reconstruction from x-rays, planning of radiation treatment, optical perception, analysis of decay processes and inertial navigation system control. Index. RJA

APPLICATIONS (BIOLOGY), T(17-18), P, L. *Neurophysics.* Alwyn C. Scott. Wiley, 1977, xii + 340 pp, $24.95. [ISBN: 0-471-02998-X] A survey of neuroscience (nerve fibres, thresholds, pulse interactions, neural networks) from the viewpoint of an electrical engineer who wishes to reverse the "lack of humility" that physical scientists often exhibit in the face of the "enormous dynamic complexity exhibited by living systems." Photocopies from typescript, the 281 page text is followed by a 50-page bibliography. LAS

APPLICATIONS (BIOLOGY), T(13-15; 1). *Mathematical Modeling of Biological Systems--An Introductory Guidebook.* Harvey J. Gold. Wiley, 1977, xv + 357 pp, $21.95. [ISBN: 0-471-02092-3] Written by a biologist for biology students, this text surveys the uses and "conceptual content" of standard elementary mathematics: dimension-probability, dynamic processes, feedback, curve fitting, computing. Great stress on the justification (and limitations) of the modelling process; little emphasis on deriving consequences by mathematical techniques. LAS

APPLICATIONS (BIOLOGY), P, L. *Lecture Notes in Biomathematics-13: Mathematical Models in Biological Discovery.* Ed: D.L. Solomon, C. Walter. Springer-Verlag, 1977, vi + 240 pp, $11 (P). Nine survey papers from a 1975 AAAS symposium, mixing history, speculation and philosophy of modelling. Topics range widely, from genetics to ecology. LAS

APPLICATIONS (BIOLOGY), S(15-18), P. *Lecture Notes in Biomathematics-12: Models of the Stochastic Activity of Neurones.* Arun V. Holden. Springer-Verlag, 1976, vii + 368 pp, $11 (P).

APPLICATIONS (BIOLOGY), P, L. *Scale Effects in Animal Locomotion*. Ed: T.J. Pedley. Acad Pr, 1977, xx + 545 pp, $35.25. A unique interdisciplinary survey of the relationship between locomotory function and animal size. Biologists, physicists, engineers and mathematicians examine all manner of locomotion, emphasizing the role of scale effects. Accessible to upperclass undergraduate science students. LAS

APPLICATIONS (BIOLOGY), P. *Some Mathematical Questions in Biology, VIII*. Ed: Simon A. Levin. Lect. on Math. in Life Sci., V. 9. AMS, 1977, vi + 186 pp, $14.40 (P). Six papers from the 10th symposium on mathematical biology held in Boston during February 1976 in conjunction with the annual meeting of AAAS. Topics: immunology, developmental biology, biomechanics. LAS

APPLICATIONS (BUSINESS), T(13: 2). *Mathematics for Management*. Gary J. and Richard Bronson. Dun-Donnelley, 1977, xiii + 490 pp, $15. Commercial problems motivate each mathematical idea presented. Applications spread throughout examples and problems. Basic techniques regarding matrices, linear programming, probability and calculus covered without proof. LH

APPLICATIONS (ECONOMICS), T(18: 1), S, P. *Production Functions and Aggregation*. Kazuo Sato. Amer Elsev, 1975, xxxiii + 313 pp, $32.95. Examines connection between micro and macro production functions by studying the aggregation process. Emphasizes efficiency of firms and heterogeneous capital. Largely theoretical but applies concepts to empirical analysis of several industries. Includes consideration of economy-wide aggregation. LH

APPLICATIONS (ECONOMICS), P. *Bidding and Auctioning for Procurement and Allocation*. Ed: Yakov Amihud. New York U Pr, 1976, xv + 220 pp, $17.50. [ISBN: 0-8147-0558-8] 18 papers on the theory and practice of bidding and auctioning--an important yet virtually unstudied form of resource allocation. From a conference held at the NYU Center for Applied Economics in October, 1975. LAS

APPLICATIONS (ENGINEERING), T(15-18: 1, 2), S, L. *Numerical Methods in Finite Element Analysis*. Klaus-Jürgen Bathe, Edward L. Wilson. P-H, 1976, xv + 528 pp, $28.95. [ISBN: 0-13-627190-1] Text is divided into three parts: required concepts from matrix and linear algebra; general principles and numerical procedures of the finite element method; solution of the finite element equilibrium equations in static and dynamic analysis. Many examples and computer program segments are included. Chapter references. Index. RJA

APPLICATIONS (ENGINEERING), S(14-16). *Advanced Engineering Mathematics*. A.C. Bajpai, L.R. Mustoe, D. Walker. Wiley, 1977, x + 578 pp, $24.95; $11.95 (P). Written for engineering students. It contains sketches of many branches of modern mathematics and discusses applications. Topics include differential equations, vector field theory, complex analysis, and statistical methods. MU

APPLICATIONS (INFORMATION PROCESSING), S(15-18), L. *Machine Takeover*. Frank George. Pergamon Pr, 1977, xiv + 193 pp, $6.50 (P). [ISBN: 0-08-021228-X] Presents an explanation and warning of the dangers inherent in the technological advancements that the computer has brought to present day society. This warning is given in strong language and centers around the growing numbers of large computer data banks and the ease of access to them by bureaucrats and criminals. Style is polemical. References. Name index. Subject index. RJA

APPLICATIONS (PHYSICS), T(18). *A First Course in Rational Continuum Mechanics, V. 1: General Concepts*. C. Truesdell. Pure and Appl. Math., V. 71-1. Acad Pr, 1977, xxiii + 280 pp, $23. Chapters: bodies, forces, motions, energies; kinematics; the stress tensor; constitutive relations. A rigorous presentation, with gaps to be closed. Exercises throughout (with solutions). Assumes familiarity with the rudiments of the subject, and with measure theory. Volume 2 will treat fluid mechanics and elasticity; Volume 3, fading memory, thermodynamics, statics, thermostatics. DFA

APPLICATIONS (PHYSICS), S(13-16), L**. *Space, Time, and Gravity: The Theory of the Big Bang and Black Holes*. Robert M. Wald. U of Chicago Pr, 1977, viii + 131 pp, $10.95. [ISBN: 0-226-87030-8] A superb layman's introduction to contemporary cosmology derived from the 1976 Compton Lectures at the University of Chicago. Well-illustrated with space-time diagrams and world-lines, but devoid of formulas to frighten non-mathematicians. An exciting recital of the cosmic repertoire that has appeared frequently in, e.g., recent articles in *Scientific American*. LAS

APPLICATIONS (SOCIAL SCIENCE), S(14-16), L**. *Some Illustrative Examples of the Use of Undergraduate Mathematics in the Social Sciences*. Ed: Samuel Goldberg. MAA, 226 pp, (P). Fifteen vignettes (with good references to social science research literature) of interesting (and, among mathematicians, little known) applications of undergraduate mathematics to, e.g., demography, politics, learning theory, social choice. Each chapter, designed merely to whet the appetite, would make an ideal enrichment lecture or special project assignment. LAS

APPLICATIONS (PHYSICS), T(15-17), L. *Quantum Mechanics for One- and Two-Electron Atoms*. Hans A. Bethe, Edwin E. Salpeter. Plenum Pr, 1977, xii + 370 pp, $8.95 (P). [ISBN: 0-306-20022-8] Paperback edition of 1957 Springer hardcover edition, itself adapted from a 1956 article for the *Encyclopedia of Physics* which was in turn based on a 1930 article written for *Handbuch der Physik*. Emphasizes older "low-brow" explicit derivations, rather than recent elegant yet more difficult formalisms. LAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jennifer R. Galovich, St. Olaf; Steven Galovich, Carleton; Jay E. Goldfeather, Carleton; Loren Haskins, Carleton; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton; Frank Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, UNIVERSITY OF NEBRASKA at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D.C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Professor Victor L. Shapiro, University of California, Riverside, has been appointed Faculty Research Lecturer for the academic year 1977-78.

Professor Edward J. Dudewicz has been awarded the 1977 Reasearch Award of the Ohio State Chapter of the Society of the Sigma Xi.

Associate Professor Seymour Kass, Boston State College, has been promoted to Professor.

Assistant Professor Frank C. Sherburne, Jr., Hope College, has been promoted to Associate Professor.

Mr. Michael H. Moore has joined Orincon Corporation in La Jolla, California, as principal engineer.

Salisbury State College: Henry J. Greene, Aberdeen Proving Grounds, has been appointed Instructor; Sandra R. Smith, Howard University, has been appointed Instructor.

Rensselaer Polytechnic Institute: Professor Carlton E. Lemke is on leave at Stanford University to January, 1978, and at Eidgenossene Technische Hochschule, Zurich, Switzerland, from February to June, 1978. Professor Carlton E. Lemke is on leave for the 1977-78 academic year. Dr. Roger Alexander, University of Colorado, has been appointed Assistant Professor. Associate Professor Norman Free has been promoted to Professor. Associate Professor Harry W. McLaughlin has been promoted to Professor.

John Carroll University: Assistant Professors Carl R. Spitznagel and David L. Stenson have been promoted to Associate Professor.

University of Nebraska at Omaha: Associate Professor J. Scott Downing has been promoted to Professor. Assistant Professor John Karlof has been promoted to Associate Professor.

Assistant Professor G.W. Pineau, University of Prince Edward Island, has been promoted to Associate Professor.

Assistant Professor Patricia Andresen, University of Alaska, Fairbanks, has been promoted to Associate Professor.

Associate Professor Richard J. Maher, Loyola University of Chicago, has been appointed Chairman of the Department of Mathematical Sciences.

Marquette University: Dr. Stephen J. Merrill, University of Iowa, Dr. Daniel F.X. O'Reilly, Simmons College, and Dr. Karl E. Byleen, University of Nebraska, have been appointed Assistant Professors. Associate Professor J. Douglas Harris has been promoted to Professor.

Eric S. Lander, A.B. candidate from Princeton University, is one of 18 students currently participating in the Mass Media Intern Program sponsored by the American Association for the Advancement of Science.

Christopher Newport College: Assistant Professors Robert Collins and Martin W. Bartell have been promoted to Associate Professor. Associate Professor Daisy D. Bright has retired with the title of Associate Professor Emeritus.

University of Alabama in Birmingham: Dr. Howard H. Campaigne, Eastern New Mexico University, has been appointed Visiting Professor. Assitant Professor Mary B. Rumsey has retired.

California State University, Northridge: Professor Tung-Po Lin was elected Chairman of the Mathematics Department. Associate Professor William E. Watkins has been promoted to Professor. Assistant Professor Joel Zeitlin has been promoted to Associate Professor.

Dr. Paul Klingsberg, Bucknell University has been appointed Assistant Professor.

Dr. Kondagunta Sundaresan, Cleveland State University, has been appointed Professor.

Dr. John D. Nichols, Presbyterian College, has been appointed Assistant Professor.

Dr. G.M. Reekie, Livingston University, has been appointed Assistant Professor.

Dr. Clarence H. Heinke, Capital University, has retired with the title of Professor Emeritus.

Assistant Professor Z.H. Chowdhury, Clinch Valley College, has been promoted to Associate Professor.

San Francisco State University: Dr. Diane Resek has been appointed Assistant Professor. Professor Arthur J. Hall has been retired. Associate Professor Robert J. Douglas has been promoted to Professor. Assistant Professor Susann J.N. Shaw has been promoted to Associate Professor.

University of Rochester: Dr. Richard Mandelbaum was appointed Assistant Professor. Assistant Professor John W. Shuck resigned to accept a position at Ursinas College. Assistant Professor Richard D. Mosak has been promoted to Associate Professor.

University of Oklahoma: Dr. Morris L. Marx, Vanderbilt University, has been appointed Chairman and Professor. Dr. Luther W. White, University of Illinois, has been appointed Assistant Professor. Assistant Professor Marilyn J. Breen has been promoted to Associate Professor. Associate Professors Andy R. Magid and Thomas J. Hill have been promoted to Professors. Professor George M. Ewing has retired with the title of George Lynn Cross Research Professor Emeritus.

Professor Edison Greer, San Jose State University, has retired with the title of Professor Emeritus.

University of North Carolina at Wilmington: Dr. Dargan Frierson, University of Arizona, has been appointed Assistant Professor. Associate Professors Thaddeus G. Dankel and Fletcher R. Norris have been promoted to Professors.

Dr. Carol G. Heines, Texas Southern University, has been appointed Assistant Professor at Rider College.

Dr. Harold H. Johnson, University of Washington, has been appointed Professor at Trinity College, Deerfield, Illinois.

Assistant Professor Stanley J. Wertheimer, Connecticut College, has been promoted to Associate Professor and Director of Academic Computing.

Professor Herman Rosenberg, Jersey City State College, has been appointed Chairman of the Department of Mathematics.

Professor George B. Ax, Virginia Military Institute, has retired.

Associate Professor Underwood Dudley, De Pauw University, has been promoted to Professor.

College of Fredonia: Dr. H. Joseph Straight, University of Western Michigan, has been appointed Assistant Professor. Assistant Professor Kenneth R. Slonneger has been promoted to Associate Professor. Associate Professor James E. McKenna has been appointed Acting Chairman.

SUNY at Buffalo: Dr. Phillip E. Parker, Oregon State University, has been appointed Visiting Assistant Professor. Dr. Jon E. Kraus, University of California, Berkeley, has been appointed Research Instructor. Assistant Professor Catherine L. Olsen has been promoted to Associate Professor.

University of Toledo: Associate Professors Edward E. Ebert and Henry C. Wente have been promoted to Professors. Assistant Professor Stephen E. Spielberg has been promoted to Associate Professor.

Associate Professor James E. Hall, University of Wisconsin, has been promoted to Professor.

Dr. Ed. W. Huffman, University of Missouri-Rolla, has been appointed Assistant Professor at Southwest Missouri State.

Virginia Commonwealth University: Dr. William E. Haver, University of Tennessee, has been appointed Associate Professor and Chairman. Associate Professor Malcolm L. Murrill, has retired with the title of Associate Professor Emeritus.

University of Southern Mississippi: Dr. Gary L. Walls, Oklahoma State University, has been appointed Visiting Assistant Professor. Associate Professor Joseph S. Morrell is on sabbatical leave. Assistant Professors Stephen A. Doblin and Wallace C. Pye have been promoted to Associate Professors. Associate Professor David J. Caveny is a Visiting Professor at Clemson University under a grant for the development of an alternative approach to graduate education in the mathematical sciences.

Assistant Professor Richard W. Billstein, University of Montana, has been promoted to Associate Professor.

East Texas State University: Professor W. G. Hill has retired. Associate Professor Bill D. Anderson has been promoted to Professor.

Professor Rene Dennemeyer, California State College - San Bernardino, is a Visiting Professor at California State University - Northridge.

Associate Professor Fred Pollack, Miami University of Ohio, has taken employment in private industry.

Associate Professor Dale M. Rognlie, South Dakota School of Mines, has been promoted to Professor.

Colorado College: Dr. Frederick C. Tinsley, University of Wisconsin, and Mr. John J. Watkins, University of Kansas, have been appointed Instructors. Assistant Professor David W. Roeder has been promoted to Associate Professor.

Dr. James T. Wood, Colorado Springs, has been appointed to a position with Ford Aero Space.

Assistant Professor David P. Mather, Regis College, has been promoted to Associate Professor.

Assistant Professor Michael L. Kovacic, Colorado State University, has retired with the title of Associate Professor.

University of Southern Colorado: Professor Stephen D. Bronn has been appointed Chairman. Dr. Roger D. Johnson, Georgia Tech, has been appointed Professor. Associate Professor Thomas J. Bartlett and Assistant Professor Albert P. Richards have retired.

Assistant Professor Ronald E. Laser, Adams State College, has been promoted to Associate Professor.

Colorado School of Mines: Associate Professor Raymond R. Gutzman has been appointed Chairman. Assistant Professor Raymond W. Mueller has been promoted to Associate Professor.

Associate Professor David S. Moore, Purdue University, has been promoted to Professor.

University of Wisconsin-Madison: Dr. Andrew F. Siegel has been appointed Assistant Professor. Mr. Erik A. Nordheim has been appointed Instructor.

University of North Carolina-Charlotte: Assistant Professors Philip E. Johnson, Nilo A. Niccolai, and Harold B. Reiter have been promoted to Associate Professors.

Dr. Steven M. Rohde, a senior research mathematician with General Motors Research Laboratories, Warren, Michigan, received the Burt L. Newkirk Award from The American Society of Mechanical Engineers. Dr. Rohde received the award for his contributions to the field of tribology and his authorship of a number of publications in that area.

Professor Emeritus John Franklin Locke, Birmingham-Southern College, died on July 13, 1977, at the age of 73. He was a member of the Association for forty-three years.

Professor Jacob T. Golightly, Jacksonville University, died on July 4, 1977, at the age of 56. He was a member of the Association for nine years.

Professor Henry L. Lucas, North Carolina State University, died on June 8, 1977. He was a member of the Association for twelve years.

Dr. Harry Levy, Urbana, Illinois, died on September 2, 1977, at the age of 75. He was a member of the Association for fifty-three years.

Mr. Jeffrey L. Rackusin, Van Nuys, California, died on June 29, 1977, at the age of 25. He was a member of the Association for two years.

Professor Will E. Edington, De Pauw University, died on March 12, 1977, at the age of 90. He was a charter member of the Association.

Dr. Joachim Weyl, New York, New York, died on July 20, 1977, at the age of 62. He was a member of the Association for thirty-six years.

Professor Franz E. Hohn, University of Illinois, died on July 10, 1977, at the age of 61. He served the Association as a Visiting Lecturer and Consultant.

Dr. John O. Blumberg, University of Pittsburgh, died on January 7, 1976. He was a member of the Association for thirty-six years.

Professor John H. Curtiss, University of Miami, died on August 13, 1977, at the age of 67. He was a member of the Association for forty-three years.

Dr. James Thomas Day, Pennsylvania State University, died on May 20, 1977, at the age of 44. He was a member of the Association for ten years.

Mr. Joseph L. Lortie, Downsview, Ontario, died on August 10, 1977. He was a member of the Association for fifteen years.

Dr. Frank M. Weida, Port Republic, Maryland, died on September 13, 1977, at the age of 86. He was a charter member of the Association in 1915 and continued his membership for fifty-two years.

Professor William Thomas Reid, retired from the University of Oklahoma and Visiting Scholar at the University of Texas, died on October 14, 1977. He was a long-time member of the Association.

THE NEW YORK ACADEMY OF SCIENCES

The New York Academy of Sciences is sponsoring the Second International Conference on Combinatorial Mathematics to be held at the Barbizon Plaza Hotel in New York City, April 4-7, 1978. The conference will bring together mathematicians in both applied and theoretical areas of combinatorics. For details write to Professor Louis V. Quintas, Department of Mathematics, Pace University, New York, New York 10038.

INTERNATIONAL CONFERENCE ON APPLIED GAME THEORY

INSTITUTE FOR ADVANCED STUDIES, VIENNA - JUNE 12-15, 1978

Papers will be delivered that involve serious applications of game theory to the social sciences, natural sciences, or humanities. All papers will involve the development and application of game-theoretic models to real-world phenomena or processes. They will be written in English, and it is expected that the Conference papers will be published in a volume after the Conference. The International Conference on Applied Game Theory is co-sponsored by the Institute for Advanced Studies, Vienna, and the New York University Center for Applied Economics. Decisions will be made only on the basis of final papers. For information on all organizational matters, please contact Dr. Gerhard Schwödiauer, Institute for Advanced Studies, Stumpergasse 56, A-1060 Vienna, Austria.

FIFTH INTERNATIONAL SYMPOSIUM ON MULTIVARIATE ANALYSIS

The Fifth International Symposium on Multivariate Analysis will be held at the University of Pittsburgh, Pittsburgh, Pennsylvania during the period of June 19-24, 1978. There will be several sessions of contributed papers on the theory and *applications* of multivariate analysis. Further details regarding the symposium may be obtained by contacting P. R. Krishnaiah, Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, Pennyslvania 15260.

TWO JUNE 1978 WORKSHOPS IN APPLICABLE MATH

The MD-DC-VA section of the MAA will sponsor two five-day workshops at Salisbury State College, Maryland. "Mathematical Models and Contemporary Problems" will be led by Professor Fred S. Roberts of Rutgers University. His interests include graph theory and the application of mathematics to problems of biology, society and the environment. This workshop will be held the week of 12-16 June.

"Statistics-A Second Look" will be led by Dr. Glenn W. Rock of Salisbury State College. His special interests are reliability theory and the role of statistics in the Math Sciences undergraduate curriculum. This workshop will be held the week of 5-9 June. These workshops are designed for teachers in two and four-year colleges. For further information, please write to Dr. B. A. Fusaro, Department of Math Sciences, SSC, Salisbury, Maryland 21801.

FACULTY EXCHANGE PROGRAM

The Faculty Exchange Center, a non-profit, faculty-administered program, helps to arrange college and university faculty exchanges within the United States, and between the United States and overseas where the language of instruction is English. For more information write to: Faculty Exchange Center, Franklin and Marshall College, P.O. Box 1091, Lancaster, Pennsylvania 17604.

MATHEMATICS MODULES

In the winter and spring of 1976-1977 academic year a survey of available modular materials in undergraduate mathematics was conducted by The Modules and Monographs in Undergraduate Mathematics and Its Applications Project (UMAP) with the help of the MAA Special Projects Office. The result is a collection of 424 Module Description Sheets bound and indexed in the *Index and Descriptions of Available Mathematical Modules*: Volume 1 (June 1977). The Module Description Sheets give information about content, Math/Applications field, prerequisites, post-options, medium, availability, cost, and more.

The *Index* is now available at a cost of $2.50 to cover printing, handling, and mailing. You may order your copy from EDC/UMAP; 55 Chapel Street, Newton, Massachusetts 02160. Checks should be made out to Education Development Center.

THE COOPERATIVE COLLEGE REGISTER

The Cooperative College Register has been re-established as a communications link and matching service for positions and position-seekers for higher education. Write for details. Cooperative College Register, 621 Duke Street, P.O. Box 298-F.A., Alexandria, Virginia 22314.

SYMPOSIUM ON RECENT ADVANCES IN NUMERICAL ANALYSIS

The Mathematics Research Center at the University of Wisconsin-Madison will hold a two and one-half day symposium on recent developments in Numerical Analysis, May 22-24, 1978, dedicated to Professor J. Barkley Rosser on the occasion of his retirement. The program will consist of 13 invited lecturers. A detailed program will be available by April 1, 1978. Further information may be obtained from C. de Boor, Mathematics Research Center, University of Wisconson, 610 Walnut Street, Madison, Wisconsin 53706.

DISTINGUISHED LECTURE SERIES IN THE MATHEMATICAL SCIENCES

The fourth annual lecture series sponsored by the Department of Mathematical Sciences of The Johns Hopkins University and the Johns Hopkins Press, will be held in Baltimore from June 12 through June 16, 1978. The subject will be the analysis of inventory systems on computers. Included will be lectures, on-line computer demonstrations, and computer workshops. For further information write to Professor Roger A. Horn, Chairman, Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, Maryland 21218.

UNIVERSITY RESIDENT RESEARCH PROGRAM

Faculty members of U.S. institutions of higher education will soon have the opportunity to conduct research in an Air Force laboratory or serve as a research manager in the Air Force Office of Scientific Research (AFOSR). The assignments are for one year but can be extended. Institutions will be required to pay the faculty member's salary, which will be reimbursed by the Air Force. Also, each assignment will be covered by a written agreement using procedures of the inter-governmental Personnel Act (IPA). The program has been set at 24 positions each year.

For more information on Air Force laboratories and research programs, interested faculty members from institutions of higher education should contact the Air Force Office of Scientific Research (AFOSR/XO); Attn: Lt. Col. Thurmon L. Deloney, AFSC University Resident Research Program, Building 410, Bolling AFB, DC 20332.

AACJC CAREER STAFFING CENTER

The American Association of Community and Junior Colleges maintains a Career Staffing Center for its member institutions and those individuals who would like to be considered for staff positions at more than 900 member colleges. Write for details to AACJC Career Staffing Center, P.O. Box 298-A, Alexandria, Virginia 22314.

MATHEMATICIAN RECEIVES TAYLOR AWARD FOR SCIENTIFIC ACHIEVEMENT

Dr. Elizabeth H. Cuthill, the Numerical Analysis Coordinator for the Computation, Mathematics, and Logistics Department of the David W. Taylor Naval Ship R&D Center (DTNSRDC), Bethesda, MD, recently received the David W. Taylor Award for Scientific Achievement for the calendar year 1976.

Dr. Cuthill was recognized for her valuable contributions in the development and exploitation of mathematical and computational techniques for significant Navy applications. She led the successful development of the widely used General Bending Response Codes which have received wide acceptance and are in general use throughout the nation. She was also the leader within the Navy in promoting the use of general purpose finite element codes for structural analysis.

The Award is named after Rear Admiral David Watson Taylor, a naval constructor with a brilliant reputation in the field of naval engineering who was the driving force behind the development and adoption of modern experimental techniques in ship and aircraft research. Originally established by the Navy in 1961, this Award has been presented annually since that time to the individual scientist whose contributions were considered truly outstanding in the field of research and development.

SHORT COURSE ON MULTIVARIATE DATA ANALYSIS

There will be a short course on Multivariate Data Analysis at the University of Pittsburgh during the period of June 16-17, 1978, just before the Fifth International Symposium on Multivariate Analysis. The topics to be covered in this course are: (i) Finite Intersection Tests for Multiple Comparisons of Univariate and Multivariate Normal Populations, (ii) Reduction of Dimensionality and (iii) Repeated-Measurements Designs. The instructors for this course are P. R. Krishnaiah, C. R. Rao and Neil H. Timm. Further details regarding the course may be obtained by contacting P. R. Krishnaiah, Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, PA 15260.

PRESIDENTIAL EXCHANGE PROGRAM AN OPPORTUNITY FOR MAA MEMBERS

A select group of high-calibre middle management executives qualify each year to participate in the President's Executive Interchange Program. This unique business/government exchange places executives from both sectors in specific one-year assignments with their opposite sector.

Members of the Mathematical Association of America are among the potential candidates for the 1978/79 group, according to Jay F. Morris, executive director of the Interchange Program.

Though not on a one-for-one direct exchange, the Program places high-potential middle managers from business, industry and higher education in senior level government posts, while simultaneously placing promising government executives in responsible private sector positions.

To-date, over 400 highly qualified executives have participated in the Interchange Program. Aimed at improving the rapport and cooperation between business and government, the President's Executive Interchange Program is designed specifically to allow the two sectors to learn from each other.

Nominees to the Interchange Program must have a proved record of management ability with significant on-the-job accomplishment, along with a high intellectural capacity and demonstrated leadership ability. The nominees should be identified as having the potential to become senior executives of the sponsoring organization. To assure the quality of the candidates, all nominations must be personally approved by the chief executive officer of the private sector sponsor or the head of the sponsoring federal department or agency.

In addition to their job assignments, the executives also participate in an extensive educational program duing their Interchange year. Their assignments officially begin in September with a week-long seminar in Washington designed to acquaint the participants with the issues and problems of their assigned areas.

There are also regularly scheduled discussion sessions with prominent national figures throughout the year and mid-year 10-day International Study Seminar in Europe where the executives meet with senior officials of various governments. Near the end of the Interchange Year, the participants again meet in Washington for a wrap-up and evaluation. Since its inception, the alumni have overwhelmingly praised the Program. Members interested in additional information should write to the President's Commission on Personnel Interchange; 1900 E Street, NW; Washington, D.C. 20415; (202) 632-6834.

STUDENT INCOME TAX SERVICE FOR THE ELDERLY

Professor Thelma E. Bradford, Mathematics Department, North Carolina Agricultural and Technical State University, is coordinator of a program which grew out of her class, "Mathematics of Business and Finance." The students learn to fill out income tax returns and apply this training to helping the elderly fill out their forms. They have established service centers at several points in the city of Greensboro, and they report gratifying results. Professor Bradford has received many inquiries regarding this activity, and our readers may find it a worthwhile service to explore for their own students and constituents.

PUBLICATION ANNOUNCEMENT - NSF

The National Science Foundation has just published a *Guide to Programs* for fiscal year 1978 for institutions or persons who wish to participate in the Foundation's programs. The 69-page publication includes a description and purpose of each program; eligibility requirements; closing date, where applicable; and addresses from which more information or application forms may be obtained. Copies of the *Guide to Programs* may be obtained from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402 for $2.20 each. Ask for Stock Number 038-000-00342-9.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE NORTH CENTRAL SECTION

The 1977 fall meeting of the North Central Section of the MAA was held at the University of Minnesota, Morris, on October 14-15, 1977. There were 96 persons in attendance including 67 MAA members.

The principal speaker was Elaine Tatham, Johnson County Community College, Overland Park, KS, who spoke on "Educational Rsearch and Planning as a Career Option for a Mathematics Major." Allan Kirch, Macalester College, St. Paul, MN, was the invited speaker for the Friday evening session. He spoke on "Random Numbers, What They Are and How They Are Used."

The following people presided at sessions: Milton Legg, Chairman of the NCS/MAA, Moorhead State University, at the Saturday morning session; Sen Fan, University of Minnesota, Morris, at the Friday evening session; and Robert Raymond, University of Minnesota, Morris, at the Saturday afternoon session.

At the business meeting a final report was given about the first North Central Section Summer Seminar on "Applications of Mathematics in Modeling Theory" held at Bemidji State University June 20-24, 1977. Forty-six people (from all parts of the country) attended the seminar. The proceedings have been reproduced and a few copies are available from the Section Secretary at a cost of $3.00.

The Saturday contributed papers included:

Embedding graphs in their complements, by Seymour Schuster, Carleton College, Northfield.

Chebyshev polynomials and Pythagorean Triples, by Kenneth Yocum, South Dakota State University, Brookings.

Power-sums and sum-powers, by Victor Feser, Mary College, Bismarck.

Tridiagonal matrices, by Gerald Bergum, South Dakota State University, Brookings.

Some thoughts on modeling and simulation, by Dan Fritze, IBM Corporation, Rochester.

The Schwarz-Christoffel transformation revisited, by William Marian, North Dakota State University, Fargo.

Factoring the groups of units in the ring of integers modulo n, by Joseph Gallian, University of Minnesota, Duluth.

How much rent should you charge? A mathematical model of sharecropping, by Leonard Shapiro, North Dakota State University, Fargo.

Pythagorean Triples a la Bergum, Burgstahler, and Gallian, by Steve Galovich, Carleton College, Northfield.

The next meeting of the North Central Section will be April 21-22, 1978, at the College of St. Thomas, St. Paul, MN.

STEVE GALOVICH, *Secretary-Treasurer*

OCTOBER MEETING OF THE OHIO SECTION

The Ohio Section of MAA held its Fall meeting at Wright State University, Dayton, Ohio, October 28 and 29, 1977. Approximately 130 people were in attendance. Section Chairman W. H. Beyer presided, assisted by Program Chairman J. H. Carney. The themes for the meeting centered on Differential Equations and Combinatorics.

Invited addresses included: *Differential Equations in the Undergraduate Curriculum*, by Richard C. DiPrima, Rensselaer Polytechnic Institute; and *Graph Models in a Combinatorial Problem-Solving Course*, by Alan C. Tucker, SUNY at Stony Brook. Additional program highlights were: *Discussion Session on the Subject of Differential Equations*, moderated by Professors DiPrima and J. Jones, Air Force Institute of Technology; *Swap Session on Metric Conversion* and *Swap Session on Classroom Use of the Hand-Held Calculator.*

The following contributed papers were also presented:

Kinematic Similarities of Systems of Differential Equations, K. Dolan, Air Force Institute of Technology.

Analysis of Coupled Differential Equations Representing Chemically Reacting Flows, H. Kerzner, Baldwin-Wallace College.

Mathieu's Equation-Solution, Stability and Applications, G. Mavrigian, Youngstown State University and P. J. Gingo, University of Akron.

Solving Linear Congruences Efficiently, R. McFarland Wright State University.

Combinatorial Chracterizations of Elliptic Quadrics in PG (5,4), S. E. Payne, Miami University.

Non-diagraphic and Non-graphic Permutation Groups, J. T. Riley, College of Stubenville.

Periodic Solutions of Liénard Systems, L. D. Sabbagh, Bowling Green State University.

A 3-Dimensional Nearly Affine Geometry, A. P. Sprague, Ohio State University.

A Quick Solution of Exact Differential Equations, G. L. Szoke, University of Akron.

Spruce Budworms and Other Applications - Even More New Modular Materials, P. M. Tuchinsky, Ohio Wesleyan University.

The agenda also included a brief Business Meeting of the Section, meeting of the Executive Committee, and meeting of *ad hoc* Committees: Committee on Co-operation among Colleges and Universities, Committee on Curriculum, and Committee on Teacher Training and Certification, and also meeting of mathematics department chairpersons and MAA representatives.

Announcements of future activity: Spring 1978 meeting of the Ohio Section to be held at the University of Akron, Akron, Ohio, April 28-29, 1978; also, Summer 1978 Short Course on *Applications of Control Theory*, to be held in June 1978 at Allegheny College, Meadville, PA, co-sponsored by The Ohio and The Allegheny Sections, MAA.

GUS MAVRIGIAN, *Secretary-Treasurer*

AUGUST MEETING OF PACIFIC NORTHWEST SECTION

The 1977 annual meeting of the Pacific Northwest Section of the MAA was held at the University of Washington, Seattle, Washington, August 14-16, 1977, in conjunction with the 57th Summer Meeting of the MAA. All papers presented were listed in the Association's announcement of the summer meetings.

JOHN O. HERZOG, *Secretary-Treasurer*

OCTOBER MEETING OF THE SEAWAY SECTION

The Fall Meeting of the Seaway Section of the MAA was held at State University of New York College at Plattsburgh, Plattsburgh, N.Y. on October 28 and 29, 1977, with a registered attendance of 55 people, including 49 members of the Association. Professor Paul Schaefer, State University College at Geneseo, Chairman of the Section, presided.

Following a dinner Friday evening at the Holiday Inn in Plattsburgh, Professor Boris Korenblum of State University of New York at Albany, a distinguished Russian mathematician, talked to the members. His topic was "From the U.S.S.R. to the U.S.A. — Two Worlds of Mathematics."

At the Saturday morning session, C. F. A. Beaumont, University of Waterloo, gave a talk, "The Mathematical Sciences: New Packages — New Careers."

Erik Hemminsen, Syracuse University, Seaway Section Governor, talked on "Concerns of the Board of Governors of MAA."

Anton Kotzig, Centre de recherches mathématiques, Université De Montréal, gave a talk entitled "Graph Theory and Combinatorics Applied to the Macroeconomy of Several Countries."

During the business meeting discussion was held as to the feasibility of expanding the activities of the Seaway Section. It was decided that for a three year trial peiod the Section would publish a Newsletter. Donald W. Trasher, State University College of Genesco, was appointed as the Newsletter Editor during this three year trial period. The members voted to establish registration fees for Section meetings at $3, with undergraduate students continuing to be exempt from registration fees. The Section dues were dropped.

During the afternoon sessions, eight contributed papers were presented:

Group Rings and Their Automorphisms, by D. C. Lantz, Colgate University.

Computational Graphics As A Teaching Aid, by David Schawe, SUNY College at Plattsburgh.

Determination of All Four Digit Kaprekar Constants, by G. D. Prichett, Hamilton College.

Generalization of Helmholtz's Harmonic Model, by Norbert Oldani, Mohawk Valley Community College.

Application of Catastrophe Theory, by A. C. Green, SUNY College at Buffalo.

The Impact of the U.S.A. Mathematical Olympiad on the Seaway Section, by Nura Turner, SUNY at Albany.

The Resultant Iteration for Determining the Stability of a Polynomial, by J. L. Howland, University of Ottawa.

Opportunities for Applications of Mathematics to Biolgoy and Medicine, by J. M. Reiner, Albany Medical College.

EMMET STOPHER, *Secretary-Treasurer*

CORRECTION

In the notice, "1977 Contributing Members and Special Gifts," published in this MONTHLY, Volume 84, page 763, the following should have been included:

The Association gratefully acknowledges a contribution of $1,000 from Harry M. Gehman.

MEETING OF THE NEW JERSEY SECTION MATHEMATICAL ASSOCIATION OF AMERICA

The Fall meeting of the New Jersey Section was held at Caldwell College, Caldwell, New Jersey, on November 5, 1977. Featured speakers during the morning session were Ronald Graham, Bell Telephone Laboratories, who spoke about "Irrational Staircases," and Erwin Just, Chairman of the Department of Mathematics, Bronx Community College, New York, who spoke on "Surprising Results of the Cantor Diagonlization Process." A stimulating panel discussion on "The Mathematical World: A Woman's Perspective," completed the afternoon session.

Minutes of the Business Meeting: The Business meeting began at 10:40. Chair H. Kurshan presented the revised by-laws, Adoption of these by-laws was moved, seconded and unanimously passed.

The Nominations Committee, R. Kurshan, M. Kiernan, E. Poiani, presented nominations for the following offices: Vice-Chair for Innovations: Ashby Foote; Vice-Chair for Speakers: Charles Lewis; Vice-Chair for Two-Year Colleges: Hernando Godderz; Secretary: Jean Lane; Treasurer: Susan Marchand. No further nominations were received from the floor. The nominations were moved, closed and the slate unanimously elected. R. Kurshan then assumed the office of Immediate Past Chair and M. Kiernan assumed the office of Chair. It was noted that the term of office for Vice-Chair of High School Contest, Samuel Greitzer, has not yet expired.

M. Kiernan announced the Spring meeting on Saturday, April 29, 1978, at Steinert High School, Trenton, N. J. in conjuction with AMTNJ. Luncheon speaker will be Norman Schaumberger; program will feature undergraduate speakers. Fall meeting will be November 4, 1978, at Saint Peter's College, Englewood Cliffs Campus; tentative program will deal with Math for the Libertal Arts Student.

A motion was made and passed to empower the Executive Committee to raise the meeting registration fee to $1.00 if necessary. S. Greitzer raised the possibility of the section using the interest from the High School contest funds, and will present this proposal at the national MAA meeting in January to obtain permission.

E. Poiani distributed a report on the Governors' meeting during the national Summer Meeting, August, 1977.

JEAN LANE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fifty-eighth Summer Meeting, Brown University, Providence, August 8–10, 1978.

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, University of Pittsburgh, Pennsylvania, April 14–15, 1978.

FLORIDA, St. Petersburg Junior College, Clearwater, March 3–4, 1978.

ILLINOIS, Western Illinois University, Macomb, May 5–6, 1978.

INDIANA, Earlham College, Richmond, April 22, 1978.

INTERMOUNTAIN

IOWA, University of Northern Iowa, Iowa Falls, April 22, 1978.

KANSAS, Wichita State University, Wichita, late March—early April 1978.

KENTUCKY, Northern Kentucky University, Highland Heights, April 7–8, 1978.

LOUISIANA–MISSISSIPPI, Buena Vista Hotel-Motel, Biloxi, Mississippi, February 17–18, 1978.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Queensborough Community College, May 7, 1978.

MICHIGAN, Michigan State University, East Lansing, May 5–6, 1978.

MISSOURI, Central Missouri State University, Warrensburg, April 7–8, 1978.

NEBRASKA, University of Nebraska at Omaha, April 14–15, 1978.

NEW JERSEY, Steinhart High School, Trenton, April 28, 1978.

NORTH CENTRAL, College of St. Thomas, St. Paul, Minnesota, April 21–22, 1978.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, College of Notre Dame, Belmont, California, February 18, 1978.

OHIO, The University of Akron, Akron, April 28–29, 1978.

OKLAHOMA–ARKANSAS, Henderson State University, Arkadelphia, Arkansas, March 31–April 1, 1978.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 16–17, 1978.

PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.

ROCKY MOUNTAIN, South Dakota School of Mines and Technology, Rapid City, April 28–29, 1978.

SEAWAY, Brock University, St. Catharines, Ontario, Canada, May 5–6, 1978.

SOUTHEASTERN, Clemson University, Clemson, South Carolina, March 31–April 1, 1978.

SOUTHERN CALIFORNIA, California State University, Fullerton, March 11, 1978.

SOUTHWESTERN, New Mexico Institute of Mining and Technology, Socorro, Spring 1978.

TEXAS, Stephen F. Austin State University, Nacogdoches, March 31–April 1, 1978.

WISCONSIN, University of Wisconsin, Whitewater, late April 1978.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Washington, February 12–17, 1978.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Stouffer's Greenway Plaza Hotel, Houston, Texas, Fall 1978.

AMERICAN MATHEMATICAL SOCIETY, Brown University, Providence, Rhode Island, August 9–12, 1978.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of British Columbia, Vancouver, June 19–22, 1978.

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC, Madison, Wisconsin, June 18–24, 1978.

ASSOCIATION FOR WOMEN IN MATHEMATICS, Brown University, Providence, Rhode Island, August 8–12, 1978.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, University of Western Ontario, London, Ontario, Canada, June 1–2, 1978.

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, San Diego, California, April 12–15, 1978.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, New York City, May 1–3, 1978 (Joint Meeting with the Institute of Management Sciences).

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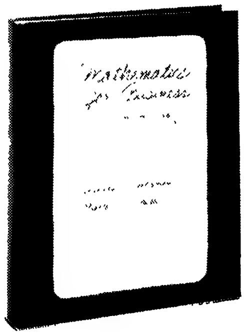

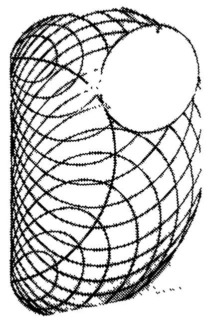

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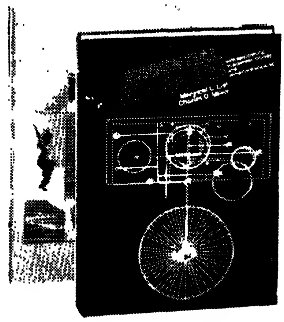

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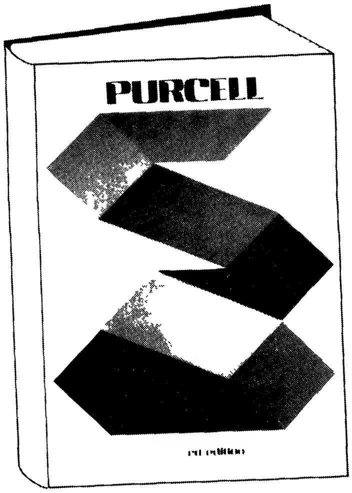

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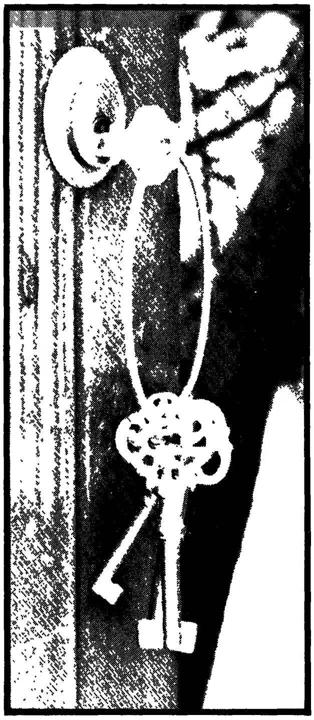

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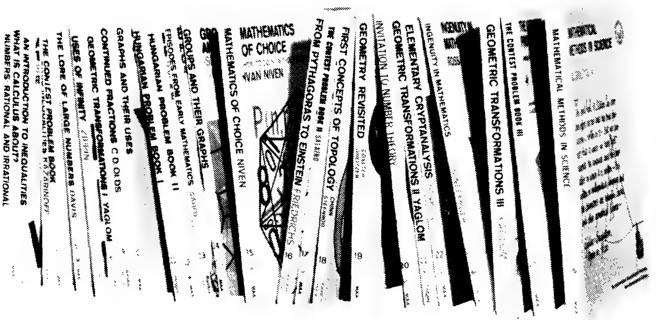

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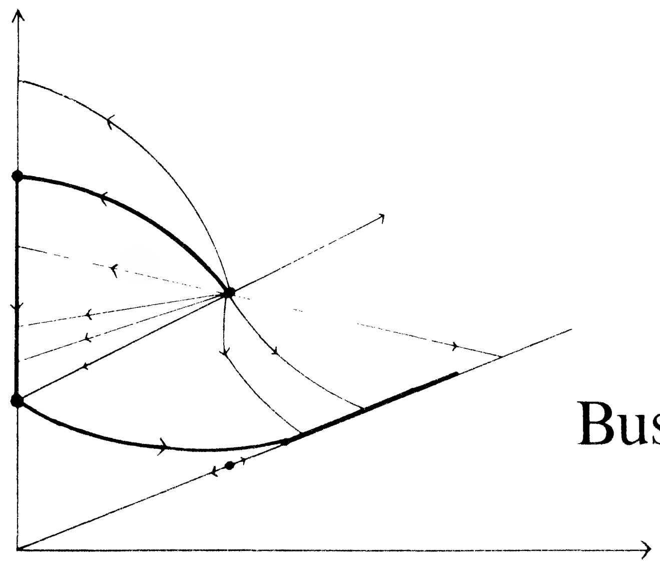

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We think there should be closer ties between the academic members and the nonacademic members of the Association. Most of our 19,000 members are college and university professors. But there are 2000 members—that is over 10%—who are employed by industry or government.

Probably most professors know relatively little of the needs and practices of nonacademic mathematicians. However, many of them do wish to learn more about the kinds of mathematics used in industry, the purposes for which it is used, the conditions under which it is practiced, and interesting research problems that arise there. Much of this information will also be of interest to students in mathematics classes. The nonacademic members have much to offer the profession and the Association. They can provide first-hand information about the nonacademic market, including details such as what mathematical knowledge and attitudes students are expected to bring with them to particular jobs.

Our industrial members, for their part, sometimes feel isolated in their work and would enjoy closer contacts with the colleges. Many are eager to confer with college teachers, make presentations at meetings, write papers for our journals, serve on committees, assist in editorial functions, serve as visiting lecturers, and collaborate with academic members.

Also, the corporations and government agencies where our nonacademic members are employed depend on academic mathematicians to educate and train students who go into government and industry. College and university mathematicians not only prepare mathematics majors at all degree levels, but also a far greater number of students in engineering and physical science and the quantitative branches of life and social sciences, business, finance, management, agriculture, and manufacturing. Therefore, nonacademic mathematicians and their institutions can expect to benefit from communicating information, understanding, and attitudes about nonacademic users of mathematics to their academic colleagues.

We encourage college mathematicians to seek out their colleagues in industry and establish personal and professional ties. The professional life of both groups will be enhanced and enriched. At the same time this Committee is actively pursuing a campaign to recruit Corporate Members among manufacturing and business corporations and government agencies, and assisting others in the MAA who are encouraging more nonacademic mathematicians to take up individual memberships.

The Committee on Corporate Members welcomes requests for information or for advice and assistance in any undertaking that contributes to closer professional relations between practitioners of mathematics outside and within the academic community. We would be glad to receive suggestions about how the Association might better achieve this object.

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A SURVEY: NON-COOPERATIVE GAMES AND A MODEL OF THE BUSINESS CYCLE

F. R. BUIANOUCKAS

A general measure of how well an economy is doing is the standard of living of the participants in terms of their personal income for, say, a particular year compared to some datum year. The following statistics for the years 1973 and 1974 for the United States are of interest. Similar figures for 1975 are preliminary and generally not available.

	1973	1974
Personal Income (billions of dollars)	1,055.0	1,150.5
1958 Prices	723.1	708.0
Disposable Personal Income	903.7	979.7
1958 Prices	619.4	602.9
Annual Percent Change:		
Real (1958 prices) Personal Income	−2.1	
Real (1958 prices) Disposable Personal Income	−2.7	

Evidently, our standard of living has declined even with bigger paychecks. Of course this is not too surprising given the continuing high rates of unemployment and inflation that the U. S. has been experiencing for the past 3 to 5 years.

We wish to address ourselves to the question: How can the general economic well-being of all participants in an economy (developed) be raised? This broad query is one of the most central which the economist can address.

I. Non-Cooperative Games. Central to our attempt to elucidate the significant question which we have raised is John Nash's notion of a non-cooperative game. Such games can be used to model certain very essential behavioural aspects of economies like that of the U.S., in which many economically vital decisions are taken in a decentralized way, and motivated by the individual entrepreneur's or corporation's attempt to maximize their expected profit. In such a game there is no communication or collaboration among players. The interaction between players is purely by observation, i.e., each player, regarding himself as an insignificant part of some overwhelming external situation (e.g., a market) observes the activity of all the other players, and adjusts his own strategy to maximize his expected payoff, given what the others are doing.

In such a model, there is a natural notion of equilibrium, namely a collection of strategies in which no player sees any reason to change his own choice of move. We may think of such an equilibrium as being attained by a series of equilibrating cycles during which players can, and do, change their moves. But the equilibrium that emerges can be studied most simply without explicit reference to the details of this equilibrating process.

The following mathematical definitions, due to Nash [1], formalize the important concepts that we have just sketched.

DEFINITION 1. An *n-person game* is a set of n functions $f_j(k_1, \dots, k_n)$, $j = 1, \dots, n$, of integer-valued variables k_j, the jth integer k_j being restricted to be in an interval $1 \leqq k_j \leqq K_j$. The integer K_j is

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called the *number of moves* open to the jth player; the function f_j is called the *payoff function* to the jth player.

DEFINITION 2. A set of *mixed* or *randomized* strategies for the n-person game is a set of vectors $\mathbf{x}_j$; $j = 1, \ldots, n$ satisfying:

(a) $\mathbf{x}_j = (x_j^{(1)}, \ldots, x_j^{(K_j)})$ is a vector in K_j-dimensional space;

(b) The components of $\mathbf{x}_j$ are non-negative and satisfy the equation

$$\sum_{k_j=1}^{K_j} x_j^{(k_j)} = 1.$$

If all but one of the $x_j^{(k)}$ are zero, the strategy is called *pure*; otherwise it is called *mixed.*

DEFINITION 3. Let $x_1, \ldots, x_n$ be a set of mixed strategies. Then the *mixed strategy payoff function* F_j to the jth player is the function

$$F_j(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{k_1=1}^{K_1} \sum_{k_2=1}^{K_2} \cdots \sum_{k_n=1}^{K_n} x_1^{(k_1)} x_2^{(k_2)} \cdots x_n^{(k_n)} f_j(k_1, \ldots, k_n).$$

Next we define the important concept of an equilibrium point of the n-person game in mixed strategies defined above.

DEFINITION 4. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a set of mixed strategies. Then this set of mixed strategies is called an *equilibrium point* for the n-person game if and only if

$$F_j(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n) \geqq F_j(\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_n) \quad j = 1, \ldots, n$$

for each vector $\mathbf{x}$ of strategies available to player j. The set of all strategies available to player j is called the jth *strategy space.*

It is evident from the definition of an equilibrium point that no *one* player can improve his payoff by changing from equilibrium. If he changes his move alone, he can only suffer for it.

The following theorem is due to Nash.

THEOREM 1. *Every game has an equilibrium point.*

Nash proved this theorem by showing that the points of equilibrium of the n-person game corresponded to the fixed points of a particular continuous transformation T acting on a closed convex subset of a vector space.

DEFINITION 5. Let $\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n$ be an equilibrium point for the n-person game. Then $\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n$ is said to be a *stable equilibrium point* if and only if

$$F_j(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n) > F_j(\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, \ldots, \mathbf{x}_n) \quad j = 1, \ldots, n$$

for each vector $\mathbf{x}$ in the jth strategy space. An equilibrium point which is not stable is called *unstable.*

The surprising scope of the preceding definitions reveals itself if we consider a number of examples.

The simple game of "Majority" furnishes us with a first example and illustrates the notion of an unstable equilibrium point.

This game is played by an odd number $n \geqq 3$ of players. Each player has two moves, "stay" and "go".

The payoff function to each player is as follows: if a player chooses "stay", his payoff is zero, independent of all other players. If a player chooses "go" he receives a positive quantity P if a majority of players also choose "go"; otherwise his payoff is $-Q$, $Q > 0$.

If we denote by p_j the probability assigned by the jth player to "stay", then the strategy spaces S_j are $\{p_j \mid 0 \leqq p_j \leqq 1\}$. There are two stable equilibrium points, namely, $p_1 = p_2 = \cdots = p_n = 0$ and $p_1 = p_2 = \cdots = p_n = 1$.

Further study of the game shows the existence of a third equilibrium point (unstable) $p_1 = p_2 = \cdots = p_n = \beta$ where β is the unique root of the equation

$$\sum_{k \leqq 2n+1} \binom{2n+1}{k} \beta^k (1-\beta)^{2n+1-k} = P/(P+Q).$$

The root β lies between 0 and 1. At this unstable equilibrium point, any player is free to change his strategy without loss. Moreover, once any player chooses to change his strategy, all others must follow. On the other hand, in the case of the two stable equilibria, i.e., when all players are playing "stay" or all players are playing "go", no player can improve his payoff by changing his own strategy. Indeed, to improve his payoff a player must secure at least half the players as his collaborators.

The preceding example, simple as it is, illustrates the very important notion of a *suboptimal equilibrium* which we now proceed to define formally.

DEFINITION 6. An equilibrium point $\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n$ is called *suboptimal* if there exists a mixed strategy $\mathbf{y}_1, \ldots, \mathbf{y}_j, \ldots, \mathbf{y}_n$ such that

$$F_j(\mathbf{y}_1, \ldots, \mathbf{y}_n) > F_j(\mathbf{x}_1, \ldots, \mathbf{x}_n) \quad \text{for all } j = 1, \ldots, n.$$

At a suboptimal equilibrium, each player receives a payoff that is inferior to, and perhaps very much inferior to, the payoff that would be available to him if an appropriate regime of inter-player cooperation or collusion were established. In spite of its seemingly paradoxical character, a suboptimal equilibrium may be quite stable, since each participant is actively, and perhaps even desperately, maximizing his own payoff, i.e., the participant may very well perceive the general situation to be deplorable, but out of his control and to which he contributes only insignificantly. In spite of its simplicity, the "majority game" model, presented above, gives considerable insight into a variety of social situations. Most people will, for example, be familiar with the situation of paralysis that can so easily develop in a group of five or more people attempting to go to lunch together, as the desire of each person to get going struggles with everybody's determination not to be an isolated breakaway from the group. More grimly, this model explains how a large group of concentration camp prisoners can be led to their collective annihilation; each desperately desires a general revolt, but none feels that he can afford to be the first to revolt. The existence of two equilibrium points in this model explains the way in which an initially docile group of prisoners in such a situation can suddenly become transformed into a very dangerous and highly active mob.

It is also worth noting that dictatorships arise out of suboptimal equilibria, since the suboptimal character of the equilibria implies that a dictator can improve the situation of every one of his followers, and even society in general, by forcing the social situation away from its suboptimal state. The attachment (perhaps even passionate) to the charismatic "leader" or dictator by most or even almost all players in such situations reflects the fear of the players that in the absence of a social conscience enforced by a "man of the hour", the condition may again revert to a disastrous suboptimal equilibrium.

Next, consider a two-person game which may be called "self-defense"; this illustrates the fact that a game may have only suboptimal equilibria. In this game each player has strategy space consisting of the points $\{1, 2, \ldots, m\}$.

The payoff function for player 1 is defined as

$$F_1(k_1, k_2) = \begin{cases} m - k_2 - 1, & k_1 < k_2 \\ m - k_2, & k_1 = k_2 \\ m - k_2 + 1, & k_1 > k_2; \end{cases}$$

the payoff function $F_2(k_1, k_2)$ for player 2 is defined symmetrically.

The payoff matrix is

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & \cdots & \cdot & & m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ \\ m \end{array} & \begin{pmatrix} m-1 & m-3 & \cdots & \cdot & 0 & -1 \\ m & m-2 & \cdots & \cdot & 0 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & 1 & -1 \\ m & m-1 & m-2 & \cdots & 2 & 0 \end{pmatrix} \end{array}$$

Obviously, each player is forced to play bigger and bigger strategies in order to avoid the -1 payoff, which is the fate of the player who chooses a smaller integer than his opponent.

The only equilibrium is seen to be (m, m) with payoff 0. Note that if the players had colluded and agreed between themselves to play $(1,1)$, the payoff to each of them would have been $m-1$.

Another example of a game with suboptimal equilibria is a long-term investment market game.

Oftentimes technological or commercial innovations spur long-term investment opportunities that yield superior rates of return. A prospective investor is faced at any given moment with a decision between investing in a long-term security that yields a given rate of return or maintaining liquidity and waiting for a better long-term investment to come along.

We shall model our long-term investment market as follows. We assume that the market operates in fixed periods, say ten years. Suppose there are n investors and suppose, further, that past experience indicates that a typical variety of unit long-term investments will be offered at premia $q_1, q_2, q_3, \ldots$ above short-term yields. Without any loss we may assume $q_1 > q_2 > q_3 > \cdots$. Each investment is awarded in a random way to *one* of the investors who has bid for it. For each investment, an investor assigned to that investment by this random process receives the stated premium; an investor not so assigned receives no premium.

It is clear that our market model defines a game. If we denote by n_k the number of investors bidding on the investment with return q_k, then the expected premium to each of the investors is q_k / n_k. At equilibrium, no investor will accept an unnecessarily low premium without changing his bid. Therefore, at equilibrium, the condition $n_k > 0$ implies that k satisfy the bounds $1 \leqq k \leqq p$, since the constant value of q_k / n_k will be equal to the best premium which any investor could obtain by shifting his bid from an investment for which there are other bidders to an investment for which there are no other bidders. Thus, if we ignore some inessential complications coming from the discreteness of our model, we may set $q_k / n_k = q_p$. The integers n_k and p may be determined by the equations

$$n_k = q_k / q_p \quad \text{and} \quad \sum_{k=1}^{p} q_k = nq_p.$$

Now $q_k > q_p$ for $1 \leqq k \leqq p$ so that $p < n$.

Furthermore, the total premium obtained by all n investors is $\Sigma_{k=1}^{p} q_k$. But the total possible premium which could be obtained is $\Sigma_{k=1}^{n} q_k$. The difference between the maximum possible amount obtainable and the actual amount obtained is

$$\sum_{k=p+1}^{n} q_k$$

which can be non-zero.

Of course, considerations of individual optimization are the cause of the loss; that is, the ranks of bidders for the best premia become overcrowded. Also, investors turn away lower yielding securities waiting for something better to come along. In the context of our market game model, the lost premium is reflected in recession-losses which occur in a real economy through the stabilization of the long-term interest rate.

It is amusing to ponder an often-used expression "there ain't no such thing as a free lunch". In light of the existence of suboptimal equilibria, the axiomatic character of this statement comes into

question. For if we agree that an economy such as that of the United States can be viewed in some sense as a non-cooperative game, we must admit to the existence of a strategy, most likely induced by some external influence, that will increase all participants' payoffs. Obviously, the increased payoff could be reflected in a "free lunch" or perhaps some other benefit(s) which lie within the capacity of the economy. There is much room for the imagination here, given the history and economic growth of the United States in the past forty-five years.

In the next section, we introduce the Schwartz [3] production-inventory model of an economy. This model provides us with an analytical description of the business cycle. Our contention is that the Schwartz model, in conjunction with Nash's notion of a non-cooperative game, can provide important insights to problems of economic analysis.

II. Multi-commodity input-output models of production and inventory. The model shall have many simplifications for the sake of time. However, the salient features of the business cycle remain. Fixed capital shall be ignored for the present. Our economy shall consist of N manufacturers, each manufacturer producing a distinct commodity $C_1, \ldots, C_N$ along with labor commodities $C_0, \ldots, C_{-L}$. The economy shall work on a day to day basis. Each morning each manufacturer takes inventory of his present stock, after which he schedules the day's production to bring the inventory to some optimal level. The optimum inventory level is defined by the equation: Optimum Inventory = Constant × Expected Sales + Basic Inventory.

The expected sales for today may be yesterday's actual sales. Now that the optimum inventory level is determined, each manufacturer must order the necessary amounts of $C_1, \ldots, C_N$ to meet today's production schedule.

The model refers basically to the production of material commodities and to the inventory of material commodities.

We shall absorb wage-payment into the formulation of the model, hence eliminating the occurrence of the various labor sectors explicitly. This elimination does *not* represent a major distortion of the effect of wage payments on sales and on inventory accumulation, for a large part of what is usually considered "personal consumption" will be hidden in the model by the transformation made. It is agreed, however, that this is a gross social distortion.

Let π_{ij} be the amount of C_j required to produce one unit of C_i, π_{0j} represents the real wage bill, i and j vary from $-L$ to N. Let $a_{-L}, \ldots, a_0, a_2, \ldots, a_N$ be the total production of all commodities $C_{-L}, \ldots, C_0, C_1, \ldots, C_N$ for a given period of time. Then the amount of C_j produced net of consumption is

$$a_j - \sum_{i=-L}^{N} a_i \pi_{ij}, \quad j = -L, \ldots, N. \tag{1}$$

As labor cannot be held in inventory, but is consumed during production, we have

$$a_j - \sum_{i=-L}^{0} a_i \pi_{ij} = \sum_{i=1}^{N} a_i \pi_{ij}, \quad j = 0, \ldots, -L.$$

Denote by $(I - \ddot{\Pi})_{ij}^{-1}$ the entries of the $(L+1) \times (L+1)$ matrix whose elements are $\delta_{ij} - \pi_{ij}$. Now we can solve for the amounts a_j of $C_0, \ldots, C_{-L}$; i.e.,

$$a_j = \sum_{i=1}^{N} \sum_{i=-L}^{0} a_i \pi_{ik} (I - \ddot{\Pi})_{ij}^{-1}. \tag{2}$$

Substituting for a_i, $i = 0, \ldots, -L$ in (1), we get

$$a_j - \sum_{i=1}^{N} a_i \tilde{\pi}_{ij}, \quad j = 1, \ldots, N$$

with

$$\tilde{\pi}_{ij} = \pi_{ij} + \sum_{k,l=-L}^{0} \pi_{ik} (I - \ddot{\Pi})^{-1}_{kl} \pi_{lj}, \quad i, j = 1, \ldots, N. \tag{3}$$

The labor sector now does not appear explicitly and all production takes place in a closed economy with input-output matrix $\tilde{\pi}_{ij}$ (this is a Leontief scheme).

Continuing, each producer is expected to supply a certain amount of his commodity; usually he will be able to supply just a fraction of the amount ordered. We shall assume that if total demand cannot be supplied, the same fraction of each order received is to be supplied. Each manufacturer determines what fraction of his various orders can be satisfied, selects the smallest fraction, and cancels orders for commodities in amounts larger than that fraction of his original order. Shipment is instantaneous and a new day begins.

Mathematically, let c_j be the number of days' sales regarded as the optimum inventory for commodity C_j. Let $a_j(t)$ be the amount of C_j produced on the tth day. Let $\tilde{\pi}_{ij}$ be as previously defined.

The sales of C_j on the $(t-1)$st day is $\Sigma_{i=1}^{N} \tilde{\pi}_{ij} a_i(t-1)$. Let $b_j(t)$ be the actual inventory on the morning of the tth day. Then

$$b_j(t) = b_j(t-1) + a_j(t-1) - \sum_{i=1}^{N} \tilde{\pi}_{ij} a_j(t-1). \tag{4}$$

The desired production for the tth day is

$$\begin{aligned} d_j(t) = \Bigg\{ & c_j \sum_{i=1}^{N} \tilde{\pi}_{ij} a_i(t-1) + \sum_{i=1}^{N} \tilde{\pi}_{ij} a_i(t-1) - b_j(t-1) \\ & - a_j(t-1) + \sum_{i=1}^{N} \tilde{\pi}_{ij} a_i(t-1) \Bigg\}^{+}, \quad (\{x\}^{+} = 0 \text{ if } x < 0). \end{aligned} \tag{5}$$

We have made an allowance for expected sales. This can be written as

$$d_j(t) = \left\{ (c_j + 2) \sum_{i=1}^{N} \tilde{\pi}_{ij} a_i(t-1) - a_j(t-1) - b_j(t-1) \right\}^{+}. \tag{6}$$

The producer of C_j must order commodities C_k in the amounts $\tilde{\pi}_{ik} d_i(t)$; hence, the producer C_k receives a total order in the amount $\Sigma_{i=1}^{N} \tilde{\pi}_{ik} d_i(t)$ units of C_k.

The fraction $u_k(t)$ of total orders received which can be filled by the manufacturer of C_k is

$$u_k(t) = b_k(t) \Big/ \sum_{i=1}^{N} \tilde{\pi}_{ik} d_i(t) \quad \text{(the } k\text{th } \textit{market strain coefficient}\text{)}. \tag{7}$$

We assume that the manufacturer cancels all orders in excess of this smallest fraction of his initial orders. This is expressed mathematically as

(8) $g_j(t) = \min_k (1, u_k(t))$, $k \in K_j$, where K_j is the set of indices for which $\tilde{\pi}_{ik} > 0$.

$g_j(t)$ is the supply strain factor for the jth producer.

The jth producer's actual production for the tth day is

$$a_j(t) = d_j(t) g_j(t). \tag{9}$$

Equations (4)–(9) determine $a_j(t)$ and $b_j(t)$ in terms of $a_j(t-1)$ and $b_j(t-1)$.

Thus, today's production and inventory levels are given recursively by yesterday's production and inventory levels. Obviously, the specification of an initial state determines the whole subsequent motion of the economy.

III. The aggregated model. Equations (4)–(9) are called the disaggregated equations for an economy. We may obtain the aggregative equations by making the assumption that the economy produces a single "commodity". Let γ be the amount of the "commodity" required to produce one unit of "commodity". Clearly, $0 < \gamma < 1$, for if $\gamma \geqq 1$, we would consume everything and wind up in a perpetual depression; $\gamma \neq 0$ is obvious. If $a(t-1)$ is the total production on the $(t-1)$st day, then the amount consumed in production is $\gamma a(t-1)$ and the desired production for the tth day (ignoring basic inventory) is $c\gamma a(t-1)$. The recursions are

$$b(t) = b(t-1) + \varepsilon a(t-1) \tag{10}$$

$$a(t) = \bar{\gamma} a(t-1) - b(t-1), \text{ with } \varepsilon = 1-\gamma \text{ and } \bar{\gamma} = ((c+2)\gamma - 1). \tag{11}$$

If e denotes personal consumption on the $(t-1)$st day (we assume this is constant each day) and h is basic inventory, (10) and (11) become

$$b(t) = b(t-1) + \varepsilon a(t-1) - e, \tag{12}$$

$$a(t) = \bar{\gamma} a(t-1) - b(t-1) + h. \tag{13}$$

The transformation (12)–(13) has a fixed point (the Keynes point).

$$a_k = e/\varepsilon, \quad b_k = (\bar{\gamma} - 1)(e/\varepsilon) + h. \tag{14}$$

The point (a_k, b_k) is by definition the level of production at which production balances consumption and the level of inventory at which actual inventory balances desired inventory. If we set $\tilde{a}(t) = a(t) - a_k$ and $\tilde{b}(t) = b(t) - b_k$ (the deviations of actual production and inventory), equations (12) and (13) become

$$\tilde{b}(t) = \tilde{b}(t-1) + \varepsilon \tilde{a}(t-1), \tag{15}$$

$$\tilde{a}(t) = \bar{\gamma} \tilde{a}(t-1) - \tilde{b}(t-1), \tag{16}$$

which are equivalent to (10) and (11). We shall, therefore, drop the symbol $\sim$.

It can be shown that the transformation (10)–(11) acts on the crosshatched region which we call the *accessible region.* The line $b = \gamma a$ is called the *scarcity line.*

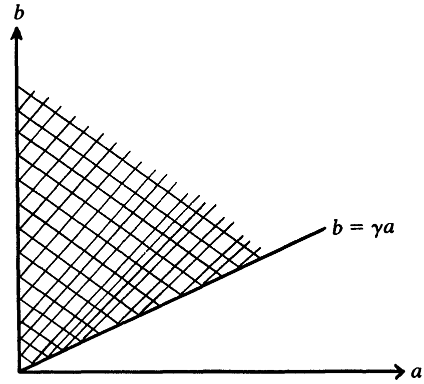

FIG. 1.
The crosshatched portion is called the accessible region. The line $b = \gamma a$ is the scarcity line.

The characteristic equation of the transformation (10)–(11) is

$$\lambda^2 - (\gamma + 1)\lambda + (\bar{\gamma} + \varepsilon) = 0.$$

The two solutions are approximately

$$\lambda_1 = \bar{\gamma} - \varepsilon/(\bar{\gamma} - 1) \quad \text{which is slightly less than 2,}$$

$$\lambda_2 = 1 + \varepsilon/(\bar{\gamma} - 1) \quad \text{which is slightly greater than 1.}$$

The corresponding eigenvectors W_i, $i = 1, 2$ satisfy the equations

$$\lambda_i a_i = \bar{\gamma} a_i - b_i, \quad \lambda_i b_i = b_i + \varepsilon a_i.$$

If (10)–(11) is diagonalized and the action of the diagonalized transformation is superimposed in the proper way onto the accessible region, the following two situations arise:

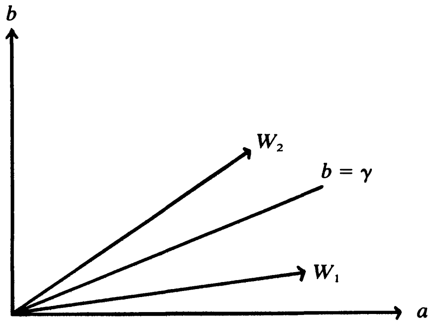

FIG. 2.
Expansive Case $(c + 1)\gamma > 1 + \lambda_2$.

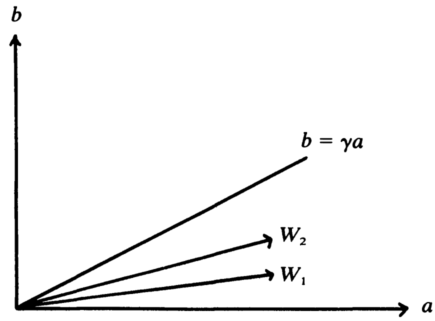

FIG. 3.
Depressive Case $(c + 1)\gamma < 1 + \lambda_2$.

The motions of the two cases are shown in Figure 4 and 5; although the orbits are discrete, they are drawn as though they were continuous. It can be seen that the cycle is one of over-production, not of disproportion.

We now return to the disaggregated model (which now will contain personal consumption explicitly).

The relation analogous to $a_k = e/\varepsilon$ is the Keynes theorem: If $\tilde{\Pi}$ is the input-output Leontief matrix with dominant eigenvalue < 1, $\bar{e}$ the consumption vector of commodities C_j, $\bar{f}$ the investment vector of commodities C_j, and $\bar{a}$ the vector of long-time average production, then

$$(I - \tilde{\Pi})\bar{a} = \bar{e} + \bar{f} \quad \text{(at equilibrium)}. \tag{17}$$

Equation (17) indicates that production adjusts to consumption; classicists argue that consumption adjusts to production.

If we solve for $\bar{a}$ in (17) we get

$$\bar{a} = (I - \tilde{\Pi})^{-1}(\bar{e} + \bar{f}). \tag{18}$$

It is obvious from (18) that if we wish to increase production over the long run (i.e., eliminate unemployment) we may increase $\bar{e}$, $\bar{f}$, or $(I - \tilde{\Pi})^{-1}$.

To increase $\bar{e}$ we can spend more on unemployment insurance, spend more on military programs, spend more on civil programs (hire more transport workers, firemen, etc.).

To increase $\bar{f}$, money could be placed in public works programs, quick tax write-offs, as well as loss guarantees for investors.

The most interesting possibility, and a major bone of contention between Keynesians and the classicists, is the following: since the dominant eigenvalue of $\tilde{\Pi}$ is less than 1, we have

$$(I - \tilde{\Pi})^{-1} = I + \tilde{\Pi} + \tilde{\Pi}^2 + \cdots$$

so that $(I - \tilde{\Pi})^{-1}$ may be increased by increasing the elements of $\tilde{\Pi}$. This can be done by increasing the real wages of the participants in one way or another.

So far, we have mentioned three points of disagreement with the classical and Neo-classical economists, namely:

(i) The business cycle is a cycle of overproduction rather than a cycle of correction of disproportions caused by the previous boom; moreover, it is decidedly real (production-inventory), not monetary.

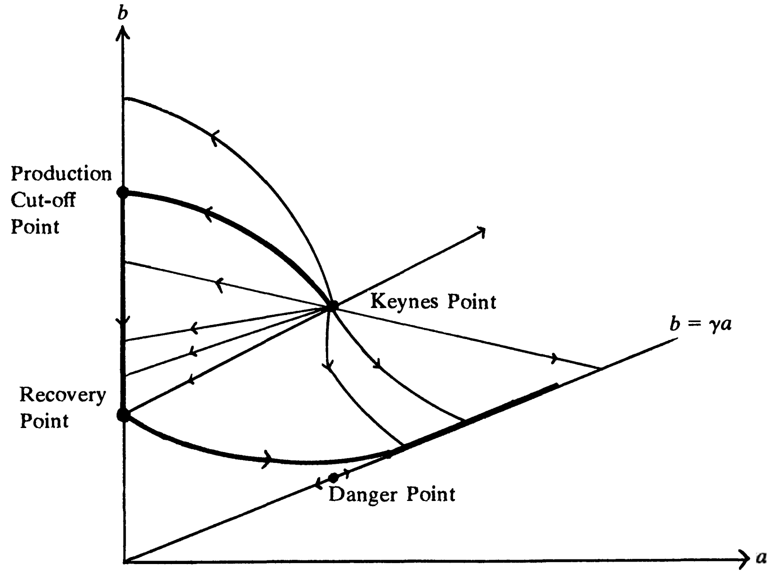

FIG. 4.
Expansive Case

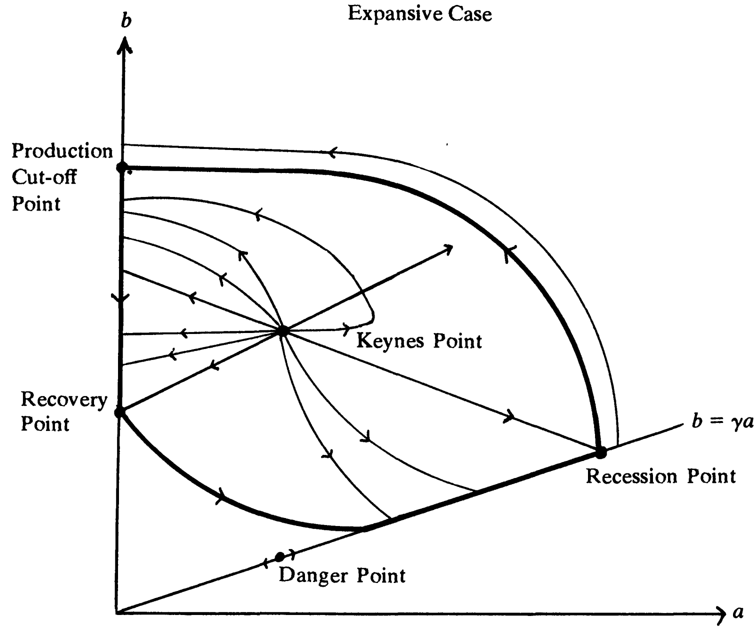

FIG. 5.
Depressive Case

(ii) Production adjusts to consumption rather than consumption adjusting to production. We quote "Business Week", August 2, 1976, page 8: "Data released today show that business inventories increased at a rapid 8.4% annual rate in May. Goods that business puts on the shelf boost production in the period in which they are produced, but when inventories pile up, it is an indication that sales farther along the line may be falling below expectation. According to John W. Kendrick, the newly appointed chief economist at the Commerce Department, the May inventory figure was 'further evidence that the rate of economic growth slowed during the second quarter'."

(iii) The classical economists argue that by decreasing wages more labor will be hired. Actual statistics indicate that production tends to use definite amounts of labor, regardless of the wages paid.

Although the (Schwartz) model indicates that real wages should be raised rather than lowered, the model assumes fixed production schemes so that comparison is not quite fair. The two theories, however, can be put on common ground and with the aid of statistics, the dispute can be settled in principle.

We now return to the concept of the suboptimal equilibrium as it relates to the (Schwartz) model. There are two motions associated with the economic model:

a) the cycle about the Keynes point and

b) the perturbation of the Keynes point in the accessible region.

It is the perturbation of the Keynes point upward in the accessible region that corresponds to raising the standard of living. This perturbation is effected by a change in strategy determined by an external influence.

Some possible suggested changes in strategy are as follows:

(i) a rapid increase in the real wage rate, or

(ii) substantial government deficits for long periods of time, or

(iii) a substantial tax increase in the upper income level, coupled with a reduction in taxes in the lower income bracket, or

(iv) a substantial increase in the volume of investment for long periods of time, or

(v) division of the large estates of the very wealthy to raise the secular autonomous personal spending level.

We feel that the theory of non-cooperative games and the phenomena of a suboptimal equilibrium will be the crux of theoretical economics. This theory will be enhanced by the theory of von Neumann and Morgenstern, i.e., theoretical economics will consist of equilibrium point analysis supplemented by coalition analysis.

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THE RADON-NIKODÝM THEOREM AS A THEOREM IN PROBABILITY

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Introduction. My purpose here is to describe the Radon–Nikodým Theorem's role as a fundamental building block in the modern theory of probability; a role assigned to it by Kolmogorov in his historic 1933 book [2].

The theorem, as used by Kolmogorov, asserts that what probabilists call *conditional expectations* exist in great generality. The implications of this for Probability Theory were enormous as we now know.

For the student of the subject, however, it is always a bit of a problem to forge an adequate link between conditional expectations in simple cases (i.e., those where they are defined constructively) and in more general cases where the definition is purely descriptive (conditional expectation = Radon–Nikodým derivative). Both before and since Kolmogorov, the desire has always been to "see" all conditional expectations as limits of simple ones. In fact they are, but the point has been obscured, partly because every textbook author I know of, beginning with Kolmogorov himself, has merely quoted the Radon–Nikodým Theorem rather than actually proving it *as a theorem in probability*. This is too bad because the proof itself *does* get the Radon–Nikodým derivative as a limit of constructively defined conditional expectations. Another obstacle to correct understanding has been what I consider to be some misleading "bad propaganda" about the validity of a certain appealing limiting operation (which I shall describe later), despite Kolmogorov's explicit demonstration that it "works."

My mission here—and my only one—is to try to illuminate this link between the "two kinds" of conditional expectations. To do so requires that I first describe the Radon–Nikodým Theorem both in general and as a theorem in probability; second, discuss simple conditional expectations at some length (with apologies in advance for putting you through such simple-minded stuff); third, motivate, by example, the need for more general conditional expectations; fourth, give a complete "probabilistic" proof of the Radon–Nikodým theorem; and finally, apply the theorem to validate some heuristics and exorcize a persistent paradox.

This is what I *will* do. What I *won't* do is to discuss any of the *other* applications of the Radon–Nikodým Theorem to Probability and Statistics—such as its use in connection with the idea of sufficiency. And I won't even *begin* to talk about its role *outside* of Probability and Statistics.

1. The Radon–Nikodým Theorem. Between Radon's paper in 1913 [4] and Nikodým's in 1930 [3], there was the work of Fréchet [1] generalizing Lebesgue's theories of measure and integration to abstract spaces; so, as you might expect, Nikodým does for Radon what Fréchet did for Lebesgue. His version in 1930 is just what we use today, except for an easy extension to infinite signed measures.

So we begin with an arbitrary σ-finite measure space; call it $(\Omega, \mathcal{G}, \mu)$ where Ω is any set, $\mathcal{G}$ a σ-field of subsets of Ω, and μ a σ-finite measure on $\mathcal{G}$.

Let f be a $\mathcal{G}$-measurable, extended real-valued function on Ω such that

$$\int_{\Omega} f d\mu$$

exists (possibly infinite). Then the indefinite integral,

$$\phi(G) = \int_G f d\mu$$

is a signed measure (i.e., countably additive) on $\mathcal{G}$ and is absolutely continuous with respect to μ (i.e., $\mu(G) = 0$ implies $\phi(G) = 0$. Nikodým used a different definition of absolute continuity which is equivalent to the currently-used one when ϕ is finite.)

Clearly f is integrable if and only if ϕ is finite; and f is finite almost everywhere with respect to μ (a.e. (μ)) if and only if ϕ is σ-finite. Also f is essentially unique in the sense that if, for all $G \in \mathcal{G}$,

$$\phi(G) = \int_G g d\mu,$$

then $f = g$ a.e. (μ).

The Radon–Nikodým Theorem is simply the statement that *all* signed measures, ϕ, which are absolutely continuous with respect to μ, are obtained in this way, as an indefinite integral of some measurable f.

Now here is the particular situation in probability. We have an underlying probability space $(\Omega, \mathcal{F}, P)$ which is just any measure space in which the measure of Ω is one. We like to think of Ω as

the set of all the possible outcomes of some experiment and we like to call the elements of $\mathcal{F}$ the "Events."

Let $\mathcal{G}$ be some non-empty sub-σ-field of $\mathcal{F}$. Intuitively, we are going to "observe" $\mathcal{G}$. That is, for each $G \in \mathcal{G}$ we will know whether or not G has "occurred." (G "occurs" means simply that ω, the outcome of the experiment, belongs to G.)

Also, let Y be an $\mathcal{F}$-measurable extended real-valued function on Ω, whose integral exists, possibly infinite. (I shall call Y a Random Variable, though "Probability Language" usually restricts the term to Y's which are finite almost everywhere.)

From now on we deal primarily with the space $(\Omega, \mathcal{G}, P_{\mathcal{G}})$ where $P_{\mathcal{G}}$ is the restriction of P to $\mathcal{G}$. And the only signed measure ϕ we shall consider is the indefinite integral of Y, restricted to $\mathcal{G}$:

$$\phi(G) = \int_G YdP.$$

(Notice that ϕ may fail to be σ-finite, even if Y is finite.)

Of course the Radon–Nikodým theorem applies and the $\mathcal{G}$-measurable function, f, satisfying, for all $G \in \mathcal{G}$,

$$\int_G fdP_{\mathcal{G}} = \phi(G) = \int_G YdP$$

is what we call "the conditional expectation of Y given $\mathcal{G}$," and denote by

$$f \equiv E(Y \mid \mathcal{G}).$$

The way I shall prove the theorem later is by showing that there is an increasing sequence, $\{\mathcal{G}_n\}$, of sub-σ-fields of $\mathcal{G}$, with each $\mathcal{G}_n$ generated by a countable partition of Ω, and a corresponding sequence, $\{f_n\}$, of $\mathcal{G}_n$-measurable solutions to

$$\int_G f_n dP_{\mathcal{G}_n} = \phi(G) \qquad \forall G \in \mathcal{G}_n$$

(these f_n's being, as we shall see, constructively defined conditional expectations) which have an a.e. ($P_{\mathcal{G}}$) limit; call it f. I will show further that $\forall G \in \mathcal{G}$, as $n \to \infty$

$$\int_G f_n dP_{\mathcal{G}} \to \int_G fdP_{\mathcal{G}} \text{ and } \int_G f_n dP_{\mathcal{G}} \to \int_G YdP,$$

which makes f a version of $E(Y \mid \mathcal{G})$, to complete the proof.

2. Discrete Conditional Probabilities and Conditional Expectations. The most primitive notion of conditional probability is the definition of "the conditional probability of event B, given that event A occurs":

$$P(B \mid A) \equiv P(AB)/P(A),$$

provided, of course, that $P(A) > 0$. One motivation for this definition comes from looking at relative frequencies: If I repeat an experiment n times, letting $n(A)$ denote the number of times event A occurs, then the proportion of occurrences of B *among those trials in which A occurs* is, of course,

$$\frac{n(AB)}{n(A)} = \frac{n(AB)/n}{n(A)/n}.$$

Now the most common conception of $P(AB)$ and $P(A)$ is as "predictors" of the "limits" of $n(AB)/n$ and $n(A)/n$, respectively, as $n \to \infty$. Hence, to be consistent, the predictor of the "limit" of the conditional relative frequency must be as we have defined $P(B \mid A)$.

Conditional probabilities can be thought of not merely as numbers but as variables, in at least two ways. First, fix A, and let B vary over $\mathcal{F}$; $P(\cdot\,|\,A)$ is clearly a probability measure on $\mathcal{F}$. Intuitively, if you tell me that A has occurred, I must "update" my predictors of all events B from $P(B)$ to $P(B\,|\,A)$. Second, fix B, and define the random variable whose values are $P(B\,|\,A)$ if $\omega \in A$ and $P(B\,|\,A^c)$ if not. Intuitively, you are *going* to tell me whether or not A has occurred so the as yet unknown value of the "update" of my predictor of $P(B)$ depends on the as yet unknown outcome of the experiment; i.e., it is the random variable I just defined.

This second notion has an immediate extension. Let $\mathcal{G}$ be any countable partition of Ω whose members $\{G_1, G_2, \ldots\}$ are events. (I shall also let $\mathcal{G}$ denote the σ-field generated by the partition.) You are going to tell me which G_i has occurred, in which case I will of course update $P(B)$ to $P(B\,|\,G_i)$. So as I await your word my "conditional probability of B given $\mathcal{G}$" is the random variable

$$P(B\,|\,\mathcal{G}) \equiv \sum_i P(B\,|\,G_i) I_{G_i}$$

where $I_G(\omega)$ is one if $\omega \in G$, zero if not.

Turning things around we can see that the, shall we say, "unconditional" probability, $P(B)$, is a weighted average of the conditional probabilities $\{P(B\,|\,G_i)\}$;

$$P(B) = \sum_i P(BG_i) = \sum_i P(B\,|\,G_i)P(G_i);$$

this formula is called the Law of Total Probability and it has the extension

$$P(BG) = \sum_{\{i:G_i \subset G\}} P(B\,|\,G_i)P(G_i) = \int_G P(B\,|\,\mathcal{G})\, dP_{\mathcal{G}}$$

for any $G \in \mathcal{G}$—since necessarily G must then be some countable disjoint union of G_i's. Thus the constructively-defined $P(B\,|\,\mathcal{G})$ is the a.e. ($P_{\mathcal{G}}$) unique Radon–Nikodým derivative, $d\phi/dP$, where $\phi(G) = P(BG)$.

Now let us do for conditional expectation what we just did for conditional probability. To begin with, the reason we call

$$\int_\Omega YdP$$

the "Expectation of Y" (denoted by EY) is because we use it to predict the limiting average gain (or loss if negative) per trial in repeated identical experiments in each of which the payoff for outcome ω is $Y(\omega)$. (Properly formulated this is a celebrated limit theorem: the Law of Large Numbers.) So exactly the same argument, based on relative frequencies which I used before, motivates the definition

$$E(Y\,|\,A) = \int_A YdP/P(A);$$

and we are led to the analogous "extended" Law of Total Expectation:

$$E(YI_G) = \sum_{\{i:G_i \subset G\}} E(Y\,|\,G_i)P(G_i) = \int_G E(Y\,|\,\mathcal{G})dP_{\mathcal{G}},$$

where $E(Y\,|\,\mathcal{G})$ is the random variable

$$\sum_i E(Y\,|\,G_i)I_{G_i}.$$

Of course conditional probability is now merely the special case of conditional expectation in which $Y = I_B$ for some event B.

3. Some examples. The first example (Best Choice Problem) will show, in a simple discrete setting, how we typically plug *known* conditional probabilities into the Law of Total Probability to derive a probability measure.

The second example (Diagnosis Problem), is designed to show how conditional probabilities can be an end-in-themselves, as well as a means to an end. In the discrete case, they are well defined (by the so-called Bayes Formula). But they are equally desired in the uncountable case as well. Hence extending the definition is mandatory.

The third example ($X - Y$ Problem) will be an uncountable space problem with an absolutely continuous probability measure which strongly suggests more than one intuitively appealing method of defining conditional probability. The validity of these methods in more general cases, however, can be appreciated only in the light of the Radon–Nikodým Theorem (Section 5).

Meanwhile, the unbridled use of intuition can lead to paradoxes as the last two examples (Sphere and Plane Problems) will show.

A. *Best Choice Problem.* I take a well-shuffled deck of n distinguishable cards (e.g. $n = 52$) which I have arbitrarily ranked (Rank $1 =$ best, to Rank $n =$ worst). As I deal the cards one by one, I will tell you only one thing: whether or not the card just dealt is the *best so far.* Hearing this, you must then decide immediately whether or not to accept the card. You can accept at most one card and you win a prize if and only if you take the *best of all n* cards.

For large n, it may appear that your chances of winning the prize are very slim, but this is not so as the following solution will show.

Choose an integer k between 1 and n and use the strategy: Let the first k cards go by, no matter what they are; afterward take the first card (if any) which is best so far. Let B be the event: "you win the prize," and, for $i = 1, 2, \ldots, n$, let G_i be the event that the best of all n cards is ith from the top of the deck. Notice that on G_i, for $i > k$, you win if and only if you don't stop too soon; that is, if and only if the best of the first $i - 1$ cards is in fact among the first k. It is then easy to see that

$$P_k(B) = \sum_i P(B \mid G_i)P(G_i) = \sum_{k+1 \leqq i \leqq n} P(B \mid G_i)P(G_i) = \sum_{k+1 \leqq i \leqq n} [k/(i-1)]\,(1/n).$$

For $k = [n/2]$, this probability is at least 1/4 no matter how large n is! The optimal $k = k(n)$ satisfies:

$$\sum_{k<i<n} i^{-1} \leqq 1 < \sum_{k \leqq i < n} i^{-1}$$

from which we conclude that

$$k(n)/n \to e^{-1}, \quad P_{k(n)}(B) \to e^{-1} \approx .36.$$

Thus conditional probabilities, as "givens" are instrumental in solving the problem.

B. *Diagnosis Problem.* Suppose a person can have one and only one of the diseases in the set $\Delta = \{\delta_0, \delta_1, \delta_2, \ldots\}$. ($\delta_0$ stands for "perfectly healthy".) Let each $\sigma \in \Sigma$ be a symptom (or collection of symptoms). A randomly chosen person will have disease δ_i with probability $Q(\delta_i)$ equal to the frequency of the disease in the population. Conditional probabilities, $R(S \mid \delta_i)$, $S \in \mathcal{S}$, where $\mathcal{S}$ is a σ-field of subsets of Σ, would (let us say) be known from clinical studies. What the doctor observes is some symptom σ, from which he tries to diagnose the patient's true condition; i.e., he tries to evaluate $\{P(\delta_i \mid \sigma)\}$.

If Σ is countable these "diagnostic" conditional probabilities are well defined:

$$P(\delta \mid \sigma) = P(\delta\sigma)/P(\sigma),$$

where

$$P(\delta\sigma) = Q(\delta)R(\sigma \mid \delta), \quad P(\sigma) = \sum_i Q(\delta_i)R(\sigma \mid \delta_i)$$

(this is known as Bayes' Formula).

If Σ is not countable, we still put the appropriate probability measure on the space $(\Omega, \mathscr{F}) = (\Delta \times \Sigma, 2^{\Delta} \times \mathscr{S})$, namely

$$P(\delta \times S) = Q(\delta)R(S \mid \delta),$$

but of course $P(\delta \mid \sigma)$ is undefined if $P(\Delta \times \{\sigma\}) = 0$, as may well be the case for *every* σ. What to do?

C. $(X - Y)$ *Problem.* First choose a number Y at random in $(0,1)$—i.e., $P(Y \leqq y) = y$ if $0 < y < 1$—then choose a number X at random in $(Y, 1)$. Only X is to be observed. Given its value, what is the (conditional) distribution of Y?

In this case, although $P(Y \in B \mid X = x)$ is—as an ordinary conditional probability—undefined since $P(X = x) = 0$ for every x, nevertheless we know exactly what to do. After we get the probability measure for the whole experiment (otherwise known as "the joint distribution of X and Y"):

$$P(X \in A, Y \in B) = \int\int_{A \times B} f(x, y)dxdy$$

with the density

$$\begin{aligned} f(x, y) &= 1/(1 - y) \qquad 0 < y < x < 1 \\ &= 0 \qquad \text{otherwise;} \end{aligned}$$

we then "know" that for all $x \in (0, 1)$

$$P(Y \in B \mid X = x) = \int_B g(y \mid x)dy$$

with the density

$$\begin{aligned} g(y \mid x) &= f(x, y) / \int_0^1 f(x, z)dz \\ &= -1/(1 - y)\ln(1 - x) \qquad 0 < y < x < 1, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

This formula for this "conditional density" is the direct analog of the one for elementary conditional probabilities; but this is not the only reason why we call it the correct answer. For one thing it satisfies the Law of Total Probability:

$$\begin{aligned} &\int_A P(Y \in B \mid X = x) P(X \in dx) \\ &= \int_A \left[\int_B g(y \mid x)dy\right]\left[\int_0^1 f(x, z)dz\right]dx \\ &= \int\int_{A \times B} f(x, y)dxdy \\ &= P(X \in A, Y \in B). \end{aligned}$$

For another, any appropriate limiting operation on elementary conditional probabilities yields it. For example

(1) $$\lim_{h \to 0} P(Y \in B \mid x \leqq X < x + h),$$

or

$$\lim_{n\to\infty} P(Y \in B \mid X_n = x_n), \tag{2}$$

where, say,

$$X_n = \sum_k (k/n) I_{\{k/n \leqq X < (k+1)/n\}}$$

$$x_n = k/n \quad \text{if} \quad k/n \leqq x < (k+1)/n.$$

Of course our particular choices of $P(Y \in B \mid X = x)$, as functions of x for fixed B, are only single members of equivalence classes satisfying the Law of Total Probability. They are the "smoothest versions".

In this example, we have been looking at a pair of random variables whose joint distribution is absolutely continuous with respect to Lebesgue measure. What happens for arbitrary X and Y will be the subject of Section 5.

D. *A Sphere Problem and a Plane Problem.* Here are two perfectly innocent-sounding problems with paradoxical overtones:

SPHERE PROBLEM: Suppose a point is chosen at random on the surface of the earth (which is assumed to be a perfect sphere). Find (a) The conditional distribution of the longitude of the point, given that it lies on the equator, and (b) the conditional distribution of the latitude of the point given that it lies on the great circle through the Greenwich meridian. (Kolmogorov [2] presented a version of this problem under the heading: "Explanation of a Borel paradox.")

PLANE PROBLEM: Let X and Y be chosen independently, each according to the standard normal distribution:

$$P(B) = \int_B (2\pi)^{-1/2} \exp(-t^2/2)dt.$$

(i.e., put the product measure $P \times P$ on the plane.) Find the conditional distribution of $R = (X^2 + Y^2)^{1/2}$, the distance of the point (X, Y) from the origin, given that $X = Y$.

We shall postpone our discussion of these two "problems" until after giving the proof of the Radon–Nikodým Theorem.

4. Probabilistic Proof of the Radon–Nikodým Theorem. First let us recall the setup. There is the underlying probability space $(\Omega, \mathcal{F}, P)$, a sub-σ-field $\mathcal{G}$, of $\mathcal{F}$, and an $\mathcal{F}$-measurable function Y, from Ω to $[-\infty, \infty]$ whose integral *exists.* From now on we look at the probability sub-space $(\Omega, \mathcal{G}, P_{\mathcal{G}})$, where $P_{\mathcal{G}}$ is P restricted to $\mathcal{G}$, and at the particular signed measure ϕ, defined on $\mathcal{G}$ by

$$\phi(G) = \int_G Y dP.$$

(The fact that ϕ is absolutely continuous with respect to $P_{\mathcal{G}}$ will be used *implicitly* in the proof.)

Here is the theorem in the form we want it:

THEOREM. *There is an increasing sequence $\{\mathcal{G}_n : n = 1, 2, \ldots\}$ of sub-σ-fields of $\mathcal{G}$, each generated by a countable partition, $\{G_{nk} : k = 0, 1, 2, \ldots, \infty\}$, of Ω such that the conditional expectations $E(Y \mid \mathcal{G}_n)$, have a (pointwise) limit f, satisfying, for every $G \in \mathcal{G}$,*

$$\int_G E(Y \mid \mathcal{G}_n) dP_{\mathcal{G}} \to \int_G f dP_{\mathcal{G}} \quad \text{and} \quad \int_G E(Y \mid \mathcal{G}_n) dP_{\mathcal{G}} \to \int_G Y dP \equiv \phi(G).$$

Thus $E(Y \mid \mathcal{G})$ exists (f is a version of it) and is a limit of constructively defined conditional expectations.

Proof. It is only necessary—and is more convenient—to prove the theorem for non-negative Y since it is easy to see that we can take $E(Y \mid \mathcal{G}) = E(Y^+ \mid \mathcal{G}) - E(Y^- \mid \mathcal{G})$.

Following the usual proof, (see e.g., Royden [5]) we look at the family of signed measures

$$\{\phi - uP_{\mathscr{G}} : u \geqq 0\}.$$

By the Hahn Decomposition Theorem, for each u there is a "maximal positive" $A_u \subset \mathscr{G}$ such that, for any $G \subset \mathscr{G}$,

$$G \subset A_u \Rightarrow (\phi - uP_{\mathscr{G}})(G) \geqq 0$$
$$G \subset A_u^c \Rightarrow (\phi - uP_{\mathscr{G}})(G) \leqq 0,$$

or equivalently,

$$G \subset A_u \Rightarrow \int_G YdP \geqq uP(G)$$
$$G \subset A_u^c \Rightarrow \int_G YdP \leqq uP(G).$$

Of course the A_u's are not unique. However, for a countable set of u's—and we shall be using the set $\{k/2^n : k, n = 0, 1, 2, \ldots\}$—we can choose them to be decreasing in u. (Start with any sequence and let

$$A_u^* = A_u - \bigcup_{v<w} (A_w - A_v).$$

This works because $P(A_w - A_v) = 0$ if $w > v$.) And, because $Y \geqq 0$, we can take $A_0 = \Omega$.

Now let A_∞ be the limit (intersection) of all of our A_u's ($P(A_\infty) = 0$ if Y is integrable) and notice that A_∞ is also, trivially, the limit of each subsequence $\{A_{k/2^n} : k = 0, 1, 2, \ldots\}$. So for each n, the sequence

$$\mathscr{G}_n = \{G_{nk} = A_{k/2^n} - A_{(k+1)/2^n};\ k = 0, 1, 2, \ldots;\ A_\infty\}$$

in a $\mathscr{G}$-measurable partition of Ω.

Now the conditional expectations

$$E(Y \mid \mathscr{G}_n) = \sum_k E(Y \mid G_{nk}) I_{G_{nk}} + \infty I_{A_\infty},$$

where $E(Y \mid G_{nk}) = \int_{G_{nk}} YdP/P(G_{nk})$, are exactly what we want. The key fact is that, for any $G \in \mathscr{G}$,

$$k2^{-n} \leqq \int_{GG_{nk}} YdP/P(GG_{nk}) \leqq (k+1)2^{-n}.$$

The $\mathscr{G}_n$'s being nested, we conclude immediately that $E(Y \mid \mathscr{G}_n)$ converges pointwise to a, necessarily, $\mathscr{G}$-measurable function f, satisfying, for every n, and k

$$k2^{-n} \leqq f \leqq (k+1)2^{-n} \quad \text{on} \quad G_{nk}.$$

If $P(GA_\infty) > 0$,

$$\int_G E(Y \mid \mathscr{G}_n) dP_{\mathscr{G}} \equiv \infty = \int_G f dP_{\mathscr{G}} = \int_G YdP;$$

while if $P(GA_\infty) = 0$, it is easy to verify that

$$\left| \int_G (E(Y \mid \mathscr{G}_n) - f) dP_{\mathscr{G}} \right| \leqq \sum_k \int_{GG_{nk}} |E(Y \mid \mathscr{G}_n) - f| \, dP_{\mathscr{G}}$$
$$= \sum_k \int_{GG_{nk}} |E(Y \mid G_{nk}) - f| \, dP_{\mathscr{G}}$$

$$\leqq \sum_k 2^{-n} P(GG_{nk}) = 2^{-n};$$

and

$$\left| \int_G (E(Y \mid \mathcal{G}_n) - Y) dP \right| = \left| \sum_k \left[\int_{G_{nk}} Y dP / P(G_{nk}) \right] P(GG_{nk}) - \sum_k \int_{GG_{nk}} Y dP \right|$$

$$\leqq \sum_k 2^{-n} P(GG_{nk}) = 2^{-n};$$

which completes the proof. (Notice how important is the finiteness of the measure $P_{\mathcal{G}}$.)

5. Constructive definitions which ARE valid. Now let us look at a special case. Let $\mathcal{G}$ be the σ-field generated by some random variable X: $\mathcal{G} \equiv \sigma(X) = \{X^{-1}(B): B \in \mathcal{B}([-\infty, \infty])\}$. We customarily write $E(Y \mid X)$ rather than $E(Y \mid \sigma(X))$, which takes note of the fact that there is a version of $E(Y \mid \sigma(X))$ which is a Borel function of X. From now on we will consider only such versions. Let us also specialize to conditional probabilities: replace Y by $I_{\{Y \in B\}}$ where B is some Borel set, so $E(Y \mid X)$ is replaced by $P(Y \in B \mid X)$. Let $P(Y \in B \mid X = x)$ denote its common value on the event $X^{-1}(\{x\})$.

Here are three appealing constructive "definitions" of conditional probability:

(1)
$$P(Y \in B \mid X = x) = \lim_{h \to 0} P(Y \in B \mid x \leqq X < x + h).$$

(2)
$$P(Y \in B \mid X = x) = \lim_{n \to \infty} P(Y \in B \mid X_n = x_n),$$

where the X_n's are countable-valued functions of X converging to X a.e. and x_n is X_n's value when $X = x$. For example, take

$$X_n = \sum_k (k/n) I_{\{k/n \leqq X < (k+1)/n\}}$$

$$x_n = \sum_k (k/n) I_{\{k/n \leqq x < (k+1)/n\}}.$$

The third "definition" applies in case the joint distribution of X and Y is absolutely continuous with respect to Lebesgue measure, i.e., for some "density" f,

$$P(X \in A, Y \in B) = \iint_{A \times B} f(x, y) dx dy$$

for every Borel Rectangle $A \times B$. The "definition" is:

(3)
$$P(Y \in B \mid X = x) = \frac{\int_B f(x, y) dy}{\int_{-\infty}^{\infty} f(x, y) dy}.$$

Definition (3) asks us to treat a density just as though it were a probability, by analogy with the definition when X and Y are countable-valued:

$$P(Y \in B \mid X = x) = \frac{P(X = x, Y \in B)}{P(X = x)}$$

$$= \frac{\sum_{y \in B} P(X = x, Y = y)}{\sum_y P(X = x, Y = y)}.$$

Many textbooks will tell you that the limit in (1) may fail to exist for a given x. But what they (but not Kolmogorov) often neglect to add is that the set of x's for which it *does* exist has $P_X(\sim P_{\mathscr{G}})$ measure one and that the resulting limit function is indeed a version of $P(Y \in B \mid X)$. Kolmogorov showed that this is an easy consequence of Lebesgue's Theorem on almost everywhere differentiability of monotone functions. But the proof assumes the existence of a conditional probability: i.e., it *follows* from the Radon–Nikodým Theorem.

It is trivial that (3) works. Just plug it in and integrate over any Borel set A.

That (2) also works is a corollary to a well-known Martingale Theorem of Lévy which says if Z is integrable and $\mathscr{G}_n \uparrow \mathscr{G}$, then $E(Z \mid \mathscr{G}_n) \to E(Z \mid \mathscr{G})$ a.e. Of course it too comes *after* the Radon–Nikodým Theorem.

How is (2) related to the proof I gave in Section 4? Are the $\mathscr{G}_n$'s there simply the σ-fields generated by the X_n's (or some minor modification of them)? No, not in general, because the G_n's in that proof depend on Y as well as on X. For example, in the joint density case it is easy to see that

$$A_u = X^{-1}\left\{x : \int_B f(x,y)dy \geqq u \int_{-\infty}^{\infty} f(x,y)dy\right\}.$$

(Notice that the set in brackets need not be an interval.)

6. Analysis of the sphere and plane problem. Finally we return to the tantalizing last pair of examples in Section 3.

To begin with, the only really correct answer in both problems is that *there is no answer*: Neither conditional distribution is defined, simply because the "given" events have probability zero. (There is no such thing as $P(B \mid A)$ when $P(A) = 0$.)

Strictly speaking, there is nothing more to say; but to stop now would be to pay no heed at all to "the idea" of conditional probability. So let us leave logic aside for a moment and consider the sphere problem "aesthetically." The equator and the Greenwich meridian circle are merely two great circles, so, since the point on the sphere is chosen according to a spherically symmetric distribution (namely the uniform distribution), *if part* (*a*) *has an answer, part* (*b*) *must have the same answer.* And "symmetry" (or formula (3) of Section 5, if you prefer) demands that the answer be: "The conditional distribution (on any great circle) is uniform." The inescapable corollary to this is that both the longitude in part (a), and the latitude in part (b), have uniform conditional distributions.

Now comes the paradox. The longitude (call it ϕ) and the latitude (call it θ), are perfectly well-defined random variables. The (unconditional) distribution of ϕ is indeed uniform, but the distribution of θ is not. (Its density is proportional to $|\cos\theta|$.) Moreover, θ and ϕ are independent, so (the smoothest versions of) the conditional distributions of each, given the other, are simply their unconditional distributions. Thus if we use these versions and interpret "given that it lies on the equator" and "given that it lies on the ... Greenwich meridian" to mean "given $\theta = 0$" and "given $\phi = 0$" respectively, then *we have answers* to parts (a) and (b) *but,* lo and behold, *they are not the same*!

Thus we can give an answer "if we regard the given circle as an element of some decomposition of the entire spherical surface..." [2], but *the answer depends on the kind of decomposition we choose*: meridian great circles with common poles on the given circle leads to a non-uniform distribution, while circles formed by intersecting the sphere's surface with planes parallel to the one determined by the given circle does lead to the uniform distribution. Of course, the possibilities for other decompositions are limitless.

The *plane problem* provides a further illuminating example of how the "answer" depends on the choice of decomposition.

On the one hand, the density of the measure $P \times P$ on the plane is

$$(2\pi)^{-1}\exp[-(x^2+y^2)/2] = (2\pi)^{-1}\exp(-r^2/2) \equiv f(r),$$

where $r^2 = x^2 + y^2$; so, "obviously," (by formula (3) of Section 5 again), the "answer" is

$$f(r)/\int_0^\infty f(u)du = (2/\pi)^{1/2}\exp(-r^2/2)$$

(the density of the absolute value of a standard normal random variable).

Another way to describe this "answer" is from the identity

$$R^2 = [(X+Y)^2 + (X-Y)^2]/2$$

and the easily proved fact that $(X+Y)$ and $(X-Y)$ are independent; so, if "$X=Y$" means "$X-Y=0$," the smoothest version of the conditional distribution of R^2 given $(X-Y)$—on the set $\{X-Y=0\}$—is simply the unconditional distribution of $(X+Y)^2/2 = [(X+Y)/\sqrt{2}]^2$.

On the other hand, we can also show that R^2 and (X/Y) are independent; so if "$X=Y$" means "$X/Y=1$," the smoothest version of the conditional distribution of R^2 given (X/Y) is simply the unconditional distribution of $R^2 = X^2+Y^2$.

Thus, whether you think that the conditional distribution of R^2, given $X=Y$, is Chi-Squared with *one* degree of freedom or Chi-Squared with *two* degrees of freedom, depends on whether the decomposition you put the line $y=x$ into is {all lines parallel to $y=x$} or {all straight lines through the origin}.

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IRREGULAR INTEGERS

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1. In this note we discuss in some detail several multiplicatively closed systems of integers in which prime factorization need not be unique.

The phenomenon of nonunique factorization was first observed in certain quadratic extensions of the integers $\mathbf{Z}$. The set $\mathbf{Z}[\sqrt{-6}] = \{x + y\sqrt{-6}; x, y \in \mathbf{Z}\}$, endowed with the obvious arithmetic, is an instance. It is easy to show that this system admits factorization into primes of the system, that 2, 3, and $\sqrt{-6}$ are primes therein, and therefore that 6 has the two distinct prime factorizations

(1) $$6 = 2\cdot 3 = (-\sqrt{-6})\cdot\sqrt{-6}.^{(1)}$$

[(1)] See Rademacher and Toeplitz [1, pp. 48–50] for a brief, elementary, and complete account.

Various writers have noted that the phenomenon may be demonstrated, more simply, in subsets of $\mathbf{Z}$, for instance the set $4\mathbf{Z}+1=\{4x+1; x\in\mathbf{Z}\}$. This set is closed under $\mathbf{Z}$-multiplication and has a prime-power factorization thereby. One sees by inspection that 9, 21, 33, and 77 are prime in $4\mathbf{Z}+1$, and

$$693 = 9\cdot 77 = 21\cdot 33.\text{[2]} \tag{2}$$

Here is a still simpler example. The system $2\mathbf{Z}=\{2x; x\in\mathbf{Z}\}$ of even integers is closed under $\mathbf{Z}$-multiplication and has a prime-power factorization thereby. It is immediate that 2, 6, and 18 are prime in $2\mathbf{Z}$, and

$$36 = 6\cdot 6 = 2\cdot 18. \tag{3}$$

The factorizations (2) and (3), far from being accidents of the numbers 4 and 2 respectively, are essentially generic. We will here characterize, partially for the systems $n\mathbf{Z}+1$, $n\geqq 3$, and completely for $n\mathbf{Z}$, $n\geqq 2$; those numbers (called "irregular") which have several prime-power factorizations. We shall show thereby that there are infinitely many such numbers in all these systems, and for the systems $n\mathbf{Z}$ we shall find that these numbers constitute "almost all" composite numbers, in the sense of density. So far as we can discover, examples of the type (2) have not been studied, and the simpler examples of the type (3), which would seem to be the simplest conceivable, appear to have been overlooked entirely.

We employ the following terminology. Given $S\subset\mathbf{Z}$ such that $S\cdot S\subseteq S$, a number in S which cannot be factored in S is an *S-prime*; a number in S whose S-prime factorization is unique apart from order is *S-regular*, and otherwise *S-irregular*. In this terminology the Fundamental Theorem of Arithmetic is the assertion that every element of $\mathbf{Z}$ is $\mathbf{Z}$-regular.

It is a pleasure to thank H. Royden and M. B. Nathanson for helpful conversations.

2. Let $n\geqq 2$, $n\in\mathbf{Z}$ be arbitrary but fixed. We shall characterize the $n\mathbf{Z}$-primes and the $n\mathbf{Z}$-irregular numbers and calculate their densities.

PROPOSITION 1. *The element $x=nk\in n\mathbf{Z}$ is $n\mathbf{Z}$-prime if and only if $k\not\equiv 0 \bmod n$.*

Proof. If nk factors in $n\mathbf{Z}$, say $nk=nu\cdot nv$, then $k=nuv\equiv 0 \bmod n$. And if $k\equiv 0 \bmod n$, say $k=nu$, then $x=nk=n\cdot nu$ is composite in $n\mathbf{Z}$. QED.

PROPOSITION 2. *The density in $n\mathbf{Z}$ of the $n\mathbf{Z}$-primes is $(n-1)/n$.*

Proof. The $n\mathbf{Z}$-composite numbers correspond to the congruence class $\{k\equiv 0 \bmod n\}$ by Proposition 1, and the present assertion follows at once from this.

PROPOSITION 3. *There are infinitely many $n\mathbf{Z}$-irregular numbers for each $n\geqq 2$.*

Proof. If $k\not\equiv 0$, $k^2\not\equiv 0 \bmod n$ then nk, nk^2 are $n\mathbf{Z}$-primes and

$$(nk)^2 = (nk)\cdot(nk) = n\cdot(nk^2), \tag{4}$$

which is to say, $(nk)^2$ is $n\mathbf{Z}$-irregular for every such k. Since there are infinitely many such k we have demonstrated the proposition.

Example (3) is of the type (4). There are other forms of irregularity in $n\mathbf{Z}$. To find them all, and to calculate their density, we introduce the following small apparatus. Let l be the number of prime factors, with repetition, of n:

$$n = \prod_{i=1}^{l} p_i, \qquad p_i \ \mathbf{Z}\text{-primes}. \tag{5}$$

[2] Davenport [2, pp. 20, 21] attributes this to Hilbert but gives no reference. We have not been able to find any mention of this matter in the collected works of Hilbert.

For $x \in n\mathbf{Z}$ define $\nu(x)$, $\mu(x)$ by

$$n^{\nu(x)} \| x, \qquad (x/n^{\nu(x)}) = \prod_{i=1}^{\mu(x)} q_i, \tag{6}$$

the latter expression being the $\mathbf{Z}$-prime factorization, with repetitions, of $x/n^{\nu(x)}$. The notation $a^b \| c$ means that b is the largest integer r such that $a^r | c$.

LEMMA 1. *The element $x \in n\mathbf{Z}$ is $n\mathbf{Z}$-prime if and only if $\nu(x) = 1$.*

Proof. For positive integers m, k we have $n^m \| k$ if and only if $n^{m+1} \| nk$, whence $\nu(nk) = 1$ if and only if $n^0 \| k$, or $k \not\equiv 0 \bmod n$, and by Proposition 1 we have the assertion.

LEMMA 2. *When $l = 1$, which is to say n is a $\mathbf{Z}$-prime, the element $x \in n\mathbf{Z}$ is $n\mathbf{Z}$-irregular if and only if $\nu(x) \geqq 2$ and $\mu(x) \geqq 2$.*

Proof. If $\nu(x) \geqq \mu(x) \geqq 2$, then x has the two $n\mathbf{Z}$-factorizations

$$x = (nq_1) \cdot (nq_2) \cdot \cdots \cdot (nq_\mu) \cdot n^{\nu-\mu} = (nq_1q_2 \cdots q_\mu) \cdot n^{\nu-1}. \tag{7}$$

By definition of μ and ν, $q_i \not\equiv 0 \bmod n$ and $\Pi q_i \not\equiv 0 \bmod n$, so all factors in the two expressions are $n\mathbf{Z}$-prime or prime-powers, whence (7) constitutes two distinct $n\mathbf{Z}$-prime factorizations of x.

If $\mu(x) \geqq \nu(x) \geqq 2$ then x has the two $n\mathbf{Z}$-factorizations

$$x = (nq_1) \cdot (nq_2) \cdot \cdots \cdot (nq_{\nu-1}) \cdot (nq_\nu \cdots q_\mu) = (nq_1q_2 \cdots q_\mu) \cdot n^{\nu-1}, \tag{8}$$

so that, as in the previous case, x is $n\mathbf{Z}$-irregular.

The remaining cases in which x is $n\mathbf{Z}$-composite are: (i) $\nu(x) \geqq 2$, $\mu(x) = 0$; and (ii) $\nu(x) \geqq 2$, $\mu(x) = 1$. In Case (i) we have $x = n^\nu$, and if also $x = (nk_1) \cdot \cdots \cdot (nk_r)$ with $k_i \not\equiv 0 \bmod n$, then $n^\nu = n^r \cdot \Pi_1^r k_i$. Evidently $r \leqq \nu$. If $r = \nu$ then $1 = \Pi_1^r k_i$, $k_i = 1$ for all 1, and the second factorization is identical with the first. If $r < \nu$ we have $n^{\nu-r} = \Pi_{i=1}^r k_i$, whence $n | \Pi k_i$. But n is a $\mathbf{Z}$-prime ($l = 1$), so by the theorem of Euclid we have $n | k_i$, $k_i \equiv 0 \bmod n$ for some i, contrary to supposition. Hence in Case (i) x is $n\mathbf{Z}$-regular.

In Case (ii) we have $x = n^\nu \cdot q = n^{\nu-1} \cdot nq$, q a $\mathbf{Z}$-prime, $(q, n) = 1$. If also $x = (nk_1) \cdot \cdots \cdot (nk_r)$ with $k_i \not\equiv 0 \bmod n$, then $n^\nu \cdot q = n^r \Pi_1^r k_i$. If $r \geqq \nu$ we would have a $\mathbf{Z}$-factorization of q. Hence $r < \nu$ and we have $n^{\nu-r} \cdot q = \Pi k_i$. But then $n | \Pi k_i$, and we conclude, as in the previous case, that x is $n\mathbf{Z}$-regular in Case (ii).

This exhausts all possibilities and so establishes the lemma.

LEMMA 3. *When $l > 1$ the condition $\nu(x) \geqq 2$, $\mu(x) \geqq 2$ implies the $n\mathbf{Z}$-irregularity of $x \in n\mathbf{Z}$.*

Proof. The factorizations (7) and (8) remain valid when n is composite.

LEMMA 4. *When $l > 1$ and $\mu(x) = 0$, the condition $\nu(x) \geqq 3$ is necessary and sufficient for the $n\mathbf{Z}$-irregularity of $x \in n\mathbf{Z}$.*

Proof. Suppose first that $l > 1$, $\mu(x) = 0$, $\nu(x) = 2$. Then $x = n^2$, and if also $x = (nk_1) \cdot \cdots \cdot (nk_r)$ with $k_i \not\equiv 0 \bmod n$, then $n^2 = n^r \Pi^r k_i$, whence $r \leqq 2$. If $r = 2$ then $k_1k_2 = 1 = k_1 = k_2$ and the second factorization is identical with the first. If $r = 1$ then $n^2 = nk_1$, $n = k_1 \equiv 0 \bmod n$, contrary to supposition. Hence x is $n\mathbf{Z}$-regular if $\nu(x) = 2$.

Suppose now that $l > 1$, $\mu(x) = 0$, $\nu(x) \geqq 3$. Then, in the notation of (5), $x = n^\nu = (np_1) \cdot (np_2 \cdots p_l) \cdot n^{\nu-3}$, each factor being an $n\mathbf{Z}$-prime (or prime power). Hence x is $n\mathbf{Z}$-irregular. This exhausts all cases (by Lemma 1) and establishes the lemma.

LEMMA 5. *When $l > 1$ and $\mu(x) = 1$ the condition $\nu(x) \geqq 3$ is necessary and sufficient for the $n\mathbf{Z}$-irregularity of $x \in n\mathbf{Z}$.*

Proof. Suppose first that $l>1$, $\mu(x)=1$, $\nu(x)=2$. Then $x=n\cdot nq$, q a **Z**-prime, $(q,n)=1$. If also $x=(nk_1)\cdot\cdots\cdot(nk_r)$ with $k_i\not\equiv 0 \bmod n$, then $n^2\cdot q=n^r\cdot\Pi_1^r k_i$. If $r\geqq 2$ then we would have **Z**-factorization of q. Hence $r=1$. But then $n^2\cdot q=nk_1$, $k_1\equiv 0 \bmod n$, contrary to supposition. Hence x is n**Z**-regular if $\nu(x)=2$.

Suppose now that $l>1$, $\mu(x)=1$, $\nu(x)\geqq 3$. Then $x=n^{\nu-1}\cdot nq$, and, writing $nq=(p_1q)\cdot(p_2\cdots p_l)$, we have $x=(np_1q)\cdot(np_2\cdots p_l)\cdot n^{\nu-3}$, each factor being an n**Z**-prime (or prime power). Hence x is n**Z**-irregular if $\nu(x)\geqq 3$. This exhausts all cases and proves the lemma.

The results of the foregoing lemmas are summarized in:

THEOREM 1. *When n is* **Z**-*prime, the n***Z**-*irregular numbers in n***Z** *are characterized by the condition*: $\mu(x)\geqq 2$, $\nu(x)\geqq 2$ (*in the notation of* (6)). *When n is* **Z**-*composite there are additional n***Z**-*irregular numbers characterized by the condition*: $\mu(x)\leqq 1$, $\nu(x)\geqq 3$.

This result permits the calculation of the density in n**Z** of the n**Z**-irregular numbers. To this we now turn.

LEMMA 6. *The density in n***Z** *of the set of n***Z**-*composite numbers x with* $\mu(x)=0$ *is* 0.

Proof. Denote the set of numbers in question by C^0, write $P(K)=\{n\cdot 1, n\cdot 2,\dots,n\cdot K\}$ for the first K positive elements of n**Z**, and let $\#(X)$ denote the cardinality of the set X. Now

$$C^0\cap P(K)=\{n^\nu; n^\nu\leqq nK\}=\left\{n^\nu; \nu-1\leqq\frac{\log K}{\log n}\right\},\ \text{whence}\ \#\{C^0\cap P(K)\}\leqq\frac{\log K}{\log n}+1.$$

Therefore $\lim_{K\to\infty}(1/K\,\#\{C^0\cap P(K)\})=0$. QED.

LEMMA 7. *The density in n***Z** *of the set of n***Z**-*composite numbers x with* $\mu(x)=1$ *is* 0.

Proof. Denote the set of numbers in question by C^1, write (as before) $P(K)$ for the first K positive elements of n**Z**, and let Π denote the set of all **Z**-primes. If we put

$$C_k=\{n^kq; q\in\Pi\}$$

then

$$C^1=\bigcup_{k=1}^{\infty} C_k,$$

a disjoint union. Now $C_k\cap P(K)=\{n^kq; n^kq\leqq nK\}=\{n^kq; q\leqq K/n^{k-1}\}$. Note that

$$k>\frac{\log(1/2)K}{\log n}+1\Leftrightarrow\frac{K}{n^{k-1}}<2 \tag{9}$$

whence $\#\{C_k\cap P(K)\}=0$ if $k>(\log(1/2)K/\log n)+1$. Hence if we put

$$N=\left[\frac{\log(1/2)K}{\log n}\right]+1, \tag{10}$$

we have

$$\#\{C^1\cap P(K)\}=\sum_{k=1}^{N}\#\{C_k\cap P(K)\}. \tag{11}$$

We may replace the number $\frac{1}{2}$ in (9), (10) by any α, $0<\alpha<1$, and note that

$$k>\frac{\log\alpha K}{\log n}+1\Leftrightarrow\frac{K}{n^{k-1}}<\alpha^{-1}.$$

The quantity corresponding to N is $M=[\log\alpha K/\log n]+1$. We now assume $0<\alpha\ll 1/2$, and partition the sum (11) as follows:

$$\sum_{k=1}^{N} \#\{C_k \cap P(K)\} = \left(\sum_{k=1}^{M-1} + \sum_{M}^{N}\right)(\#\{C_k \cap P(K)\}). \tag{12}$$

For the first term we use the prime number theorem. We have $\#\{C_k \cap P(K)\} = \#\{q \in \Pi; q \leqq K/n^{k-1}\} \leqq (1+\delta(\alpha))\cdot(K/n^{k-1})/(\log K/n^{k-1})$, where $\delta(\alpha)$ depends only upon α, and $\delta(\alpha) \leqq 1$ for α sufficiently small. Therefore

$$\sum_{k=1}^{M-1} \#\{C_k \cap P(K)\} \leqq 2\sum_{k=1}^{M-1} (K/n^{k-1})/(\log K/n^{k-1}) = 2K\sum_{k=1}^{M-1} 1/n^{k-1}(\log K/n^{k-1}).$$

In the range of this summation we have $K/n^{k-1} > \alpha^{-1}$, $\log K/n^{k-1} > \log \alpha^{-1}$, whence this sum is dominated by

$$2(K/\log \alpha^{-1})\sum_{k=1}^{M-1} 1/n^{k-1} < 2(K/\log \alpha^{-1})\sum_{l=0}^{\infty}(1/n)^l = 2(K/\log \alpha^{-1})\cdot n/(n-1).$$

For the second sum in (12) we observe that

$$N - M \leqq \left[\frac{\log(1/2) - \log \alpha}{\log n}\right] + 1,$$

and note that this number is independent of K. We denote it by $\lambda(\alpha)$. In the range of this summation we have $K/n^{k-1} < \alpha^{-1}$, so $\#\{C_k \cap P(K)\} \leqq \#\{q \leqq \alpha^{-1}; q \in \Pi\}$, which is a number independent of K. We denote it by $p(\alpha)$. The second sum in (12) is thus dominated by $\lambda(\alpha)p(\alpha)$, independent of K. Now

$$\frac{1}{K}\#\{C^1 \cap P(K)\} \leqq \frac{n}{n-1}\cdot\frac{2}{\log \alpha^{-1}} + \frac{1}{K}p(\alpha)\lambda(\alpha)$$

so that in the limit as $K \to \infty$ this quantity is dominated by $[n/(n-1)]\cdot[2/\log \alpha^{-1}]$. By prior choice of α this bound can be made arbitrarily small. Hence the density in $n\mathbf{Z}$ of C^1 is 0.

THEOREM 2. *The density in $n\mathbf{Z}$ of the $n\mathbf{Z}$-irregular numbers is $1/n$.*

Proof. Let C^0 and C^1 have the meanings given them in Lemmas 6 and 7; let C_3^0, C_3^1 denote the sets of $x \in n\mathbf{Z}$ with $\nu(x) \geqq 3$ and $\mu(x) = 0$, $\mu(x) = 1$ respectively (thus $C_3^0 \subset C^0$, $C_3^1 \subset C^1$); let D denote the set of $x \in n\mathbf{Z}$ with $\nu(x) \geqq 2$ and $\mu(x) \geqq 2$; let I denote the set of $n\mathbf{Z}$-irregular numbers, and C the set of $n\mathbf{Z}$-composite numbers; finally let $\delta(X)$ denote the density in $n\mathbf{Z}$ of $X \subset n\mathbf{Z}$. By Lemma 1 we have

$$C = D \cup C^1 \cup C^0,$$

a disjoint union. By Lemmas 6 and 7 we have $\delta(C^0) = \delta(C^1) = 0$, whence $\delta(C) = \delta(D)$. By Theorem 1 we have

$$I = D \quad \text{or} \quad I = D \cup C_3^0 \cup C_3^1,$$

according as n is $\mathbf{Z}$-prime or $\mathbf{Z}$-composite, respectively. Since $C_3^0 \subset C^0$ and $C_3^1 \subset C^1$, whence $\delta(C_3^0) = \delta(C_3^1) = 0$, we have in either case $\delta(I) = \delta(D)$, so that $\delta(I) = \delta(C)$. Now $\delta(C) = 1/n$ by Proposition 2, and we have the result.

COROLLARY. *The density in $n\mathbf{Z}$ of the $n\mathbf{Z}$-regular composite numbers is* 0.

Proof. Denote by R the set of numbers in question. Then $C = R \cup I$ and $\delta(C) = \delta(I) = 1/n$, so $\delta(R) = 0$.

It might be supposed that the numbers of $n\mathbf{Z}$-primes in any two factorizations of an $n\mathbf{Z}$-irregular number should be equal. This appears to be the case (though it is not proved) for numbers in D: the factorizations (7), (8) each contain exactly ν $n\mathbf{Z}$-primes, counting repetitions. But this is not the case for numbers in C_3^0 and C_3^1. In Lemmas 4 and 5 we found, for $x \in C_3^0$ or $x \in C_3^1$ respectively, a factorization with exactly $\nu(x)$ $n\mathbf{Z}$-prime factors, and a second factorization with only $\nu(x)-1$. For

example, let $n = 6$. Then $x_0 = 216 = 6^3 \in C_3^0$, $x_1 = 1080 = 6^3 \cdot 5 \in C_3^1$, and

$$216 = 6 \cdot 6 \cdot 6 = (6 \cdot 2) \cdot (6 \cdot 3),$$

$$1080 = 6 \cdot 6 \cdot (6 \cdot 5) = (6 \cdot 10) \cdot (6 \cdot 3).$$

This concludes our discussion of the systems $n\mathbf{Z}$, and we turn now to sets of the form $n\mathbf{Z} + 1$.

3. Algebraically speaking, the next simplest systems of integers after $n\mathbf{Z}$ are the arithmetic sequences $A = n\mathbf{Z} + m$. Now $A \cdot A \subset A$ if and only if $m^2 \equiv m \mod n$; (for if to each pair $x, y \in \mathbf{Z}$ there is $z \in \mathbf{Z}$ such that $(nx + m) . (ny + m) = (nz + m)$ then $m^2 \equiv m \mod n$; and if $m^2 \equiv m \mod n$ then the equation can be solved for $z \in \mathbf{Z}$ in terms of x, y, m, n). If also $(n, m) = 1$, then $n \;\, m - 1$, say $m - 1 = kn$, and then the substitution $z = z' - k$ transforms A to $n\mathbf{Z} + 1$, which is to say, $n\mathbf{Z} + m = n\mathbf{Z} + 1$. Thus the systems $n\mathbf{Z} + 1$ arise naturally in the present context, by virtue of being the most general arithmetic sequences with relatively prime coefficients which are closed under multiplication.

For the study of prime factorization in $n\mathbf{Z} + 1$ we need consider only $n \geqq 3$ because the situation is that of $\mathbf{Z}$ when $n = 1, 2$. We employ the following (mostly standard) apparatus. $\mathbf{Z}/(n)$ denotes the ring of congruence classes mod n, and $(\mathbf{Z}/(n))^*$ its group of units (that is, the subset consisting of those congruence classes which are relatively prime to n). We represent each element of $\mathbf{Z}/(n)$ by its least non-negative member. In these terms the theorem of Dirichlet on primes in arithmetic sequences is the assertion that, for every $n > 0$ and every $t \in (\mathbf{Z}/(n))^*$, the set of all $\mathbf{Z}$-primes $q \in (n\mathbf{Z} + t)$, or $q \equiv t \mod n$, is infinite (and in particular, not empty). We write $(x)_n$ for the least non-negative residue mod n of $x \in \mathbf{Z}$. By a *word* α on $\mathbf{Z}/(n)$ we mean a list $\alpha = \{t_1, t_2, \ldots, t_r\}$ of elements $t_i \in \mathbf{Z}/(n)$, without regard to order, and with repetition permitted. Let W denote the set of all words on $\mathbf{Z}/(n)$. We define a map $w : \mathbf{Z} \to W$ as follows. Let $x = \pm \Pi q_i$ be the $\mathbf{Z}$-prime factorization, with repetitions, of $x \in \mathbf{Z}$. Then the image $w(x) \in W$ of x, which we call the *word* of x, is

$$w(x) = \{(q_1)_n, (q_2)_n, \ldots\}.$$

The *weight* $\bar{\alpha}$ of a word $\alpha = \{t_1, t_2, \ldots\}$ on $\mathbf{Z}/(n)$ is $\bar{\alpha} = (\Pi t_i)_n$, and α is a *unit word* if $\bar{\alpha} = 1$. Let W^* denote the set of unit words on $\mathbf{Z}/(n)$. Evidently an integer x belongs to $(n\mathbf{Z} + 1)$ if and only if $w(x) \in W^*$.

LEMMA 8. *The product of a non-unit in $\mathbf{Z}/(n)$ by any element of $\mathbf{Z}/(n)$ is a nonunit in $\mathbf{Z}/(n)$.*

Proof. $t \in \mathbf{Z}/(n)$ is a nonunit if and only if $(t, n) > 1$, and then $(tx, n) > 1$ for all $x \in \mathbf{Z}/(n)$, whence tx is a nonunit.

Thus every "letter" of a unit word on $\mathbf{Z}/(n)$ must be a unit in $\mathbf{Z}/(n)$. In symbols,

PROPOSITION 4. *If $\{t_1, t_2, \ldots\} \in W^*$ then $t_i \in (\mathbf{Z}/(n))^*$.*

A unit word α is called *irreducible* if no proper subword of α is a unit word.

PROPOSITION 5. *An integer $x \in (n\mathbf{Z} + 1)$ is $(n\mathbf{Z} + 1)$-prime if and only if $w(x)$ is an irreducible unit word on $\mathbf{Z}/(n)$.*

Proof. This is clear.

EXAMPLE 1. Fix $t \in (\mathbf{Z}/(n))^*$, choose $\mathbf{Z}$-primes $q_1, q_2, \ldots, q_r$ with $q_i \equiv t \mod n$, and put $x = \Pi q_i$. Then $w(x) = \{t, t, \ldots, t\}$, $w(x) = (t^r)_n$, so that x is $(n\mathbf{Z} + 1)$-prime precisely when r is the order of t in $(\mathbf{Z}/(n))^*$. For instance, take $n = 5$. Then $x = 136 = 2^3 \cdot 17$ belongs to $(5\mathbf{Z} + 1)$, $w(x) = \{2, 2, 2, 2\}$, and the order of 2 in $(\mathbf{Z}/(5))^*$ is 4, so that 136 is $(5\mathbf{Z} + 1)$-prime.

EXAMPLE 2. If the $\mathbf{Z}$-primes p, q are such that $pq \equiv 1 \mod n$, then clearly $x = pq$ is $(n\mathbf{Z} + 1)$-prime. For instance, with $n = 5$, $x = 141 = 3 \cdot 47 \in (5\mathbf{Z} + 1)$, $w(x) = \{3, 2\}$, and $3 \cdot 2 \equiv 1 \mod 5$, so that 141 is $(5\mathbf{Z} + 1)$-prime.

There are infinitely many $(n\mathbf{Z}+1)$-primes for every $n \geqq 3$: any $\mathbf{Z}$-prime $q \equiv 1 \bmod n$ is $(n\mathbf{Z}+1)$-prime, and it is known, without recourse to Dirichlet's theorem, that there are infinitely many such $\mathbf{Z}$-primes (see, for instance, [3]). The structure of the most general $(n\mathbf{Z}+1)$-prime is easily described: let $\alpha = \{t_1, t_2, \dots\}$ be a generic irreducible unit word on $\mathbf{Z}/(n)$, let any $\mathbf{Z}$-primes $q_i \equiv t_i \bmod n$ be given, and put $x = \Pi q_i$. Then x is the generic $(n\mathbf{Z}+1)$-prime. More concisely, the number x just described is an arbitrary element of $w^{-1}(\alpha)$, and the union $\cup\{w^{-1}(\alpha);\ \bar{\alpha}=1,\ \alpha \text{ irreducible}\}$ comprises all $(n\mathbf{Z}+1)$-primes. By Dirichlet's theorem, for each $n \geqq 3$ and for each irreducible unit word α on $\mathbf{Z}/(n)$, the set $w^{-1}(\alpha)$ is infinite. It is perhaps surprising, though almost immediate, that the converse assertion is true. The proof is as follows. Given $n \geqq 3$ and $t \in (\mathbf{Z}/(n))^*$, let $\alpha = \{t, t, \dots, t\}$, ν-fold, where ν is the order of t in $(\mathbf{Z}/(n))^*$. Then (see Example 1) α is an irreducible unit word on $\mathbf{Z}/(n)$, and

$$w^{-1}(\alpha) = \{x \in \mathbf{Z};\ x = \pm \Pi q_i,\ q_i\, \mathbf{Z}\text{-prime},\ q_i \equiv t \bmod n\}.$$

Now if $\{q;\ q \text{ a } \mathbf{Z}\text{-prime},\ q \equiv t \bmod n\}$ were finite, then $w^{-1}(\alpha)$ would be finite, contrary to supposition. We state this result formally as

THEOREM 3. *The set of $\mathbf{Z}$-primes $q \equiv t \bmod n$ is infinite, for every $n \geqq 3$ and every $t \in (\mathbf{Z}/(n))^*$ (Dirichlet's theorem) if and only if the set of $(n\mathbf{Z}+1)$-primes on each irreducible unit word α on $\mathbf{Z}/(n)$ is infinite, for every $n \geqq 3$, and every such α.*

Turning now to irregularity, we shall give a construction for $(n\mathbf{Z}+1)$-irregular numbers which is probably not a complete algorithm. The idea of the construction is this: if $x \in (n\mathbf{Z}+1)$ has a factorization $x = u_1 \cdot u_2$ into $(n\mathbf{Z}+1)$-primes u_i such that there is a proper $\mathbf{Z}$-factor of u_1 which is congruent mod n to, but unequal to, a proper $\mathbf{Z}$-factor of u_2, then by exchanging these factors between u_1 and u_2 one hopes to get a second $(n\mathbf{Z}+1)$-factorization of x. The number 693 of (2), the so-called Hilbert example, is an instance of this procedure. We have there $n = 4$, and $x = 693 = 3^2 \cdot 7 \cdot 11 \in (4\mathbf{Z}+1)$. Put $u_1 = 3 \cdot 3$, $u_2 = 7 \cdot 11$. Both are $(4\mathbf{Z}+1)$-primes, by the construction of Example 1. The factor 3 of u_1 is congruent mod 4 to, and unequal to, the factor 7 of u_2. Exchanging these factors we have $x = v_1 \cdot v_2$, where $v_1 = 3 \cdot 7$ and $v_2 = 3 \cdot 11$ are both $(4\mathbf{Z}+1)$-primes by the construction of Example 1. The procedure is precisely formulated in

PROPOSITION 7. *If $x \in (n\mathbf{Z}+1)$ is the product $x = u_1 \cdot u_2$ of $(n\mathbf{Z}+1)$-primes u_1, u_2 for which there exist $a, b, c, d \in \mathbf{Z}$, all distinct from 1, such that*
(i) $u_1 = ab$, $u_2 = cd$
(ii) $a \neq d$, $b \neq c$,
(iii) $b \equiv c \bmod n$,
then x is $(n\mathbf{Z}+1)$-irregular.

Proof. Put $v_1 = ac$, $v_2 = bd$. Now $\overline{w(v_1)} \equiv \overline{w(a)} \cdot \overline{w(c)} \equiv \overline{w(a)} \cdot \overline{w(b)} \equiv 1 \bmod n$ by (iii), whence $w(v_1)$ is a unit word, or $v_1 \in (n\mathbf{Z}+1)$; and similarly $v_2 \in (n\mathbf{Z}+1)$. If either (or both) of v_1, v_2 are $(n\mathbf{Z}+1)$-composite, then the expression $x = v_1 \cdot v_2$ will constitute, after $(n\mathbf{Z}+1)$-factorization of the v_i, an $(n\mathbf{Z}+1)$-factorization for x requiring at least three $(n\mathbf{Z}+1)$-primes, which thereby differs from the given factorization. Hence in this case x is $(n\mathbf{Z}+1)$-irregular. If both v_1 and v_2 are $(n\mathbf{Z}+1)$-prime, then by (ii) we have $u_i \neq v_j$, $i, j = 1, 2$, and so $x = v_1 \cdot v_2$ is as it stands a second $(n\mathbf{Z}+1)$-prime factorization of x. Thus in either case x is $(n\mathbf{Z}+1)$-irregular and we have established the proposition.

THEOREM 4. *For every $n \geqq 3$ there are infinitely many $(n\mathbf{Z}+1)$-irregular numbers.*

Proof. This follows at once from Proposition 7 and the theorem of Dirichlet.

The condition $n \geqq 3$ is of course necessary in the theorem, and it is entertaining to see how the apparatus excludes $n = 2$: the group $(\mathbf{Z}/(2))^*$ is trivial, so all unit words have the form $\alpha = \{1, 1, \dots, 1\}$, and such α is irreducible only when $\alpha = \{1\}$, whence a $(2\mathbf{Z}+1)$-prime is a $\mathbf{Z}$-prime.

J. Simms [4] has found, by means of a miniature calculator, the smallest irregular number in $(n\mathbf{Z}+1)$ for $3 \leqq n \leqq 25$. The first few are:

$$\begin{aligned} n &= 3 \quad & 100 &= 10\cdot 10 = 4\cdot 25 \\ n &= 4 & 441 &= 9\cdot 49 = 21\cdot 21 \\ n &= 5 & 336 &= 6\cdot 56 = 16\cdot 21 \\ n &= 6 & 3025 &= 25\cdot 121 = 55\cdot 55. \end{aligned}$$

All are instances of the procedure of Proposition 7, as are all $(n\mathbf{Z}+1)$-irregular numbers known to us. We note that 693 is not the smallest $(4\mathbf{Z}+1)$-irregular number.

We conclude with a partial converse to Proposition 7.

PROPOSITION 8. *If $x \in (n\mathbf{Z}+1)$ has two $(n\mathbf{Z}+1)$-prime factorizations $x = u_1 \cdot u_2 = v_1 \cdot v_2$, each pair $\{u_i\}$, $\{v_j\}$ of $(n\mathbf{Z}+1)$-primes being relatively prime in $\mathbf{Z}$, then each pair satisfies the conditions of Proposition* 7.

Proof. Assume for simplicity that $x > 0$, $u_i > 0$, $v_j > 0$. Since the two factorizations are distinct, $u_i \neq v_j$ for $i, j = 1, 2$. Since $x = u_1 \cdot u_2 = v_1 \cdot v_2$, all $\mathbf{Z}$-primes in u_1 occur in $v_1 \cdot v_2$. If none occur in v_1, which is to say, if $(u_1, v_1) = 1$, then they all occur in v_2. If u_1 were a proper $\mathbf{Z}$-divisor of v_2 then $w(u_1)$ would be a unit proper subword of $w(v_2)$, contrary to the supposition that v_2 is $(n\mathbf{Z}+1)$-prime. Therefore $(u_1, v_1) \neq 1$; and, by symmetry, $(u_i, v_j) \neq 1$ for $i, j = 1, 2$. Define $a, b, c, d \in \mathbf{Z}$ by

$$a = (u_1, v_1), \qquad b = (u_1, v_2),$$

$$c = (u_2, v_1), \qquad d = (u_2, v_2).$$

Since $(v_1, v_2) = 1$ and $u_1 \mid v_1 \cdot v_2$, we have $ab = (u_1, v_1)\cdot(u_1, v_2) = u_1$; and similarly $cd = u_2$, $ac = v_1$, $bd = v_2$. Since $ab = u_1 \neq v_1 = ac$, $b \neq c$; and similarly $ab = u_1 \neq v_2 = bd$, $a \neq d$. It remains only to show that $b \equiv c \bmod n$. To this end, observe that because $u_1 \equiv 1 \bmod n$ and $1 \in (\mathbf{Z}/(n))^*$, it follows by Lemma 8 from the equation $ab = u_1$ that $(a)_n \in (\mathbf{Z}/(n))^*$, $(b)_n \in (\mathbf{Z}/(n))^*$; and similarly, from the equation $ac = v_1$, that $(c)_n \in (\mathbf{Z}/(n))^*$. Now $(a)_n(b)_n \equiv 1 \equiv (a)_n(c)_n \bmod n$, whence $(b)_n \equiv (c)_n \bmod n$, which is equivalent to $b \equiv c \bmod n$.

Postscript. We thank the referee for calling our attention to the paper [5] of James and Niven. Their result anticipates part of our work, as follows. They consider all multiplicatively closed systems S of integers which are also arithmetically closed, in the sense that, for some $n > 0$, if $x \in S$ and $y \equiv x(n)$ then $y \in S$. Letting m be the least such n for a given S, they show that factorization is unique in S if and only if $S = [x : (x, m) = 1]$. The proof employs factorizations resembling our (7), (8). The systems $n\mathbf{Z}$, $n\mathbf{Z}+1$ studied in the present paper fall within the compass of this theorem but do not satisfy its condition, clearly, whence all of them must contain irregular numbers.

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ALGEBRAIC TRANSFORMATION GROUPS AND THE SIMILARITY PROBLEM

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The purpose of this paper is to give an exposition of the geometric ideas and techniques which underlie the following theorem [2, Th. 3.6], and to compare it to other solutions of the similarity problem.

THEOREM. *Let K be a field of characteristic zero and let A and B be n by n K-matrices. Then A is similar to B if and only if*

(i) *A and B have the same characteristic polynomial, and*

(ii) *for all even $k \leqq 2n-2$ one has*

$$\operatorname{rank}(A \otimes I_n - I_n \otimes A)^k = \operatorname{rank}(B \otimes I_n - I_n \otimes B)^k = \operatorname{rank}(A \otimes I_n - I_n \otimes B)^k.$$

The framework in which to discuss this result is the theory of algebraic transformation groups, the essential ingredients being affine algebraic geometry and algebraic group actions. So the first order of business is to lay a foundation in these areas.

1. Algebraic geometry in K^n. Let K be an infinite field. K^n is given the so-called *Zariski topology* by defining a subset F to be closed if it is the set of common solutions of a collection of polynomial equations in n variables. Closed sets in K^1 are just the finite subsets. In K^2, closed sets are finite unions of points and algebraic curves (e.g., the parabola $y = x^2$ is an algebraic curve as it is the solutions of the polynomial equation $y - x^2 = 0$). The most unusual feature of this topology is that the open sets are all large. To be precise, each open set is dense, or equivalently, any two open sets have a non-empty intersection. This topology is not a T_2-topology, although it is T_1.

In any topology, the subsets constructed from the open sets (or closed sets) by lattice operations (union, intersection, and complementation) are known as the *constructible subsets*. For the Zariski topology on K^n, this Boolean algebra has a particularly nice interpretation, since if S is a constructible subset of K^n, then to decide the membership of a point $x = (x_1, \ldots, x_n)$ in S, one must simply check a finite number of polynomial equalities and inequalities involving the coordinates x_i of x. For example, identify $M_n(K)$ (n by n matrices over K) with K^{n^2} and let S be the set of all matrices having some fixed rank r. Let $x = (x_{ij})$ and let $m_{k,\alpha}(x)$ be a kth order minor determinant of x determined by the multi-index α. Then x belongs to S if and only if all $m_{r+1,\alpha}(x) = 0$ and some $m_{r,\beta}(x) \neq 0$. Thus the somewhat unusual looking second condition of the theorem mentioned above is connected with the constructibility of a certain set of matrices.

On K^n with the Zariski topology, the polynomial ring $K[X_1, \ldots, X_n]$ (denoted simply by $K[X]$ most often) acts as a ring of continuous K-valued functions. This topological space, together with the ring of functions, is called *affine n-space*. More generally, a closed subset V in K^n is an *affine variety* if it is given the induced topology and a ring of continuous functions by restricting the polynomial functions on K^n to V. Each point of V has coordinates, and certain continuous functions are computed algebraically in terms of those coordinates. The ring of continuous functions is denoted by $K[V]$ and is called the *coordinate ring* of V. Certain open subsets of K^n are affine varieties by the following process. If $f(X)$ belongs to $K[X]$ then it defines a *principal open subset* $K_f^n = \{x \text{ in } K^n \mid f(x) \neq 0\}$. K_f^n is homeomorphic to the closed set in K^{n+1} defined by the polynomial equation $1 - X_{n+1} f(X_1, \ldots, X_n) = 0$, under the mapping

$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_n,\ 1/f(x_1, \ldots, x_n)).$$

Under this identification K_f^n is viewed as an affine variety. Every open subset in K^n is a finite union of principal open subsets. The most useful example for us in what follows is $GL_n(K)$—the non-singular n

by n matrices. It may be naturally viewed as a principal open subset of K^{n^2} since it is the set of non-zeroes of the determinant, a polynomial function.

The *dimension* of an affine variety V is a non-negative integral measure of its size analogous to the linear dimension of a vector space. Algebraically, the dimension of V, $\dim V$, is the transcendence degree over K of the coordinate ring $K[V]$ ($\dim K^n = n$). Geometrically, viewing V as a surface in K^n, at each point x of V one has a space of *tangent vectors*. If x is a *simple point* (the surface is "smooth" in a neighborhood of x; see J. D. Emerson [4] for the connection between the Jacobian of the defining equations and $\dim V$ in the characterization of simple points), then $\dim V$ is equal to the linear dimension of the tangent space to V at x. Thus, for example, the circle $X^2 + Y^2 = 1$ and the parabola $Y - X^2 = 0$ are 1-dimensional subsets of K^2. Every affine variety has an open subset of simple points.

The type of mappings one allows between affine varieties are those whose coordinate functions are polynomial functions. These mappings are called polynomial mappings. In particular, if $V \subseteq K^n$ and $W \subseteq K^m$ are varieties, then a *morphism* $F: V \to W$ is the restriction of a polynomial map F between the ambient spaces, i.e., $F: K^n \to K^m$. These mappings are continuous and differentiable, so if $F: V \to W$ is a morphism of affine varieties and v is in V, then dF_v, the *differential of* F *at* v, may be regarded as a linear mapping from the tangent space to V at v, to the tangent space of W at $F(v)$. As a matrix, dF_v is just the Jacobian of F at v, defined as in calculus as a matrix of partial derivatives. If V and W are smooth, then $\dim F(V) = \operatorname{rank} dF_v$ for any point v. Just as in calculus, the rank of the differential gives local dimension of the image. An excellent treatment of affine algebraic geometry and algebraic groups is available in Humphreys [8]. One might also consult Mumford [10] or Shafarevich [11] for the fundamentals of modern algebraic geometry.

2. Algebraic transformation groups. A subgroup of $\operatorname{Perm}(S)$ is called a *group of transformations* on S. More generally, a group G is called a transformation group on the set S if there is a mapping $G \times S \to S$, $(g, s) \to g(s)$, satisfying $1_G(s) = s$ for all s in S and $g_1(g_2(s)) = (g_1g_2)(s)$ for all g_i in G.

For any s in S, the set $G(s) = \{g(s) | g \text{ in } G\}$ is called the *orbit* of s and will be denoted by $\mathcal{O}_s$. Evidently, the orbits partition S into equivalence classes. The collection of orbits or equivalence classes is denoted by S/G and is called the *orbit space.*

Many algebraic classification problems amount to determining the orbit space for the action of some transformation group, that is, settling such questions as: when do two points lie in the same orbit? how many orbits are there? what structure, if any, can be given to the set of orbits? For example, *similarity classes* in $M_n(K)$ are $GL_n(K)$-orbits under the action $GL_n(K) \times M_n(K) \to M_n(K)$ given by $(g, A) \to gAg^{-1}$. Recall that two symmetric (skew-symmetric) n by n matrices are called *congruent* if there is an invertible matrix g with $gAg^t = B$, where t denotes transpose. These matrices arise in the study of orthogonal and symplectic geometries on a vector space of n-dimensions, and, of course, the action of $GL_n(K)$ on them corresponds to a change of basis in the vector space. Thus, congruence classes are also $GL_n(K)$-orbits for natural actions of $GL_n(K)$. Very little can be said of a transformation group $G \times S \to S$ unless the sets are given some additional structure. Making G and S affine varieties (complex or real manifolds) leads to the notion of an algebraic transformation group (analytic transformation group).

An *affine algebraic group* is a variety having group operations compatible with its variety structure. Such a group is isomorphic to a Zariski-closed subgroup of some $GL_n(K)$, which is itself an algebraic group. Since $\det(X_{ij})$ is a polynomial in the X_{ij}'s, the *special linear group*

$$SL_n(K) = \{g \text{ in } GL_n(K) | \det(g) - 1 = 0\}$$

is an affine algebraic group. If B is an n by n symmetric (skew-symmetric) non-singular matrix, then the *orthogonal* (*symplectic*) *group*

$$G = \{A \text{ in } GL_n(K) | ABA^t = B\}$$

is also an affine algebraic group. Note, the equations defining G are quadratic in the coordinates of A.

From a more geometric point of view, B arises from a non-degenerate symmetric (skew-symmetric) bilinear form $(,)$ on an n-dimensional vector space V, and

$$G = \{g \text{ in } GL(V) \mid (g(v), g(w)) = (v, w), \text{for all } v, w \text{ in } V\}.$$

Since angles and lengths in V are computed with respect to $(,)$, G is the group of linear transformations on V which preserve the geometry.

To say that $\alpha : G \times V \to V$ is an *algebraic transformation group*, is to assume that G is an affine algebraic group, V is an affine variety, and α is a morphism. If g is in G and v is in V, then $\alpha(g, v)$, written $g(v)$ usually, can be computed coordinatewise as polynomials in the coordinates of g and v. By definition, the multiplicative action of an algebraic group G on itself,

$$\mu : G \times G \to G,$$

is an algebraic transformation group on G. Along the same lines, $G_a(K)$, the additive group of K, can be viewed as an algebraic group acting on K^2 by

$$\alpha : G_a(K) \times K^2 \to K^2,$$

where

$$G_a(K) = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a \text{ in } K\right\}, \text{ and } \alpha\left(a, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Notice that the orbit of $\binom{x}{y}$ is just the horizontal line through it when $y \neq 0$. On the other hand, if $y = 0$ then each $\binom{x}{0}$ is a *fixed point* for the action; i.e.,

$$G_a(K)\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

We should remark that this action is the simplest instance of the ancient theory of binary semi-invariants.

Before investigating more examples of algebraic transformation groups, we'll define an important object which may be associated to any algebraic action. Recall that each variety V has a ring of continuous K-valued polynomial functions, $K[V]$. If $G \times V \to V$ is an algebraic transformation group, then G acts naturally on this ring by $(g, f) \to f^g$ where

$$f^g(v) \equiv f(g^{-1}(v)).$$

The subring $K[V]^G$ of all polynomials fixed by this action is called the *ring of invariant functions* on V. For example, $K[X, Y]^{G_a(K)} = K[Y]$ is the ring of invariants for the action of $G_a(K)$ on K^2 described above. To see this, notice that to say f is invariant is to say that f is constant on each orbit. In fact, one always knows that f is constant on the closure of each orbit as well (a continuity argument), and thus if $\mathscr{C}\ell(\mathcal{O}_v) \cap \mathscr{C}\ell(\mathcal{O}_w) \neq \varnothing$ then $f(v) = f(w)$ for all invariants f ($\mathscr{C}\ell$ denoting Zariski closure).

EXAMPLE A. Let

$$G = \left\{g_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \, \theta \text{ in } \mathbf{R}\right\}$$

and let $V = \mathbf{R}^2$, with the action $\alpha : G \times V \to V$ given by matrix multiplication. $G = SO(2, \mathbf{R})$ is called the *group of plane rotations*, as each g_θ is a counter clockwise rotation through an angle of θ radians. Thus, for any $v = \binom{x}{y}$ in $\mathbf{R}^2$, $\mathcal{O}_v$ is the circle in $\mathbf{R}^2$ centered at the origin and having radius $(x^2 + y^2)^{1/2}$. This example explains in a satisfactory way the term orbit, in the sense that an orbit may be viewed as

a path traced out in V with G acting as the parameter space. Viewing G as the set of all 2 by 2 matrices satisfying the equations

$$a_{11} = a_{22}, \quad a_{12} = -a_{21}, \quad \text{and} \quad a_{11}a_{22} - a_{12}a_{21} = 1,$$

G is easily seen to be an affine algebraic group. On the ring level, the action of G has a pleasant interpretation. The function $p(X, Y) = X^2 + Y^2$ is obviously an invariant, and in fact, it is not hard to see that it is a generator for the ring of invariants; i.e., $\mathbf{R}[X, Y]^G = \mathbf{R}[p]$. For suppose that $f(X, Y)$ is some invariant. Since G acts by graded automorphisms on $\mathbf{R}[X, Y]$, each homogeneous piece of f is also an invariant. So suppose that f is homogeneous. Since $(-x, y)$, $(x, -y)$ and (x, y) all lie in the same orbit, it is clear that X and Y must always occur to even powers in f. Let $d = 2r$ be the degree of f, and write

$$f(X, Y) = a_0X^d + a_1X^{d-2}Y^2 + \cdots + a_rY^d.$$

The fact that $(0, (x^2 + y^2)^{1/2})$ is always in the orbit of (x, y) yields

$$f(x, y) = f(0, (x^2 + y^2)^{1/2}) = a_r(x^2 + y^2)^r.$$

Thus $f(X, Y) = a_rp(X, Y)^r$. Of course the level curves of p are just the orbits of the action.

EXAMPLE B. If $G = GL_n(K)$ and $V = K^n$, the vector space of n by 1 column matrices, then G acts naturally on V by matrix multiplication. There are just two orbits in this case, the orbit of the zero column, and the orbit of any non-zero column. This latter orbit is open in K^n and thus, by continuity (or, what is the same, by Weyl's principle of the irrelevancy of algebraic inequalities, see Dieudonné and Carrell [3]) every invariant is constant.

EXAMPLE C. Let $G = GL_n(K)$ and let V be the space of n by n symmetric matrices. $\alpha : G \times V \to V$ is defined by congruence, $\alpha(g, A) = gAg^t$. Evidently the zero matrix is a fixed point for this action. However, in contrast to Example B there are usually more than two orbits. Explicitly, if K is algebraically closed there are $n+1$ orbits, $\mathcal{O}_{B_i}$, $i = 0, \ldots, n$, where $B_i = (b_{kt})$ is defined by

$$b_{kk} = 1, \text{for } k = 1, \ldots, i \text{ and } b_{kt} = 0 \text{ otherwise.}$$

As is well known, the B_i's are a complete set of canonical forms for the n by n symmetric matrices over K. The non-singular matrices comprise the orbit $\mathcal{O}_{B_n}$. Since $\mathcal{O}_{B_n}$ is Zariski open, $K[V]^G = K$.

EXAMPLE D. Let $\mathcal{S}_n = \mathrm{Perm}\{1, \ldots, n\}$ and consider the action of $\mathcal{S}_n$ on K^n, defined by

$$\alpha(\sigma, (x_1, \ldots, x_n)) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Each orbit is closed, since it is finite, and the absence of open orbits suggests that there exist non-constant invariant functions.

Examples of such functions are provided by the elementary symmetric functions in the variables $X_1, \ldots, X_n$ and, moreover, it is shown in [5] that these polynomials generate the ring $K[X_1, \ldots, X_n]^{\mathcal{S}_n}$.

If $G \times V \to V$ is an algebraic transformation group and G is a reductive group (in characteristic zero, one whose linear representations are direct sums of simple representations, for example, finite groups, $GL_n(K)$, $SL_n(K)$, $SO_n(K)$, or an orthogonal or symplectic group) the ring of invariants, $K[V]^G$, is finitely generated.

Now our main interest in the statement $K[V]^G = K[f_1, \ldots, f_r]$ is in the determination of an answer to the classification problem: given v and w, when do they lie in the same orbit? Assuming finite generation, an obvious necessary condition is $f_i(v) = f_i(w)$ for all i. For example, the reader may already have come in contact with the classical proposition which asserts that two n-tuples differ by a permutation if and only if each elementary symmetric function takes the same value on both n-tuples. That is, for Example D, the necessary condition cited above is also sufficient. The sufficiency of this condition may be formulated in several useful ways,

1. If $\mathcal{O}_v \neq \mathcal{O}_w$, then there is an i with $f_i(v) \neq f_i(w)$—*the ring of invariants separates orbits*, or

2. $\mathcal{O}_v = \Phi^{-1}(\Phi(v))$, where $\Phi: V \to K^r$ is defined by $\Phi(v) = (f_1(v), \ldots, f_r(v))$—*the orbits are the fibers of* Φ.

In this case, the orbit space V/G can be identified with $\Phi(V) \subseteq K^r$, which by the Hilbert's *Nullstellensatz* naturally has the structure of a variety, provided that K is algebraically closed. This latter condition on K is absolutely necessary, as Example A shows. In this example, the ring of invariants $\mathbf{R}[X^2 + Y^2]$ separates points, but $\Phi(\mathbf{R}^2) = \mathbf{R}^+ \cup \{0\}$ is not a variety, although it is a manifold with boundary. Notice that the second formulation requires that all orbits be Zariski closed, since for any continuous map, $\Phi: X \to Y$, of T_1-spaces, $\Phi^{-1}(y)$ is closed for all y in Y. This pathology is often the ruin of a complete classification by polynomial invariants (see Examples B and C). On the other hand, as the example of binary semi-invariants easily shows, there can be (infinitely) many closed orbits in a single fiber. Mumford has shown that this cannot occur when the group acts reductively. The following seems to be the best result available (see [3]).

THEOREM. (Mumford–Nagata) *If G is a reductive algebraic group, and if the orbits of the action $\alpha: G \times V \to V$ are closed in V, then the orbits are separated by the ring of invariant functions, $K[V]^G$, which is also the coordinate ring of the affine algebraic variety V/G. In particular, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are distinct orbits, there is an invariant which is* 1 *on $\mathcal{O}_1$ and* 0 *on $\mathcal{O}_2$.*

In case there exist non-closed orbits one must study the fibers of Φ more closely; $\Phi^{-1}(\Phi(v))$ always contains $\mathcal{O}_v$ and its closure. In Example B, there exists one fiber which is the union of an open dense orbit and a closed orbit, the origin. Although these orbits are inseparable by polynomial invariants, they can be geometrically and algebraically distinguished by their dimensions, that is, the open orbit is n-dimensional while the closed orbit is zero-dimensional.

The situation in Example C is similar, and we illustrate how the dimension of an orbit is calculated algebraically. For any v there is a surjective morphism $\rho_v: G \to \mathcal{O}_v$ defined by $\rho_v(g) = g(v)$. Thus $\dim \mathcal{O}_v$ can be computed in several ways. If S_v denotes the *isotropy group of G at v*,

$$S_v = \{g \text{ in } G \mid g(v) = v\},$$

then $\mathcal{O}_v = G/S_v$, and hence $\dim \mathcal{O}_v = \dim G - \dim S_v$. On the other hand, if $d\rho_v(1_G)$ is the differential of ρ_v at the identity, 1_G, of G, one also obtains $\dim \mathcal{O}_v = \operatorname{rank} d\rho_v(1_G)$. In Example C, $d\rho_{B_i}(1_G)$ is the linear transformation from $M_n(K)$ into $M_n(K)$ defined by

$$A \to AB_i + B_iA.$$

The rank of this linear map may be computed as

$$d(i) = i(i+1)/2 + i(n-i).$$

Noting that $d'(i) = -i + n + 1/2$ is positive for i in $[o, n]$, $d(i)$ is easily shown to be an increasing function of i. Again, the orbits are separated by their dimension.

Returning to the general transformation group $G \times V \to V$, by a continuity argument, $\mathcal{C}\ell(\mathcal{O}_v)$ is stable under G and hence is a union of orbits. It can be shown that $\mathcal{O}_v$ is open and dense in $\mathcal{C}\ell(\mathcal{O}_v)$. In fact, the boundary of $\mathcal{C}\ell(\mathcal{O}_v)$ is a union of orbits of strictly smaller dimension. Now the fibers of Φ are G-stable closed sets, and often a fiber is the closure of the largest dimensional orbit which it contains. The invariants clearly separate orbits in distinct fibers, and one (perhaps naively) hopes that within each fiber there are finitely many orbits distinguishable by their dimensions (as in Example C). In this situation, one has a solution to the orbit space problem which is truly algebraic; i.e., one can decide, by finitely many polynomial equalities and inequalities, whether $\mathcal{O}_v$ equals $\mathcal{O}_w$. The equalities arise from invariants and the inequalities from orbital dimension calculations involving minor determinants of Jacobian matrices. In the next section we subject the similarity problem to this type of analysis, calculating the invariants and orbital dimension.

Of course, the success of such a program depends rather critically on the existence of only finitely many orbits in a given fiber. We should remark that one of the most fundamental geometric objects,

projective space, arises as an orbit space for an action which is not of this type. Explicitly, one may consider the action by scalar multiplication of the multiplicative group, $G_m(K)$, on the space K^n. Recall that $G_m(K) = K - \{0\}$, so that the orbit of any non-zero vector x is the line through x and the origin with the origin deleted. Since the action is by linear transformations, the origin is a fixed point. Any invariant function must satisfy the equation

$$f(\lambda x) = f(x), \text{ for } \lambda \text{ in } G_m(K).$$

In other words, each invariant is homogeneous of degree zero. Trivially, the only such polynomials are constant. Hence, there is only one fiber containing infinitely many orbits of dimension one together with a fixed point. It is well known, however, that after discarding the fixed point (or, more generally, retaining only orbits of maximal dimension) the action $G_m(K) \times K^n - \{0\} \to K^n - \{0\}$ gives rise to an orbit space, viz., $\mathbf{P}_{n-1}(K)$, which is locally affine. In particular, one might suspect that, by enlarging the category of varieties under consideration (say from affine to those spaces which are locally affine) and by discarding "bad" orbits, a satisfactory partial solution to the orbit space problem can be obtained. Such a program has in fact been carried out by Mumford in his treatise, *Geometric Invariant Theory* [**9**]. A very readable account of the affine case may be found in Fogarty [**5**]. The reader might also consult the survey article, Dieudonné and Carrell [**3**], and the classical treatise, Weyl [**12**]. Algebraic groups and algebraic transformation groups are analyzed in detail in Borel [**1**] and Humphreys [**8**].

3. Similarity as an orbit space problem. Here K is an algebraically closed field of characteristic zero. The similarity class of A in $M_n(K)$ is just $\mathcal{O}_A$—the orbit of A under the algebraic transformation group $GL_n(K)$ acting by conjugation. If K/k is any extension of fields and A and B in $M_n(k)$ are similar over K, then they can also be shown to be similar over k ([7, p. 269]). Thus no generality is lost in assuming K to be algebraically closed. We begin by calculating the ring of invariant functions and dim $\mathcal{O}_A$ relying on Jordan canonical form. Then we do a fiber analysis as in the preceding section.

Identifying $M_n(K)$ with K^{n^2}, the coordinate ring is $K[X_{11}, X_{12}, \ldots, X_{nn}]$. For A in $M_n(K)$, $\det(\lambda I_n - A) = \lambda^n + c_1(A)\lambda^{n-1} + \cdots + c_n(A)$ is the *characteristic polynomial* of A. Similar matrices have identical characteristic polynomials, so the functions $c_1, \ldots, c_n$ are invariants. Putting A in triangular form, it can be seen that $c_i(A)$ is $(-1)^i$ times the ith elementary symmetric function in the eigenvalues of A. We show next that $K[c_1, \ldots, c_n]$ is the ring of invariant functions.

For a typical matrix $A, A \wedge I_n - I_n \wedge A$ can be viewed as the matrix of a linear transformation on $\wedge^2(K^n)$. $d(A) = \det(A \wedge I_n - I_n \wedge A)$ is the *discriminant* of A. If $d(A) \neq 0$, then A has n distinct eigenvalues; A is similar to a diagonal matrix. We let $D = D_n(K)$ be the n by n diagonal matrices. D is identified with K^n and its coordinate ring with $K[X_{11}, \ldots, X_{nn}]$. The symmetric group $\mathcal{S}_n$ (viewed as n by n permutation matrices—each row and column having precisely one non-zero entry, a one) stabilizes D by conjugation. There is a natural homomorphism from the coordinate ring of K^{n^2} to the coordinate ring $K[D]$ given by restricting a function f to the set of diagonal matrices. If f is invariant under $GL_n(K)$, the $f|_D$ is invariant under $\mathcal{S}_n$. Thus we have a homomorphism $\gamma : K[\{X_{ij}\}]^{GL_n(K)} \to K[D]^{\mathcal{S}_n}$ given by $\gamma(f) = f|_D$. To see that γ is injective let f be an invariant such that $f|_D$ is identically zero. Let $U = K^{n^2}|_d$—the principal affine open subset consisting of the nonzeroes of the discriminant. For every A in U there is a g in $GL_n(K)$ with $g^{-1}Ag$ in D. Thus $0 = f|_D(g^{-1}Ag) = f^g(A) = f(A)$. Thus $f|_U$ is identically zero. By a continuity argument f is zero.

Now $K[D]^{\mathcal{S}_n} = K[s_1, \ldots, s_n]$ where s_i is the ith elementary symmetric function in $X_{11}, \ldots, X_{nn}$ (see Example D). Also $\pm c_i|_D = s_i$, so γ is surjective. By means of the isomorphism γ one sees that $K[c_1, \ldots, c_n]$ is the ring of invariants.

For α in K we denote by $J(\alpha; r)$ the r by r *Jordan block matrix* having α's on the diagonal, 1's on the superdiagonal, and zeroes elsewhere. For an r by r matrix A and an s by s matrix B, we denote by $A \oplus B$ the $r+s$ by $r+s$ matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Every matrix A is similar to a direct sum of Jordan block matrices via its Jordan form. Let the Jordan form of A be given by $\bigoplus_{i=1}^{s}\bigoplus_{j=1}^{r_i} J(\gamma_i; n_{ij})$ where for each i one has $n_{i1} \leqq \cdots < n_{ir_i}$. To compute $\dim \mathcal{O}_A$ we compute the rank of the differential $d\rho_A(I_n)$, where $\rho_A: GL_n(K) \to \mathcal{O}_A$, $\rho_A(g) = gAg^{-1}$. The tangent space to $GL_n(K)$ at I_n is $M_n(K)$, and $d\rho_A(I_n)(B) = BA - AB$. Thus $\dim \mathcal{O}_A = n^2 - \dim Z(A)$, $Z(A)$ being the centralizer of A in $M_n(K)$. Hence

$$\dim \mathcal{O}_A = n^2 - \sum_i \sum_j (2r_i + 1 - 2j)n_{ij},$$

by a classical formula for $\dim Z(A)$ (see **[2]**, section 2; it is also shown in **[2]** that $\dim \mathcal{O}_A = \operatorname{rank}(A \otimes I_n - I_n \otimes A)$, thus condition (ii) of the theorem mentioned at the outset is connected with orbital dimension). As should be expected, orbital dimension is directly influenced by the block sizes, the number of blocks, and the multiplicities of the eigenvalues. In particular, in view of what is displayed in the Jordan form, there is some hope that if the invariants alone do not separate orbits, then the invariants plus dimension will.

As in section two, let $\Phi: K^{n^2} \to K^n$ by $\Phi(A) = (c_1(A), \ldots, c_n(A))$. $\Phi^{-1}(\Phi(A))$—the fiber of A—is the set of all matrices having the same characteristic polynomial, hence the same eigenvalues (counting multiplicities). Each fiber is the set of all matrices whose Jordan forms have identical diagonals, that is, semi-simple parts. The ring of invariant functions does not separate orbits, but does not miss that goal by much. Each fiber is a finite union of similarity classes. For the data determining a particular fiber, a given distribution of eigenvalues and multiplicities λ_i, r_i $i = 1, \ldots, s$ $(n = \Sigma r_i)$, there is a *regular* element A of $M_n(K)$ given by $J(\lambda_1; r_1) \oplus \cdots \oplus J(\lambda_s; r_s)$, and a *semi-simple* element A_s, the diagonal of A. It turns out that the fiber of A is $\mathcal{C}\ell(\mathcal{O}_A)$. Also $\mathcal{O}_{A_s}$ is the unique closed orbit in this fiber.

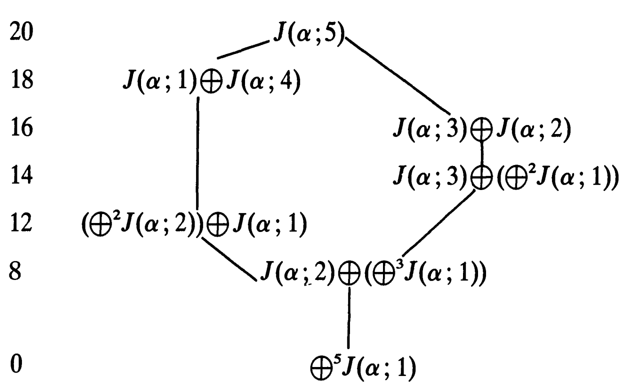
In fact, one can show that if A and B are in the same fiber, then $\mathcal{O}_A \subset \mathcal{C}\ell(\mathcal{O}_B)$ if and only if the Jordan form of A can be obtained by deleting 1's on the superdiagonal of the Jordan form of B. The computation

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-2} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-2} \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & t \\ 0 & \alpha \end{pmatrix} \equiv A_t$$

is suggestive of this behavior. Explicitly, if $A = J(\alpha; 2)$, then A_t is in $\mathcal{O}_A$ for all t, thus, letting "t approach zero", one obtains

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \text{ in } \mathcal{C}\ell(\mathcal{O}_A).$$

For each fiber, one thus has in mind a lattice where each similarity class in the fiber is put on a level determined by its dimension, and is joined by a path of upward lines to a class above it in the lattice only if it lies in the closure of this class. For example:



$$\begin{tikzcd}
12 & & J(\alpha;2)\oplus J(\beta;2) \arrow[dl, dash] \arrow[dr, dash] & \\
10 & (\oplus^2 J(\alpha;1))\oplus J(\beta;2) \arrow[dr, dash] & & J(\alpha;2)\oplus(\oplus^2 J(\beta;1)) \arrow[dl, dash] \\
8 & & \oplus^2(J(\alpha:1)\oplus J(\beta;1)) &
\end{tikzcd}$$

The second example is a crushing blow to our hopes—within certain fibers, orbital dimension is insufficient to separate orbits.

For a moment, we return to the general algebraic transformation group $G\times V\to V$. From one point of view all one needs are criteria for deciding when two points lie in the same orbit. Thus one is led naturally to consider in $V\times V$ the set $D=\{(u,v)\mid \mathcal{O}_u=\mathcal{O}_v\}$. D is the image of the morphism $G\times V\to V\times V$ defined by $(g,v)\to(v,g(v))$, and due to a theorem of Chevalley, as such it is a constructible subset of $V\times V$. An explicit description of the equalities and inequalities defining D is called a *decidability criterion* for the orbit space problem.

This is the nature of the theorem with which these notes began. Conditions (i) and (ii) of this theorem give explicitly the polynomial equalities and inequalities describing the subset $\{(A,B)\mid A$ is similar to $B\}$ (see section three of [2]). As such it can be called an "algebraic" solution to the similarity problem. Before comparing this theorem to other solutions of the similarity problem, we pause to give an example of an orbit space problem with a decidability criteria of a similar nature, and to sketch the proof of our result.

EXAMPLE E. Let $V=M_n(K)$ and let $G=GL_n(K)$ act by left multiplication, that is, G acts on elements of V by row operations. $\mathcal{O}_A=\mathcal{O}_B$ if and only if A and B have the same row space. The only invariant functions are constant, so there is one fiber and infinitely many orbits. $\dim\mathcal{O}_A=(\operatorname{rank}(A))(n)$; $\mathcal{O}_{I_n}$ is open, and consists of all non-singular matrices. One can show that $\mathcal{O}_A\subset\mathcal{C}\ell(\mathcal{O}_B)$ if and only if the row space of A is contained in that of B. If for A and B in $M_n(K)$ we construct the $2n$ by n matrix $\begin{pmatrix}A\\B\end{pmatrix}$, then $\mathcal{O}_A=\mathcal{O}_B$ if and only if $\operatorname{rank}(A)=\operatorname{rank}\begin{pmatrix}A\\B\end{pmatrix}=\operatorname{rank}(B)$. The left hand (right hand) equality says that the row space of B is contained in that of A (the row space of A is contained in that of B). Thus, the rank conditions alone (after all, the only invariants are constant) give the equalities and inequalities describing $\{(A,B)\mid A$ is row equivalent to $B\}$ in $M_n(K)\times M_n(K)$.

THEOREM (Byrnes–Gauger). *Let K be a field of characteristic zero, and let A and B be n by n K-matrices. Then A is similar to B if and only if*

(i) *A and B have the same characteristic polynomial, and*

(ii) *for all even $k\leqq 2n-2$ one has*

$$\operatorname{rank}(A\otimes I_n-I_n\otimes A)^k=\operatorname{rank}(B\otimes I_n-I_n\otimes B)^k=\operatorname{rank}(A\otimes I_n-I_n\otimes B)^k.$$

Sketch of Proof. The necessity of the conditions over any field is obvious. Furthermore, there is no harm in assuming K to be algebraically closed—K-matrices similar over an extension field are also similar over K. The sufficiency is established by an induction argument based on the following lemmas.

LEMMA 1. *rank $(J(0;n))^k=n-k$ if k is less than or equal to n, and the rank is zero otherwise.*

LEMMA 2. *If λ_1,λ_2 belong to K, then $J(\lambda_1;n)\otimes I_m-I_n\otimes J(\lambda_2;m)$ is similar to*

$$\bigoplus_{t=1}^{\min(n,m)}J(\lambda_1-\lambda_2;m+n+1-2t).$$

REMARK. Lemma 2 is essentially a consequence of the representation theory of the simple Lie algebra sl_2 of 2 by 2 trace-zero matrices, and it is only true in characteristic zero. It is what presently restricts our similarity theorem to characteristic zero, see [6].

LEMMA 3. *Let A and B be n by n matrices and suppose D is a square matrix such that $A=A'\oplus D$,*

$B = B' \oplus D$. *Suppose also that*

$$(*)\ \operatorname{rank}(A \otimes I_n - I_n \otimes A)^k + \operatorname{rank}(B \otimes I_n - I_n \otimes B)^k - 2\operatorname{rank}(A \otimes I_n - I_n \otimes B)^k = 0.$$

Then one has a similar identity in replacing A by A' and B by B'.

In proving the sufficiency, condition (ii) is replaced by (*) since it goes through the induction step more readily. Condition (i) says that A and B have the same eigenvalues (counting multiplicities), and hence, up to a permutation, their Jordan forms have identical diagonals. Next, one replaces A and B by their direct sums in terms of Jordan blocks, and expands $A \otimes I_n - I_n \otimes A$, $B \otimes I_n - I_n \otimes B$, and $A \otimes I_n - I_n \otimes B$ using Lemma 2. Each of the latter can be written as $P_i \oplus N_i$ where P_i is an r by r non-singular matrix and N_i is an s by s nilpotent matrix. Condition (*) then yields

$$(**) \qquad \operatorname{rank}(N_1^k) + \operatorname{rank}(N_2^k) - 2\operatorname{rank}(N_3^k) = 0$$

for all even $k \leqq 2n-2$. Let b_1 be the largest Jordan block size in A, and let b_2 be the largest block size in B. Suppose $b_1 \geqq b_2$ and

(***) there is no eigenvalue having the same largest block size in both A and B.

Then Lemma 2 says that the largest block in N_1 is of size $2b_1 - 1$, and the largest block size in N_2 is $2b_2 - 1$, while (***) and Lemma 2 indicate that the largest block size in N_3 is $\leqq 2b_1 - 2$. By Lemma 1, (**) is false for $k = 2b_1 - 2$. Hence (***) is false. So for some eigenvalue, A and B have the same largest block size. Apply Lemma 3 with D equal to this common block, and proceed by induction on n.

There are philosophical as well as mathematical distinctions between the solution to the similarity problem mentioned here, and the best known classical solutions, Jordan and rational canonical form. Given two matrices A and B, the classical determination of similarity is to calculate individually their canonical forms and to compare them. Our method has as an essential ingredient certain algebraic calculations which simultaneously employ the entries of A and B. It does not give the similarity invariants of a single matrix.

While we have exhibited the "algebraic" nature of our solution to the similarity problem, the classical solutions are not "algebraic." The process of obtaining a Jordan form is not "algebraic" in that it involves the transcendental and generally impossible problem of solving a 1-variable polynomial equation. To put it another way, one cannot express the eigenvalues of a matrix as polynomial functions of the entries. On the other hand, the process of obtaining a rational form, or what is equivalent, the invariant factors, is algorithmic, although the mapping associating to a matrix its list of invariant factors cannot be "algebraic" (a morphism). If it were, by continuity, similarity classes would necessarily be Zariski-closed. We have seen that this is generally untrue.

In **[13]** we have eliminated the characteristic-zero restriction and all but $k = 1$ from the 2nd condition of our theorem.

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PROGRESS REPORTS

EDITED BY P. R. HALMOS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but some new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

INVARIANT SUBSPACES

P. R. HALMOS

The main reason for the success of finite-dimensional complex linear algebra is the existence of eigenvalues and eigenvectors. An eigenvector of a transformation spans a 1-dimensional subspace invariant under the transformation; it makes possible a study of the transformation on the whole space via a study of its behavior on smaller and therefore more manageable subspaces. The existence of eigenvalues is a deep fact, derived by techniques far from the spirit of linear algebra. What it comes down to is that an eigenvalue, a geometric concept, is the same as a zero of the characteristic polynomial, an algebraic concept, and the existence of such zeroes is guaranteed by the fundamental theorem of algebra, an analytic tool.

Neither the methods nor the results hinted by the preceding paragraph extend to Hilbert space, which is the simplest, most natural, and most useful infinite-dimensional generalization of finite-dimensional vector spaces. There is no simple generalization of chracteristic polynomials, what there is has no useful relation to eigenvalues, there are transformations that have no eigenvectors at all, and, for all anyone knows, non-trivial invariant subspaces may fail to exist.

That, in fact, is the invariant subspace problem. If A is an operator (bounded linear transformation) on a separable infinite-dimensional Hilbert space H, does there exist a subspace M of H (closed linear manifold) different from both O and H and invariant under A ($AM \subset M$)? If the topological restrictions (boundedness and closure) are removed, the answer (yes) becomes easy. If the space is too small (finite-dimensional), the answer (yes) is classical; if the space is too large (non-separable), the answer (yes again) is trivial.

The most important class of operators for which the answer has been known to be yes for a long time is the class of Hermitian (and, more generally, normal) operators; this fact is a corollary of the infinite-dimensional version of the process of diagonalizing a Hermitian matrix (the spectral theorem).

The next most important class is connected with the concept of compactness (complete continuity). An operator on a Hilbert space is *compact* if it maps the unit ball onto a compact set. The first major invariant subspace theorem was obtained by Aronszajn and Smith (1954): using an idea of von Neumann, they proved that every compact operator has non-trivial invariant subspaces. The method depends on approximation by operators of finite rank.

The Aronszajn–Smith technique seemed to be so sharply focused on its particular purpose that for a dozen years it resisted even mild generalization; it was, for instance, not known whether the conclusion remained true for operators whose square is compact. Now that is known; the extension to polynomially compact operators was obtained by Bernstein and Robinson (1966). They presented their result in the metamathematical language called non-standard analysis, but, as it was realized very soon, that was a matter of personal preference, not necessity.

One of the next problems in this connection that received much attention can be expressed this way: if two compact operators commute, must they have a non-trivial invariant subspace in common? The Aronszajn–Smith–Bernstein–Robinson technique could be stretched to do more than it did at first, but it did not seem to be elastic enough to stretch this far. The final advance (which may close the subject of invariant subspaces of compact operators) was made by Lomonosov (1973); he proved that for every non-zero compact operator there is a non-trivial subspace that is invariant not only under that operator but under every operator that commutes with it. The main tool in the proof is the Schauder fixed point theorem for continuous mappings of a compact convex set into itself.

The invariant subspace problem for operators that have nothing to do with compact ones is still open.

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MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

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A GAME WITH n NUMBERS

ROBERT MILLER

In his book [1], R. Sprague considers the following mathematical recreation. We start with four integers a, b, c, d, and form their absolute differences, $|a-b|, |b-c|, |c-d|, |d-a|$. We then apply

the same operation to these four integers, etc. Sprague shows that after a finite number of steps, one always reaches four zeroes.

The obvious generalization of this is to start with a sequence of n integers $a_1, a_2, \ldots, a_n$, and form their absolute differences, $|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_n - a_1|$. We then iterate this transformation S, and ask whether the sequence $0, 0, \ldots, 0$ must always be reached. In this paper we will show that the answer is affirmative if and only if n is a power of 2.

To the sequence $a_1, a_2, \ldots, a_n$ we associate the vector $(b_1, b_2, \ldots, b_n)$ over $\mathbf{F}_2$, where $a_i \equiv b_i \pmod 2$. The above transformation S induces the linear transformation $T(b_1, \ldots, b_n) = (b_1 + b_2, b_2 + b_3, \ldots, b_n + b_1)$ on $\mathbf{F}_2^n$. The matrix of T is

$$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{pmatrix}$$

and one readily finds that its characteristic polynomial $|\lambda I - T| = (\lambda + 1)^n + 1$. If every sequence $a_1, a_2, \ldots, a_n$ is mapped to $0, 0, \ldots, 0$ by some iterate of S, then T is nilpotent, and hence

$$(*) \qquad (\lambda + 1)^n + 1 \equiv \lambda^n \pmod 2,$$

where the coefficients are from $\mathbf{F}_2$. Let $n = 2^k m$ where m is odd. Then $(*)$ becomes

$$(\lambda^{2^k} + 1)^m + 1 \equiv \lambda^{2^k m} \pmod 2.$$

If $m > 1$, the coefficient of $\lambda^{2^k(m-1)}$ on the left is $\binom{m}{1} = m \equiv 1 \pmod 2$, since m is odd, but on the right it is 0. Hence in order for T to be nilpotent, it is necessary that $m = 1$, i.e., $n = 2^k$. Conversely, if $n = 2^k$, then the characteristic polynomial of T is $(\lambda + 1)^{2^k} + 1 \equiv \lambda^{2^k} + 1 + 1 \equiv \lambda^{2^k} \pmod 2$. Hence all of the eigenvalues of T are zero, and so T is nilpotent.

Now if T is nilpotent, then T^n maps any binary sequence $b_1, b_2, \ldots, b_n$ onto the sequence $0, 0, \ldots, 0$. Hence if $a_1, a_2, \ldots, a_n$ is any sequence of integers with $a_i \equiv b_i \pmod 2$, $i = 1, 2, \ldots, n$, then

$$S^n(a_1, a_2, \ldots, a_n) \equiv T^n(b_1, b_2, \ldots, b_n) \equiv (0, 0, \ldots, 0) \pmod 2.$$

Thus $S^n(a_1, a_2, \ldots, a_n) = (2a_1^{(1)}, 2a_2^{(1)}, \ldots, 2a_n^{(1)})$ for some $a_i^{(1)} \in \mathbf{Z}$, $i = 1, 2, \ldots, n$.

Iterating and using linearity, we obtain

$$\begin{aligned} S^{2n}(a_1, a_2, \ldots, a_n) &= S^n(2a_1^{(1)}, 2a_2^{(1)}, \ldots, 2a_n^{(1)}) \\ &= 2S^n(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}) \\ &= 2^2(a_1^{(2)}, a_2^{(2)}, \ldots, a_n^{(2)}), \end{aligned}$$

where $a_i^{(2)} \in \mathbf{Z}$, $i = 1, 2, \ldots, n$. Proceeding by induction, we see that S^{rn} maps $a_1, a_2, \ldots, a_n$ onto a sequence of the form $2^r a_1^{(r)}, 2^r a_2^{(r)}, \ldots, 2^r a_n^{(r)}$.

On the other hand, each of the numbers $2^r a_1^{(r)}, 2^r a_2^{(r)}, \ldots, 2^r a_n^{(r)}$ is $\leqq \max |a_i - a_j|$. Thus unless all $a_i^{(r)} = 0$ for some r, the sequence $2^r a_1^{(r)}, 2^r a_2^{(r)}, \ldots, 2^r a_n^{(r)}$ is unbounded, since if $a_k^{(r)} \neq 0$, then $2^r a_k^{(r)} \geqq 2^r$. This violates the constraint that $2^r a_k^{(r)} \leqq \max |a_i - a_j|$.

The above argument shows that if $n = 2^k$, then by repeating the transformation S, any sequence of integers, $a_1, a_2, \ldots, a_n$ will eventually be mapped to the sequence $0, 0, \ldots, 0$. However, the number of

repetitions of S required can be arbitrarily large. R. Sprague [1] proves that given $N > 0$, there exists a sequence of 4 integers requiring at least N iterations before being transformed into $0, 0, 0, 0$. If $n = 2^k$, $k \geqq 2$, then we need only take 2^{k-2} copies of the sequence Sprague uses to obtain a sequence of 2^k integers requiring at least N iterations of the transformation to become $0, 0, \ldots, 0$.

Added in proof: This problem was posed and solved by B. Freeman (*The four numbers game*, Scripta Math. 14 (1948), 35–47) and was posed by J. M. Hammersley (Problem 69–1, SIAM Review 11 (1969), 73–74) with a solution given by L. Carlitz and R. Scoville (SIAM Review 12 (1970), 247–300). — *Ed.*

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A SHORT PROOF OF THE CHEVALLEY-JACOBSON DENSITY THEOREM, AND A GENERALIZATION

LOUIS H. ROWEN

Suppose a ring R has an irreducible left module M, which is "faithful" in the sense that $rM \neq 0$ for all $r \neq 0$ in R. By Schur's lemma, $D = \mathrm{End}_R M$ is a division ring. A fundamental theorem in the structure of rings is

DENSITY THEOREM (Chevalley–Jacobson). *If $x_1, \ldots, x_u$ are D-independent elements of M and if $x'_1, \ldots, x'_u$ are arbitrary, then there exists r in R, such that $rx_i = x'_i$, $1 \leqq i \leqq u$.*

Proof. Induction. The theorem is obvious if $u = 1$; assume the theorem holds for $u - 1$, and let $x_1, \ldots, x_u$ be independent elements of M. By induction, $M^{u-1} = R(x_1, \ldots, x_{u-1})$.

Claim: There exists r_u such that $r_u x_u \neq 0$ and $r_u x_i = 0$ for all $i \neq u$. Otherwise the map $\phi: R(x_1, \ldots, x_{u-1}) \rightarrow M$ given by $\phi(r(x_1, \ldots, x_{u-1})) = rx_u$ is a well-defined module homomorphism. Hence $\phi \in \mathrm{Hom}_R(M^{u-1}, M) \approx (\mathrm{End}_R M)^{u-1} = D^{u-1}$. Writing ϕ as $(d_1, \ldots, d_{u-1}) \in D^{u-1}$, we have $x_u = \phi(x_1, \ldots, x_{u-1}) = \sum_{i=1}^{u-1} x_i d_i$, contrary to the assumed D-independence of $x_1, \ldots, x_u$.

The claim is established. By symmetry, for each j, $1 \leqq j \leqq u$, there exist r_j such that $r_j x_j \neq 0$ and $r_j x_i = 0$ for $i \neq j$. Since M is irreducible, we have r'_j such that $r'_j r_j x_j = x'_j$. Let $r = \sum_{j=1}^{u} r'_j r_j$. For each i, $rx_i = r'_i r_i x_i = x'_i$. Q.E.D.

Proving the density theorem via the claim is standard cf. [2, 3]; the new element in this proof is the map ϕ. This proof of the claim can be put in a more general setting, as pointed out to me by C. Faith and B. Osofsky. Setting $A = \mathrm{End}_R M$, define the "double annihilator condition" as, "For each $x_1, \ldots, x_{u-1}$ in M and x in $M - \sum_{i=1}^{u-1} x_i A$, there exists r in R such that $rx \neq 0$ and $rx_i = 0$ for all $i \leqq u - 1$."

PROPOSITION. *The double annihilator condition (for M) holds if we have the following condition:*

(∗) For all $t > 0$ and for each nonzero element y of M^t, every homomorphism $Ry \rightarrow M$ extends to a homomorphism from M^t to M.

Proof. Same as the proof of the claim, using property (∗) instead of induction. Q.E.D.

The best example of a module satisfying (∗) is a QI-module M, cf. [1, pp. 173–174], i.e., a module for which every homomorphism from a submodule to M lifts to an endomorphism of M. Indeed, it is well known that if M is QI then M^t is QI for all $t \in \mathbf{Z}^+$. (Proof: Write M_k = {elements of M^t whose only nonzero components are in the kth position} $\approx M$, $M^{(0)} = \{(0, \ldots, 0)\} \subset M^t$, and inductively $M^{(k)} = M^{(k-1)} + M_k \subseteq M^t$. Using induction, given a homomorphism $\phi: N \rightarrow M^t$, N a submodule of M^u, it suffices to extend the domain of ϕ from $N + M^{(k-1)}$ to $N + M^{(k)}$ for each $k \geqq 1$. But for each $i \leqq t$, ϕ induces a homomorphism ψ_i from $(N + M^{(k-1)}) \cap M_k \rightarrow M^t \rightarrow M_i$, where the last map is the

canonical projection. Since $M_i \approx M \approx M_k$ canonically and M is QI, each ψ_i can be lifted to a homomorphism from M_k to M_i; clearly, by using $\Sigma_{i=1}^{t}\psi_i$ we can extend the domain of ϕ from $N + M^{(k-1)}$ to $N + M^{(k)}$, as desired.) Thus we have

THEOREM (Artin–Tate–Jacobson–Johnson–Wong [4]) QI *modules satisfy the double annihilator condition.*

(To get this more general result without knowing already that QI modules satisfy (*), the quickest proof is still the original Johnson–Wong proof.)

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HOW TO CONSTRUCT A REGULAR POLYGON

WAYNE BISHOP

In place of the customary existential proof of constructibility of a regular polygon, the author gives an explicit method of construction in general and all the steps in the construction of a 257-gon in particular.

Introduction. One ingredient in the recipe for "mathematician" is Gauss's result: for prime p, a regular p-gon is constructible by straight edge and compass if and only if p is a Fermat prime, $p = 2^k + 1$, [2], [4]. For general n, this becomes: a regular n-gon is constructible if and only if n can be factored as $n = 2^l \Pi p_i$ where the p_i are distinct Fermat primes.

That a single prime p must be Fermat in order that a regular p-gon can be constructed, follows at once from the fact that vertices of a regular p-gon can be thought of as complex roots of the polynomial $x^p - 1$ where $x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + 1)$ with the last factor irreducible by Eisenstein's Criterion [3]. To be constructible, $p - 1$ must be a power of two [3] and since p is prime, the exponent of 2 must also be a power of 2. The converse requires a bit of Galois theory. One notes that for any integer j, the map which sends the root $e^{2\pi i/p}$ to $e^{2\pi ij/p}$ determines a unique automorphism of K, the splitting field of $x^p - 1$ over Q, and any automorphism of K over Q can be viewed as such. Thus the Galois group $G(K/Q)$ is of order 2^k and is solvable. Reducing to a composition series and using the Galois correspondence, one has a tower of fields, each of degree two over its predecessor, exactly what is needed for constructibility.

Because this approach is so slick, few people realize that Gauss's procedure was constructive and, with a little reflection, straightforward. By using the Galois theory only to simplify the argument that the construction will always work (it is not needed to verify a given construction once made), there is a routine construction for any Fermat prime and primitive root corresponding to that prime. So routine in fact that a program has been written which describes completely what constructions to make and in what order. Furthermore, the constructions are simply a repetition of two simple constructions: construction of the roots of a monic polynomial of degree 2, where the coefficients have previously been constructed, and summation of integer multiples of line segments. Since these constructions lead to degree 2 extensions if the polynomial is irreducible, we need only k of them where $p = 2^k + 1$, a description of the rest of the construction, and some rational dependence upon the result. This requires creativity, however, and the promise is for a completely routine construction technique.

The method. Let g be a primitive root for the Fermat prime $p = 2^k + 1$; e.g., 3 for $p \neq 3$, let $\theta = e^{2\pi i/p}$, let

$$\alpha_{i,j} = \sum_{m \equiv j(2^i)} \theta^{g^m}, \quad (i = 1, \ldots, k; j = 0, 1, \ldots, 2^i - 1),$$

and $B_i = \{\alpha_{i,j}\}$. Note that B_k is just the set of pth roots of 1. The construction rests on the following fact.

The coefficients of $x^2 - (\alpha_{i,j} + \alpha_{i,j+2^{i-1}})x + \alpha_{i,j}\alpha_{i,j+2^{i-1}}$ are integer linear combinations of the elements of B_{i-1} and integers in case $i = 1$. To compute the coefficients needed, $\alpha_{i,j} + \alpha_{i,j+2^{i-1}} = \alpha_{i-1,j}$ and $\alpha_{i,j}\alpha_{i,j+2^{i-1}}$ can be computed by simply multiplying the defining sums and replacing $\theta^0 = 1$ by $-\Sigma_j \alpha_{i-1,j}$ should that term appear in the product. Grouping according to the elements of B_{i-1}, the computation is then complete. Thus each element of B_i is the root of some monic polynomial of degree 2 with coefficients integrally dependent on B_{i-1} and easily computed by the quadratic formula giving rise to the construction. Which of the two roots is $\alpha_{i,j}$ can be determined by summing rough approximations to the defining terms.

The construction is thus described as follows. Construct the 2 elements of B_1 over the rational complex plane Q^2. Then construct the 4 elements of B_2 using the result of B_1. Continue in this fashion until all elements of each B_i have been constructed. The result desired is just B_k together with the point $(1, 0)$.

For large values of k (e.g., $k = 8$) this process involves considerable redundancy which can be eliminated as follows. Since $\alpha_{k,0} = \theta = e^{2\pi i/p}$ will quickly determine the construction, we need only to keep track of that element of B_k. Since $\alpha_{k,0}$ is a root of $x^2 - (\alpha_{k,0} + \alpha_{k,2^{k-1}})x + \alpha_{k,0}\alpha_{k,2^{k-1}}$, keep track of the subset of B_{k-1} needed to re-express the coefficients as described above; i.e. the elements $\alpha_{k-1,0}$ and the terms which arise in $\alpha_{k,0}\alpha_{k,2^{k-1}}$. For each of these, find the needed elements of B_{k-2}, etc. Finally, start at the bottom with the elements determined and work up as before, but omit those which have been shown unnecessary.

The proof. Let $g, \theta, \alpha_{i,j}$, and B_i be as described above and let $F_i = Q[\alpha_{i,0}], i = 1, \ldots, k$. Then $F_k = Q[\theta]$ is the splitting field of $x^p - 1$. Define $\phi\colon F_k \to F_k$ by $\phi(\theta) = \theta^g$ extended to an automorphism of F_k.

THEOREM. *For $i = 1, \ldots, k$, F_i is the fixed field of $\langle \phi^{2^i} \rangle$ the subgroup of the Galois group $G(F_k / Q) = \langle \phi \rangle$ generated by ϕ^{2^i}, and B_i is a basis for F_i over Q.*

Proof. The set B_i is linearly independent over Q because in any vector space, if $\{b_i \mid i \in S\}$ is independent and $S_j \subseteq S$ are pairwise disjoint, then $\{\Sigma_{i \in S_j} b_i\}$ is independent. That B_i is in the fixed field of $\langle \phi^{2^i} \rangle$ follows from the fact that ϕ^{2^i} only permutes the terms of an element of B_i. The degree of the fixed field over Q is $[G(F_k / Q) \colon \langle \phi^{2^i} \rangle] = 2^k / 2^{k-i} = 2^i$. Since $|B_i| = 2^i$ as well, $Q[B_i]$ is the fixed field of $\langle \phi^{2^i} \rangle$. Clearly $F_i \subseteq Q[B_i]$. Inductively assuming that $F_{i-1} = Q[B_{i-1}]$, to show $F_i = Q[B_i]$ we only show $F_i \neq F_{i-1}$ because the subfields are linearly ordered and $[Q[B_i] \colon Q[B_{i-1}]] = 2$. This is trivial because $\phi^{2^{i-1}}(\alpha_{i,0}) \neq \alpha_{i,0}$.

COROLLARY. *For each i, j, $\alpha_{i,j}, \alpha_{i,j+2^{i-1}} \in F_{i-1}$ and in terms of the basis B_{i-1}, has coefficients from* **Z**. *The irreducible monic polynomial for $\alpha_{i,j}$ over F_{i-1} is $x^2 - (\alpha_{i,j} + \alpha_{i,j+2^{i-1}})x + \alpha_{i,j}\alpha_{i,j+2^{i-1}}$.*

Proof. By definition, $\alpha_{i,j} + \alpha_{i,j+2^{i-1}} = \alpha_{i-1,j} \subseteq B_{i-1}$. That the product is also in F_{i-1} follows from the fact that $\phi^{2^{i-1}}(\alpha_{i,j}\alpha_{i,j+2^{i-1}}) = \alpha_{i,j+2^{i-1}}\alpha_{i,j}$. From the description in The Method, it is clear that the coefficients are integral.

An example. We conclude with some results of the aforementioned computer program. Unfortunately, core limitations have prevented me from generating the sequence for the grandaddy $2^{2^4} + 1$, though by time of publication, I will have done that as well. This somewhat updates the bit of folklore stated by Coxeter [**1**, p. 27]. For the case $p = 2^{2^3} + 1$, here are the polynomials, their roots and hence the construction. The sign columns represent which root of the polynomial is to be constructed to obtain the indicated $\alpha_{i,j}$. For example, in the second line, the "$-$" after $\alpha_{2,2}$ indicates that $\alpha_{2,2} = (\alpha_{1,0} - \sqrt{\alpha_{1,0}^2 - 64(\alpha_{1,0} + \alpha_{1,1})})/2$ which is easily constructed after $\alpha_{1,0}$ and $\alpha_{1,1}$ have been.

Polynomial	Root	Sign	Root	Sign
$x^2 + x - 64$	$\alpha_{1,0}$	+	$\alpha_{1,1}$	−
$x^2 - \alpha_{1,0}x + 16(\alpha_{1,0} + \alpha_{1,1})$	$\alpha_{2,0}$	+	$\alpha_{2,2}$	−
$x^2 - \alpha_{1,1}x + 16(\alpha_{1,0} + \alpha_{1,1})$	$\alpha_{2,1}$	+	$\alpha_{2,3}$	−
$x^2 - \alpha_{2,0}x + (2\alpha_{2,0} + 5\alpha_{2,1} + 4\alpha_{2,2} + 5\alpha_{2,3})$	$\alpha_{3,0}$	+	$\alpha_{3,4}$	−
$x^2 - \alpha_{2,1}x + (5x_{2,0} + 2\alpha_{2,1} + 5\alpha_{2,2} + 4\alpha_{2,3})$	$\alpha_{3,1}$	−	$\alpha_{3,5}$	+
$x^2 - \alpha_{2,2}x + (4x_{2,0} + 5\alpha_{2,1} + 2\alpha_{2,2} + 5\alpha_{2,3})$	$\alpha_{3,2}$	+	$\alpha_{3,6}$	−
$x^2 - \alpha_{2,3}x + (5x_{2,0} + 4\alpha_{2,1} + 5\alpha_{2,2} + 2\alpha_{2,3})$	$\alpha_{3,3}$	−	$\alpha_{3,7}$	+
$x^2 - \alpha_{3,0}x + (2\alpha_{3,0} + 2\alpha_{3,2} + \alpha_{3,4} + 2\alpha_{3,5} + \alpha_{3,6}$	$\alpha_{4,0}$	+	$\alpha_{4,8}$	−
$x^2 - \alpha_{3,1}x + (2\alpha_{3,1} + 2\alpha_{3,3} + \alpha_{3,5} + 2\alpha_{3,6} + \alpha_{3,7})$	$\alpha_{4,1}$	+	$\alpha_{4,9}$	−
$x^2 - \alpha_{3,2}x + (\alpha_{3,0} + 2\alpha_{3,2} + 2\alpha_{3,4} + \alpha_{3,6} + 2\alpha_{3,7})$	$\alpha_{4,2}$	+	$\alpha_{4,10}$	−
$x^2 - \alpha_{3,3}x + (2\alpha_{3,0} + \alpha_{3,1} + 2\alpha_{3,3} + 2\alpha_{3,5} + \alpha_{3,7})$	$\alpha_{4,3}$	+	$\alpha_{4,11}$	−
$x^2 - \alpha_{3,4}x + (\alpha_{3,0} + 2\alpha_{3,1} + \alpha_{3,2} + 2\alpha_{3,4} + 2\alpha_{3,6})$	$\alpha_{4,4}$	+	$\alpha_{4,12}$	−
$x^2 - \alpha_{3,5}x + (\alpha_{3,1} + 2\alpha_{3,2} + \alpha_{3,3} + 2\alpha_{3,5} + 2\alpha_{3,7})$	$\alpha_{4,5}$	+	$\alpha_{4,13}$	−
$x^2 - \alpha_{3,6}x + (2\alpha_{3,0} + \alpha_{3,2} + 2\alpha_{3,3} + \alpha_{3,4} + 2\alpha_{3,6})$	$\alpha_{4,6}$	−	$\alpha_{4,14}$	+
$x^2 - \alpha_{3,7}x + (2\alpha_{3,1} + \alpha_{3,3} + 2\alpha_{3,4} + \alpha_{3,5} + 2\alpha_{3,7})$	$\alpha_{4,7}$	+	$\alpha_{4,15}$	−
$x^2 - \alpha_{4,0}x + (\alpha_{4,0} + \alpha_{4,1} + \alpha_{4,2} + \alpha_{4,5})$	$\alpha_{5,0}$	+		
$x^2 - \alpha_{4,1}x + (\alpha_{4,1} + \alpha_{4,2} + \alpha_{4,3} + \alpha_{4,6})$	$\alpha_{5,1}$	+		
$x^2 - \alpha_{4,7}x + (\alpha_{4,7} + \alpha_{4,8} + \alpha_{4,9} + \alpha_{4,12})$	$\alpha_{5,23}$	+	$(\alpha_{4,23} = \alpha_{4,7})$	
$x^2 - \alpha_{4,8}x + (\alpha_{4,8} + \alpha_{4,9} + \alpha_{4,10} + \alpha_{4,13})$	$\alpha_{5,24}$	+	$(\alpha_{4,24} = \alpha_{4,8})$	
$x^2 - \alpha_{4,9}x + (\alpha_{4,9} + \alpha_{4,10} + \alpha_{4,11} + \alpha_{4,14})$	$\alpha_{5,25}$	+	$(\alpha_{4,25} = \alpha_{4,9})$	
$x^2 - \alpha_{4,15}x + (\alpha_{4,15} + \alpha_{4,0} + \alpha_{4,1} + \alpha_{4,4})$	$\alpha_{5,15}$	+	$(\alpha_{5,56} = \alpha_{5,24})$	
$x^2 - \alpha_{5,0}x + (\alpha_{5,1} + \alpha_{5,23})$	$\alpha_{6,0}$	+		
$x^2 - \alpha_{5,24}x + (\alpha_{5,15} + \alpha_{5,25})$	$\alpha_{6,56}$	+		
$x^2 - \alpha_{6,0}x + \alpha_{6,56}$	$\alpha_{7,0}$	+		
$x^2 - \alpha_{7,0}x + 1$	$\alpha_{8,0}$	+		

Since $\alpha_{8,0} = \theta = e^{2\pi i/257}$, the construction is finished.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

PERMANENTAL PAIRS OF DOUBLY STOCHASTIC MATRICES

EDWARD T. H. WANG

For an $n \times n$ matrix $A = (a_{ij})$, the permanent of A is defined to be

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n denotes the symmetric group of degree n. An $n \times n$ matrix with nonnegative real entries is called doubly stochastic (d.s.) if each row sum and column sum is 1. We denote by Ω_n the set of all $n \times n$ d.s. matrices, and by J_n, the $n \times n$ d.s. matrix with all entries equal to $1/n$. In [3], the following problem was suggested by M. Marcus:

If A, $B \in \Omega_n$ are such that $\operatorname{per}(\alpha A + (1-\alpha)B) =$ *constant for* $\alpha \in [0, 1]$, *is it true that* $A = B$?

For $n = 2$, it is easy to verify that the answer is in the affirmative. For $n = 3$, however, the following surprising counterexample was first found by M. Newman [5].

$$\text{Let } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then straightforward calculations show that $\operatorname{per}(\alpha A + (1-\alpha)B) = \frac{1}{4}$ for all $\alpha \in [0, 1]$.

DEFINITION. Two matrices $A, B \in \Omega_n$ where $A \neq B$ are said to form a permanental pair (p.p.) if $\operatorname{per}(\alpha A + (1-\alpha)B) = \text{constant}$ for $\alpha \in [0, 1]$.

It follows from the definition that there exist no p.p. when $n = 2$. For $n = 3$, p.p. exist by Newman's example. In fact, if we let P_i denote the six permutation matrices of order 3, $i = 1, 2, \ldots, 6$, and let $A_i = \frac{1}{2}(3J_3 - P_i)$, then it is straightforward to check that A_i and A_j form a p.p. for all $i \neq j$. Indeed, up to permutation of rows and columns, $\alpha A_i + (1-\alpha)A_j$ equals one of the following two matrices, depending on whether A_i and A_j have a zero entry in common or not.

$$M_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \alpha/2 & (1-\alpha)/2 \\ \frac{1}{2} & (1-\alpha)/2 & \alpha/2 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} \frac{1}{2} & (1-\alpha)/2 & \alpha/2 \\ \alpha/2 & \frac{1}{2} & (1-\alpha)/2 \\ (1-\alpha)/2 & \alpha/2 & \frac{1}{2} \end{bmatrix}$$

we compute:

$$\operatorname{per} M_1 = \tfrac{1}{8}\{\alpha + \alpha + (1-\alpha) + (1-\alpha)\} = \tfrac{1}{4}$$

$$\operatorname{per} M_2 = \tfrac{1}{8}\{1 + \alpha^3 + (1-\alpha)^3 + 3\alpha(1-\alpha)\} = \tfrac{1}{4}.$$

The next theorem includes some of the facts about p.p. that can be proved easily.

THEOREM.

(1) *For every $n \geqq 3$, there exist infinitely many p.p.'s.*

(2) *No permutation matrix can form a p.p. with any d.s. matrix.*

(3) *If van der Waerden's conjecture* [4] *is true, then J_n does not form a p.p. with any d.s. matrix. In particular, J_n does not form a p.p. with any d.s. matrix for $n =$* 1, 2, 3, 4 *and* 5.

Proof:

(1) The fact that p.p's exist for all $n \geqq 3$ is seen by taking direct sum of the known examples and the identity matrix of order $n - 3$. Since convex combinations of convex combinations are again convex combinations, the existence of infinitely many p.p's is clear.

(2) Suppose the permutation matrix P forms a p.p. with the d.s. matrix Q. Then $\operatorname{per}(Q) = \operatorname{per}(P) = 1 \Rightarrow Q$ is itself a permutation matrix. But $\operatorname{per}(\frac{1}{2}(P+Q)) = 1 \Rightarrow \frac{1}{2}(P+Q)$ is also a permutation matrix. Hence $P = Q$.

(3) The first assertion clearly follows from the statement of van der Waerden's conjecture. Since this conjecture has been verified for $n \leqq 5$ [4, 1, 2], the second assertion follows.

The problem of finding all p.p.'s seems to be a difficult one. In fact, for $n = 3$, we do not even know

whether there are p.p.'s of which the d.s. matrices are not $\frac{1}{2}(3J_3 - P_i)$, mentioned earlier, or the convex combinations of them. In particular, we propose the following 2 problems for $n \geqq 3$.

PROBLEM 1: Is it true that J_n and the permutation matrices are the only ones that cannot form a p.p. with any other d.s. matrix? i.e., is it true that if $A \in \Omega_n$ is not a permutation matrix and that $A \neq J_n$, then there exists $B \in \Omega_n$ such that A and B form a p.p.?

PROBLEM 2: Is it true that if A and B form a p.p. then $A = PBQ$ for some permutation matrices P and Q? (For all the known examples, this is the case.)

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CLASSROOM NOTES

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WEAK DERIVATIVES AND INTEGRATION BY PARTS

BENT E. PETERSEN

Sometimes some distribution theory is incorporated in a senior analysis course. The notion of weak derivatives makes an interesting supplement to the study of L^p spaces. In this case, it is desirable to have simple examples of how to manipulate the notion of weak derivatives. We present here an integration by parts result, which is surely known, but which deserves to be better known. As an application, we give essentially W. Pauli's proof of the Heisenberg uncertainty principle. Aside from basic L^p facts, the only tools we use are Leibnitz' formula and a regularization argument.

We denote by $E(R^n)$ the linear space of infinitely differentiable complex valued functions on R^n and by $D(R^n)$ the space of functions in $E(R^n)$ which are compactly supported. The linear space $D(R^n)$ is topologized in an appropriate way, and the continuous linear functionals on $D(R^n)$ are then called *distributions* on R^n. We don't need to know the topology explicitly. For our purpose, two facts are sufficient. The reader who wants more detail should consult L. Schwartz' treatise **[1]** or any of a number of other texts. The first fact is that if f is a locally integrable function on R^n then the linear functional $\phi \to \langle f, \phi \rangle = \int f\phi\, dx$, $\phi \in D(R^n)$, is a distribution. We denote this distribution also by f and note that it determines the *function* f up to almost everywhere equivalence. Thus, we may regard the locally integrable functions as a subspace of the space of distributions. The second fact is that the partial derivative D_j with respect to the jth coordinate function in R^n, is a continuous linear operator on $D(R^n)$. Thus if F is a distribution, then $\langle D_jF, \phi \rangle = -\langle F, D_j\phi \rangle$, $\phi \in D(R^n)$, defines a distribution D_jF, called the *distribution derivative* of F with respect to the jth coordinate function. In case F and

D_jF are both locally integrable functions, then classically D_jF is also called a *weak derivative* of F. In this note, we will always take derivatives in the distribution sense. There are a number of results relating the distribution derivative and the ordinary derivative. For many applications, it suffices to know that if f and D_jf are both continuous functions on R^n then D_jf coincides with the ordinary derivative.

LEMMA 1. *If f and D_jf both belong to $L^1(R^n)$ then $\int D_jf\,dx = 0$.*

Proof. Choose ϕ in $D(R^n)$, $0 \leqq \phi \leqq 1$ such that $\phi = 1$ in a neighborhood of the origin. If $k \geqq 1$ is an integer, let $\phi_k(x) = \phi(k^{-1}x)$. From the definition of the distribution derivative, we have

$$\left| \int (D_jf)\phi_k dx \right| \leqq k^{-1}\|f\|_1 \sup |D_j\phi|.$$

Since D_jf is in $L^1(R^n)$ the dominated convergence theorem implies that $\int (D_jf)\phi_k\,dx$ converges to $\int D_jf\,dx$ as $k \to \infty$.

THEOREM 2. *Suppose $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and f and D_jf are in $L^p(R^n)$ and g and D_jg are in $L^q(R^n)$. Then $D_j(fg) = (D_jf)g + f(D_jg)$ almost everywhere and $\int f(D_jg)\,dx = -\int (D_jf)g\,dx$.*

Proof. If we already knew that Leibnitz' formula holds in the present case, then the last part would follow from Lemma 1. The proof, however, will proceed by establishing the last part first. Choose a mollifier ρ i.e., a function ρ in $D(R^n)$ such that $\rho \geqq 0$ and $\int \rho\,dx = 1$. If $k \geqq 1$ is an integer, let $\rho_k(x) = k^n\rho(kx)$. If $f_k = f_*\rho_k$ then f_k is in $E(R^n) \cap L^p(R^n)$, and by Minkowski's inequality and the continuity of translation in $L^p(R^n)$, we have $f_k \to f$ in $L^p(R^n)$ as $k \to \infty$. Moreover, from the definition of the distribution derivative, $D_jf_k = (D_jf)_*\rho_k$, and so $D_jf_k \to D_jf$ in $L^p(R^n)$ as $k \to \infty$. Since f_k is in $E(R^n)$, we may apply Leibnitz' formula to obtain $D_j(f_kg) = (D_jf_k)g + f_k(D_jg)$, which implies $D_j(f_kg)$ is in $L^1(R^n)$. Since we also have f_kg in $L^1(R^n)$, Lemma 1 implies $\int (D_jf_k)g\,dx = -\int f_k(D_jg)\,dx$. Now $f_k \to f$ and $D_jf_k \to D_jf$ in $L^p(R^n)$, and so an application of Hölder's inequality yields the last conclusion.

If ϕ is in $D(R^n)$, then $\langle D_j(fg), \phi\rangle = -\int fg D_j\phi\,dx$ and by Leibnitz' formula $D_j(\phi g) = (D_j\phi)g + \phi(D_jg)$. Thus $D_j(\phi g)$ is in $L^q(R^n)$ and

$$\langle D_j(fg), \phi\rangle = -\int f D_j(\phi g)\,dx + \int f(D_jg)\,\phi\,dx. \tag{1}$$

Since ϕg and $D_j(\phi g)$ are in $L^q(R^n)$, the first part of the proof shows $-\int f D_j(\phi g)\,dx = \int (D_jf)\,g\,\phi\,dx$, which together with (1) yields $D_j(fg) = (D_jf)g + f(D_jg)$.

We will now give, without any extraneous differentiability hypotheses, essentially E. Pauli's proof of Heisenberg's uncertainty principle. (See [2] p. 77 and p. 393.)

THEOREM 3. *Suppose g, D_jg and x_jg belong to $L^2(R^n)$. Then*

$$\tfrac{1}{4}\left(\int |g|^2 dx\right)^2 \leqq \int x_j^2 |g|^2 dx \int |D_jg|^2 dx.$$

Proof. By Theorem 2 we have $x_jD_j(g\bar{g}) = x_j\bar{g}(D_jg) + x_jg(D_j\bar{g})$ a.e. which shows that $x_jD_j(g\bar{g})$ is in $L^1(R^n)$. Since x_j is in $E(R^n)$, Leibnitz' formula gives $D_j(x_jg\bar{g}) = x_jD_j(g\bar{g}) + g\bar{g}$ and therefore $D_j(x_jg\bar{g})$ is in $L^1(R^n)$. Since $x_jg\bar{g}$ also belongs to $L^1(R^n)$, by Lemma 1 we have $\int D_j(x_jg\bar{g})\,dx = 0$. But then

$$-\int g\bar{g}\,dx = \int x_jD_j(g\bar{g})\,dx = \int x_j\bar{g}(D_jg)\,dx + \int x_jg(D_j\bar{g})\,dx = 2\,Re\,(x_jg, D_jg).$$

Then by Schwartz' inequality $\|g\|_2^2 \leqq 2\|x_jg\|_2\,\|D_jg\|_2$.

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A "MORE TOPOLOGICAL" PROOF OF THE TIETZE–URYSOHN THEOREM

BRIAN M. SCOTT

In the standard textbook proof of the Tietze–Urysohn Theorem, the desired extension is obtained as the uniform limit of a sequence of continuous approximations. (See e.g., [1]–[7].) This proof is admittedly quite easy, but it must often seem a bit "magical" (or at least unnatural) to the student who has just learnt the construction used in the proof of Urysohn's lemma. For that reason, and because it seems more in the spirit of general topology than the usual proof—which, after all, is just glorified advanced calculus! I prefer the following argument, which uses a more sophisticated version of the Urysohn's lemma construction. (I make no claim to priority of invention; although I have never seen it in print, the proof must be part of the folklore.)

The heart of the proof is the following lemma, which is of some interest in its own right. Although we know that it is not in general possible in a T_4-space to find disjoint open sets about two separated sets, it *is* possible to do so if the separated sets are "sufficiently like" closed sets: specifically, if they are F_σ-sets.

LEMMA 1. *Let X be T_4, and let A_0 and A_1 be separated F_σ-sets in X; then there are disjoint, open V_0, $V_1 \subseteq X$ such that $A_0 \subseteq V_0$ and $A_1 \subseteq V_1$.*

Proof. For $i \in \{0, 1\}$, let $A_i = \bigcup \{F_i(n) : n \in N\}$, where, for each $n \in N$, $F_i(n)$ is closed, and $F_i(n) \subseteq F_i(n+1)$. Using normality repeatedly, construct, by induction on n, open sets $V_i(n)$ such that

(a) $F_0(0) \subseteq V_0(0) \subseteq \mathrm{cl}\, V_0(0) \subseteq X \backslash \mathrm{cl}\, A_1$;
(b) $F_1(0) \subseteq V_1(0) \subseteq \mathrm{cl}\, V_1(0) \subseteq X \backslash (\mathrm{cl}\, A_0 \cup \mathrm{cl}\, V_0(0))$;
(c) for each $n \in N$, $F_0(n+1) \cup \mathrm{cl}\, V_0(n) \subseteq V_0(n+1) \subseteq \mathrm{cl}\, V_0(n+1) \subseteq X \backslash (\mathrm{cl}\, A_1 \cup \mathrm{cl}\, V_1(n))$;
(d) for each $n \in N$, $F_1(n+1) \cup \mathrm{cl}\, V_1(n) \subseteq V_1(n+1) \subseteq \mathrm{cl}\, V_1(n+1) \subseteq X \backslash (\mathrm{cl}\, A_0 \cup \mathrm{cl}\, V_0(n+1))$.

Now let $V_i = \bigcup \{V_i(n) : n \in N\}$, $i \in \{0, 1\}$; clearly $A_i \subseteq V_i$ for $i \in \{0, 1\}$, and $V_0 \cap V_1 = \varnothing$.

(The proof of Lemma 1 provides another nice example of the technique of "climbing a chimney" encountered in the proof that "regular + Lindelöf implies normal.")

THEOREM 1.(Tietze–Urysohn). *Let X be a T_4-space, and let K be a (non-empty) closed subset of X. If $f: K \to I\,(=[0,1])$ is continuous, there is a continuous $F: X \to I$ such that $F \mid K = f$.*

Proof. The idea is to try to mimic the usual proof of the Urysohn lemma. Recall that in the proof one begins by constructing open sets $W(q)$, one for each rational $q \in I$, satisfying the condition

(i) $p < q$ implies $\mathrm{cl}\, W(p) \subseteq W(q)$.

The desired function is then defined by

$$F(x) = \begin{cases} 1, & \text{if } x \notin W(1) \\ \inf\{q \in Q : x \in W(q)\}, & \text{otherwise}, \end{cases} \tag{1}$$

(where Q is the set of rationals in I,) and is shown to satisfy the requirements of the Urysohn lemma.

Those requirements are not very stringent: F must be continuous and must assume specified values (0 and 1) in just two "places". The Tietze theorem is more demanding, since it "ties down" *many* values of F, so we impose additional conditions on the sets $W(q)$:

(ii) $K \cap W(q) = V(q)$, and
(iii) $K \cap \mathrm{cl}\, W(q) = C(q)$,

where $V(q) = f^{-1}[[0, q)]$, and $C(q) = f^{-1}[[0, q]]$. Condition (ii) ensures that F, defined by (1), does extend f. Condition (iii) is technical: the sets $W(q)$ will be constructed recursively, using an enumeration of Q, and (iii) is needed to make the recursion work.

We begin by defining $W(0) = \varnothing$ and $W(1) = X \backslash f^{-1}[\{1\}]$. At the general step of the construction,

we shall have rationals $p, q, r \in I$ with $W(p)$ and $W(q)$ already defined so as to satisfy (i)–(iii) and with $p < q < r$, and we shall construct $W(q)$ as follows.

$V(q)$ and $K \setminus C(q)$ are separated F_σ-sets in X, so, by Lemma 1, there is an open $H \subseteq X$ such that $V(q) \subseteq H$ and $K \cap \operatorname{cl} H \subseteq C(q)$. Let $G = H \cap [(X \setminus K) \cup V(q)]$, and note that G is open and satisfies (ii) and (iii). Next, using normality twice, find an open $H_0 \subseteq X$ such that $\operatorname{cl} W(p) \cup \operatorname{cl} V(q) \subseteq H_0 \subseteq \operatorname{cl} H_0 \subseteq W(r)$ and an open $H_1 \subseteq X$ such that $\operatorname{cl} W(p) \subseteq H_1 \subseteq \operatorname{cl} H_1 \subseteq H_0 \setminus (K \setminus V(q))$. Put $W(q) = H_1 \cup (G \cap H_0)$; it is then easily checked that $W(q)$ satisfies (i)–(iii).

Actually, the full statement of the Tietze theorem is slightly stronger than Theorem 1: *any* continuous $f: K \to E^1$ can be extended to a continuous $F: X \to E^1$ in such a way that any open ray containing the range of f also contains the range of F. This strengthening follows from Theorem 1 as in the usual proof (cf. [1]).

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MATHEMATICAL EDUCATION

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A MODELING PROBLEM FOR THE CLASSROOM

J. Glenn Brookshear

The increase in course offerings in the area of mathematical modeling has resulted in a universal interest in problems for students to solve on their own. These problems must be involved enough to be challenging yet simple enough to be stated in a few pages. Moreover, the diverse disciplines represented by the enrollment in most modeling classes dictate that these problems be based on a background common to a wide variety of students. Such problems are successfully being used to develop a student's ability to isolate the key points of a problem, to make meaningful assumptions in order to obtain a manageable model, and to evaluate the effects these assumptions have on the model's predictions. It is my hope that the Monthly can serve as a forum for the exchange of such problems. The following "prerequisite problem" has succeeded in stimulating interest in my students, and I hope it can do the same for others.

> In 1969 an increase in tuition at Bunker University and a strong recruiting campaign by other nearby universities resulted in an unusually small freshman class at Bunker University. The enrollment problem was solved the next year. In fact, admission applications have soared to the extent that the university has been able to adopt more rigorous entrance requirements and still fill

biochemistry class size reduces this expression to B. Thus, the capacity of the biology facilities will become the maximum average throughput of the pre-med program.

I have found that students are anxious to state and defend the assumptions they made in approaching a problem. Such a discussion in class allows the students to see a variety of approaches and encourages them to think for themselves. The following questions have developed during such classroom discussions.

What supports the assumption that the drop rate among freshmen is constant from year to year? How could a model be constructed if the drop rate from year to year were a random variable? What is a reasonable distribution to assume for this random variable? How might the predictions of this "random" model compare to the predictions of the "constant rate" model? Can this system be modeled as a Markov chain? How about a simple flow network? How can feedback be handled?

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INVESTIGATING MATHEMATICAL MODELS

DAVID A. FIELD

It has become fashionable to call upon mathematical models to illustrate, at an elementary level, fundamental mathematical concepts and modern applications of mathematics. Finding a problem whose mathematical solution appeals to average college students and at the same time confronts them with its underlying assumptions is indeed challenging. A problem which lends itself easily to a variety of sound pedagogies is the following.

At the ground floor p persons enter an elevator. If each person is equally likely to get off at any one of n floors, determine $E(k)$, the expected number of stops until the elevator is emptied.

A solution to this problem is easily understood by non-science majors in an elementary probability course. Rather than presenting a solution, a fruitful strategy is to offer two plausible mathematical models with expressions for $E(k)$ which clearly yield different numerical evaluations. The expressions $E(k)=n(1-(1-1/n)^p)$ and $E(k)=np/(n+p-1)$ are given in the problems and solutions section in [2]. Attention is now focused on the mathematical models and their adherence to the underlying assumptions of the models. The probability space in the second solution is erroneously assumed to be uniform.

An excellent opportunity to reinforce this focus without giving away the correct solution is to simulate an elevator on a time-sharing computer. In addition to the advantage of interactive computer programs, the simulation promotes much needed discussion on the roles of mathematics and computers. Simulations strongly hint at the correct solution, e.g. a typical simulation for $n=20, p=20, n(1-(1-1/n)^p)=12.83, (np)/(n+p-1)=10.26$, yielded a mean value of 12.7 for 20 elevator runs; however, the student can obtain a healthy realization of the pitfalls and advantages of computers. By performing numerous routine calculations and recording the progress of simulations, the computer can be very effectively employed in classroom discussions.

The inverse of the elevator problem is not only of special scientific interest but also provides an interesting contrast. Suppose an elevator with p passengers, each equally likely to get off on any one of n floors, stops k times. Given the values for n and k, what can be said about p?

A standard answer is to choose σ, $0<\sigma<1$, and k' such that $1-\text{prob}\{p[k,k']\}<\sigma$. The expression

$$\text{prob}\{p=k\}=\frac{1}{n^p}\left[\binom{n}{k}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}(k-j)^p\right]$$

can be derived, see [1, 101–102], to determine the cumulative distribution $\Sigma_{i=k}^{k'}\text{prob}\{p=i\}$.

Whatever is said about p must reflect the uncertainty of the mathematical model. Moreover, numerical evaluation of formulas such as the one above are difficult enough to warrant use of a computer. There just isn't a simple analysis of this problem as there is with the first elevator problem.

The desire to sharpen the estimates of p leads to many interesting questions. For example, if allowed two elevator runs, would it be better to fix or vary n? The open-ended nature of the inverse problem can be useful in illustrating mathematics at work in further investigations of both problems.

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PROBLEMS AND SOLUTIONS

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All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before June 30, 1978.

E 2701. *Proposed by Richard Stanley, Massachusetts Institute of Technology*

Find the volume of the convex polytope determined by $x_i \geqq 0$ $(1 \leqq i \leqq n)$, and $x_i + x_{i+1} \leqq 1$ $(1 \leqq i \leqq n-1)$.

E 2702. *Proposed by David Jackson, University of Waterloo, Ontario*

Let $a = (a_1, a_2, \ldots, a_{2m})$ be a non-decreasing sequence of positive integers with $a_{2m} \leqq n$. Let S denote the set of sequences obtained from a by permuting its terms. Let A, B, C be the subsets of S consisting of those sequences $s = (s_1, s_2, \ldots, s_{2m})$ which satisfy

$$s_1 < s_2 \geqq s_3 < s_4 \geqq \cdots \geqq s_{2m-1} < s_{2m},$$

$$\prod_{i=1}^{2m} (s_i - a_i) > 0, \qquad \prod_{i=1}^{2m} (s_i - a_i) < 0,$$

respectively. Show that $|A|$ is equal to the absolute value of $|B| - |C|$.

E 2703. *Proposed by David Jackson, University of Waterloo, Ontario*

Let J be the $n \times n$ matrix whose entries are all ones and write $J = L + U$ where L (resp. U) is lower (resp. upper) triangular matrix and the diagonal entries of L are zeros. Let $X = \operatorname{diag}(x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are variables. Prove that

$$\det(I - (XU)^{k-1}XL) = \sum_{s \geqq 0} (-1)^s a_{sk} \qquad (k = 1, 2, 3, \dots),$$

where the a_j are defined by

$$\prod_{i=1}^{n} \frac{1 - (tx_i)^k}{1 - tx_i} = \sum_{j \geqq 0} a_j t^j$$

where t is a new variable.

E 2704. *Proposed by S. Collins, S. M. Reddy and N. J. A. Sloane, University of Iowa and Bell Telephone Laboratories, Murray Hill, N.J.*

Find the number of solutions of $x^2 = x$ in the ring of integers modulo n.

E 2705. *Proposed by Clark Kimberling, University of Evansville, Indiana*

For an experiment having m equally probable outcomes, find the expected number of independent trials for k consecutive occurrences of at least one of these outcomes.

E 2706. *Proposed by David L. Lovelady, Florida State University*

Let

$$f(t) = g(t) \int_0^t g(s)^{-\alpha} ds,$$

where $\alpha > 1$ and g is a positive continuous function on $[0, \infty)$.
Prove that f is unbounded. Is this true if $\alpha = 1$?

SOLUTIONS OF ELEMENTARY PROBLEMS

Separately Continuous Functions

E 2610 [1976, 567]. *Proposed by Hugh L. Montgomery, University of Michigan*

Let f be a real valued function defined on the unit square $[0,1] \times [0,1]$. Suppose that $f(x, y)$ is continuous in x for each fixed y and continuous in y for each fixed x. Show that if $f^{-1}(0)$ is dense in the unit square then $f = 0$.

Solution by D. J. Newman, Temple University. Suppose $f \neq 0$, say $f(a, b) > 0$ for some (a, b). By continuity of $f(x, b)$ there exists $\varepsilon > 0$ such that $f(x, b) \geqq \frac{1}{2} f(a, b)$ for each x in the interval $J_\varepsilon = [a - \varepsilon, a + \varepsilon] \cap [0,1]$. For $y \in [0,1]$ let $S_y = \{x \in [0,1] \mid f(x, y) \geqq \frac{1}{4} f(a, b)\}$. Since $f(x, y)$ is continuous in x, each S_y is closed. For each integer $n \geqq 1$ let

$$I_n = [b - 1/n, b + 1/n] \cap [0,1] \quad \text{and} \quad T_n = \bigcap\nolimits_{y \in I_n} S_y.$$

It follows from the continuity of $f(x, y)$ in y that $J_\varepsilon \subset \bigcup_{n=1}^{\infty} T_n$. Since each T_n is closed, it follows from the Baire category theorem that for some m, T_m has non-empty interior as a subset of $[0,1]$. Hence, $T_m \times I_m$ also has non-empty interior. Since $T_m \times I_m$ does not meet $f^{-1}(0)$ we have a contradiction.

Also solved by N. J. Fine, Gustaf Gripenberg (Finland), Doug Hensley, J. R. Kuttler, O. P. Lossers (Netherlands), Robert McCoy, Stephen Noltie, Adam Riese, Thomas Sellke (Germany), Arthur Solomon, Paul Vojta, and the proposer.

L. E. Mattics and M. A. McKiernan (Canada) observe that the problem is easily solved by using a theorem of Baire given in E. W. Hobson, *Theory of Functions of a Real Variable*, vol. 1, p. 449.

McCoy proves the following more general result. Let X and Y be metric spaces, at least one of them being a Baire space, and let Z be a Hausdorff space. If $f, g: X \times Y \to Z$ are separately continuous and coincide on a dense set then $f = g$.

A Characterization of Primes

E 2611 [1976, 656]. *Proposed by C. A. Nicol, University of South Carolina*

A long standing conjecture of D. H. Lehmer asserts that if $n \geqq 2$ is a natural number and $\phi(n) \mid (n-1)$ then n is prime.

Show that a natural number $n \geqq 2$ is prime if and only if $\phi(n) \mid (n-1)$ and $(n+1) \mid \sigma(n)$.

Solution by Paul Vojta, student, University of Minnesota. If n is prime then $\phi(n) = n-1$ and $\sigma(n) = n+1$.

Conversely, let $n > 2$ and assume that $\phi(n) \mid n-1$ and $n+1 \mid \sigma(n)$. Since $\phi(n)$ is even, n must be odd. If p is an odd prime such that $p^r \mid n$, $r \geqq 2$, then $p^{r-1} \mid \phi(n)$ which implies $p^{r-1} \mid n-1$, a contradiction. Hence n is a product $p_1 p_2 \cdots p_k$ of distinct odd primes. We have $\phi(n) = (p_1 - 1) \cdots (p_k - 1)$, $\sigma(n) = (p_1 + 1) \cdots (p_k + 1)$ and thus 2^k divides both $\phi(n)$ and $\sigma(n)$. If $k \geqq 2$ then $4 \mid \phi(n) \mid n-1$ and so $4 \nmid n+1$. It follows that 2^{k-1} divides $\sigma(n)/(n+1)$ and hence

$$2^{k-1} < \frac{\sigma(n)}{n} = \left(1 + \frac{1}{p_1}\right) \cdots \left(1 + \frac{1}{p_k}\right) < \left(\frac{4}{3}\right)^k.$$

This is a contradiction. Hence $k = 1$ and n is a prime.

Also solved by Mangho Ahuga, Robert Breusch, Peter Bundschuh (W. Germany), Robert Dressler, Thomas Elsner, Irving Gerst, M. G. Greening (Australia), Eli Isaacson, Mark Kleiman, L. Kuipers (Switzerland), J. R. Kuttler, Jordan Levy, Graham Lord (Canada), L. E. Mattics, Jerry Metzger, R. G. Nath, Chester Palmer, Leonard Palmer, Sita Ramaiah (India), Bart Rice, Martin Schechter, H. N. Sharpe, Temple University Problem Group, E. W. Trost (Switzerland), Gillian Valk, and the proposer.

Diamond Packing of a Chinese Checkerboard

E 2612 [1976, 656]. *Proposed by Sidney Penner, Bronx Community College, CUNY*

How many diamonds can be packed in a Chinese checkerboard? This board consists of 2 order 13 triangular arrays of holes, overlapping in an order 5 hexagon, 121 holes in all. A *diamond* consists of four marbles that fill four adjacent holes.

Solution by Paul Vojta, student, University of Minnesota. Color the rows of the board red and blue, alternately, with the top hole being red. Since there are 56 blue holes, and each diamond must cover exactly two blue holes, no more than 28 diamonds will fit onto the board.

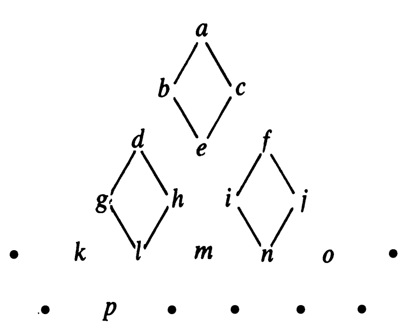

FIG. 1.

Assume that 28 diamonds can be packed onto the board. Then all blue holes must be covered, with exactly 9 red holes left uncovered. Let one corner of the board be as illustrated in Figure 1. Since b and c are both blue, both must be covered. They cannot be covered by separate diamonds, so they must be covered by the same diamond, which can be moved to position $abcd$ without causing any overlap. Assuming that blue holes g and h are covered by distinct diamonds requires diamonds to be in positions $gklp$ and $himn$, leaving blue j uncoverable, a contradiction. Thus one diamond covers both g and h; it can be moved to $dghl$. Similarly, a diamond will cover $fijn$ in a 28–covering.

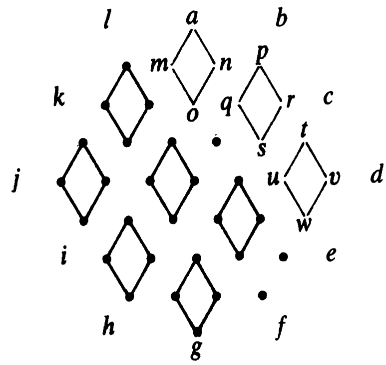

FIG. 2.

Fill each of the six corners with three diamonds as just described. Removing them from the board gives Figure 2. Holes b, d, f, h, j and l cannot be covered; thus at most three of a, c, e, g, i and k can remain unfilled. Assume, without loss of generality, that a is covered. This gives $amno$; p is blue, hence $pqrs$; t is blue, hence $tuvw$. Thus c and e cannot be filled; similarly, neither can k or i. Thus 10 holes cannot be covered, a contradiction. Thus 28 diamonds will not fit onto the board.

Filling in the remaining holes as shown in Figure 2, however, gives a way of fitting 27; therefore 27 is the optimum.

Also solved by David Berman, Marguerite Gerstell, Eli Isaacson, Allen Schwenk, Temple University Problem Group, and the proposer.

Convergent and Divergent Series

E 2626 [1976, 812]. *Proposed by Richard Johnsonbaugh, Chicago State University*

Is there a positive continuous function f on $[1,\infty)$ such that

$$\sum_{n=1}^{\infty} f(n)=\infty \quad \text{but} \quad \sum_{n=1}^{\infty} a^n f(a^n)<\infty, \quad \text{for all } a>1?$$

Solution by Paul Erdös, Hungarian Academy of Sciences (details supplied by the editor using the solution by J. Komlós & A. Meir). The answer is positive. Let S be the set of positive integers which are not powers. Let $\varepsilon_n \to 0$ sufficiently fast, say

$$\varepsilon_n = 1/2^{2^{2^n}}.$$

For $n \in S$ we put $f(n)=1/n \log n$, $f(n-\varepsilon_n)=f(n+\varepsilon_n)=0$. Further let $f(1)=0$ and define $f(x)$ for $x \geqq 1$ by requiring that it is piecewise linear and continuous. Thus $f(x)$ is zero outside the union of the intervals $I_n=(n-\varepsilon_n, n+\varepsilon_n)$, $n \in S$ and $f(n)$ is the maximum of $f(x)$ in I_n.

Now let $a>1$ be fixed. We claim that if $n \in S$ is sufficiently large and $a^p \in I_n$ for some integer p then $a^q \notin I_m$ for all integers $q \neq p$ and $m \in S$ such that $n \leqq m \leqq n^2$. Otherwise we have

$$(n-\varepsilon_n)^q < a^{pq} < (n+\varepsilon_n)^q, \qquad (m-\varepsilon_m)^p < a^{pq} < (m+\varepsilon_m)^p,$$

and so

$$(n+\varepsilon_n)^q > (m-\varepsilon_m)^p, \qquad (m+\varepsilon_m)^p > (n-\varepsilon_n)^q.$$

Since $a^p < n+\varepsilon_n$, $a^q < m+\varepsilon_m$ we have

$$p < \log(n+1)/\log a, \qquad q < \log(m+1)/\log a.$$

By the mean value theorem, we have for sufficiently large n,

$$(n+\varepsilon_n)^q - (n-\varepsilon_n)^q = 2\varepsilon_n q(n+\sigma\varepsilon_n)^{q-1} < \tfrac{1}{2},$$
$$(m+\varepsilon_m)^p - (m-\varepsilon_m)^p = 2\varepsilon_m p(m+\theta\varepsilon_m)^{p-1} < \tfrac{1}{2},$$

where σ, $\theta \in (-1,1)$.

Hence $|m^p - n^q| < 1$ and we obtain a contradiction $m^p = n^q$ since $m, n \in S$, $p \neq q$. Thus our claim is proved.

Let $p < q$ be two positive integers such that $f(a^p) \neq 0 \neq f(a^q)$ and $f(a^r) = 0$ for $p < r < q$. If $a^p \in I_n$ $(n \in S)$ and $a^q \in I_m$ $(m \in S)$, then for sufficiently large p we have $m > n^2$ (as proved above). This and

$$a^p f(a^p) < (n+\varepsilon_n)f(n) < \frac{2}{\log n}$$

imply that for sufficiently large r, the series

$$\sum_{p \geq r} a^p f(a^p)$$

is majorized by

$$2\sum_{k \geq 0} \frac{1}{\log(r^{2^k})}.$$

It follows that $\Sigma_{p=1}^{\infty} a^p f(a^p)$ is convergent.

By adding $1/x^2$ to $f(x)$ one obtains a positive function with the same property.

Also solved by William Emerson, J. Komlós & A. Meir (Canada), L. E. Mattics, Harold Shapiro, William Skaggs, and Peter Ungar.

Comments. Komlós and Meir remark that if $\varphi(x)$ is an increasing continuous function with $\Sigma\varphi(n)^{-1} = \infty$ then there exists a positive continuous function $f(x)$ such that $\Sigma f(n) = \infty$, but $\Sigma\varphi(a^n)f(a^n) < \infty$ for all $a > 1$.

Erdös believes that if we replace the condition $\Sigma f(n) = \infty$, in the problem, by $\Sigma f(n+\varepsilon) = \infty$ for all $\varepsilon > 0$, then $\Sigma a^n f(a^n)$ cannot converge for all $a > 1$. In a similar vein, Ungar conjectures that if $\Sigma a^n f(a^n) < \infty$ for all $a > 1$ then there exists an $\varepsilon > 0$ such that $\Sigma f(n\varepsilon) < \infty$.

Quadratic Residues and Squares

E 2627 [1976, 812]. *Proposed by Ron Evans, University of Wisconsin, Madison*

Let m and n be fixed integers greater than 1, n odd. Suppose n is a quadratic residue modulo p for all sufficiently large prime numbers $p \equiv -1 \pmod{2^m}$. Show that n is a square.

Solution by University of South Alabama Number Theory Class. Let $n = ab$, where b is a square and a is square-free. Assume $a \neq 1$ and let $a = p_1 p_2 \cdots p_k$ where p_i are distinct odd primes. Let $\delta = \frac{1}{2}\Sigma_{i=1}^{k}(p_i - 1)$ and choose positive integers d_i such that

$$\left(\frac{d_1}{p_1}\right)\left(\frac{d_2}{p_2}\right)\cdots\left(\frac{d_k}{p_k}\right) = -(-1)^{\delta}. \tag{1}$$

By the Chinese Remainder Theorem and Dirichlet's Theorem there exists a prime p such that $p \equiv -1 \pmod{2^m}$ and $p \equiv d_i \pmod{p_i}$ for $1 \leqq i \leqq k$. By Quadratic Reciprocity Theorem we have

$$\left(\frac{n}{p}\right) = \left(\frac{a}{p}\right) = \prod_{i=1}^{k} \left(\frac{p_i}{p}\right) = (-1)^{\delta} \prod_{i=1}^{k} \left(\frac{p}{p_i}\right)$$
$$= (-1)^{\delta} \prod_{i=1}^{k} \left(\frac{d_i}{p_i}\right) = -1 \quad \text{by} \quad (1).$$

This is a contradiction and hence we must have $a = 1$, $n = b$.

Also solved by D. M. Bloom, Robert Breusch, Lorraine Foster, Irving Gerst, L. Kuipers (Switzerland), S. C. Locke (Canada), Jerry Metzger, Blair Spearman, and the proposer.

Roots in Arithmetic Progression

E 2628 [1976, 813]. *Proposed by Richard Hall, London, England*

Let a, b, c be distinct positive integers, at least two of which are prime. Show that $\sqrt[n]{a}$, $\sqrt[n]{b}$, and $\sqrt[n]{c}$ cannot be terms of an arithmetic progression.

Solution by I. M. Isaacs, University of Wisconsin, Madison. The following is a more general result.

THEOREM. *Let x, y, z be distinct nonzero real numbers which lie in an arithmetic progression. Assume that x^m, y^n, z^k are rational for some positive integers m, n, k. Then x/y is rational.*

For the proof we need two lemmas.

LEMMA 1. *Let x, y, z be nonzero reals and α, β, γ complex numbers such that $|\alpha| = |\beta| = |\gamma| = 1$ and*

(1) $$x + y + z = 0, \qquad (2) \quad \alpha x + \beta y + \gamma z = 0.$$

Then $\alpha = \beta = \gamma$.

Proof. We may assume that $x, y > 0$, $z < 0$ and $\gamma = 1$. From (1) and (2) we then obtain $(1-\alpha)x + (1-\beta)y = 0$. Since $\mathrm{Re}(\alpha) \leqq 1$, $\mathrm{Re}(\beta) \leqq 1$ and $x, y > 0$ we conclude that $\alpha = \beta = 1$.

LEMMA 2. *Let $\bar{Q}$ be the algebraic closure of Q (the rationals) in C (the complex numbers). If $u \in \bar{Q}$ is fixed by all automorphisms of $\bar{Q}$ then $u \in Q$.*

Proof of this is well known.

Proof of the theorem. Since x, y, z lie in an arithmetic progression and are distinct, we have $z - x = r(y - x)$ for some rational number $r \neq 0, 1$. Let σ be an automorphism of $\bar{Q}$. Since $x^m \in Q$ we have $x \in \bar{Q}$ and $\sigma(x) = \alpha x$ where $\alpha^m = 1$. Similarly, $\sigma(y) = \beta y$, $\sigma(z) = \gamma z$, $\beta^n = \gamma^k = 1$. By applying σ to $(r-1)x - ry + z = 0$ we get $\alpha(r-1)x - \beta r y + \gamma z = 0$ and Lemma 1 gives $\alpha = \beta = \gamma$. Thus

$$\sigma\left(\frac{x}{y}\right) = \frac{\alpha x}{\beta y} = \frac{x}{y}$$

and Lemma 2 implies that $x/y \in Q$.

The problem as stated follows from the theorem since it is clear that if a and b are distinct primes and $n > 1$ is an integer, then $(a/b)^{1/n}$ is irrational.

Also solved by Irving Gerst, O. P. Lossers (Netherlands), L. E. Mattics, Jerry Metzger, William Vélez, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before June 30, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6198. *Proposed by Sanford S. Miller, State University College, Brockport, New York*

Let $u(z)=u(x,y)$ be harmonic in the unit disk D with $u(0)=1$, and let $g(t)$ be a real-valued function satisfying $g(1)>0$ and $g(0)\leqq\frac{1}{2}$. Show that if u satisfies $g(u)+xu_x+yu_y>0$, for $z\in D$, then $u(z)>0$, for $z\in D$. In particular, if $g(t)=\frac{1}{2}$, or $g(t)=t+\frac{1}{2}$ we obtain respectively

$$xu_x+yu_y>-\tfrac{1}{2}\Rightarrow u>0,\quad u+xu_x+yu_y>-\tfrac{1}{2}\Rightarrow u>0.$$

6199. *Proposed by Hugh L. Montgomery, University of Michigan*

Suppose that q is a positive integer, and that $(a,q)=1$. Put

$$\mathcal{S}=\{n:1\leqq n\leqq q,\{an/q\}\leqq(n/q)^2\},$$

where $\{\theta\}=\theta-[\theta]$ is the fractional part of θ. Show that

$$\sum_{n\in\mathcal{S}}n^{-2}\leqq 9/q.$$

6200. *Proposed by Brian Conrey, David Leep and Gerry Myerson, University of Michigan*

Define $\left(\frac{a}{b}\right)_r$ to be the least positive integer x such $bx\equiv a\pmod r$. Let k,m,n be positive integers with $(m,n)=1$, $k<n$, $m<n$. Show (1) $m<\left(\frac{k}{m}\right)_n+\left(\frac{k}{n}\right)_m\leqq n$. Show also

(2) $$\left(\frac{1}{m}\right)_n+\left(\frac{1}{n}\right)_m=\frac{m+n}{2}\quad\text{if and only if } n-m=2.$$

Thus, for m,n prime, (2) characterizes twin primes.

6201. *Proposed by Daniel D. Anderson, University of Missouri, Columbia*

Let $GF(p^n)$ be the finite field of order p^n. For which positive integers k is every element of $GF(p^n)$ a sum of kth powers?

6202. *Proposed by A. A. Jagers, Twente University of Technology, Netherlands*

Let S be a set of generators of a finite group G. For $g\in G$, let $m(g)$ be the least number of terms in a representation of g as a product of elements of S. Let $n_1,n_2,\ldots,n_k$ be the degrees of the irreducible characters of G. Prove that $m(g)\leqq n_1+n_2+\cdots+n_k-1$.

6203. *Proposed by Albert Wilansky, Lehigh University*

Let X, Y, Z be Banach spaces and $T\colon X\to Z$, $S\colon Y\to Z$ continuous and linear. Show that the (equivalent) conditions

(i) $\overline{TD_1}\supset SD_\varepsilon$ for some $\varepsilon>0$ (D_ε is the disk of radius ε),

(ii) $\|S'(f)\|\leqq k\|T'(f)\|$ for all $f\in Z'$ for some $k>0$,

do not imply that $TX\supset SY$. (See M. Embry: Proc. A.M.S. 38 (1973), 587–589.)

SOLUTIONS OF ADVANCED PROBLEMS

Generators for some Non-Abelian Groups

6099 [1976, 489]. *Proposed by Jerome Minkus, Berkeley, California*

For $n \geqq 3$ let G_n denote the group generated by $a_1, a_2, \ldots, a_n$ subject to the relations

$$a_1 a_2^{-1} a_3 = a_2 a_3^{-1} a_4 = \cdots = a_{n-2} a_{n-1}^{-1} a_n = a_{n-1} a_n^{-1} a_1 = a_n a_1^{-1} a_2 = 1.$$

(Compare Problem 5327 [1967, 91].) Show that

(i) G_5 is isomorphic to the binary dodecahedral group $\{a, u : a^5 = u^3 = (au)^2\}$.

(ii) G_n is nonabelian for all $n \geqq 3$.

Solution by A. M. Brunner, Sussex University, England. Our solution is based on an embedding of Minkus' group as a subgroup of finite index in a factor group of the trefoil knot group $(c, d\,; c^3 = d^2)$. Let $n \geqq 3$ and

$$G_n = (a_0, a_1, \ldots, a_{n-1} : a_{i+3} = a_{i+2} a_{i+1}^{-1}\,; i = 0, \ldots, n-1),$$

where the subscript j in a_j is taken modulo n. Let $E_n = (a, b\,; b^n = 1, a^{b^2} = a^b a^{-1})$. (We use the notation k^u to mean conjugation, $u^{-1}ku$.) It is a straightforward consequence of the Reidemeister–Schreier subgroup rewriting process (see, e.g., Magnus, Karrass and Solitar, *Combinatorial Group Theory*, Academic Press, 1966, p. 94) that G_n is embedded in E_n as the least normal subgroup containing a (and $a_i = b^{-i} a b^i$ for each i). In particular, G_n is generated by a and $a_1 = a^b$.

If $c = ab^2$, $d = ab^2ab$ then

$$E_n = (c, d\,; c^3 = d^2, (cd^{-1})^n = 1).$$

Of course G_n is abelian if and only if $aa^b = a^b a$, and $aa^b = a^b a$ holds if and only if $d^2 = (c^{-1}d)^6$.

Now E_n has an epimorph, the triangle group $(c, d\,; c^3 = d^2 = (cd)^n = 1)$; here n is uniquely determined by geometric considerations (see Coxeter and Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, 1957, p. 68). Consequently $d^2 \neq (c^{-1}d)^6$ in E_n unless, possibly, $n = 3$ or $n = 6$. In these two cases it is easy to find representations to show that G_n is not abelian; by using known properties of the triangle groups, the argument shows also that G_n is infinite when $n \geqq 6$.

We now proceed generally. Let λ be the automorphism of the free group generated by x and $y : x\lambda = y,\ y\lambda = yx^{-1}$. Thus $x\lambda = y$, $x\lambda^2 = yx^{-1}$, $x\lambda^3 = yx^{-1}y^{-1}$, $x\lambda^4 = yx^{-1}y^{-1}xy^{-1}$; $x\lambda^5 = yx^{-1}y^{-1}x^2y^{-1}$, and $x\lambda^6 = cxc^{-1}$ where $c = yx^{-1}y^{-1}x$ and $c\lambda = c$. It follows that if $n = 6k + r$, where $0 \leqq r < 6$, then $x\lambda^n = c^k x\lambda^r c^{-k}$.

Now it follows from the subgroup rewriting process that

$$G_n = (x, y\,; x\lambda^n := x, y\lambda^n := y);$$

so that

$$G_n = (x, y\,; x\lambda^r := c^{-k}xc^k, y\lambda^r := c^{-k}yc^k).$$

For example, we may write down that $G_3 = (x, y\,; y^{-1}xy = x^{-1}, x^{-1}yx = y^{-1})$, the quaternion group of order 8, and G_6 as the free nilpotent of class 2 group (2-generator); both are nonabelian. Again we may write $G_5 = (x, y : yx^{-1}y^{-1}x^2y^{-1} = x, xyx = yxy)$, and a Tietze transformation ($w = xy$) shows that $G_5 = (w, y : y^5 = w^3 = (wy)^2)$.

The analysis of groups of the type considered by Minkus has a rather long history. The method employed in the above solution has been used in a more general setting in the author's paper, *On groups of Fibonacci type*, Proc. Edinburgh Math. Soc., 1977.

Also solved by R. B. J. T. Allenby (England), John Cantwell, George Whitson & Charles Holmes, and the proposer.

Editor's notes. As pointed out by the proposer, subsequent to the submission of the proposal, what is essentially a solution of part (i) appeared in the Journal of the Australian Math. Society, (Series A) Part 2, V. 20, p. 201 by Johnson and Mawdesley.

Allenby proves a generalization to part (ii); Let p be an odd prime, $m \geqq p$, and set

$$G_{m,p} = \langle x_0, x_1, \ldots, x_{m-1}\,;\, x_i x_{i+1}^{-1} x_{i+2} \cdots x_{i+p-2}^{-1} x_{i+p-1}\rangle$$

where there are m relators, i runs over all integers from 0 to $m-1$, all indices being reduced modulo m. Then $G_{m,p}$ is nonabelian.

Whitson and Holmes note that $G_5, G_{15}, G_{25}, \ldots$ give examples of an infinite number of perfect, two generated, nonsimple groups and thus answer a question stated in Coxeter's article (page 84) in the AMS Symposia on Finite Groups (1959).

Continuous Bijections on $\mathbb{R}$

6100 [1976, 490]. *Proposed by Eric Chandler, North Carolina State University*

For fixed integer $n > 1$ find a bijection T on the real numbers such that T^m is a contraction if and only if $m = kn$ for $k = 1, 2, \ldots$. Can T be continuous?

Solution by Mark D. Meyerson, University of Illinois. Let m_r be the midpoint of $I_r = [2^m, 2^{m+1}]$ for $r = 0, 1, \ldots$. For $F : \mathbb{R} \to \mathbb{R}$ we will denote $|F(x) - F(y)|/|x-y|$ by $F\langle x, y\rangle$.

Let $f : I_0 \to I_0$ be a C^∞ bijection such that $f(m_0) = m_0$, $f'(1) = f'(2) = 1$, and $0 < f'(x) \leqq 2^n$ with equality on the right precisely for $x = m_0 = 3/2$. The existence of f can be shown by standard methods, like those employed in partition of unity problems.

Now define $S : \mathbb{R} \to \mathbb{R}$ by $S(x) = x$ if $x \notin \bigcup I_r$, $S(x) = 2^m f(x/2^m)$ if $x \in I_r$. Then $0 < S'(x) \leqq 2^n$ with equality on the right precisely for $x = m_r$ for any r. One may check that for $x \neq y$, $S\langle x, y\rangle < 2^n$ and that f may be chosen so that if x and y do not lie in a single I_r, $S\langle x, y\rangle < 2$. Now let $T(x) = S(x)/2$; $T(x)$ is continuous.

First we show that T^m is not a contraction if $m = kn + s$, $0 < s < n$. We may choose x_0 close to m_k so that $S\langle m_k, x_0\rangle$ is close to 2^n. If p is not a multiple of n, neither $T^p(m_k)$ nor $T^p(x_0)$ lies in any I_r. But if $p = qn$ $(q \leqq k)$, then $T^p(m_k) = m_{k-q}$ and $T^p(x_0)$ is close to m_{k-q}. Hence

$$T^m\langle m_k, x_0\rangle = 2^{-nk-s} \prod_{q=1}^{k} S\langle T^{qn}(m_k), T^{qn}(x_0)\rangle > 1$$

if we choose x_0 close enough to m_k.

Finally we show that T^{kn} is a contraction. Choose $x_0 \neq y_0$. Consider the sequence of pairs: $(x_0, y_0), (T(x_0), T(y_n)), \ldots, (T^{kn}(x_0), T^{kn}(y_0))$. Suppose one of the pairs lies in an interval I_r. Then all but at most k of the pairs lie outside $\bigcup I_r$, and $T^{kn}\langle x_0, y_0\rangle < 2^{-kn} 2^{kn} = 1$. The other possibility is that none of the pairs lie in a single I_r, but at most k of the first elements and at most k of the second elements meet $\bigcup I_r$. Then $T^{kn}\langle x_0, y_0\rangle < 2^{-kn} 2^{2k} \leqq 1$.

Also solved by Ralph Gellar, L. E. Mattics, and the proposer. Partial solution by Gustaf Gripenberg (Finland).

Sequences of Independent Random Variables in a Vector Space

6103 [1976, 572]. *Proposed by Gérard Letac, Université Paul Sabatier, France*

Let $(X_n)_{n=1}^\infty$ be independent and identically distributed random variables, valued in a real vector space E of finite dimension d. Let Y be a random linear form on E such that $\lim_{n\to\infty} Y(X_n)$ exists almost surely (a.s.). If $d = 1$, it is easily proved that either $Y = 0$ a.s. or $X_n =$ constant a.s. What happens if $d > 1$?

Solution by the proposer. Let F be the smallest subspace of the dual E^* of E such that $\Pr[Y \in F] = 1$, and let M be the smallest affine linear manifold of E such that $\Pr[X_n \in M] = 1$.

We prove that $\lim_{n\to\infty} Y(X_n)$ exists if and only if F and M are orthogonal. The "if" part is

obvious. We prove the "only if" part by induction on d. The result is trivial for $d = 0$, and we suppose it is true for $d-1$.

Without loss of generality, we may suppose that M is a subspace of E and that $\dim F > 0$. For convenience we equip E with a euclidean structure with scalar product $\langle\ ,\ \rangle$, and thereby identify E^*. Since $\lim_{n\to\infty}\langle Y, X_{2n} - X_{2n-1}\rangle = 0$ (a.s.), we can assert that if $(y_k)_{k=1}^{\infty}$ is a sequence in the unit ball B of F, dense in B, then

$$\inf_k \overline{\lim_{n\to\infty}} |\langle y_k, X_{2n} - X_{2n-1}\rangle| = 0 \quad \text{(a.s.)}.$$

Since the random variables $(X_{2n} - X_{2n-1})_{n=1}^{\infty}$ are independent,

$$\overline{\lim_{n\to\infty}} |\langle y_k, X_{2n} - X_{2n-1}\rangle| \leqq \varepsilon \quad \text{(a.s.)}$$

implies $|\langle y_k, X_{2n} - X_{2n-1}\rangle| \leqq \varepsilon$ (a.s.) and since B is compact, there exists y_0 in B such that

$$\langle y_0, X_{2n} - X_{2n-1}\rangle = 0 \quad \text{a.s.}$$

Hence y_0 is orthogonal to M.

Denote by E' the $d-1$ dimensional subspace orthogonal to y_0. Note that $M \subset E'$ and $Y = \lambda y_0 + Y'$ with Y' in E' and λ a real random variable. Hence $\lim_{n\to\infty} Y'(X_n)$ exists and we may apply the induction hypothesis.

The Random Variable X/Y, X, Y Normal

6104 [1976, 573]. *Proposed by Leonard W. Deaton, California State University at Los Angeles*

If X and Y are independent normal random variables with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 respectively, then the probability density function f of the random variable $Z = X/Y$ is given by

$$f(z) = \frac{\sigma_1\sigma_2}{\pi(\sigma_1^2 + \sigma_2^2 z^2)}, \qquad -\infty < z < \infty.$$

Prove or disprove that f is the probability density function of Z.

Solution by Dana B. Kamerud, Saint Louis University. The function f is an even function, and so implies that Z is positive with probability equal to $\frac{1}{2}$. However, if the normal random variables X and Y both had positive means, Z would be positive with probability greater than $\frac{1}{2}$; thus the given function f is incorrect. We can get the density function for Z as follows.

Let $U = X/\sigma_1$, $V = Y/\sigma_2$, and $W = (U/V)$. Then $U \sim \text{Normal}(\mu_1, 1)$, $V \sim \text{normal}(\mu_2, 1)$, and $Z = (\sigma_1/\sigma_2)W$. If we find the density g of W, then the density of Z is given by $f(z) = (\sigma_2/\sigma_1)g(\sigma_2 z/\sigma_1)$. To find g, write $P(W \leqq w) = P(U \leqq wV$ and $V > 0) + P(U \geqq wV$ and $V < 0) =$

$$\int_0^{\infty} P(U \leqq wv) f_V(v)dv + \int_{-\infty}^{0} P(U \geqq wv) f_V(v)dv.$$

Taking d/dw on both sides and differentiating under the integral sign yields

$$g(w) = \left(\int_0^{\infty} - \int_{-\infty}^{0}\right)[v f_U(vw) f_V(v)]dv.$$

To find this integral, simplify the integrand:

$$v f_U(vw) f_V(v) = (2\pi)^{-1}\exp(M)v \cdot \exp(-(v-k)^2/2s^2),$$

where $s = (w^2+1)^{-1/2}$, $k = (\mu_1 w + \mu_2)\cdot s^2$, and $M = -\frac{1}{2}(\mu_2 w - \mu_1)^2 s^2$. Using the substitution $t = (v-k)/s$, we obtain $g(w) = (2\pi)^{-1} Q \cdot \exp(M)$, where $Q = ks(2\pi)^{1/2}[1 - 2\Phi(-k/s)] + 2s^2 \cdot \exp(-k^2/2s^2)$ and Φ is the standard normal cumulative distribution function.

Several properties of the density f are evident. First of all, k/s is a function of w whose inverse can be expressed in radicals, while Φ is a function that cannot be written in terms of elementary functions. Thus Q (and hence, $g(w)$) cannot be written as an elementary function, unless k/s is a constant function of w. This happens only if $\mu_1 = \mu_2 = 0$, and in this case $g(w) = 1/(\pi + \pi w^2)$, a well-known result. If either μ_1 or μ_2 is zero, we can at least say that g is a function which is even and satisfies $g(w) \sim a/w^2$ as $w \to \infty$, for some constant a. Clearly, these statements apply to the density of Z as well.

Also solved by A. J. Bosch (Netherlands), Michael Driscoll & Dennis Young, Edward Gbur, C. Ann Goodsell, Ellen Hertz, G. S. Rogers, Michael Skalsky, Wolfe Snow, Philip Young, and the proposer.

$$\Sigma(-1)^{[n\alpha/\sqrt{2}]}n^{-1}$$

6105 [1976, 573]. *Proposed by Harry D. Ruderman, Hunter College Campus School*

Prove that the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{[n\sqrt{2}]}}{n}.$$

Estimate its value.

Solution by David Borwein, University of Western Ontario. Let

$$s_n = \sum_{k=1}^{n} (-1)^{[k\sqrt{2}]}, \quad \sigma_n = \sum_{k=1}^{n} \frac{(-1)^{[k\sqrt{2}]}}{k}.$$

Then $s_n = 2e_n - n$ where e_n is the number of integers k for which $1 \leqq k \leqq n$ and $[k\sqrt{2}]$ is even. Since $[k\sqrt{2}]$ is even if and only if the fractional part of $k\sqrt{2}/2$ is in the interval $(0, 1/2)$ (and since the continued fraction expansion of $\sqrt{2}/2$ is $[0, 1, 2, 2, \ldots]$), it follows from a known result [L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, 1974, Theorem 3.4, p. 125] on the *discrepancy* of the sequence $(n\sqrt{2}/2)$ that

$$|s_n| = 2n\left|\frac{e_n}{n} - \frac{1}{2}\right| \leqq 6 + M\log n,$$

where

$$M = \frac{2}{\log((1+\sqrt{5})/2)} + \frac{4}{\log 3} < 7.8.$$

Hence, by Abel's partial summation formula,

$$\sigma_n = \sum_{k=1}^{n-1} \frac{s_k}{k(k+1)} + \frac{s_n}{n} \to \sigma \quad \text{as} \quad n \to \infty,$$

where σ is finite. The series $\Sigma_{k=1}^{\infty}(-1)^{[k\sqrt{2}]}/k$ is thus convergent, and its sum σ can be estimated to a desired degree of accuracy as follows. We have

$$\begin{aligned}\left|\sigma - \sigma_n + \frac{s_n}{n}\right| &= \left|\sum_{k=n}^{\infty} \frac{s_k}{k(k+1)}\right| \leqq \sum_{k=n}^{\infty} (6 + M\log k)\left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \frac{6}{n} + M\sum_{k=n}^{\infty}\left(\frac{\log k}{k} - \frac{\log(k+1)}{k+1}\right) + M\sum_{k=n}^{\infty}\frac{\log(k+1) - \log k}{k+1} \\ &< \frac{6 + M\log n}{n} + M\sum_{k=n}^{\infty}\frac{1}{k(k+1)} = \frac{6 + M + M\log n}{n} \equiv \varepsilon_n\end{aligned}$$

and so

$$\sigma_n - \frac{s_n}{n} - \varepsilon_n < \sigma < \sigma_n - \frac{s_n}{n} + \varepsilon_n.$$

Computer calculations yield $s_n = 0$, $\sigma_n \doteq -.5154184551$, $\varepsilon_n < .00000896$ for $n = 1599428$. Hence

$$-.515428 < \sigma < -.515409.$$

NOTE. The theory of the discrepancy of the sequence (αn) shows, as above, that $\sum_{n=1}^{\infty}(-1)^{[2\alpha n]}/n$ is convergent when α is an irrational number of finite type γ [*op. cit.*, p. 121], for then we have [*op. cit.*, p. 123, Theorem 3.2] that, for every $\mu > 1-(1/\gamma)$,

$$s_n(\alpha) = \sum_{k=1}^{n}(-1)^{[2\alpha k]} = O(n^{\mu}).$$

All algebraic irrational numbers have type $\gamma = 1$.

Also solved by William Adams, Martin Beumer (Netherlands), P. Bloemendaal (Netherlands), Leonard Borucki, P. Bundschuh (W. Germany), John Davison (Canada), Doug Hensley, Leo Levine, O. P. Lossers (Netherlands), A. McD. Mercer (Canada), R. W. K. Odoni (England), Joseph Rosenblatt & B. Davis, Eric Rosenthal, J. van de Lune (Netherlands), L. van Hamme (Belgium), and Andy Vance.

Editor's notes. (1) van de Lune and Beumer show that n may be replaced by n^s, $s > 0$, in the series and convergence obtains when α is a quadratic irrationality.

(2) Levine shows that there are irrationals α for which the generalized series $\sum(-1)^{[n\alpha/\sqrt{2}]}$ diverges.

(3) For a value of α for which the series diverges, Lossers offers the Liouville number $\alpha = 2\sum_{k=1}^{\infty}3^{-t_k}$ where $t_k \to \infty$ sufficiently rapidly.

(4) Odoni, noting that $\sum(-1)^{[n\alpha/\sqrt{2}]}n^{-1}$ has period equal to 2, obtains the following:

(i) the series converges almost everywhere in $(1,3)$,

(ii) convergence results when x is algebraic irrational or a reduced rational number with odd numerator and diverges if the numerator is even (also proved by Levine).

(5) Rosenblatt considers other generalizations of the problem and communicates the following:

For a series of the form $\sum_{n=1}^{\infty}c_n f\{nx\}$ with f a mean zero function in $L_1[0,1]$, it is difficult to give conditions for a.e. convergence or L_p-norm convergence. If $c_n \geqq 0$ and $\sum_{n=1}^{\infty}c_n = \infty$, then the series will fail to converge in L_p-norm on a dense G_δ subset of the mean zero functions in $L_p[0,1]$. There are mean zero functions $f \in L_\infty[0,1]$ such that $\sum_{n=1}^{\infty}(1/n)f\{nx\}$ [converges at no point; if $\varphi(n)\uparrow\infty$ and $\varphi(n) = O(n^{\delta})$] for some $0<\delta<1$, then there is a mean zero continuous function f with $\sup_N|\sum_{n=1}^{N}(\varphi(n)/n)f\{nx\}| = \infty$ a.e. However, if f is a mean zero function of bounded variation or in some Lipschitz class, then there are many examples of series of this form which do converge a.e. and in some or all $L_p[0,1]$. If f is a mean zero function of bounded variation or in some Lip(α) with $\frac{1}{2} < \alpha \leqq 1$, then the series $\sum_{n=1}^{\infty}(1/n)f\{nx\}$ converges unconditionally in $L_2[0,1]$ as well as a.e. Using the estimates for the discrepancy of $(\{nx\})$ in terms of the continued fraction expansion of x, one can prove that for any mean zero function f of bounded variation, the series $\sum_{n=1}^{\infty}(1/n)f\{nx\}$ converges a.e. and has its maximal function in all $L_p[0,1]$ with $1 \leqq p < \infty$. Other examples and their relationship to certain problems in ergodic theory are possible.

A Functional Equation

6106 [1976, 573]. *Proposed by D. S. Mitrinović and P. M. Vasić, University of Belgrade, Yugoslavia*

Find the general solution to the functional equation

$$\sum_{k=1}^{n} f(x^k) = \sum_{k=1}^{n} f(x^{-k}).$$

Solution by Christopher Henley, California Institute of Technology. First we note that this problem is not well posed, because the proposers do not say what f is to have for range or domain; nor is it certain whether they assume any restrictions such as continuity or differentiability.

Let us assume only $f: \mathbb{R}\backslash\{0\} \to \mathbb{R}$. Let $g(x) = (f(x)+f(1/x))/2$, $h(x) = ((f(x)-f(1/x))/2$, so that $f(x) = g(x)+h(x)$. Then the given equation becomes

$$\sum_{k=1}^{n} h(x^k) = 0 \tag{1}$$

and we can choose for $g(x)$ any function satisfying $g(x)=g(1/x)$; for example, take $g(x)=g_1(|\log x|)$ for $x>0$ and $g(x)=g_2(|\log(-x)|)$ for $x<0$, where g_1 and g_2 are arbitrary real functions.

Now say we have an h satisfying (1). Write $h(-x)=h(x)+g(x)$, so that

$$\sum_{k=1}^{n} h((-x)^k)-\sum_{k=1}^{n} h(x^k)=0,$$

$$\sum_{\substack{k \text{ odd}\\ 1\leq k\leq n}} [h(-x^k)-h(x^k)]=\sum_{\substack{k \text{ odd}\\ 1\leq k\leq n}} g(x^k)=0 \tag{2}$$

giving another functional equation which g must satisfy. Also, if $h(x)$ satisfies (1) and $g(x)$ satisfies (2) for $x>0$, and we define $h(x)$ for negative x by $h(-x)=h(x)+g(x)$ (for $x>0$), then h will satisfy (1) for all nonzero x. For if $x<0$,

$$\sum_{k=1}^{n} h(x^k)=\sum_{k=1}^{n} h(|x|^k)=\sum_{\substack{k \text{ odd}\\ 1\leq k\leq n}} g(|x|^k)=0+0.$$

So the problem is reduced to solving (1) and (2) for $x>0$. In fact, $h(1/x)=-h(x)$ for all x, so $g(1/x)=-g(x)$ for $x>0$. Consequently $h(1)=g(1)=0$ and we only need to consider solving (1) and (2) for $x>1$, since we can then extend h and g to $(0,1)$.

Perform the change of variables

$$h(x)=s(\ln\ln x),\quad g(x)=t(\ln\ln x).$$

The functions $s(y)$ and $t(y)$ are well defined on $(-\infty,+\infty)$ since the function $\ln\ln x$ is one-to-one from $(1,\infty)$ to $(-\infty,+\infty)$. Then (1) and (2) become

$$\sum_{k=1}^{n} s(y+\ln k)=0, \tag{1$'$}$$

$$\sum_{\substack{k \text{ odd}\\ 1\leq k\leq n}} t(y+\ln k)=0. \tag{2$'$}$$

The difficulty of solving (1$'$) and (2$'$) depends on the value of n.

For $n=2$ we have from (1$'$) and (2$'$)

$$s(y)+s(y+\ln 2)=0,\quad t(y)=0.$$

It is easy to find $s(y)$. For example take $s(y)=\cos(\pi y/\ln 2)$. More generally, take $s(y)=a(y)-a(y+\ln 2)$ where $a(y)$ is any function with the period $2\ln 2$. (Of course, this representation is not unique.)

Note that $h(x)=0$ is the only solution continuous on $(0,\infty)$ since in that case

$$\begin{aligned} s(y) &= \lim_{n\to\infty} s(y-2n\ln 2)\\ &= \lim_{n\to\infty} h(e^{\exp(y-2n\ln 2)})\\ &= \lim_{n\to\infty} h((e^{\exp y})^{1/2^{2n}})=h(1)=0 \end{aligned}$$

by continuity at $x=1$. If $s(y)$ is continuous, then $h(x)=s(\ln\ln x)$ will be continuous everywhere except at $x=1$.

For $n=3$ we have

$$s(y)+s(y+\ln 2)+s(y+\ln 3)=0 \tag{1$'$}$$

$$t(y)+t(y+\ln 3)=0. \tag{2$'$}$$

Obviously (2′) can be solved exactly as (1′) was in the case $n = 2$. We can construct some solutions for (1′) of the form $s(y) = \mathrm{Re}(e^{cy}) = e^{c_1 y}\cos c_2 y$ where $c = c_1 + c_2 i$ is complex. This will satisfy (1′) if

$$1 + e^{c \ln 2} + e^{c \ln 3} = 0.$$

This has a sequence of solutions near the imaginary axis, of which the first is $c \cong -.2 + 3.9i$. Unfortunately all seem to have negative real part and therefore

$$\lim_{x \to 1^+} h(x) = \lim_{y \to -\infty} s(y)$$

diverges, unless $s(y) = 0$. Thus there are no continuous solutions, except $s = t = 0 \Rightarrow h(x) = 0$ for all nonzero x.

Also solved, partially, by P. G. Kirmser.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Differential Topology. By Victor Guillemin, Alan Pollack. Prentice-Hall, Englewood Cliffs, New Jersey, 1974. xvi + 222 pp. $16.95. (Telegraphic Review, January 1975.)

I used this book in a graduate course consisting of first and second year graduate students most of whom were M.A. candidates. We all found that the book is an excellent text for the usual reasons. It contains significant mathematics presented clearly. There are many good exercises and many figures which illustrate the concepts being discussed. In addition, the book is almost completely free of typographical errors. (I found only two — both insignificant.)

Readability and mathematical accuracy as well as the number and quality of exercises are certainly important things to consider when choosing a text. There are however two other very important criteria that often go ignored and these criteria distinguish an excellent book such as this from a merely good one. The first is the accessibility of the contents of the book and the second (and more important) is the mathematical insight or "flair" with which the book is written.

By the accessibility I mean that by the end of the semester the students should have covered enough material so that they get a good idea of what the subject is about. It is most frustrating for teachers and students alike to go through a lot of preliminary work to develop some machinery for solving certain problems and then to run out of time before having a chance to use the machinery. We covered the first two chapters of this book in our course, which was an eight week summer course. However, as I show below, a reader who was only able to cover the first chapter (e.g., a reading course for an undergraduate or a faculty seminar in which the faculty members ran out of time at the end of the semester) would still cover a significant amount of mathematics.

The first chapter starts with the ideas of a manifold and its tangent space. The authors assume that their manifolds are embedded in some Euclidean space. This helps the exposition because it enables the idea of tangent space to be defined much more easily than by the usual abstract approach. As the authors remark "the most serious objection to working in Euclidean space is that it obscures the

difference between properties intrinsic to the manifold and properties of the embedding. We have endeavored to make the student aware of this distinction, yet we have not scrupled to use the ambient space to make proofs more comprehensible." They succeed admirably in making this distinction.

In the next section the authors state the Inverse Function Theorem: "Suppose that $f: X \to Y$ is a smooth map whose derivative df_x at the point x is an isomorphism. Then f is a local diffeomorphism at x." Unfortunately, in this form it is not familiar to most students. The authors claim that "you should easily be able to translate the Euclidean result to the manifold setting by using local parametrizations." This is true of an experienced mathematician but it would have been nice to include a more classical statement of the Inverse Function Theorem and then derive quickly the one quoted above. The Inverse Function Theorem in this version and the Implicit Function Theorem form the only real prerequisite for the first three chapters.

In the next section the authors define immersions and submersions (maps $f: X \to Y$ in which df_x is onto for all $x \in X$). They define regular value ($y \in Y$ is a regular value if df_x is onto for every $x \in f^{-1}(y)$) and then prove the preimage theorem (if y is a regular value of $f: X \to Y$ then the preimage $f^{-1}(y)$ is a submanifold of X with $\dim f^{-1}(y) = \dim X - \dim Y$). They use this theorem to prove the nontrivial fact that the group of orthogonal matrices ($0(n)$) is an $n(n+1)/2$ manifold. They use it then to motivate the notion of a map being transversal to a submanifold. ($f: X \to Y$ is transversal to Z if Image df_x + Tangent space to Z at $f(x)$ = Tangent space to Y at $f(x)$.) "To provide cohesiveness to the elementary material," the authors "have tried to emphasize the 'stable' and 'generic' quality of our definitions." This they do in section 6 which my students found especially exciting. (A property is stable if any homotopy which starts with a map having the property must be a homotopy through maps having that property for small values of the homotopy parameter. A property is generic if any map may be deformed slightly by a homotopy to get a map having that property.) The first chapter ends with a discussion of Sard's theorem and its applications to both Morse theory and the Whitney embedding theorem (every n-manifold can be embedded in R^{2n+1}). There is indeed a significant amount of mathematics in the first chapter!

The second chapter deals with transversality and intersection theory. It presents the intersection number mod 2 and uses it to present the winding number, the Jordan-Brouwer separation theorem, the Borsuk–Ulam theorem (any map that is symmetric about the origin must wind around it an odd number of times) and the Fundamental Theorem of Algebra. The Jordan-Brouwer theorem is presented as a collection of exercises (with hints). Chapter 3 presents oriented intersection theory from which the authors derive the Lefschetz fixed point theorem, Poincaré–Hopf theory on vector fields, and the Hopf degree theorem. The last chapter, which is more difficult than the first three, presents integration of differential forms and a proof of the Gauss–Bonnet theorem.

All this material is presented with considerable "flair." The authors have a very strong geometric insight and they are able to communicate this to the reader. For example, in discussing the immersion of the Klein bottle in R^3 they say, "to envision an embedding in R^4, represent the fourth dimension by density of red coloration and allow the bottle in the drawing to blush as it passes through itself." For a more substantive example, consider the explanation of the difference between a stable property and a generic one "... transversality is a generic quality: any smooth map no matter how bizarre its behaviour with respect to a given submanifold Z in Y may be deformed by an arbitrary small amount into a map that is transversal to Z. Physically, stability means that transversal maps are actually observable. The fact that transversality is generic says that *only* transversal maps are observable." Passages like the two above are sprinkled liberally throughout the book. They are a joy to read and greatly clarify the mathematics involved. This kind of geometric insight also allows a nonspecialist to teach out of this book—an opportunity not frequently enough available or used.

My students enjoyed the book very much. They especially appreciated the wealth of examples in the text itself as well as the insights provided in the text. The students were intrigued by some of the exercises which lead to some very important kinds of mathematical theories (e.g., p. 25, #7 as a hint of catastrophe theory or p. 84, #14 and #15 which talk about cobordism). The bibliography is

excellent, with each of the books in it rated G, GP, R or X depending upon background necessary. (I only wish that there were a reference to catastrophe theory in it.) There are 141 exercises in Chapter 1 alone which enables the instructor to pick and choose as desired. The only complaint that I have about the exercise sets is that some of the earlier sets have too many starred problems (indicating that they will be used later in the text). This means either assigning long problem sets or omitting some of the starred problems. It would also be helpful if the more difficult problems were marked.

This book should certainly be in the library of every college and university. It is also ideal for faculty seminars and reading courses as well as for use as a graduate text. The authors say that the book could be used for undergraduates, but I would only attempt this with very good undergraduates.

RICHARD MILLMAN, Southern Illinois University at Carbondale

FILMS

Regular Homotopies in the Plane: Parts I and II. Produced by the Educational Development Corporation for the Topology Films Project with support from the National Science Foundation. Project Director: Nelson L. Max. Part I 14 minutes; Part II 18.5 minutes; both parts have sound, color and computer-animation. Guides are available. Sale and rental by the International Film Bureau.

This pair of films offers the viewer a delightful short excursion into elementary differential topology. The trip is more than simply entertaining. It proceeds with deliberateness and efficiency along a route well chosen to unfold to the viewer some of the major features of the landscape. Equipped with the knowledge of some elementary vector calculus, the viewer is likely to come away from these films with a clear intuitive picture of the "lay of the land," a sense of the geometric *style* of modern topology.

The two films divide the task of arguing the "if" and the "only if" parts of the Whitney-Graustein theorem. The statement of the theorem is this: *Two C^1 closed curves in the plane are regularly homotopic if and only if they have the same rotation number.* The various topological conditions which the curves and homotopies must satisfy are explained (and cleverly illustrated with computer graphics) in elementary and intuitive terms. The "rotation number" of a curve (which is, essentially, the winding number of the derivative about the origin) is described via an ingenious heuristic device which the films call the "tangent graph." This graph simultaneously plots the argument of the derivative and the trajectory of a point while it traverses the curve. The technique has immediate visual and intuitive appeal and provides very accessible illustrations of simple pathology such as discontinuity of the derivative, or continuity but non-differentiability of the derivative.

In the course of the demonstration, the film succeeds in communicating in a natural and motivated way the style and the flavor of certain important constructions and techniques in modern topology. This, it appears, is its major strength, and justifies the intuitive and easy-going approach which the films faithfully maintain. For example, the theorem itself calls attention to the powerful but simple technique (which is applied almost everywhere in modern topology) of classifying mappings and characterizing the classes by means of algebraic or numerical invariants. More specific techniques like the use of Lebesgue number arguments, the construction of homotopies of maps of the same degree, and the first steps in the reconstruction of a curve by integration of its derivative are discussed and illustrated.

While the heuristic level is intuitive, the films succeed in presenting in an honest way some of the knottier problems that arise in the second half (the "if" part) of the Whitney–Graustein theorem. These films have something to offer almost anyone who is familiar with the rudiments of vector calculus. It is one of the best mathematical rides that I've had in a long time.

JAMES WHITE, Carleton College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, L*. *Encyclopaedic Dictionary of Mathematics for Engineers and Applied Scientists*. Ed: I.N. Sneddon. Pergamon Pr, 1976, viii + 800 pp, $100. [ISBN: 0-08-016767-5] Over 2000 alphabetically arranged entries, from brief definitions to general introductions (2000-3000 words), on applied mathematics, theoretical physics and engineering. Limited in emphasis (e.g., "algorithm" receives only one sentence) and dated (e.g., all but one reference on graph theory is from 1962 or earlier, splines and finite elements appear only in a last-minute appendix), it is, nevertheless, a useful compendium of information, especially concerning named formulas and theorems: Debye's equation, Hankel's symbol, Pexider's equations, Schlömilch series, etc. A subject index at the end of the volume is useful for locating information within the main entries. LAS

GENERAL, S, L. *Chess and Computers*. David Levy. Computer Sci Pr, 1976, vi + 145 pp, $11.95; $7.95 (P). [ISBN: 0-914894-03-X; 0-914894-02-1] Inside look at the world of computer chess written by one of the world's foremost authorities. LCL

GENERAL, S(13), L. *The Canadian Mathematics Olympiads, 1969-1975, Second Edition*. W. Moser, E. Barbeau. Canadian Math Congress, 1976, 64 pp, (P). The first seven Canadian Mathematical Olympiads--problems, solutions, and names of winners. LCL

GENERAL, S. *An Annotated Bibliography of Cryptography*. David Shulman. Garland Pub, 1976, xvi + 372 pp, $35. [ISBN: 0-8240-9974-5] A chronologically listed bibliography on the subject of cryptography going back to 1518. Also includes an alphabetical list of authors. Over 3000 entries. CEC

BASIC, T(9-13), S. *Accent on Algebra, Revised*. Pat Boyle, Bill Juarez. Creative Pub, 1977, 140 pp, $4.50 (P). [ISBN: 0-88488-046-X] Crossword and crossnomial puzzles, word scrambles, alphametics, matchstick puzzles, picture plotting activities, reinforcing vocabulary and concepts of beginning algebra. Pages specifically designed to be reproduced for classroom use. LCL

BASIC, S*(9-13). *Successful Problem Solving Techniques*. Carole Greenes, John Gregory, Dale Seymour. Creative Pub, 1977, vii + 119 pp, $5 (P). [ISBN: 0-88488-086-9] Ideas and suggestions for teaching problem solving of all types--emphasis on method rather than mathematical content. Examples illustrate main principles, followed by problem worksheets for development and practice. LCL

PRECALCULUS, T(13: 1). *College Algebra, A Skills Approach*. J. Louis Nanney, John L. Cable. Allyn, 1978, v + 634 pp, $13.95 (P). [ISBN: 0-205-05913-2] A workbook approach which provides lots of practice for the student. Chapters are: basic principles of algebra, equations and inequalities (includes complex numbers), relations and functions (with conic sections), systems of equations and inequalities, logarithms, polynomials, sequences and series, and binomial theorem. LLK

PRECALCULUS, T(13: 1). *Pre-Calculus Mathematics*. Michael Payne. Saunders, 1977, viii + 429 pp, $12.95. [ISBN: 0-7216-7122-5]; *Instructor's Manual*, ii + 54 pp, (P). Functions (polynomial, rational, exponential, logarithmic, trigonometric) including systems of linear equations and conics. Attractive, inviting format. LCL

EDUCATION, T(16: 1), S*, P, L. *Readings in Secondary School Mathematics, Second Edition*. Ed: Douglas B. Aichele, Robert E. Reys. Prindle, 1977, ix + 582 pp, $9.95 (P). [ISBN: 0-87150-202-X] Most of the articles from the first edition reappear, but there are many changes. The extensive bibliography has been updated. This is a valuable resource for mathematics educators. CEC

EDUCATION, P. *An In-Service Handbook for Mathematics Education*. Ed: Alan Osborne. NCTM, 1977, xii + 259 pp, $5.50 (P). [ISBN: 0-87353-119-1] A collection of articles which consider the goals, purposes, perceptions of, policies, processes, procedures and the future of in-service education for mathematics teachers. CEC

COMBINATORICS, P. *Applied Graph Theory Bibliography with Forward Citations*. Ed: Gerald Berman. Computer Lab, U. of Waterloo, 1977, 175 pp, (P). A bibliography complementing the author's *Forward Citations in Graph Theory*. Contents include over 2100 references to papers, index of key words, and forward citations for approximately half the papers. SS

COMBINATORICS, T**(13-14: 1), S*, L**. *Graphs as Mathematical Models*. Gary Chartrand. Prindle, 1977, x + 294 pp, $14.95. [ISBN: 0-87150-236-4] This well-organized, highly readable book studies some very interesting applications of graph theory. Suitable as a text for people with no background in college mathematics. Lots of good problems and references. CEC

COMBINATORICS, S(17), P. *Higher Combinatorics*. Ed: Martin Aigner. Reidel, 1977, xiii + 256 pp, $24. [ISBN: 90-277-0795-2] A collection of sixteen papers which constitutes the Proceedings of the NATO Advanced Study Institute held in Berlin during September 1976. CEC

NUMBER THEORY, T*(17: 1), S, P, L. *Number Fields*. Daniel A. Marcus. Springer-Verlag, 1977, viii + 279 pp, $12 (P). [ISBN: 0-387-90279-1; 3-540-90279-1] A well-written and well-motivated introduction to algebraic number theory which is fairly sophisticated and fast moving. A marvelous collection of exercises. CEC

LINEAR ALGEBRA, T(14-15: 1, 2). *Linear Algebra.* Georgi E. Shilov. Trans: Richard A. Silverman. Dover, 1977, xi + 387 pp, $6 (P). [ISBN: 0-486-63518-X] Begins with elementary material but goes quickly to more advanced areas. In writing for students of mathematics and physics, the author has pulled together ideas from algebra, geometry and analysis. Covers linear spaces, systems of linear equations, linear functions of a vector argument, coordinate transformations, the canonical form of the matrix of a linear operator, bilinear and quadratic forms, unitary spaces, quadratic forms in Euclidean and unitary spaces, and finite-dimensional algebras and their representations. LLK

ALGEBRA, T(17: 1), S*, P, L. *Examples of Groups.* Michael Weinstein. Polygonal Pub, 1977, x + 307 pp, $12; $6 (P). A detailed analysis of 44 important examples from group theory is given in a no-nonsense style. Standard construction techniques, lots of exercises, references, and indices of examples and counterexamples are included. A fairly high degree of familiarity with the techniques and notation of modern algebra is assumed. CEC

ALGEBRA, S(18), P. *Punctual Hilbert Schemes.* Anthony A. Iarrobino. Memoirs No. 188. AMS, 1977, viii + 112 pp, $7.60 (P). [ISBN: 0-8218-2188-1] On the structure of families of open ideals in rings of power series in two variables. JD-B

ALGEBRA, P. *Lecture Notes in Mathematics-590: Analyse Harmonique dans les Systèmes de Tits Bornologiques de Type Affine.* Hideya Matsumoto. Springer-Verlag, 1977, 219 pp, $11 (P). [ISBN: 0-387-08249-2; 3-540-08249-2]

ALGEBRA, S(18), P. *Decomposition and Dimension in Module Categories.* Jonathan S. Golan. Lect. Notes in Pure and Appl. Math., V. 33. Dekker, 1977, xii + 185 pp, $18.75 (P). [ISBN: 0-8247-6643-1] The notions of primary decomposition and Krull dimension extended to module categories. Emphasis on the similarities between these two concepts in their most general form. JRG

FINITE MATHEMATICS, T(13: 1). *Modelling by Mathematics: Block 1, Graphs and Symbols.* Open U, 1977, 179 pp, £4.50 (P). [ISBN: 0-335-06290-3] A self-study format--good discussion of selected models. This block includes histograms and Gaussian distribution, linear equations, and compound interest. LLK

CALCULUS, T(13: 1). *Elementary Calculus for Business, Economics, and Social Sciences.* Chaney Anderson, R.C. Pierce, Jr. HM, 1975, xi + 251 pp, $11.95. [ISBN: 0-395-18960-8] Watered-down treatment of standard first-term calculus topics, plus Lagrangian multipliers and minus trigonometry. Only minimal high school algebra prerequisite. LCL

CALCULUS, T(14-16: 1), S, L. *Foundations of the Calculus.* Henry F. DeBaggis, Kenneth S. Miller. Krieger, 1977, xiii + 232 pp, $10.50. [ISBN: 0-88275-348-7] Emphasis on theory rather than manipulative technique. Formal presentation: real numbers, limits, derivatives, integrals, antidifferentiation. Reprint of original 1966 edition. LCL

CALCULUS, T*(14-16: 1), S, L. *Two-Dimensional Calculus.* Robert Osserman. Krieger, 1977, xvii + 456 pp, $17.50. [ISBN: 0-88275-473-4] Unified presentation of two-dimensional calculus. Provides context for intuitive and geometrical understanding of essential distinctions between one-variable and several variable calculus; also eases incorporation of linear algebra into multivariable calculus. Reprint of original 1968 edition. LCL

CALCULUS, S(13). *How to Enjoy Calculus.* Eli S. Pine. Steinlitz-Hammacher, 1975, 128 pp, $3.95 (P). [ISBN: 0-917208-01-3] The author declares this appropriate for a large audience: 1) anyone with a curiosity about calculus, 2) anyone who took calculus and didn't understand it, 3) students about to start the subject, 4) high school students, and 5) those who have been away from mathematics and want a "quick calculus." It is three chapters, slope finding (differentiation), area finding (integration), and the wedding of slope finding to area finding and the resulting baby, the differential equation. LLK

COMPLEX ANALYSIS, T?(17-18: 2). *Lectures on the Theory of Functions of a Complex Variable.* George W. Mackey. Krieger, 1977, 266 pp, $12.50. [ISBN: 0-88275-531-5] A reprint (TR, May 1968) of the author's lectures in an introductory function theory course. A very formal approach which de-emphasizes the geometric aspects of the subject. TRS

COMPLEX ANALYSIS, T**(18), S, P*, L*. *Applied and Computational Complex Analysis, V. 2: Special Functions--Integral Transforms--Asymptotics--Continued Fractions.* Peter Henrici. Wiley, 1977, ix + 662 pp, $32.50. [ISBN: 0-471-01525-3] Continuing in the algorithmic spirit (*V. 1*, TR, June-July 1975), this second volume focuses on the evaluation and manipulation of solutions of analytic differential equations. Successive chapters: series expansions (including Jacobi's triple product identity and the prime number theorem), ordinary differential equations, integral transforms, asymptotic analysis, continued fraction expansions. An abundance of applications, examples, and exercises. TRS

DIFFERENTIAL EQUATIONS, P. *Nonlinear Diffusion.* Ed: W.E. Fitzgibbon, H.F. Walker. Pitman, 1977, 232 pp, $13.50 (P). [ISBN: 0-273-01066-2] Contributed papers from the N.S.F.-C.B.M.S. regional conference on non-linear diffusion equations held at the University of Houston in June, 1976. TRS

DIFFERENTIAL EQUATIONS, P. *Lifting Properties in Skew-Product Flows with Applications to Differential Equations.* Robert J. Sacker, George R. Sell. Memoirs No. 190. AMS, 1977, iv + 67 pp, $7.20 (P). [ISBN: 0-8218-2190-3]

DIFFERENTIAL EQUATIONS, P. *Explicit a priori inequalities with applications to boundary value problems.* V.G. Sigillito. Pitman, 1977, 103 pp, £5.50 (P). [ISBN: 0-273-01022-0]

NUMERICAL ANALYSIS, P. *Approximation of Functions and Operators.* Ed: S.B. Stečkin. Proc. of Steklov Inst. of Math., No. 138. AMS, 1977, iii + 211 pp, $28.80 (P). [ISBN: 0-8218-3038-4] Seven papers, including three by V.V. Arestov, on approximation of operators, on splines, and on Kolmogorov's inequalities in $L_2[0,\infty)$. LAS

NUMERICAL ANALYSIS, T(14-17; 1), S, L. *Mathematical Elements of Scientific Computing.* Ramon E. Moore. HR&W, 1975, x + 237 pp, $12.95. [ISBN: 0-03-088125-0] Presents common numerical methods used in applied mathematical work. Begins with computer arithmetic and the problem that it poses. Majority of text focuses on approximation, both continuous and discrete. Eigenvalues and eigenvectors are introduced in the context of specific applied problems. All necessary linear algebra is included. Chapter problems emphasize the implementation of solutions to problems on computers. Appendices. References. Index. RJA

NUMERICAL ANALYSIS, S(15-18), P. *Lecture Notes in Computer Science-51: Matrix Eigensystem Routines--EISPACK Guide Extension.* B.S. Garbow, *et al.* Springer-Verlag, 1977, viii + 343 pp, $14.30 (P). [ISBN: 0-387-08254-9; 3-540-08254-9] Second of two volumes on EISPACK, a collection of Fortran routines for finding eigenvalues and/or eigenvectors for matrix problems. The present text stresses the symmetric band eigenproblem, the generalized symmetric eigenproblem, the generalized real eigenproblem, and the singular value decomposition of a rectangular matrix and solution of an associated linear least squares problem. Information on the EISPACK control program is included. Illustrative examples with source code. Tables of execution times for various hardware. References. RJA

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-606: Mathematical Aspects of Finite Element Methods.* Ed: I. Galligani, E. Magenes. Springer-Verlag, 1977, 362 pp, $14.30 (P). [ISBN: 0-387-08432-0; 3-540-08432-0] Proceedings of the conference held in Rome, December 10-12, 1975. JAS

FUNCTIONAL ANALYSIS, T**(16-18: 1, 2), S, L*. *Introductory Functional Analysis with Applications.* Erwin Kreyszig. Wiley, 1978, xiv + 688 pp, $21.50. [ISBN: 0-471-50731-8] Elementary approach to metric, normed, and Hilbert spaces with three chapters on spectral theory and two on unbounded operators and quantum mechanics. Excellent for students of applied science, the text is full of examples and applications (e.g., numerical integration, contractive maps, approximation theory) and adeptly displays the interplay between "abstract" and "concrete." Presumes no measure theory, topology, or complex analysis. TRS

FUNCTIONAL ANALYSIS, T(18: 1). *Bornologies and Functional Analysis.* Henri Hogbe-Nlend. Trans: V.B. Moscatelli. Math. Stud., V. 26. North-Holland, 1977, xii + 144 pp, $19.50 (P). [ISBN: 0-7204-0712-5] An introductory text on the theory of bornology and its use in functional analysis. Topics: bornological constructions, completeness, compactness, internal and external duality of topology-bornology, applications to partial differential equations. Presumes basic facts from topology and normed space theory. Many exercises but few examples. TRS

FUNCTIONAL ANALYSIS, T(14-17: 1), S. *Grundlagen der Funktionalanalysis und Approximationstheorie.* Ernst Kühner, Peter Lesky. Vandenhoeck & Ruprecht, 1977, 216 pp, DM 30 (P). [ISBN: 3-525-40539-1] Metric spaces. Normed vector spaces. Scalar product spaces. Sequences, convergence, and continuity. Approximation in finite and infinite dimensional spaces. Exercises. References. Index.RJA

FUNCTIONAL ANALYSIS, T(18: 2), S, P. *Topological Algebras.* Edward Beckenstein, Lawrence Narici, Charles Suffel. Math. Stud., V. 24. North-Holland, 1977, xii + 370 pp, $24.50 (P). [ISBN: 0-7204-0724-9] An advanced text, dealing with general topological algebras, algebras of continuous functions, and such algebras provided with the compact-open topology. Exercises, many difficult but with hints.JD-B

FUNCTIONAL ANALYSIS, S(18), P. *Classical Banach Spaces I, Sequence Spaces.* Joram Lindenstrauss, Lior Tzafriri. Ergebnisse der Math., B. 92. Springer-Verlag, 1977, xiii + 190 pp, $24. [ISBN: 0-387-08072-4; 3-540-08072-4] The first of a four-volume sequence containing the main results, as well as current research trends and open problems, in the theory of Banach spaces. MU

FUNCTIONAL ANALYSIS, S(18), P. *Approximation of Vector Valued Functions.* João B. Prolla. Math. Stud., V. 25. North-Holland, 1977, xiii + 219 pp, $22.50 (P). [ISBN: 0-444-85030-9] A study of the many variations of the Stone-Weierstrass Theorem for vector valued functions and some applications. Readable and well motivated. MU

FUNCTIONAL ANALYSIS, S(18), P. *Lecture Notes in Mathematics-600: Value Distribution of Parabolic Spaces.* Wilhelm Stoll. Springer-Verlag, 1977, viii + 216 pp, $11.40 (P). [ISBN: 0-387-08341-3; 3-540-08341-3]

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-578: Séminaire Pierre Lelong (Analyse) Année 1975/76.* Ed: Pierre Lelong. Springer-Verlag, 1977, 327 pp, $13.70 (P). [ISBN: 0-387-08256-5; 3-540-08256-5] A continuation of the Lecture Notes publication of seminar presentations of previous years (less three papers published elsewhere) together with the proceedings of the special *Journées sur les Fonctions Analytiques* in May 1976. JAS

FUNCTIONAL ANALYSIS, T(18: 1), S, P. *Generalized Inverses of Linear Operators: Representation and Approximation.* C.W. Groetsch. Pure and Appl. Math., V. 37. Dekker, 1977, viii + 165 pp, $19.50. [ISBN: 0-8247-6615-6] Presentation of a unified treatment of approximation theory of generalized inverses of bounded linear operators in Hilbert space. Much of the recent literature is surveyed, and a comprehensive bibliography is included. Exercises. JRG

OPTIMIZATION, T?(17), P. *Stochastic Programming.* V.V. Kolbin. Theory and Decision Lib., V. 14. Reidel, 1977, xii + 195 pp, $26. [ISBN: 90-277-0750-2] Systematic theoretical presentation. with a good bibliography. After an introductory chapter, it contains six chapters on chance-constrained, two-stage, and multi-stage stochastic programming problems, the game approach, problems of the existence of a solution and its optimality, and problems of the stability of solutions. No numerical examples and no exercises. RSK

OPTIMIZATION, T*(15-18: 1, 2), S, L. *Combinatorial Optimization: Networks and Matroids.* Eugene L. Lawler. HR&W, 1976, x + 374 pp, $25. [ISBN: 0-03-084866-0] Solution of optimization problems formulated as networks and matroids. Topics: shortest paths, network flows, bipartite and nonbipartite matching, matroids and intersections, the greedy algorithm, matroid parity problem. A detailed background chapter on graphs is included. Problems. Chapter references. Subject and author indices. RJA

ANALYSIS, P. *Lecture Notes in Mathematics-602; Harmonic Analysis on Compact Solvmanifolds.* Jonathan Brezin. Springer-Verlag, 1977, vii + 179 pp, $8.30 (P). [ISBN: 0-387-08354-5; 3-540-08354-5] An exposition of the work of L. Auslander, R. Howe, and the author on inductive methods in harmonic analysis on compact homogeneous spaces. Includes an introductory chapter which is devoted entirely to examples and which may serve as an accessible overview of the field. TRS

ANALYSIS, T(18), P. *Treatise on Analysis, V. V.* J. Dieudonné. Trans: I.G. Macdonald. Pure and Appl. Math., V. 10-V. Acad Pr, 1977, xii + 243 pp, $21.50. [ISBN: 0-12-215505-X] Chapter 21 (on compact and semisimple Lie groups) of the planned 25 chapter *Treatise,* translated from the 1975 French original edition. LAS

ANALYSIS, P. *Lecture Notes in Mathematics-604: Banach Spaces of Analytic Functions.* Ed: J. Baker, C. Cleaver, J. Diestel. Springer-Verlag, 1977, v + 141 pp, $8.30 (P). [ISBN: 0-387-08356-1; 3-540-08356-1] The proceedings of the Pelczynski Conference held at Kent State University, July 12-16, 1976. Pelczynski's lectures will appear in the CBMS series and some other hour lectures will appear as individual papers elsewhere. JAS

ANALYSIS, P. *American Mathematical Society Translations, Series 2, V. 110.* AMS, 1977, iii + 188 pp, $28. [ISBN: 0-8218-3060-0] Nine papers on analysis originally published (in Russian) between 1953 and 1969. LAS

ALGEBRAIC GEOMETRY, T*(17-18: 1, 2), L. *Elementary Algebraic Geometry.* Keith Kendig. Grad. Texts in Math., V. 44. Springer-Verlag, 1977, viii + 309 pp, $18.80. [ISBN: 0-387-90199-X; 3-540-90199-X] An introductory algebraic geometry text that does not require a staggering amount of sophistication. Informal style, emphasis on development of intuition. Lots of examples and pictures. Rigorous study of plane curves is followed by generalization to varieties of arbitrary dimension. Ends with Riemann-Roch theorem. JRG

ALGEBRAIC GEOMETRY, T*(16-18: 2), P. *Basic Algebraic Geometry.* I.R. Shafarevich. Trans: K.A. Hirsch. Springer-Verlag, 1977, xv + 439 pp, $19.80 (P). [ISBN: 0-387-08264-6; 3-540-08264-6] Paperback "study edition" of the 1974 hardcover English edition (TR, April 1975) of the 1972 Russian original edition. LAS

ALGEBRAIC GEOMETRY, P. *On Degenerations of Algebraic Surfaces.* Ulf Persson. Memoirs No. 189. AMS, 1977, xv + 144 pp, $8 (P). [ISBN: 0-8218-2189-X] Presentation of methods and techniques for the study of degenerations of algebraic surfaces over a disc. Structure and classification of surfaces and structure of degenerations are discussed. JRG

ALGEBRAIC GEOMETRY, T*(18: 1, 2), P*, L. *Algebraic Geometry.* Robin Hartshorne. Grad. Texts in Math., V. 52. Springer-Verlag, 1977, xvi + 496 pp, $24.50. [ISBN: 0-387-90244-9; 3-540-90244-9] Introductory text using methods of schemes and cohomology; assumes solid graduate algebra course. Includes appendices on intersection theory, transcendental methods, the Weil conjectures. Many exercises. Substantial bibliography. JRG

DIFFERENTIAL GEOMETRY, T(16-18: 1), S, P. *Geometry of Groups of Transformations.* André Lichnerowicz. Trans: Michael Cole. Noordhoff, 1977, xiv + 234 pp, $29.25. [ISBN: 90-286-0506-1] Topics: infinitesimal transformations, groups of transformations, homogeneous spaces, reductive homogeneous spaces, reductive Lie algebras, affine transformations and isometries, conformal and analytic transformations. Assumes knowledge of some elementary differential geometry. Bibliography. Subject index. List of symbols. RJA

GEOMETRY, T?(15: 1). *Transformation Geometry.* J.N. Kapur. Affiliated East-West Pr, 1976, ix + 257 pp, (P). Detailed analytic introduction to the affine transformation group, its subgroups and symmetry groups; a good reference for a transformation geometry course but poor binding and unpolished lecture note style make it a questionable text. JNC

GEOMETRY, P, L. *Unsolved Problems Concerning Lattice Points.* J. Hammer. Fearon-Pitman, 1977, 101 pp, $9 (P). [ISBN: 0-273-01103-0] Detailed outline of current research in lattice point problems related to Minkowski's theorem, combinatorics and compactness theorems. Many unsolved problems are mentioned. The list of references is outstanding. CEC

TOPOLOGY, T(15-17), S, P, L**. *Set Theory and Metric Spaces, Second Edition.* Irving Kaplansky. Chelsea, 1977, xii + 140 pp, $8.50. [ISBN: 0-8284-0298-1] Essentially a reprinting of a beautiful little text containing most of the jewels of set theory and metric space topology. The text is well motivated by historical tidbits and informal discussions of the key ideas. (TR, June/July 1972; ER, October 1973.) MU

TOPOLOGY, P. *Lecture Notes in Mathematics-597: Geometry and Topology.* Ed: Jacob Palis, Manfredo do Carmo. Springer-Verlag, 1977, vi + 866 pp, $23 (P). [ISBN: 0-387-08345-6; 3-540-08345-6] Texts of a series of research talks presented at Rio de Janeiro in April 1977 as part of the third Latin American School of Mathematics. JAS

TOPOLOGY, T(15-18: 1), S, P, L. *Lie Groups and Compact Groups.* John F. Price. London Math. Soc. Lect. Note Ser., No. 25. Cambridge U Pr, 1977, ix + 177 pp, $8.95 (P). [ISBN: 0-521-21340-1] Provides a coordinate-free introduction to Lie group theory and presents the structure of compact connected groups and of Lie groups in a manner accessible to those merely wishing to utilize this material as a tool. Contains all results necessary for the structure theorems plus chapters on the geometry of Lie groups and on the necessary and sufficient conditions for a compact group to be Lie. End of chapter notes of historical and motivational nature. Chapter exercises. Appendix on harmonic analysis. Bibliography. Index. RJA

TOPOLOGY, P. *Lecture Notes in Mathematics-603: Complex Surfaces and Connected Sums of Complex Projective Planes.* Boris Moishezon. Springer-Verlag, 1977, 234 pp, $11.50 (P). [ISBN: 0-387-08355-3; 3-540-08355-3] Specific bounds are found for certain connected sums of 4-manifolds (which admit complex structures) to be decomposable. This improves on an existence theorem of C.T.C. Wall. JAS

TOPOLOGY, P. *Topological Semifields and Their Applications to General Topology.* M. Ja. Antonovskiĭ, V.G. Boltjanskiĭ, T.A. Sarymaskov. Amer. Math. Soc. Transl., Ser. 2, V. 106. AMS, 1977, vi + 142 pp, $21.60. [ISBN: 0-8218-3056-2] A "considerably edited and amplified" translation (by Edwin Hewitt) of three related Russian papers intended to unify a series of heterogeneous problems in general topology. LAS

PROBABILITY, T(14-15: 1), S. *Elementary Probability.* Edward O. Thorp. Krieger, 1977, x + 152 pp, $6.50. [ISBN: 0-88275-389-4] Reprint, with corrections, of the 1966 Wiley book by (Beat-the-Dealer) Thorp. Contains three major sections, dealing with finite, discrete, and continuous probability. Intended to be either a brief text or a supplementary text in a two-year calculus sequence. Although the mathematical sophistication required is rather high, it is better suited to the latter, since its coverage of standard topics, particularly continuous probability, is minimal. RSK

PROBABILITY, T(13: 1), S. *Elementary Probability Theory.* Melvin Hausner. Plenum, 1971, viii + 310 pp, $8.95 (P). [ISBN: 0-306-20026-0] Precalculus introduction includes counting, expectation, law of large numbers, binomial and normal distributions. Well motivated with lots of examples and problems. LCL

PROBABILITY, T(16: 1), S*, L*. *Computational Probability and Simulation.* Sidney J. Yakowitz. Appl. Math. and Comp., No. 12. A-W, 1977, xxii + 240 pp, $12.50 (P); $22.50. [ISBN: 0-201-08892-4; 0-201-08893-2] Designed to be a companion text for a first course in probability theory. Its purpose is partly to illustrate facts and models of probability theory and partly to introduce the subject of probabilistic simulation. Contains many interesting applications, including gambler's ruin problems with extensions to inventory control, time series simulation, and Monte Carlo integration techniques. RSK

PROBABILITY, T(18), P. *Elements of Probability Theory.* Robert Fortet. Gordon, 1977, xix + 524 pp, $45. [ISBN: 0-677-02110-0] Translation of text published in 1960 by the Centre National de la Recherche Scientifique. Sophisticated treatment employing concepts of measure and Hilbert space throughout. LCL

PROBABILITY, P*. *Decomposition of Random Variables and Vectors.* Ju. V. Linnik, I.V. Ostrovskiĭ. Transl. Math. Mono., V. 48. AMS, 1977, ix + 380 pp, $38.80. [ISBN: 0-8218-1598-9] Concerned with the problem of characterizing the components of a random variable when it is decomposed into a sum of independent random variables. Presents the state of the art as of 1971. RSK

PROBABILITY, T(17), P*. *Probability Theory I, Fourth Edition.* M. Loève. Grad. Texts in Math., V. 45. Springer-Verlag, 1977, xvii + 425 pp, $19.80. [ISBN: 0-387-90210-4; 3-540-90210-4] Revision of the 1963 Van Nostrand *Third Edition,* split into two volumes. This volume covers, in addition to the elementary introduction, the first three parts of previous editions: Notions of Measure Theory, General Concepts and Tools of Probability Theory, and Independence. New material includes a section on convergence of probabilities on metric spaces and a chapter containing sections on domains of attraction and random walks. RSK

PROBABILITY, P. *Probability in B-Spaces.* J. Hoffmann-Jørgensen. Lect. Notes Ser., No. 48. Aarhus U, 1977, 185 pp, (P).

PROBABILITY, P. *Random Measures.* Olav Kallenberg. Akademie-Verlag, 1976, 104 pp, $10. [ISBN: 012-394950-5]

PROBABILITY, P. *Innovation Processes.* Yuriy A. Rozanov. Trans: A.V. Balakrishnan. V.H. Winston, 1977, vii + 136 pp, $14.50. [ISBN: 0-470-99127-5] "A new approach to the basic problems which arise in prediction, filtering, and estimation of random processes." FLW

PROBABILITY, T(16-17: 1), P. *Brownian Motion and Classical Potential Theory.* Murali Rao. Lect. Notes Ser., No. 47. Aarhus U, 1977, 300 pp, $5.50 (P).

STATISTICS, T*(13-14: 1, 2). *Elementary Business Statistics, The Modern Approach, Third Edition.* John E. Freund, Frank J. Williams. P-H, 1977, xv + 560 pp, $14.95. [ISBN: 0-13-253062-7] Revision of the 1972 *Second Edition* (TR, August-September 1972). Emphasis is on decision making, with a substantial amount on Bayesian methods. In addition to the usual general topics, it contains chapters on index numbers, elementary decision analysis, Bayes' theorem and the revision of probabilities, time-series analysis, planning business research, and an introduction to operations research. RSK

STATISTICS, S. *Errors of Observation and Their Treatment, Fourth Edition.* J. Topping. Chapman and Hall, 1975, 119 pp, $3.75 (P). [ISBN: 412-21040-1] Reprinting of the 1972 *Fourth Edition,* which was a modest revision of the 1962 *Third Edition* (TR, March 1974). Primarily concerned with effect of errors of observation on computed results. Also includes some dated statistical material, in order to discuss the Gaussian law of error, and some applications of least squares. RSK

STATISTICS, T, S*(14-16), L. *Regression Analysis by Example.* Samprit Chatterjee, Bertram Price. Wiley, 1977, xiv + 228 pp, $16.95. [ISBN: 0-471-01521-0] In the Wiley Series in Probability and Mathematical Statistics. A non-formal approach to exploratory regression analysis, relying heavily on graphical representations of data, particularly residuals, and illustrated with many examples. Mathematical prerequisites are low (most matrix algebra is in appendices) considering the coverage, which includes ridge regression and principal components methods. May be suitable as a text for non-statistics students, or as a supplement to a theoretical course. RSK

STATISTICS, T*(1, 2), S, *Introductory Statistics, Third Edition.* Thomas H. and Ronald J. Wonnacott. Wiley, 1977, xxii + 650 pp, $15.95. [ISBN: 0-471-95982-0] Earlier editions reviewed in October 1969 and August-September 1972. This edition features a number of new examples and problems (e.g., illustrations of sampling theory by Monte Carlo simulation) and a complete rewriting of the chapter on hypothesis testing. LCL

STATISTICS, T(13: 1), *Elementary Statistics.* Gene R. Sellers. Saunders, 1977, vii + 433 pp, $12.50 [ISBN: 0-7216-8049-6]; *Instructor's Manual,* 46 pp, (P); *Student's Guide,* 287 pp, $3.95 (P) [ISBN: 0-7216-8052-6]. Standard topics of descriptive and inferential statistics, including probability and nonparametric tests, but not including analysis of variance. The text (including examples and exercises) is built around numerous newspaper and magazine clippings which illustrate how we are exposed to statistical thinking in our everyday lives. LCL

STATISTICS, T(13: 1), S, *Statistics from Scratch, Pilot Edition.* Peter Nemenyi, *et al.* Holden-Day, 1977, 630 pp, $11.95 (P). [ISBN: 0-8162-6384-1] Informal, folksy presentation written to appeal to notions of common sense. The approach is spiral, beginning and often returning to nonparametric statistics as the simplest of tests to understand and apply. Coverage includes some probability, regression and correlation, ANOVA. LCL

STATISTICS, P**, L*, *A Bibliography of Multivariate Statistical Analysis.* T.W. Anderson, Somesh Das Gupta, George P.H. Styan. Krieger, 1977, x + 642 pp, $28.50. [ISBN: 0-88275-477-7] Reprint of the 1972 Wiley book (TR, October 1973). RSK

STATISTICS, T(13: 1), *Statistics, Tool of the Behavioral Sciences.* Marcia K. Johnson, Robert M. Liebert. P-H, 1977, xiv + 237 pp, $10.95. [ISBN: 0-13-844704-7] Well-written introductory text, more concerned with experimental design and the analysis of variance than other books at this level. Coverage of some standard topics is minimal (probability) or non-existent (confidence intervals, tests of a single mean), which may make it unsatisfactory for many courses. RSK

STATISTICS, T(1), P*, L*, *The Analysis of Cross-Classified Categorical Data.* Stephen E. Fienberg. MIT Pr, 1977, x + 151 pp, $10.95. [ISBN: 0-262-06063-9] Well-written simplified presentation, designed for nonstatisticians, of many of the key ideas in Bishop, Fienberg and Holland's comprehensive *Discrete Multivariate Analysis* (TR, December 1975). Uses log-linear models to analyze multidimensional contingency tables, and includes some newer results. Presumes a general familiarity with the analysis of two-dimensional contingency tables, regression analysis, and analysis-of-variance models. Contains many illustrative examples using real data (but no exercises) and a good set of references. RSK

STATISTICS, T*(1), S, L, *Statistics, A Biomedical Introduction.* Byron Wm. Brown, Jr., Myles Hollander. Wiley, 1977, xiii + 456 pp, $15.95. [ISBN: 0-471-11240-2] Statistics seen as a means for analyzing questions arising in the biomedical sciences--hypothesis testing and estimation, contingency, regression, ANOVA. No mathematics beyond algebra. Examples and problems taken from actual case studies in medicine and biology, discussed in an informal, open style that makes interesting reading even apart from the mathematics (e.g., "Ethics and Statistical Aspects of the Clinical Trial"). LCL

STATISTICS, T(17: 1), P*, *Foundations of Inference in Survey Sampling.* Claes-Magnus Cassel, Carl-Erik Särndal, Jan Håkan Wretman. Wiley, 1977, xi + 192 pp, $18.95. [ISBN: 0-471-02563-1] In the Wiley Series in Probability and Mathematical Statistics. Presents the "first reasonably complete account" of the theory of statistical inference about the mean of a finite population of known size. Uses two approaches: the fixed population approach, in which it is assumed the quantity being measured is fixed for each unit in the population and can be observed without error, and the superpopulation approach, in which this quantity is considered to be the outcome of a random variable. Good set of references. RSK

STATISTICS, T(18: 2), P, *Multivariate Statistical Inference.* Narayan C. Giri. Acad Pr, 1977, xvi + 319 pp, $25. [ISBN: 0-12-285650-3] In their Probability and Mathematical Statistics series. Primarily theoretical presentation, emphasizing results for multivariate normal distributions using the invariance (symmetry with respect to certain transformations) approach. Good sets of exercises and references. RSK

STATISTICS, S, P*, L, *I.J. Bienaymé: Statistical Theory Anticipated.* C.C. Heyde, E. Seneta. Stud. in History of Math. and Phy. Sci., No. 3. Springer-Verlag, 1977, xiv + 172 pp, $19.80. [ISBN: 0-387-90261-9; 3-540-90261-9] Contributions of the French academician Bienaymé summarized and placed in historical perspective. Helps to fill a gap in the history of probability and statistics, and gives recognition to a man whose discoveries have largely been ignored or credited to others (e.g., the Chebyshev inequality). RSK

STATISTICS, P*, L, *The Analysis of Survey Data.* Ed: Colm A. O'Muircheartaigh, Clive Payne. Wiley, 1977. *V. I: Exploring Data Structures,* xv + 273 pp, $19.95 [ISBN: 0-471-01706-X]; *V. II: Model Fitting,* xv + 255 pp, $23.95. [ISBN: 0-471-99426-X] Practical up-to-date computer-oriented guide to the processing and analysis of social survey data for practitioners and students. *V. I* is most appropriate for dealing with relatively uncharted substantive areas. Two general down-to-earth chapters on statistical analysis and computer processing are followed by six chapters presenting various multivariate techniques. *V. II* primarily deals with model fitting and testing procedures. Also included is a general chapter on statistical estimation and hypothesis testing and two specialized chapters on effects of complex survey designs and response errors. Does not cover survey design, sampling theory or statistical theory. RSK

STATISTICS, T(13: 1, 2), S, L, *Introductory Statistics for Business and Economics, Second Edition.* Thomas H. and Ronald J. Wonnacott. Wiley, 1977, xxii + 753 pp, $15.95. [ISBN: 0-471-95980-4] Basic material on probability, estimation, and hypotheses testing plus multiple regression, nonlinear regression, nonparametric statistics, Bayesian methods, time series, index numbers, and game theory. Presupposes only high school algebra. FLW

STATISTICS, P. *Selected Tables in Mathematical Statistics, V. V.* Ed; D.B. Owen, R.E. Odeh. AMS, 1977, viii + 263 pp, $16.80. [ISBN: 0-8218-1905-4] Fifth in a series of specialized tables sponsored by the Institute of Mathematical Statistics. Contains three sets of tables: means, variances and covariances of the normal order statistics ($n \leq 50$); means, variances and covariances of the normal order statistics in the presence of an outlier ($n \leq 20$); tables for obtaining optimal confidence intervals involving the chi-square distribution. RSK

STATISTICS, P*. *Nonparametric Methods in Communications.* Ed: P. Papantoni-Kazakos, Dimitri Kazakos. Elec. Eng. and Electronics, No. 2. Dekker, 1977, viii + 293 pp, $25. [ISBN: 0-8247-6660-1] Primarily concerned with the problem of detecting signals in noise, when the noise is not assumed to have a specified (usually Gaussian) distribution. Written by six professors of electrical engineering, each an expert in the field, this book presents a comprehensive up-to-date treatment of nonparametric detection theory. Good sets of references. RSK

STATISTICS, P?. *A Handbook of Numerical and Statistical Techniques, with Examples Mainly from the Life Sciences.* J.H. Pollard. Cambridge U Pr, 1977, xvi + 349 pp, $24.95. [ISBN: 0-521-21440-8] Designed for experimental scientists who wish to solve numerical or statistical problems on a programmable calculator, mini-computer or interactive terminal. A collection of techniques, mostly statistical, each illustrated by at least one example. Explanations are sparse, however, and those without sufficient mathematical or statistical background would have difficulty using the book. RSK

STATISTICS, T(13: 1, 2). *Introductory Statistics for the Behavioral Sciences, Third Edition.* Robert K. Young, Donald J. Veldman. HR&W, 1977, xii + 594 pp, $12.50. [ISBN: 0-03-089677-0] Modest revision of the authors' 1972 *Second Edition* (TR, February 1974). Chapter on testing and prediction has been deleted, while a chapter on recent trends has been added. The latter briefly touches on nonparametric statistics, Bayesian statistics, and the possible impact of calculators and computers. Also includes new exercises involving analysis of student generated data. Unique feature is still its programmed exercises. RSK

STATISTICS, T(13: 1, 2). *Statistics and Probability in Modern Life, Second Edition.* Joseph Newmark. HR&W, 1977, xiii + 516 pp, $13.95. [ISBN: 0-03-018881-4] Presupposes only high school mathematics. Each section begins with newspaper clippings using the ideas in that section. Self-study guides and mastery tests at the end of each chapter. The usual topics. FLW

COMPUTER SCIENCE, T*(15-16: 1). *Combinatorial Algorithms: Theory and Practice.* Edward M. Reingold, Jurg Nievergelt, Narsingh Deo. P-H, 1977, xii + 433 pp, $18.95. [ISBN: 0-13-152447-X] An excellent development of most of the usual techniques in computational combinatorics. Considers integers, sequences, combinations, permutations, trees and graphs, their computer representations, their usefulness in computational problems and algorithms for operations on them. Enumeration, generation, estimation, backtrack and sieves. Applications to searching and sorting. NP-completeness. Problems, references. RWN

COMPUTER SCIENCE, P. *The Eighth Annual ACM Symposium on Theory of Computing.* ACM, 1976, iv + 246 pp, (P). Proceedings of a conference held May 3-5, 1976 at Hershey, Pennsylvania. Index. RJA

COMPUTER SCIENCE, T(13: 1). *A First Course in Data Processing.* J. Daniel Couger, Fred R. McFadden. Wiley, 1977, xiv + 557 pp, $13.50 (P). [ISBN: 0-471-17738-5] How the computer works and how it is used, with an early introduction to programming. Text punctuated with review questions, exercises, answers, summaries, review tests. LCL

COMPUTER SCIENCE, T(13: 1), S, L. *Computers and Their Societal Impact.* Martin O. Holoien. Wiley, 1977, xiii + 264 pp, $10.95. [ISBN: 0-471-02197-0] Expository presentation includes an historical survey, a description of Basic computer programming, applications of computers in our lives (education, business, industry, government, health services), and projections on future prospects. Exercises and selected references follow each chapter. LCL

COMPUTER SCIENCE, P. *Software Reliability.* Infotech Inter, 1977. *Part 1: Analysis and Bibliography,* v + 288 pp; *V. 2: Invited Papers,* ii + 410 pp, (P). [ISBN: 8553-9380-7] Discussion of methods of software development, validation, and fault-tolerance and their effect on software reliability. Eighteen invited papers on related topics. Bibliography of reliable software. Contributors' references. Contributor and subject indices. RJA

COMPUTER SCIENCE, P. *Program Optimization.* Infotech Inter, 1976, viii + 448 pp. [ISBN: 8553-9300-9] Program optimization and its relationship to program development, translation, and execution. Seventeen invited papers on related issues. Annotated bibliography. Contributors' references. Cumulative subject, contributor, and subject indices. RJA

COMPUTER SCIENCE, S*(14-16), L. *Using Computers.* Brian Meek, Simon Fairthorne. Ellis Horwood, Ltd. (U.S. Distr: Wiley), 1977, 208 pp, $16.50. [ISBN: 0-470-98932-7] In Wiley's Mathematics and Its Applications Series. Written for users of electronic digital computers rather than professionals, it provides a good overall picture of the nature of computing, indicating both potentialities and limitations. Chapter headings are all "The Computer as a ..." and include: concept, machine, problem solver, number-cruncher, data handler, watchdog, entertainment, and social force. RSK

COMPUTER SCIENCE, T(13-18: 1), S. *Structured PL/Zero plus PL/One.* Michael Kennedy, Martin B. Solomon. P-H, 1977, xxxiv + 695 pp, $10.95 (P). [ISBN: 0-13-854901-X] Text is divided into 5 parts: (1) structured PL/Zero, an 8-statement subset of PL/One which is easy to learn and facilitates writing structured programs for many routine problems; (2) those additional features which, with PL/Zero, comprise the basics of PL/One; (3) fifteen modules that explain and elucidate various programming features of PL/One; (4) forty-six lab problems in 4 sets; (5) twenty appendices on various PL/One compilers and various computing-related topics. Questions and exercises and answers. Index. RJA

COMPUTER SCIENCE, T(15-17), S, P, L. *Decision Table Languages and Systems.* John R. Metzner, Bruce H. Barnes. Acad Pr, 1977, viii + 172 pp, $14.50. [ISBN: 0-12-492050-0] Treatment of the nature of decision tables and ways in which they can be adapted to a wide variety of purposes. Provides conceptual basis for further research; comprehensive bibliography. LCL

COMPUTER SCIENCE, T(15-16: 1), S, P, L. *Macro Processors.* A.J. Cole. Cambridge U Pr, 1976, viii + 230 pp, $8.95 (P). [ISBN: 0-521-29024-4] Discusses various macro processors, their use, and, especially, their structure. Includes macro assemblers, TRAC, GPM, MP/3, WISP, among others. Examples include string handling, list processing and systems programming. Syntax macros and their application to extensible compilers. RWN

COMPUTER SCIENCE, T(14-17: 1), S, P, L. *Microcomputer Handbook.* Charles J. Sippl. Petrocelli, 1977, xix + 454 pp, $19.95. [ISBN: 0-88405-324-5] Begins with a description of standard computer systems which is followed by treatment of minicomputer and microcomputer systems. The middle portion of the text discusses various available microcomputer systems, products, kits, and software. The last third presents microcomputer applications. Two appendices: one contains a microcomputer product analysis, and the other an analysis of software support systems. Term definition index. Subject index. RJA

COMPUTER SCIENCE, P. *Mathematical Foundations of Computer Science, V. 2.* Ed: Antoni Mazurkiewicz, Zdzisław Pawlak. PWN, 1977, 259 pp. Four papers: algorithmic logic; theory of computing systems; analysis of programs; information storage and retrieval systems. RJA

SYSTEMS THEORY, P. *Distributed Systems.* Infotech Inter, 1976, viii + 488 pp. [ISBN: 8553-9310-6] Description, design, implementation, software, and survey of distributed computer systems. Distributed data bases and 15 invited papers on related topics. Bibliography and contributors' references. Cumulative subject, author, and subject indices. RJA

APPLICATIONS, T*(15-16: 1), S*, L*. *Mathematical Models, Mechanical Vibrations, Population Dynamics, and Traffic Flow (An Introduction to Applied Mathematics).* Richard Haberman. P-H, 1977, xiii + 402 pp, $18.95. [ISBN: 0-13-561738-3] An introduction to applied mathematics, presupposing calculus and some basic facts about differential equations. Considers mechanical vibrations, population dynamics, and traffic flow. FLW

APPLICATIONS (CHEMISTRY), S(17-18), P. *Computers in Polymer Sciences.* Ed: James S. Mattson, Henry B. Mark, Jr., Hubert C. MacDonald, Jr. Dekker, 1977, xi + 374 pp, $34.75. [ISBN: 0-8247-6368-8] First six chapters are devoted to the simulation of polymerization reactions. Next two chapters deal with the use of the computer in controlling experimental systems. Last two chapters discuss two data reduction algorithms. Chapter references. Author index. Subject index. RJA

APPLICATIONS (DATA ANALYSIS), T(15-18: 1), S, P, L. *Computational Methods for Data Analysis.* John M. Chambers. Wiley, 1977, xi + 268 pp, $15.95. [ISBN: 0-471-02772-3] Programming, program evaluation, design and structure of programs used in data analysis. Data management, sorting, searching. Numerical methods: various approximations, linear and nonlinear models, simulation. Graphics for data analysis. Chapter problems. Chapter references. Appendix on availability of algorithms related to methods discussed. Index. RJA

APPLICATIONS (ECONOMICS), S(16-18), P, L. *Mathematical Theory of Economic Dynamics and Equilibria.* V.L. Makarov, A.M. Rubinov. Trans: Mohamed El-Hodiri. Springer-Verlag, 1977, xv + 252 pp, $29.80. [ISBN: 0-387-90191-4; 3-540-90191-4] Models which describe the motion of an economy (production, distribution, and consumption of goods) through time. Requires some mathematical sophistication, a good knowledge of finite dimensional vector spaces, and some functional analysis. LCL

APPLICATIONS (ELECTRICAL ENGINEERING), T(16: 2), S, P. *Applied Graph Theory, Graphs and Electrical Networks.* Wai-Kai Chen. Appl. Math. and Mech., V. 13. North-Holland, 1976, xvi + 542 pp, $24.50. [ISBN: 0-7204-23716] This second edition is an enlarged and improved work. Basic theory of graphs and digraphs and their application to electrical network theory. New material is primarily in the area of linear systems and networks treated in a fresh chapter on state equations of networks. SS

APPLICATIONS (ENGINEERING), S(18), P. *Creep and Relaxation of Nonlinear Viscoelastic Materials, with an Introduction to Linear Viscoelasticity.* William N. Findley, James S. Lai, Kasif Onaran. Appl. Math. and Mech., V. 18. North-Holland, 1976, xiii + 367 pp, $34.50. [ISBN: 0-444-10775-4] Historical review of literature on creep, background material on stress, mechanics of stress analysis, linear viscoelasticity, multiple integral representations of nonlinear creep and relaxation, incompressibility, conversion from creep to relaxation and experimental methods and apparatus for creep and relaxation. A substantial list of references. CEC

APPLICATIONS (PHYSICS), T*(16-18: 1, 2), P, L. *Mathematics Applied to Continuum Mechanics.* Lee A. Segel, G.H. Handelman. Macmillan, 1977, xviii + 590 pp, $18.95. [ISBN: 0-02-408700-9] An introduction to continuum models of fluid flow and solid deformation. Four main parts: tensor algebra and calculus; continuum mechanics of viscous fluids and elasticity; water waves; variational methods and extremum problems. Section on water waves leads the reader to the research frontiers. An excellent text for a second course in applied mathematics. Exercises. TRS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jennifer R. Galovich, St. Olaf; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D.C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Professor Phillip E. Bedient, Franklin and Marshall College, represented the Mathematical Association of America at the inauguration of Mark E. Ebersole as President of Elizabethtown College on November 5, 1977.

Professor A. Everett Pitcher, Lehigh University, has announced his intention to retire in June, 1978. He has been a member of the Lehigh faculty since 1938, and in 1963 he was named distinguished professor of mathematics for his "outstanding teaching and service."

Professor Kenneth O. May, University of Toronto, died on December 1, 1977, at the age of 62. He was a member of the Association for thirty years and served it well on many committees and programs.

NCTM TO SURVEY MATHEMATICS REQUIREMENTS FOR THE '80s

What will be the mathematics curriculum for the 1980s? A study to survey the priorities for mathematics in the '80s is now under way. Teachers, parents, and mathematics educators will be asked their preferences and priorities regarding school mathematics curricula. Priorities in School Mathematics (PRISM) is a project designed by the NCTM to highlight the differences between "what is" and "what ought to be" in the mathematics curricula for grades K-14. PRISM is being funded by a $192,000 grant from the National Science Foundation.

The project, under the directorship of Alan Osborne of Ohio State University, will be developing one of NCTM's top priorities—making specific curriculum recommendations suitable for mathematics in the 1980s.

In addition to the impact the PRISM project will have on activities of the NCTM, reports resulting from the project will have implications for other groups interested in making relevant changes to meet mathematics requirements for the 1980s. Using data from existing research, the project should be helpful in suggesting priorities for curriculum development and directions for future research.

INTERNATIONAL STUDY GROUP ON THE RELATIONS BETWEEN THE HISTORY AND THE PEDAGOGY OF MATHEMATICS

There were International Congresses on Mathematical Instruction in 1968, 1972, and 1976. At the 1972 Congress, held in Exeter, England, there was a "working group" (EWG II) concerned with the relations between the history and the teaching of mathematics. The activities of this group were continued at the 1976 Congress in Karlsruhe under the co-chairmanship of Professors P.S. Jones (University of Michigan, U.S.A.) and R.J.K. Stowassser (University of Bielefeld, F.R.G.). The Executive Committee of ICMI has now approved the affiliation of this Study Group under the title, "International Study Group on Relations Between History and Pedagogy of Mathematics, cooperating with the International Commission of Mathematical Instruction." The principle aims of this Study Group are as follows:

1. To promote international contacts and exchange information concerning:
   (a) Courses in History of Mathematics in Universities, Colleges and Schools.
   (b) The use and relevance of History of Mathematics in mathematics teaching.
   (c) Views on the relation between History of Mathematics and Mathematical Education at all levels.
2. To promote and stimulate interdisciplinary investigation by bringing together all those interested, particularly mathematicians, historians of mathematics, teachers, social scientists and other users of mathematics.
3. To further a deeper understanding of the way mathematics evolves, and the forces which contribute to this evolution.
4. To relate the teaching of mathematics and the history of mathematics teaching to the development of mathematics in ways which assist the improvement of instruction and the development of curricula.
5. To produce materials which can be used by teachers of mathematics to provide perspectives and to further the critical discussion of the teaching of mathematics.
6. To facilitate access to materials in the history of mathematics and related areas.
7. To promote awareness of the relevance of the history of mathematcis for mathematics teaching in mathematicians and teachers.
8. To promote awareness of the history of mathematics as a significant part of the development of cultures.

There will be a programme of lectures, seminars and discussions on these themes at ICM Helsinki (15-23 August 1978). Any person attending ICM 1978, or who is interested in the activities of this group, is asked to contact: Leo F. Rogers, Secretary, International Study Group on Relations Between History and Pedagogy of Mathematics, DIGBY STUART COLLEGE, ROEHAMPTON INSTITUTE OF HIGHER EDUCATION, LONDON SW 15 5 PH ENGLAND.

W. T. REID SCHOLARSHIP IN MATHEMATICS

Hardin-Simmons University has established the W. T. Reid Scholarship in Mathematics in memory of Professor W. T. Reid, a 1926 graduate of the University. Each year's recipient will be an entering student who expects to major in mathematics and who, in the judgment of the faculty, has the potential to become a mathematician. The sustaining endowment for this Scholarship consists of gifts by Professor Reid's relatives, friends, students, and former classmates. Persons wishing to add to the endowment should correspond with Professor Charles D. Robinson, Department of Mathematics, Hardin-Simmons University, Abilene, Texas 79601.

MATHEMATICIAN RECEIVES TAYLOR AWARD FOR SCIENTIFIC ACHIEVEMENT

Dr. Elizabeth H. Cuthill, the Numerical Analysis Coordinator for the Computation, Mathematics, and Logistics Department of the David W. Taylor Naval Ship R&D Center (DTNSRDC), Bethesda, Maryland, recently received the David W. Taylor Award for Scientific Achievement for the calendar year 1976.

Dr. Cuthill was recognized for her valuable contributions in the development and exploitation of mathematical and computational techniques for significant Navy applications. She led the successful development of the widely used General Bending Response Codes which have received wide acceptance and are in general use throughout the nation. She was also the leader within the Navy in promoting the use of general purpose finite element codes for strutural analysis.

Originally established by the Navy in 1961, this Award has been presented annually since that time to the individual scientist whose contributions were considered truly outstanding in the field of research and development.

1978 KODAK PROGRAM CATALOG INCLUDES NINE NEW TEACHING PROGRAMS

The 1978 "Your Programs from Kodak" catalog, listing films and slide/tape shows and including nine new teaching programs, available on a free-loan basis, is now being distributed on request to schools and other educational institutions. It describes movies and slide shows, which may be ordered for group presentations.

"Your Programs from Kodak 1978" may be obtained without charge by writing Eastman Kodak Company, Department 841, Rochester, N. Y. 14650.

SHORT COURSE ON RELIABILITY TESTING

The University of Arizona College of Engineering and Hughes Aircraft Company, Tucson, Arizona Operations will conduct a short course April 3-7, 1978, at the University of Arizona, Tucson. For further information contact Dr. Dimitri Kececioglu, Institute Director, Aerospace and Mechanical Engineering Department, The University of Arizona, Building #16, Tucson, Arizona 85721.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held at Indiana Central University, Indianapolis, on Saturday, November 5, 1977, with approximately 60 persons in attendance. The Chairman of the Section, G. J. Sherman, Rose Hulman Institute of Technology, presided.

The program consisted of the following:

Measures of commutativity for finite groups, by G. J. Sherman, Rose Hulman Institute of Technology.

Computational complexity in the solution of problems in graph-theory, by G. J. Minty, Indiana University.

Gaussian probability distributions and population dynamics, by W. H. Fleming, Brown University.

Panel discussion on Perspectives on the Job Market for Mathematicians. Panelists: Moderator, M. J. Mansfield, IUPU-Fort Wayne; W. H. Fleming, Brown University; V. F. Rickey, University of Bowling Green; S. C. Geller, Purdue University; H. B. Hanes, Earlham College.

At the business meeting it was moved, seconded and passed that

(a) The Section would obtain a bulk mailing permit,

(b) The Section would initiate a newsletter; D. R. McCarthy IUPU-Fort Wayne was appointed editor.

D. E. WILSON, Secretary

NOVEMBER MEETING OF THE SOUTHERN CALIFORNIA SECTION

For the third year the Southern California Section of the MAA held a joint meeting with another mathematical organization, this time with the American Mathematical Society at California State Polytechnic University at San Luis Obispo, California on November 11 and 12, 1977. The registered attendance was in excess of 260. Local arrangements were made by Professor Euel Kennedy of Cal Poly. The Program Chairman was Donald Babbitt of UCLA.

The Program for the MAA meeting was as follows:

Application of calculus and differential equations to the social sciences, H. L. Resnikoff, UC Irvine (Hour invited address).

Application of calculus to the biological sciences, L. Pamela Cook-loannidis, UCLA (Hour invited address).

Panel Discussion: *What should the content of the basis calculus course(s) be?* Moderator: David Outcult, UC Santa Barbara.

The answer to the question I am always asked. Luncheon address by Constance Reid. The excellent talk by Mrs. Reid answered the question "How did you come to write the book about Hilbert?"

EDMUND I. DEATON, Secretary-Treasurer

NOVEMBER MEETING OF THE MARYLAND DISTRICT OF COLUMBIA-VIRGINIA SECTION

The 1977 Fall Meeting (November Meeting) of the Maryland-District of Columbia-Virginia, Section of the MAA was held at The American University on Saturday November 19, 1977. There were 86 persons in attendance including 76 members.

The principal speaker was Dr. H. O. Pollak of the Bell Laboratories and Past President of the MAA. The title of Dr. Pollak's Invited Address was *On the Relationship Between the Applications of Mathematics and the Teaching of Mathematics*.

The participants were graciously welcomed to the campus of the American University by the Mathematics Department and the Administration of the University.

The chairman of the Section, Professor Orville M. Thomas, presided over a brief business session.

The contributed papers included:

Entire functions of bounded index - A survey of some recent results, Benjamin Lepson, U.S. Naval Research Laboratory.

A transfer device for matrix properties, William P. Wardlaw, U.S. Naval Academy.

Topological spaces which can be characterized as the images of a metric space, David A. Schedler, Virginia Commonwealth University.

A new method for numerically integrating ordinary differential equations, Thomas S. Schreiber, Fairfâx, Virginia.

Some innovations in a PSI Precalculus course, J. F. Kent, University of Richmond.

Chi-square with small expected numbers, Clifford J. Maloney, Food and Drug Administration.

A simplified sieve method to determine the set of primes less than the product of the first n primes, Inda Lepson, University of Maryland (Retired).

Brouwer and Cowell orbit generation, M. Fan, A. Kopoor, and J. Kennedy, Computer Sciences Corporation.

REUBEN C. DRAKE, Secretary

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The twenty-third annual meeting of the Northeastern Section of the MAA was held at Merrimack College, North Andover, Massachusetts on November 26, 1977; there were 74 people in attendance. The Section Chairman, Ernest C. Schlesinger, presided at the morning session at which the following talks were given:

Ancient ideas for modern times, Donna L. Beers, Wellesley College

A brief survey of exploratory data analysis, by David C. Hoaglin, Harvard University and Abt Associates.

Super perfect numbers, Grattan P. Murphy, University of Maine, Orono

Prearrangements of series, H. William Oliver, Williams College.

At the afternoon business meeting two By-law Amendments were presented and passed. The first amendment makes the immediate past chairman of the Section a member of the Executive Committee. The second amendment lengthens to two years the terms of office for the Section officers.

The following officers were elected for two-year terms: Chairman, Donald B. Small, Colby College; Vice-Chairman, Roger Cook, University of Vermont; Secretary-Treasurer, George W. Best, Phillips Academy.

Chairman Schlesinger announced that the fall '78 meeting of the Section would be held on the Saturday prior to Thanksgiving, November 18th.

GEORGE W. BEST, Secretary-Treasurer

CALENDAR OF FUTURE MEETINGS

Fifty-eighth Summer Meeting, Brown University, Providence, August 8–10, 1978.

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, University of Pittsburgh, Pennsylvania, April 14–15, 1978.

FLORIDA, St. Petersburg Junior College, Clearwater, March 3–4, 1978.

ILLINOIS, Western Illinois University, Macomb, May 5–6, 1978.

INDIANA, Earlham College, Richmond, April 22, 1978.

INTERMOUNTAIN

IOWA, University of Northern Iowa, Iowa Falls, April 22, 1978.

KANSAS, Wichita State University, Wichita, late March-early April, 1978.

KENTUCKY, Northern Kentucky University, Highland Heights, April 7–8, 1978.

LOUISIANA-MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Queensborough Community College, May 7, 1978.

MICHIGAN, Michigan State University, East Lansing, May 5–6, 1978.

MISSOURI, Central Missouri State University, Warrensburg, April 7–8, 1978.

NEBRASKA, University of Nebraska at Omaha, April 14–15, 1978.

NEW JERSEY, Steinhart High School, Trenton, April 28, 1978.

NORTH CENTRAL, College of St. Thomas, St. Paul, Minnesota, April 21–22, 1978.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, The University of Akron, Akron, April 28–29, 1978.

OKLAHOMA-ARKANSAS, Henderson State University, Arkadelphia, Arkansas, March 31-April 1, 1978.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 16–17, 1978.

PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.

ROCKY MOUNTAIN, South Dakota School of Mines and Technology, Rapid City, April 28–29, 1978.

SEAWAY, Brock University, St. Catharines, Ontario, Canada, May 5–6, 1978.

SOUTHEASTERN, Clemson University, Clemson, South Carolina, March 31–April 1, 1978.

SOUTHERN CALIFORNIA, California State University, Fullerton, March 11, 1978.

SOUTHWESTERN, New Mexico Institute of Mining and Technology, Socorro, Spring 1978.

TEXAS, Stephen F. Austin State University, Nacogdoches, March 31-April 1, 1978.

WISCONSIN, University of Wisconsin, Whitewater, late April 1978.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Chicago, January 3–8, 1979.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Stouffer's Greenway Plaza Hotel, Houston, Texas, Fall 1978.

AMERICAN MATHEMATICAL SOCIETY, Brown University, Providence, Rhode Island, August 9–12, 1978.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of British Columbia, Vancouver, June 19–22, 1978.

ASSOCIATION FOR COMPUTING MACHINERY

ASSOCIATION FOR SYMBOLIC LOGIC, Madison, Wisconsin, June 18–24, 1978.

ASSOCIATION FOR WOMEN IN MATHEMATICS, Brown University, Providence, Rhode Island, August 8–12, 1978.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, University of Western Ontario, London, Ontario, Canada, June 1–2, 1978.

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, San Diego, California, April 12–15, 1978.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, New York City, May 1–3, 1978 (Joint Meeting with the Institute of Management Sciences).

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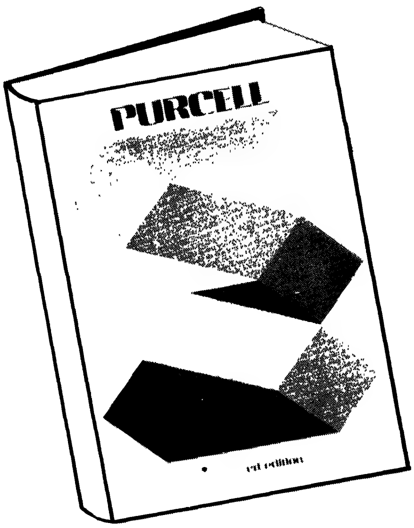

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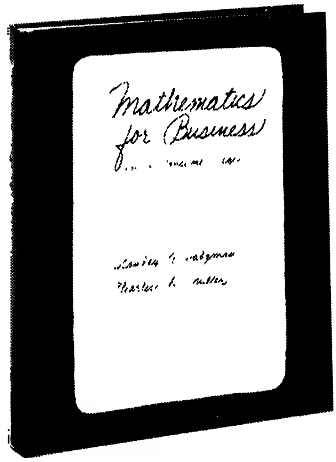

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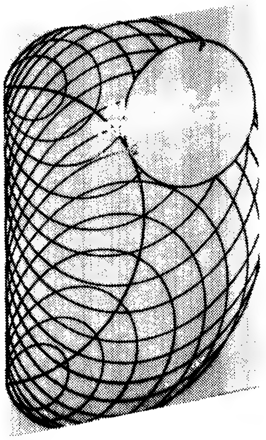

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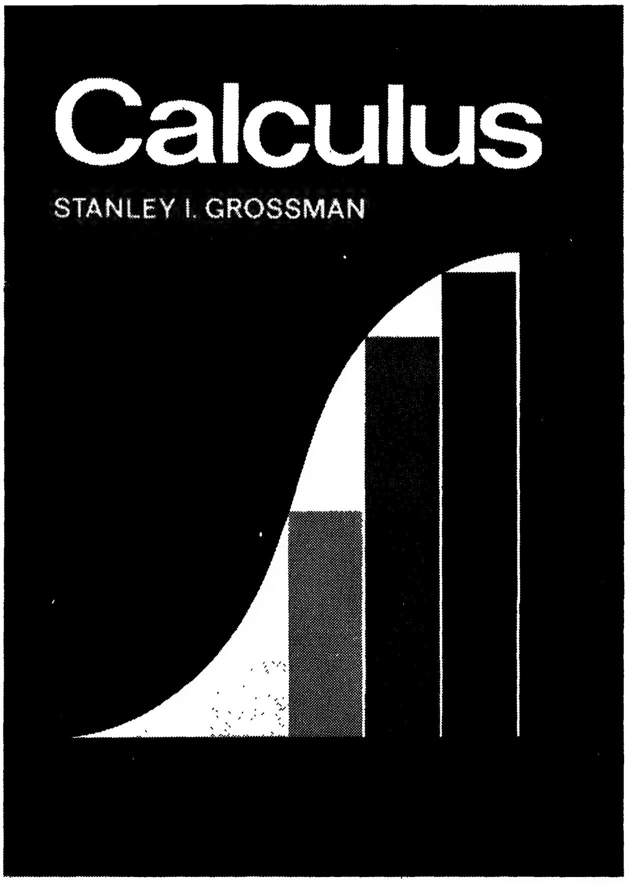

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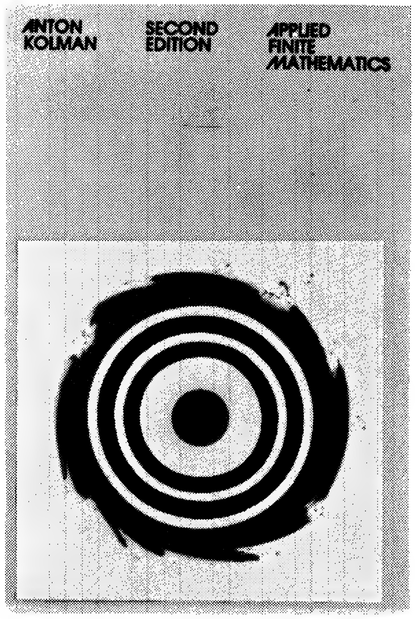

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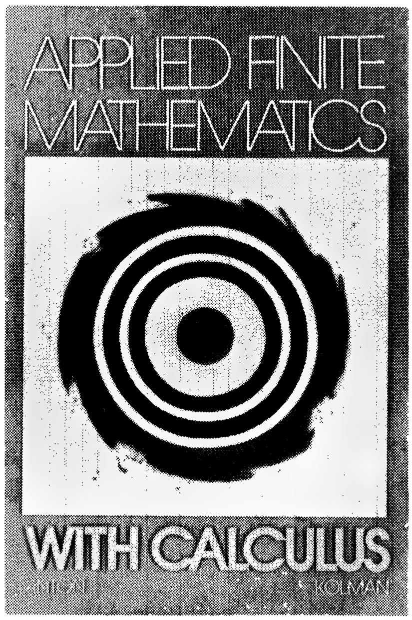

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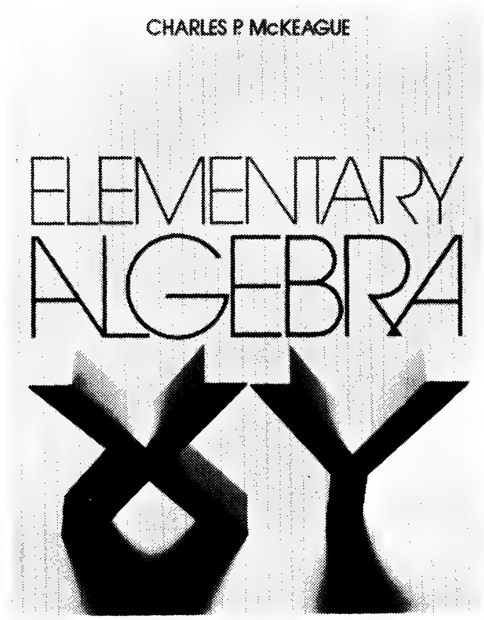

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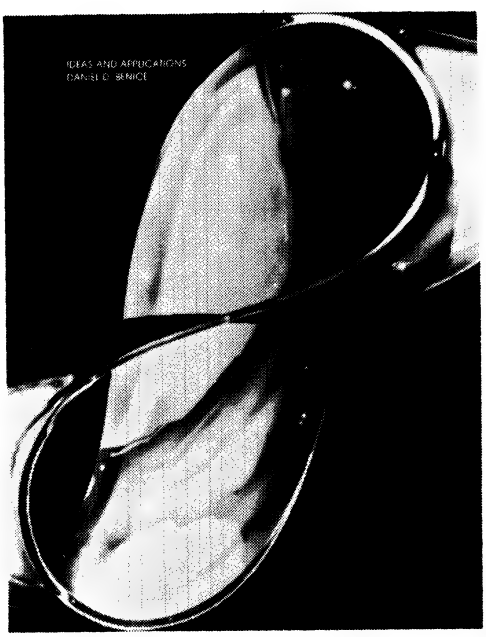

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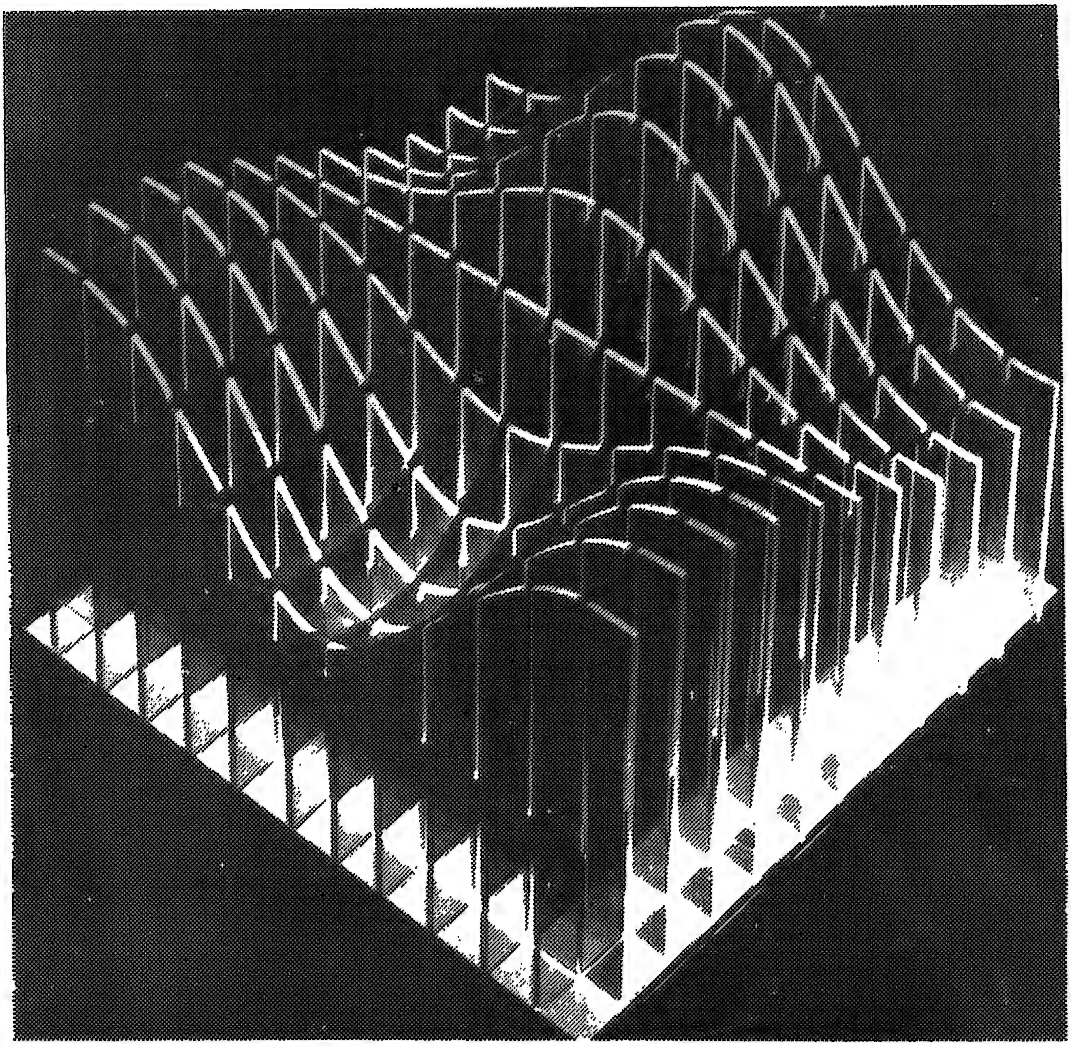

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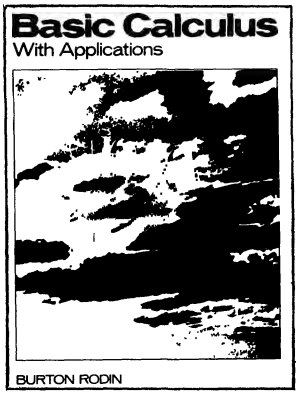

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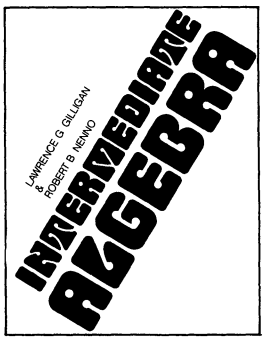

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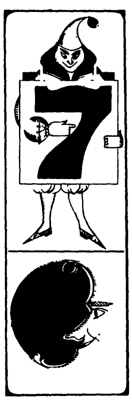

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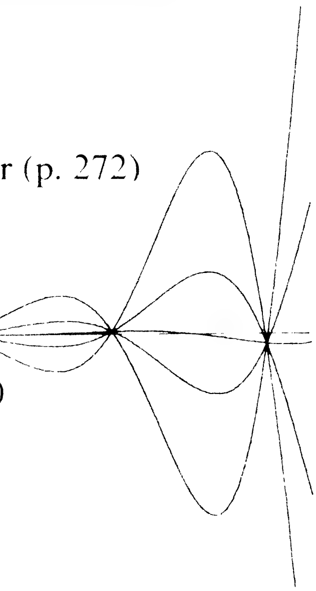

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THE COLLEGE LEVEL EXAMINATION PROGRAM IN MATHEMATICS

To the Membership of the Association: Your President and Board of Governors believe current practices in the granting of course credit or exemption on the basis solely of performance on an examination has serious implications for undergraduate mathematics. Examinations constructed and monitored by the mathematics department of a college or university for its own purposes in granting course credit or exemption have always been standard procedure. What has become increasingly widespread, especially in the last decade, is the quite different practice of granting course credit or exemption on the basis of an examination constructed by an external body, with performance standards established by reference to norms obtained with a national population sample. There appears to be a growing and disturbing tendency for the granting of credit or exemption to be mandated at a particular college or university by the administration, or sometimes state authority, without any meaningfully genuine involvement of the mathematics department in these policy decisions. There is evidence that the CLEP General Examination in Mathematics of the College Entrance Examination Board, and possibly other examinations as well, have been used in questionable as well as defensible ways for the granting of credit or exemption.

The College Entrance Examination Board, as a national association of schools, colleges, and education associations, has for many years exercised a responsibility to its membership to provide in many fields reliable and valid tests and examinations for admission, placement, and credit. The College Level Examination Program (CLEP) was created some ten years ago to afford an opportunity to adults to resume or begin their formal higher education at an age later than the usual, and to receive recognition, in the form of credit or exemption, for knowledge acquired by whatever means in academic areas covered by the CLEP examinations.

The College Board shares with the Association the concern that all CLEP examinations be appropriately and responsibly used, and the conviction that where the CLEP General Examination in Mathematics or any other is used for granting credit or exemption the judgment of the mathematics faculty is essential in this decision. A committee of the two organizations, following full discussions among representatives of both, has drafted a statement commenting on appropriate and inappropriate uses of the CLEP examinations in mathematics and outlining a policy respecting their use for granting credit or exemption. Your President and Board of Governors commend this statement to all members of the Association as a guide to the wise use of the CLEP examinations, or other similar examinations, for the granting of course credit or exemption. The practices urged appear to us likely to conserve the real gains for undergraduate mathematics education from the responsible granting of credit and exemption by examination, and to discourage possible abuses.

CEEB/MAA STATEMENT ON THE CLEP EXAMINATION

Concerns expressed by various members of the Mathematical Association of America, regarding the use of CLEP tests as a basis for awarding credit toward meeting college mathematics requirements, were brought to the attention of the CEEB/MAA Committee on Mutual Concerns.* The Committee examined these concerns, most of them centering about possible misuse of the CLEP General Examinations in various disciplines, and concluded that it would be beneficial at this time to issue a statement on the philosophy and use of the CLEP General Examination in Mathematics.

The College Level Examination Program (CLEP), established in 1967, is a program of the College Entrance Examination Board. It consists of a package of tests of introductory college level performance in various disciplines. The purpose of the program is to provide nationally normed tests

*The CEEB/MAA Committee on Mutual Concerns consists of Richard Anderson (MAA), Betty J. Hinman (MAA, Co-Chairman), Donald L. Kreider (CEEB), Henry O. Pollak (MAA), Alfred L. Putnam (CEEB, Co-Chairman), Howard E. Taylor (CEEB), and Alfred B. Willcox (MAA, ex officio).

which may be used by individuals to gain college credit by examination for introductory college courses in which they have acquired proficiency through means other than taking the courses in college. The program was originally aimed primarily toward individuals who are beginning or returning to college some years after completing their secondary school education. As indicated below, its use is now broader. Since CLEP was established, the various examinations have been constructed and periodically revised by committees of teachers and professionals in the specific disciplines.

In recent years the granting of credit by examination has become a major factor in education. Indeed, in several states there is a movement toward large scale use of proficiency examinations to measure achievement in schools and colleges and to determine credit and/or exemption from college courses for recent high school graduates. This is seen by some as an important part of providing a varied access to the educational process. It is seen by others as leading to a lowering of academic standards or a diminution of the traditional role of the college and of the teacher-student relationship.

Though not taking a stand as such on the general issue of credit by examination, the CEEB/MAA Committee on Mutual Concerns feels obligated to give attention to both proper and improper uses of such examinations. In preparing suggestions as to their proper use, the CEEB/MAA Committee found that the College Board had anticipated many of the difficulties and had already published a useful document: *Guidelines on the Uses of College Board Test Scores and Related Data.* This publication is available from the College Board; however, we find it useful to excerpt and paraphrase from these guidelines, especially as they related to mathematics and to the specific problems brought to our attention.

In mathematics the CLEP program includes a General Examination in Mathematics as well as specific subject matter examinations in algebra, trigonometry, and calculus. The concerns brought to the CEEB/MAA Committee are almost entirely confined to the General Examination in Mathematics, the main problem being the apparent use of this examination for purposes that were not intended and for which the examination was not designed. The CLEP General Mathematics Examination is described in the CEEB *Guide to Examinations in Mathematics* as follows:

> "Scores on the General Mathematics Examination may be used by colleges to award credit for or to exempt students from a general education requirement in mathematics. The examination does not test the more specialized college mathematics taken by students majoring in mathematics or the physical sciences. The 40 questions in Part A deal with basic skills and concepts in arithmetic, elementary algebra, data interpretation, and informal intuitive geometry. The remaining 40 questions in Part B test the body of mathematical knowlege that is usually taught in a college-level mathematics course designed for non-mathematics majors. Content areas covered include sets, elementary logic, the real number system, functions and their graphs, and elementary probability and statistics. No questions from trigonometry and calculus are included on the examination. Total testing time is 60 minutes, and 3 scores are reported: a score for each of Parts A and B on a 20–80 scale and a total score on a 200–800 scale. (New editions of the examination which are currently being developed will contain 40 questions in Part A, 50 questions in Part B, and will require a total of 90 minutes of testing time.) Normative data are available for both a sample of college sophomores and a more restrictive sample consisting only of those students who have had a mathematics course designed for non-mathematics majors."

In light of this description, and after examining actual test copies and the relationship between raw scores and normed scores, the CEEB/MAA Committee has reached the following conclusion:

Where the purpose of introductory college mathematics courses or general area requirements is to insure on the part of the student a general level of proficiency in mathematics and understanding of its role in society, it seems appropriate to use the CLEP General Mathematics Examination — or comparable test — as a means of assessing or determining levels of proficiency and understanding which would justify credit or exemption for such courses or requirements.

In this context, we find appropriate uses of the CLEP General Mathematics Examination to be:

A. Granting credit for or exemption from courses in general mathematics for non-science majors where courses with objectives comparable to those of the General Mathematics Examination are

given by the institution.

B. Granting credit or proficiency exemption for the first of the sequence of courses recommended by CUPM for elementary education majors. (Credit for one mathematics course would be appropriate, recognizing, however, that the CLEP General Mathematics Examination tests only elementary mathematics and not the methods and other specialized matters that are normally included in the sequence. It is important that prospective teachers experience in their further mathematics courses the back and forth verbal communications about mathematics so vital to successful teaching.)

C. Meeting suitable general mathematics area requirements as required by some institutions. Here some effort should be made to determine that individuals granted credit through the General Mathematics Examination have acquired a level of mathematical maturity comparable to students who successfully complete the institution's courses designed to meet the general mathematics requirement.

Among possible misuses of the CLEP General Mathematics Examination we find the following:

A. Granting credit for courses in algebra, trigonometry, or precalculus mathematics. The General Examination is not intended to test proficiency in these subjects and is not designed to test the skills required for successfully entering a calculus course. (The CLEP subject matter examinations in algebra and trigonometry *are* intended for this purpose.)

B. Granting credits in mathematics in cases where no course with objectives comparable to those of the General Mathematics Examination is given by the institution.

C. Requiring performance levels in the examination that are not consistent with those required in comparable existing courses. The misuse here could be in requiring performance levels that are either too low or too high compared to the existing courses.

Finally, the Committee finds one of the most serious misuses of the CLEP examinations more generally to be their use and interpretation by an institution *without including input from the mathematics faculty* in setting standards and awarding credit. It is crucial to the acceptability of the program, as the College Board's own guidelines point out, that local determination be made about the comparability of examination objectives and those of courses offered, and that local standards be taken into account in setting the cutoff line for adequate performance on the examination.

We conclude by referring to the College Board's Guidelines, paraphrasing several of them which seem to us pertinent to the concerns outlined above:

When colleges use tests, such as the College Board CLEP tests in mathematics, for exemption and credit purposes, they should:

1. Require the mathematics department or the responsible mathematics faculty members to review the examination to be used for appropriateness.

2. Develop their practices, policies, and procedures, in full consultation with the mathematics faculty, in the light of local performance levels and ranges of enrolled students. Levels of performance required on the examinations for granting of credit should not be significantly higher or lower than for students who do successful work in the relevant courses. (Here we consider successful work to be more than minimal passing work.) A most useful way to accomplish this would be to administer the test to a representative sample of students who have successfully completed courses with comparable objectives and content. The Committee, after examining actual editions of the General Mathematics Examination, normative data, raw scores, and courses at their own individual institutions, felt that a score of 500 on the 200–800 scale on this examination demonstrates proficiency above the minimal passing level and for which awarding credit would be appropriate. This may be a useful guide for other departments as they determine their own cut-off score.

3. Develop and publicize a clearly stated policy, readily available to prospective and enrolled students, on granting credit by examination. Then, consistent with such policies, the institution should accept the transfer of credits by examination when score levels meet the normal standards of the

receiving institution. Departments accordingly should hesitate to give credit that would be transferable to another institution unless they are convinced that the performance is comparable to that of ordinary transferable credit.

4. Ensure that the students exempted or given credit by examination are placed in the most appropriate next course, if a sequential course is available and chosen by the student.

It is apparent that the above guidelines cannot be met without the full involvement of the mathematics faculty. Hence we conclude by stating once again that an institution cannot successfully introduce a credit by examination program in mathematics solely at the decision-making level or within the student testing office. The judgment of the mathematics faculty is essential in maintaining standards, determining appropriateness, and ensuring fairness to both students who take the examination and those who follow the standard route of taking the comparable courses.

RECOMMENDATIONS FOR THE PREPARATION OF HIGH SCHOOL STUDENTS FOR COLLEGE MATHEMATICS COURSES

In March, 1976, the MAA and NCTM established a joint committee to prepare a statement on the mathematics needed by students planning to take collegiate mathematics and to make a list of recommendations to strengthen the preparation of students for collegiate level courses. The Committee consisted of the following: Henry L. Alder, Gerald L. Alexanderson (Chairman), Betty J. Hinman, David R. Johnson, Katherine P. Layton, Ron McCully, and Robert C. Meacham. Mrs. Layton has served also on the CEEB Advisory Panel on Score Decline. The Committee was assisted by George B. Pedrick and Kathy Magann of the MAA Special Projects Office. The statement and recommendations prepared by the Committee were unanimously approved by the MAA Board of Governors in September 1977 and the NCTM Board of Directors on September 16, 1977 in the following form:

> The following statement, adopted by the Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics, is a brief outline of the basic ingredients of adequate preparation for collegiate level mathematics.* The statement does not break new ground; it reflects standards that have been generally accepted for over a decade. It is intended to support the continuing efforts of conscientious teachers everywhere to provide their students with sound and stimulating mathematical training. It is specifically designed to provide a benchmark for our efforts and those of others to assess and react to recent reports of a general decline in the performance of students in mathematics.
>
> A joint committee of the Mathematical Association of America and the National Council of Teachers of Mathematics consulted with secondary school and college teachers in various parts of the country to study recent trends in the preparation of students. The comments from these consultations on which there was strongest consensus are the basis for this statement and its ten recommendations.
>
> The Mathematical Association of America and the National Council of Teachers of Mathematics wish to emphasize that the statement and recommendations, as they refer to secondary school programs, are addressed only to those for students planning to go to college and that they are not intended to be more comprehensive. During the past twenty years many

*Collegiate mathematics refers to courses in calculus (or calculus and analytic geometry), probability and statistics, finite mathematics, and higher level mathematics courses.

important changes have taken place in both the content and teaching of mathematics at the secondary school level. Many excellent new programs were adopted and taught effectively by teachers in elementary and secondary schools. Nevertheless, any consideration of the relative merits of new vs. traditional school curricula has been deliberately avoided. A study of this issue would have exceeded both the charge to the committee and its limited resources. This statement and these recommendations incorporate many of the best features of both of these curricula and are addressed to all mathematics programs regardless of pedagogical heritage.

Necessary course work. Mathematics is a highly structured subject in which various concepts and techniques are highly dependent upon each other. The concepts of arithmetic and algebra, however, are basic to all of mathematics. Further work in mathematics and in all areas in which mathematics is used as a tool requires correct performance, with understanding, of basic arithmetic operations, manipulation of algebraic symbols and understanding of what the manipulations mean.

Any student who is unable to perform arithmetic calculations and algebraic operations with accuracy and reasonable speed, to understand which operations to use in a given problem, and to determine whether the results have meaning, is severely handicapped in the study and applications of mathematics. The prevalence of inexpensive pocket calculators makes the performance of complicated calculations less tedious, but the use of calculators does not lessen the need for students to understand which concepts and operations are needed to solve a problem, to make sensible estimates, and to analyze their results.

For further work in mathematics, and in many other areas, from business to psychology, from biology to engineering, the ability to use algebra with skill and understanding is also essential. Having a passing grade in algebra is not enough. Both understanding and competence in the skills of algebra are necessary. Neither conceptual understanding alone nor technical skill alone will suffice in today's world, let alone in tomorrow's. Algebra is a useful subject which will help to solve problems in the real world. Opportunities to apply algebraic skills should be provided whenever possible, especially to problems that show the utility of mathematics.

Algebra courses in secondary school should include, in addition to the basic topics:

(A) polynomial functions,
(B) properties of logarithms,
(C) exponential and logarithmic functions and equations,
(D) arithmetic and geometric sequences and series,
(E) the binomial theorem,
(F) infinite geometric series,
(G) linear and quadratic inequalities.

For most students adequate coverage of the topics in algebra requires at least two years of study.

Students who will take calculus—and this now includes very many students who will take college work in business, premedicine, economics, biology, statistics, engineering, and physical science—may or may not need trigonometry, depending on the type of calculus course appropriate for their particular programs, but they will need a good deal of what is often called "pre-calculus," including especially a sound understanding of the concept of a function, which is also fundamental for work beyond the most elementary level in probability and computing.

Those students needing trigonometry should study:

(A) trigonometric functions and their graphs,
(B) degree and radian measure,
(C) trigonometric identities and equations,
(D) inverse trigonometric functions and their graphs.

For such students, the equivalent of one semester should be devoted to the study of the topics in trigonometry.

All students who go on to take collegiate mathematics will find their college work easier if they

have been introduced to some axiomatic system and to deductive reasoning. Traditionally this has been accomplished in a geometry course. Geometry courses in secondary school should include, in addition to basic topics:

(A) fundamental properties of geometric figures in three dimensions,
(B) applications of formulas for areas and volumes,
(C) experience in visualizing three-dimensional figures.

Other courses* beyond algebra, trigonometry, and geometry should be available to students who have the adequate background and the time to take them. A course in coordinate (or analytic) geometry is ideal since it combines algebra with geometry and provides a useful preparation for calculus. In addition to coordinate geometry, courses in the following topics are valuable: probability, statistics, elementary finite mathematics (or linear algebra), an introduction to computers and computing, and applications of mathematics.

If coordinate geometry is offered, it should include in addition to the basic topics:

(A) conic sections,
(B) rational functions and their graphs,
(C) polar coordinates,
(D) parametric equations and their graphs.

Inductive as well as deductive reasoning, techniques of estimation and approximation, and an awareness of problem solving techniques, with special emphasis on the transition from the verbal form to the language of mathematics, should be emphasized in all courses.

Calculus, where offered in secondary schools, should be at least a *full year* course and be taken only by those students who are strongly prepared in algebra, geometry, trigonometry, and coordinate geometry.

We recognize that many secondary schools have a curriculum similar to that outlined above. We emphasize again that, in order to be properly prepared for collegiate level courses in mathematics, students need to develop skills

(i) in applying standard techniques, and
(ii) in understanding of important concepts.

Recommendations. The Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics make the following recommendations:

1. Proficiency in mathematics cannot be acquired without individual practice. We, therefore, endorse the common practice of making regular assignments to be completed outside of class. We recommend that parents encourage their children to set aside sufficient time each day to complete these assignments and that parents actively support the request of the teacher that homework be turned in. Students should be encouraged to develop good study habits in mathematics courses at all levels and should develop the ability to read mathematics.
2. Homework and drill are very important pedagogical tools used to help the student gain understanding as well as proficiency in the skills of arithmetic and algebra; but students should not be burdened with excessive or meaningless drill. We, therefore, recommend that teachers and authors of textbooks step up their search for interesting problems that provide the opportunity to apply these skills. We realize that this is a difficult task, but we believe that providing problems that reinforce manipulative skills as a by-product should have high priority, especially those that show that mathematics helps solve problems in the real world.
3. We are aware that teachers must struggle to maintain standards of performance in courses at all levels from kindergarten through college and that serious grade inflation has been observed.

* The word "course" refers here and elsewhere in this statement to a semester course unless otherwise noted.

An apparent growing trend to reward effort or attendance rather than achievement has been making it increasingly difficult for mathematics teachers to maintain standards. We recommend that mathematics departments review evaluation procedures to insure that grades reflect student achievement. Further, we urge administrators to support teachers in this endeavor.

4. In light of 3 above, we also recognize that advancement of students without appropriate achievement has a detrimental effect on the individual student and on the entire class. We, therefore, recommend that school districts make special provisions to assist students when deficiencies are *first* noted.
5. We recommend that cumulative evaluations be given throughout each course, as well as at its completion, to *all* students. We believe that the absence of cumulative evaluation promotes short-term learning. We strongly oppose the practice of exempting students from evaluations.
6. We recommend that computers and hand calculators be used in imaginative ways to reinforce learning and to motivate the student as proficiency in mathematics is gained. Calculators should be used to supplement rather than to supplant the study of necessary computational skills.
7. We recommend that colleges and universities administer placement examinations in mathematics prior to final registration to aid students in selecting appropriate college courses.
8. We encourage the continuation or initiation of joint meetings of college and secondary school mathematics instructors and counselors in order to improve communication concerning mathematics prerequisites for careers, preparation of students for collegiate mathematics courses, joint curriculum coordination, remedial programs in schools and colleges, an exchange of successful instructional strategies, planning of in-service programs, and other related topics.
9. Schools should frequently review their mathematics curricula to see that they meet the needs of their students in preparing them for college mathematics. School districts that have not conducted a curriculum analysis recently should do so now, primarily to identify topics in the curriculum which could be either omitted or de-emphasized, if necessary, in order to provide sufficient time for the topics included in the above statement. We suggest that, for example, the following could be de-emphasized or omitted if now in the curriculum:
   (A) logarithmic calculations that can better be handled by calculators or computers,
   (B) extensive solving of triangles in trigonometry,
   (C) proofs of superfluous or trivial theorems in geometry.
10. We recommend that algebraic concepts and skills be incorporated wherever possible into geometry and other courses beyond algebra to help students retain these concepts and skills.

CONTROVERSIES IN THE FOUNDATIONS OF STATISTICS

BRADLEY EFRON

1. Introduction. Statistics seems to be a difficult subject for mathematicians, perhaps because its elusive and wide-ranging character mitigates against the traditional theorem-proof method of presentation. It may come as some comfort then that statistics is also a difficult subject for statisticians. We are now celebrating the approximate bicentennial of a controversy concerning the basic nature of statistics. The two main factions in this philosophical battle, the Bayesians and the frequentists, have

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alternated dominance several times, with the frequentists currently holding an uneasy upper hand. A smaller third party, perhaps best called the Fisherians, snipes away at both sides.

Statistics, by definition, is uninterested in the special case. Averages are the meat of statisticians, where "average" here is understood in the wide sense of any summary statement about a large population of objects. "The average I.Q. of a college freshman is 109" is one such statement, as is "the probability of a fair coin falling heads is 1/2." The controversies dividing the statistical world revolve on the following basic point: just *which* averages are most relevant in drawing inferences from data? Frequentists, Bayesians, and Fisherians have produced fundamentally different answers to this question.

This article will proceed by a series of examples, rather than an axiomatic or historical exposition of the various points of view. The examples are artificially simple for the sake of humane presentation, but readers should be assured that real data are susceptible to the same disagreements. A counter-warning is also apt: these disagreements haven't crippled statistics, either theoretical or applied, and have as a matter of fact contributed to its vitality. Important recent developments, in particular the empirical Bayes methods mentioned in Section 8, have sprung directly from the tension between the Bayesian and frequentist viewpoints.

2. The normal distribution. All of our examples will involve the normal distribution, which for various reasons plays a central role in theoretical and applied statistics. A normal, or Gaussian, random variable x is a quantity which possibly can take on any value on the real axis, but not with equal probability. The probability that x falls in the interval $[a, b]$ is given by the area under Gauss' famous bell-shaped curve,

$$\text{Prob}\{a \leqq x \leqq b\} = \int_a^b \phi_{\mu,\sigma}(x)dx, \tag{2.1}$$

where

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]. \tag{2.2}$$

For convenience we indicate such a random variable by

$$x \sim \mathcal{N}(\mu, \sigma^2), \tag{2.3}$$

with σ^2 instead of σ as the second argument by convention.

Figure 1 illustrates the normal distribution. The high point of $\phi_{\mu,\sigma}(x)$ is at $x = \mu$, the curve falling off quickly for $|x - \mu| > \sigma$. Most of the probability, 99.7%, is within ± 3 σ-units of the central value μ. We can write $x \sim \mathcal{N}(\mu, \sigma^2)$ as $x = \mu + \varepsilon$, where $\varepsilon \sim \mathcal{N}(0, \sigma^2)$; adding the constant μ merely shifts $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ μ units to the right.

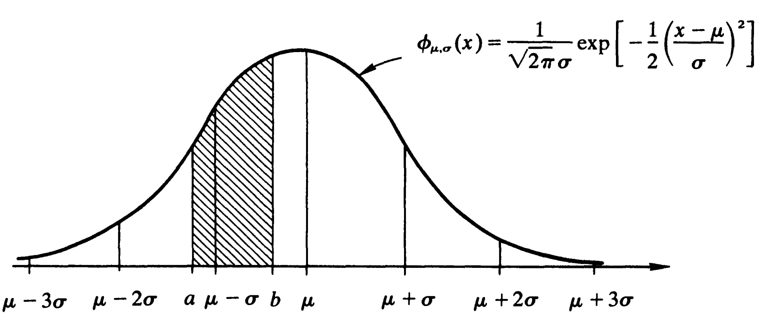

FIG. 1. The normal distribution. The random quantity $x \sim \mathcal{N}(\mu, \sigma^2)$ occurs in $[a, b]$ with probability equal to the shaded area. 68% of the probability is in the interval $[\mu - \sigma, \mu + \sigma]$, 95% in $[\mu - 2\sigma, \mu + 2\sigma]$, 99.7% in $[\mu - 3\sigma, \mu + 3\sigma]$.

The parameter μ is the "mean" or "expectation" of the random quantity x. Using "E" to indicate expectation,

$$\mu = E\{x\} \equiv \int_{-\infty}^{\infty} x\phi_{\mu,\sigma}(x)dx. \tag{2.4}$$

The reader may wish to think of $E\{g(x)\}$ for an arbitrary function $g(x)$ as just another notation for the integral of $g(x)$ with respect to $\phi_{\mu,\sigma}(x)dx$,

$$E\{g(x)\} \equiv \int_{-\infty}^{\infty} g(x)\phi_{\mu,\sigma}(x)dx. \tag{2.5}$$

Intuitively, $E\{g(x)\}$ is the weighted average of the possible values of $g(x)$, weighted according to the probabilities $\phi_{\mu,\sigma}(x)dx$ for the infinitesimal intervals $[x, x+dx]$. In other words, $E\{g(x)\}$ is a theoretical average of an infinite population of $g(x)$ values, where the x's occur in proportion to $\phi_{\mu,\sigma}(x)$.

It is easy to see, by symmetry, that μ is indeed the theoretical average of x itself when $x \sim \mathcal{N}(\mu, \sigma^2)$. A more difficult calculation (though easy enough for friends of the gamma function) gives the expectation of $g(x) = (x-\mu)^2$,

$$E\{(x-\mu)^2\} = \int_{-\infty}^{\infty} (x-\mu)^2\phi_{\mu,\sigma}(x)dx = \sigma^2. \tag{2.6}$$

The parameter σ, called the "standard deviation," sets the scale for the variability of x about the central value μ, as Figure 1 shows. A $\mathcal{N}(1, 10^{-6})$ random variable will have almost no perceptible variability under repeated trials, 997 out of 1000 repetitions occurring in [.997, 1.003], since $\sigma = 10^{-3}$. A $\mathcal{N}(1, 10^6)$ random variable is almost all noise and no signal, in the evocative language of communications theory.

The normal distribution has a very useful closure property that makes it as easy to deal with many observations as with a single one. Let $x_1, x_2, x_3, \ldots, x_n$ be n independent observations, each of which is $\mathcal{N}(\mu, \sigma^2)$, μ and σ being the same for all n repetitions. Independence means that the value of x_1, say, does not affect any of the other values: observing $x_1 > \mu$ does not increase or decrease the 34% probability that $x_2 \in [\mu, \mu+\sigma]$, etc. A familiar (non-normal) example of independent variables $x_1, x_2, x_3, \ldots$ is given by successive observations of a well-rolled die.

Let

$$\bar{x} \equiv \sum_{i=1}^{n} x_i/n \tag{2.7}$$

be the observed average of the n independent $\mathcal{N}(\mu, \sigma^2)$ variables. It is easy to show that

$$\bar{x} \sim \mathcal{N}(\mu, \sigma^2/n). \tag{2.8}$$

The distribution of $\bar{x}$ is the same as that for the individual x_i except that the scaling parameter has been reduced from σ to $\sigma/\sqrt{n}$. By taking n sufficiently large we can reduce the variability of $\bar{x}$ about μ to an arbitrarily small level, but of course in real problems n is limited and $\bar{x}$ retains an irreducible component of random variability.

In all of our examples σ will be assumed known to the statistician. The unknown parameter μ will be the object of interest, the goal being to make inferences about the value of μ on the basis of the data $x_1, x_2, x_3, \ldots, x_n$. In 1925 Sir Ronald Fisher made the fundamental observation that in this situation *the average $\bar{x}$ contains all possible information about μ*. For any inference problem about μ, knowing $\bar{x}$ is just as good as knowing the entire data set $x_1, x_2, x_3, \ldots, x_n$. In modern parlance, $\bar{x}$ is a "sufficient statistic" for the unknown parameter μ.

It is easy to verify sufficiency in this particular case. Given the observed value of $\bar{x}$, a standard

probability calculation shows that the random quantities $x_1 - \bar{x}, x_2 - \bar{x}, x_3 - \bar{x}, \ldots, x_n - \bar{x}$ have a joint distribution which does not depend in any way on the unknown parameter μ. In other words, what's left over in the data after the statistician learns $\bar{x}$ is devoid of information about μ. (This deceptively simple principle eluded both Gauss and Laplace!)

3. Frequentist estimation of the mean. The statistician may wish to estimate the unobservable parameter μ on the basis of the observed data $x_1, x_2, x_3, \ldots, x_n$. "Estimate" usually means "make a guess $\hat{\mu}(x_1, x_2, x_3, \ldots, x_n)$ depending on $x_1, x_2, \ldots, x_n$, with the understanding that you will be penalized an amount which is a smooth increasing function of the error of estimation $|\hat{\mu} - \mu|$." The usual penalty function, which we shall also use here, is $(\hat{\mu} - \mu)^2$, the squared-error loss function originally introduced by Gauss.

Fisher's sufficiency principle says that we need only consider estimation rules which are a function of $\bar{x}$. The most obvious candidate is $\bar{x}$ itself,

$$\hat{\mu}(x_1, x_2, \ldots, x_n) = \bar{x}. \tag{3.1}$$

This estimation rule is "unbiased" for μ; no matter what the true value of μ is,

$$E\bar{x} = \mu. \tag{3.2}$$

Unbiasedness is by no means a necessary condition for a good estimation rule, as we shall see later, but it does have considerable intuitive appeal as a guarantee that the statistician is not trying to slant the estimation process in favor of any particular μ value.

The expected penalty for using $\hat{\mu} = \bar{x}$ is, according to (2.6) and (2.8),

$$E(\hat{\mu} - \mu)^2 = \sigma^2 / n. \tag{3.3}$$

Gauss showed that among all unbiased estimation rules $\hat{\mu}(x_1, x_2, \ldots, x_n)$ which are linear in $x_1, x_2, x_3, \ldots, x_n$, the rule $\hat{\mu} = \bar{x}$ uniformly minimizes $E(\hat{\mu} - \mu)^2$ for every value of μ. In the early 1940's this result was extended to include any unbiased estimator at all, linear or nonlinear. The proof, which depends on ideas Fisher developed in the 1920's, was put forth separately by H. Cramér in Sweden and C. R. Rao in India.

If we agree to abide by the unbiasedness criterion and to use squared-error loss, $\bar{x}$ seems to be the best estimator for μ. It is helpful for the statistician to provide not only a "point estimator" for μ, $\bar{x}$ in this case, but also a range of plausible values of μ consistent with the data. From (2.8) and Figure 1 we see that

$$\text{Prob}\{|\bar{x} - \mu| \leqq 2\sigma / \sqrt{n}\} = .95, \tag{3.4}$$

which is equivalent to the statement

$$\text{Prob}\{\bar{x} - 2\sigma / \sqrt{n} \leqq \mu \leqq \bar{x} + 2\sigma / \sqrt{n}\} = .95. \tag{3.5}$$

The interval $[\bar{x} - 2\sigma / \sqrt{n}, x + 2\sigma / \sqrt{n}]$ is called a "95% confidence interval" for μ. The theory of confidence intervals was developed by J. Neyman in the early 1930's. As an example, suppose $n = 4$, $\sigma = 1$, and we observe $x_1 = 1.2$, $x_2 = 0.3$, $x_3 = 0.7$, $x_4 = 0.2$. Then $\bar{x} = 0.6$ and the 95% confidence interval for μ is $[-.04, 1.6]$.

All of this seems so innocuous and straightforward that the reader may wonder where the grounds for controversy lie. The fact is that all of the results presented so far are "frequentist" in nature. That is, they relate to theoretical averages with respect to the $\mathcal{N}(\mu, \sigma^2 / n)$ distribution of $\bar{x}$, with μ assumed fixed at its true value, whatever that may be. Unbiasedness itself is a frequentist concept; the theoretical average of $\hat{\mu}$ with μ held fixed, $E\hat{\mu}$, equals μ. Results (3.3) and (3.5), and the Cramér–Rao theorem, are frequentist statements. For example, the proper interpretation of (3.5) is that the interval $[\bar{x} - 2\sigma / \sqrt{n}, \bar{x} + 2\sigma / \sqrt{n}]$ covers the true value of μ with frequency 95% in a long series of independent repetitions of $\bar{x} \sim \mathcal{N}(\mu, \sigma^2 / n)$.

Nobody doubts that these results are true. The question raised by Bayesians and Fisherians is whether frequentist averages are really relevant to the process of inference scientists use in reasoning from noisy data back to the underlying mathematical models. We turn next to the Bayesian point of view.

4. Bayesian estimation of the mean. So far we have considered μ to be a fixed, albeit unknown, quantity. Suppose though that μ itself is a random variable, known to have the normal distribution with mean m and standard deviation s,

$$\mu \sim \mathcal{N}(m, s^2), \tag{4.1}$$

m and s being constants known to the statistician. For example, if μ is the true I.Q. of a person randomly chosen from the population of the United States, (4.1) holds with $m = 100$ and $s = 15$ (approximately). About 68% of I.Q.'s are between 85 and 115, about 95% between 70 and 130, etc. Information like (4.1), a "prior distribution for μ" in the language of the Bayesians, changes the nature of the estimation process.

Standard I.Q. tests are constructed so that if we test our randomly chosen person to discover his particular μ value, the overall test score*, say $\bar{x}$, is an unbiased normally distributed estimator of μ as in Section 3,

$$\bar{x} \mid \mu \sim \mathcal{N}(\mu, \sigma^2/n), \tag{4.2}$$

with $\sigma/\sqrt{n}$ about 7.5. We can expect $\bar{x}$ to be within 7.5 I.Q. points of μ 68% of the time, etc. The notation "$\bar{x} \mid \mu$" emphasizes that the $\mathcal{N}(\mu, \sigma^2/n)$ distribution for $\bar{x}$ is *conditional* on the particular value taken by the random quantity μ. The reason for this change in notation will be made clearer soon.

Bayes' theorem, originally discovered by the remarkable Reverend Thomas Bayes around 1750, is a mathematical formula for combining (4.1) and (4.2) to obtain the conditional distribution of μ given $\bar{x}$. In this case the formula gives

$$\mu \mid \bar{x} \sim \mathcal{N}(m + C(\bar{x} - m), D), \tag{4.3}$$

where

$$C = \frac{n/\sigma^2}{1/s^2 + n/\sigma^2} \quad \text{and} \quad D = \frac{1}{1/s^2 + n/\sigma^2}. \tag{4.4}$$

For example, if $\bar{x} = 160$ (and $m = 100$, $s = 15$, $\sigma/\sqrt{n} = 7.5$) then

$$\mu \mid \bar{x} \sim \mathcal{N}(148, (6.7)^2). \tag{4.5}$$

Expression (4.5), or more generally (4.3), is the "posterior distribution for μ given the observed value of $\bar{x}$." It is possible to make such a statement in the Bayesian framework because we start out assuming that μ itself is random. In the Bayesian framework the averaging process is reversed; the data $\bar{x}$ is assumed fixed at its observed value while it is the parameter μ which varies. In (4.5) for example, the conditional average of μ given $\bar{x} = 160$ is seen to be 148. If we randomly selected an enormous number of people, gave them each an I.Q. test, and considered the subset of those who scored 160, this subset would have an average true I.Q. of 148; 68% of the true I.Q.'s would be in the interval $[148 - 6.7, 148 + 6.7]$, etc.

How should we estimate μ in the Bayesian situation? It seems natural to use the estimator $\mu^*(\bar{x})$ which minimizes the conditional expectation of $(\mu - \mu^*)^2$ *given the observed value of* $\bar{x}$. From (4.3) it is

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* The symbols $\bar{x}$ for the test score and $\sigma/\sqrt{n}$ for its standard deviation are chosen to agree with our previous notation, even though real I.Q. scores aren't actually the average of n independent test items. Perfect normality, as expressed in (4.2), is an ideal only approximated by actual test scores.

easy to derive that this "Bayes estimator" is

$$\mu^*(\bar{x}) = m + C(\bar{x} - m), \tag{4.6}$$

the mean of the posterior distribution of μ given $\bar{x}$. Having observed $\bar{x} = 160$, the Bayes estimate is 148, not 160. Even though we are using an unbiased I.Q. test, so many more true I.Q.'s lie below 160 rather than above that it lowers the expected estimation error to bias the observed score toward 100. Figure 2 illustrates the situation.

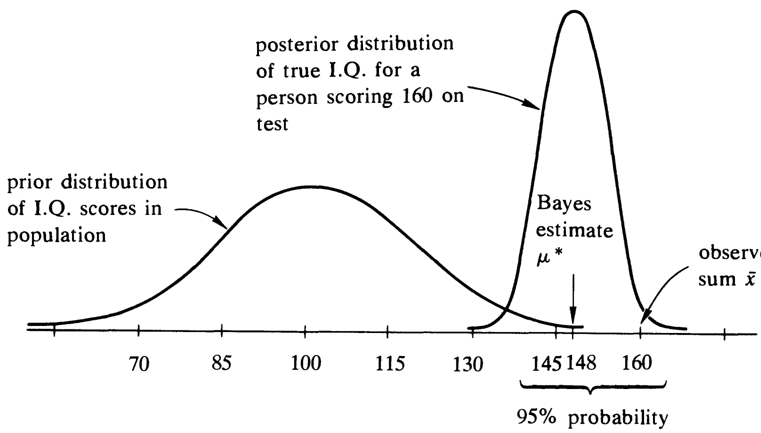

FIG. 2. I.Q. scores have a $\mathcal{N}(100, (15)^2)$ distribution in the population as a whole. A randomly selected person scoring 160 on a normal unbiased I.Q. test with standard deviation 7.5 points is estimated to have a true I.Q. of 148. The probability is 95% that the person's true I.Q. is in the interval [134.6, 161.4].

Confidence intervals have an obvious Bayesian analogue, from (4.3),

$$\text{Prob}\{\mu^*(\bar{x}) - 2\sqrt{D} \leqq \mu \leqq \mu^*(\bar{x}) + 2\sqrt{D} \mid \bar{x}\} = .95. \tag{4.7}$$

The notation $\text{Prob}\{\cdot \mid \bar{x}\}$ indicates probability conditional on the observed value of $\bar{x}$. In the I.Q. example, $\text{Prob}\{134.6 \leqq \mu \leqq 161.8 \mid \bar{x} = 160\} = .95$.

Nobody (well, almost nobody) disagrees with the use of Bayesian methods in situations like the I.Q. problem where there is a clearly defined and well-known prior distribution for μ. The Bayes theory, as we shall see, offers some striking advantages in clarity and consistency. These advantages are due to the fact that Bayesian averages involve only the data value $\bar{x}$ actually seen, rather than a collection of theoretically possible other $\bar{x}$ values.

Difficulties and controversies arise because Bayesian statisticians wish to use Bayesian methods when there is no obvious prior distribution for μ, or going even further, when it is clear that the unknown μ is a fixed constant with no random character at all. (For example, if μ is some physical constant, such as the speed of light, being experimentally estimated.) It is not perversity that motivates this Bayesian impulse, but rather a well-documented casebook of unpleasant inconsistencies in the frequentist approach.

As an example of the kind of difficulties frequentists experience, let us reconsider the I.Q. estimation problem, but without assuming knowledge of the prior distribution (4.1) for μ. In other words, assume only that we observe $\bar{x} \sim \mathcal{N}(\mu, \sigma^2/n)$, $\sigma/\sqrt{n} = 7.5$, and wish to estimate μ. Having observed $\bar{x} = 160$, the results of Section 3 tell us to estimate μ by $\hat{\mu} = 160$, with 95% confidence interval $[\hat{\mu} - 2\sigma/\sqrt{n}, \hat{\mu} + 2\sigma/\sqrt{n}] = [145, 175]$.

Suppose now that the frequentist receives a letter from the company which administered the I.Q. test: "On the day the score of $\bar{x} = 160$ was reported, our test-grading machine was malfunctioning. Any score $\bar{x}$ below 100 was reported as 100. The machine functioned perfectly for scores $\bar{x}$ above 100."

It may seem that the frequentist has nothing to worry about, since the score he received, $\bar{x} = 160$, was correctly reported. However, the reason he is using $\hat{\mu} = \bar{x}$ to estimate μ is that it is the best unbiased estimator. The malfunction of the grading machine implies that $\hat{\mu}$ is no longer even unbiased!

If the true value of μ equals 100, the machine functioning as described in the letter produces $E\bar{x} = 103$, a bias of $+3$ points. To regain unbiasedness the frequentist must replace the estimation rule $\hat{\mu} = \bar{x}$ with $\hat{\mu}' = \bar{x} - \Delta(\bar{x})$, where the function $\Delta(\bar{x})$ is chosen to remove the bias caused by the machine malfunction.

The correction term $\Delta(\bar{x})$ will be tiny for $\bar{x} = 160$, but it is disturbing that any change at all is necessary. The letter from the grading company contained no new information about the score actually reported, or about I.Q.'s in general. It only concerned something bad that might have happened but didn't. Why should we change our inference about the true value of μ? Bayesian methods are free from this defect; the inferences they produce depend only on the data value $\bar{x}$ actually observed, since Bayesian averages such as (4.6), (4.7) are conditional on the observed $\bar{x}$.

How can a Bayesian analysis proceed in the absence of firm prior knowledge like (4.1)? Two different approaches are in use. The "subjectivist" branch of Bayesian statistics attempts to assess the statistician's subjective probability distribution for the unknown parameter μ, before the data is collected, by a series of hypothetical wagers. These wagers are of the form "would you be willing to bet even money that $\mu > 85$ versus $\mu \leqq 85$? Would you be willing to bet two-to-one that $\mu < 150$ versus $\mu \geqq 150$?..." The work of L. J. Savage and B. deFinetti shows that a completely rational person should always be able to arrive at a unique (for himself) prior distribution on μ by sufficiently prolonged self-interrogation.

The subjectivist approach can be very fruitful in cases where the statistician (usually in collaboration with the experimenter, of course) has some vague prior opinions about the true value of μ, which he is trying to update on the basis of the observed data $\bar{x}$. Because it is subjective, the method is not much used where objectivity is the prime consideration, for example in the publication of controversial new scientific results.

Another line of Bayesian thought, which might be (but usually isn't) called "objective Bayesianism," attempts, in the absence of prior knowledge, to produce a prior distribution that everyone would agree represents a completely neutral prior opinion about μ. In the I.Q. problem, such a "flat" prior might take the form $\mu \sim \mathcal{N}(0, \infty)$, whereby we mean $\mu \sim \mathcal{N}(0, s^2)$ with s^2 going to infinity. From (4.3), (4.4) we get

$$\mu \mid \bar{x} \sim \mathcal{N}(\bar{x}, \sigma^2/n). \tag{4.8}$$

This result has a lot of appeal. The Bayes estimator μ^* equals the frequentist estimator $\hat{\mu} = \bar{x}$. The 95% Bayes probability interval (4.7) is the same as the 95% frequentist confidence interval (3.5). Moreover, because (4.8) is a Bayesian statement, the letter from the I.Q. testing company has no effect on it. We seem to be enjoying the best of both the frequentist and Bayesian worlds.

An enormous amount of effort has been expended in codifying the objective Bayesian point of view. Bayes himself put forth this approach (apparently with considerable reservations—his paper appeared posthumously and only through the efforts of an enthusiastic friend) which was adopted unreservedly by Laplace. It fell into disrepute in the early 1900's, and has since been somewhat revived by the work of Harold Jeffreys. One difficulty is that a "flat" prior distribution for μ is not at all flat for μ^3, say, so expressing ignorance seems to depend on which function of the unknown parameter one is interested in. A more pernicious difficulty is discussed in Section 8; in problems involving the estimation of several unknown parameters at once, what appears to be an eminently neutral prior distribution turns out to imply undesirable assumptions about the parameters.

5. Fisherian estimation of the mean. Ronald Fisher was one of the principal architects of frequentist theory. However, he was a lifelong critic, often vehemently so, of the standard frequentist

approach. His criticisms moved along the same lines as those of the Bayesians: why should we be interested in theoretical averages concerning what happens if infinitely many $\bar{x}$ values are randomly generated from $\mathcal{N}(\mu, \sigma^2/n)$, with μ fixed? We only have one observed value of $\bar{x}$ in any one inference problem, and the inference process should concentrate on just that observed value.

Fisher was also opposed to the Bayesian approach, perhaps because the type of data analysis problems he met in his agricultural and genetical work were not well suited to the assessment of prior distributions. With characteristic ingenuity he produced another form of inference, neither Bayesian nor frequentist.

The relation $\bar{x} \sim \mathcal{N}(\mu, \sigma^2/n)$ may be written

$$\bar{x} = \mu + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2/n). \tag{5.1}$$

We obtain the observation $\bar{x}$ by adding normal noise, $\varepsilon \sim \mathcal{N}(0, \sigma^2/n)$, to the unobservable mean μ. Expression (5.1) can also be written as

$$\mu = \bar{x} - \varepsilon. \tag{5.2}$$

It is obvious, or at least was obvious to Fisher, that in a situation where we know nothing a priori about μ, observing $\bar{x}$ tells us nothing about ε. As a matter of fact, said Fisher, if we can learn something about ε from $\bar{x}$ then model (5.1) by itself must be missing some important aspect of the statistical situation. We shall see this argument again, in more concrete form, in the next section.

If $\varepsilon \sim \mathcal{N}(0, \sigma^2/n)$ then $-\varepsilon \sim \mathcal{N}(0, \sigma^2/n)$ because of the symmetry of the bell-shaped curve about its central point. Fisher's interpretation of (5.2) was

$$\mu \mid \bar{x} \sim \mathcal{N}(\bar{x}, \sigma^2/n). \tag{5.3}$$

This looks just like the objectivist Bayesian statement (4.8), but has been obtained without recourse to prior distributions on μ. The interval statement following from (3.3) is

$$\text{Prob}\{\bar{x} - 2\sigma/\sqrt{n} \leqq \mu \leqq \bar{x} + 2\sigma/\sqrt{n} \mid \bar{x}\} = .95. \tag{5.4}$$

This is a "fiducial" probability statement, in Fisher's terminology.

In the fiducial argument randomness resides neither in the data $\bar{x}$, as in frequentist calculations, nor in μ, as in Bayesian calculations. Rather it lies in the mechanism which transforms the unobservable μ to the observed $\bar{x}$. (In the case at hand, this mechanism is the addition of $\varepsilon \sim \mathcal{N}(0, \sigma^2/n)$ to μ.) Fiducial statements such as (5.4) are obtained as averages over the random transformation mechanism.

The fiducial argument has fallen out of favor since its heyday in the 1940's. Most, though not all, contemporary statisticians consider it either a form of objective Bayesianism, or just plain wrong. Applied to the simultaneous estimation of several parameters, the fiducial argument can lead to disaster, as shown in Section 8.

Lest the reader feel sorry for Fisher, two other of his novel ideas on averaging, conditional inference and randomization, are still very much in vogue, and are the subjects of the next two sections.

6. Conditional inference. We return to the frequentist point of view, but with a twist, "conditioning," introduced by Fisher in 1934. Conditional inference illustrates another major source of ambiguity in the frequentist methodology, the choice of the collection of theoretically possible data values averaged over to obtain a frequentist inference.

Suppose again that we have independent normal variables $x_1, x_2, x_3, \ldots, x_n$, each $x_i \sim \mathcal{N}(\mu, \sigma^2)$, but that before observation begins the number n is randomly selected by the flip of a fair coin,

$$n = \begin{cases} 10 & \text{with probability } 1/2 \\ 100 & \phantom{\text{with probability }} 1/2. \end{cases} \tag{6.1}$$

We still wish to estimate μ on the basis of the data $x_1, x_2, x_3, \ldots, x_n$, and n with σ a known constant as before.

The conditional distribution of $\bar{x}$ *given the observed value of n* is

$$\bar{x} \mid n \sim \mathcal{N}(\mu, \sigma^2/n) \tag{6.2}$$

as at (2.8). The observed average $\bar{x}$ by itself is not a sufficient statistic in this situation. We also need to know whether n equals 10 or 100. Without this knowledge we still have an unbiased estimator of μ, namely $\hat{\mu} = \bar{x}$, but we don't know the standard deviation of $\hat{\mu}$.

What is the expected squared error of $\hat{\mu} = \bar{x}$ in this situation? Averaging (3.3) over the two values of n gives

$$E(\hat{\mu} - \mu)^2 = \frac{1}{2}\frac{\sigma^2}{10} + \frac{1}{2}\frac{\sigma^2}{100}. \tag{6.3}$$

Fisher pointed out that this is a ridiculous calculation. It is obviously more appropriate to assess the accuracy of $\hat{\mu}$ conditional on the value of n actually observed,

$$E\{(\hat{\mu} - \mu)^2 \mid n\} = \begin{cases} \sigma^2/10 & \text{if} \quad n = 10 \\ \sigma^2/100 & \text{if} \quad n = 100. \end{cases} \tag{6.4}$$

There is nothing wrong with (6.3), except that the average squared error it computes is irrelevant to any particular value of n and $\bar{x}$ actually observed! If $n = 100$ then (6.3) is much too pessimistic about the accuracy of $\hat{\mu}$, while if $n = 10$ it is much too optimistic.

This may all seem so obvious that it is hardly worth saying. Fisher's surprise was to show that exactly the same situation arises, more subtly, in other problems of statistical inference. We will illustrate this with an example involving the estimation of two different normal means, say μ_1 and μ_2, on the basis of independent unbiased normal estimates for each of them,

$$\bar{x}_1 \sim \mathcal{N}(\mu_1, 1), \qquad \bar{x}_2 \sim \mathcal{N}(\mu_2, 1), \tag{6.5}$$

$\bar{x}_1$ and $\bar{x}_2$ independent of each other. (For simplicity we have assumed that both estimates have $\sigma^2/n = 1$.) The two dimensional data vector $(\bar{x}_1, \bar{x}_2)$ can take on any value in the plane, but with high probability lies no more than a few units away from the vector of means (μ_1, μ_2).

Given no further information we would probably estimate (μ_1, μ_2) by $(\bar{x}_1, \bar{x}_2)$. (But see Section 8!) However, we now add the assumption that (μ_1, μ_2) is known to lie on the circle of radius 3 centered at the origin,

$$(\mu_1, \mu_2) = 3(\cos\theta, \sin\theta) \qquad -\pi < \theta \leqq \pi. \tag{6.6}$$

The statistical problem, as illustrated in Figure 3, is to estimate the unknown parameter θ on the basis of $(\bar{x}_1, \bar{x}_2)$.

Let us indicate the polar coordinates of $(\bar{x}_1, \bar{x}_2)$ by

$$\hat{\theta} \equiv \arctan(\bar{x}_2/\bar{x}_1), \qquad r \equiv \sqrt{\bar{x}_1^2 + \bar{x}_2^2}. \tag{6.7}$$

Then $\hat{\theta}$ is the obvious estimator of θ. It is unbiased, $E\hat{\theta} = \theta$, with expected squared error

$$E(\hat{\theta} - \theta)^2 = .12 \tag{6.8}$$

(obtained by numerical integration; (6.8) makes the convention that $\hat{\theta} - \theta$ ranges from $-\pi$ to π for any value of θ, the largest possible estimation error occurring if $(\bar{x}_1, \bar{x}_2)$ is antipodal to (μ_1, μ_2). This convention is unimportant because the probability of $|\hat{\theta} - \theta| > \pi/2$ is only .0014).

The unobvious fact pointed out by Fisher is that r plays the same role as did "n" in examples (6.1)–(6.4).

(i) The distribution of r does not depend on the true value of θ. (For readers familiar with the

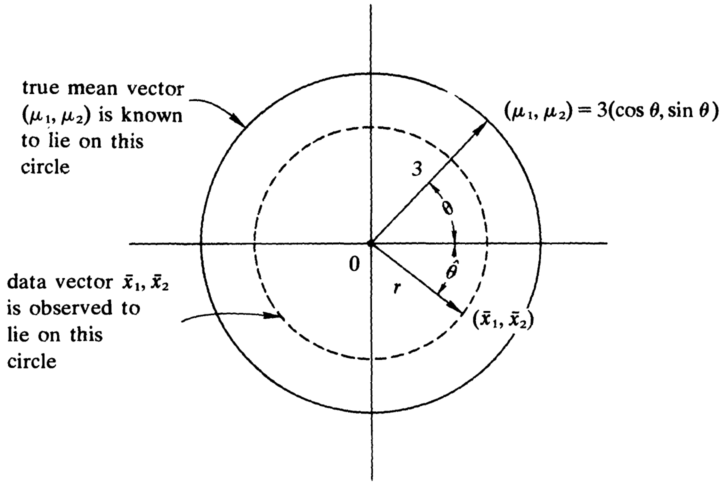

FIG. 3. The model $\bar{x}_1 \sim \mathcal{N}(\mu_1, 1)$ independent of $\bar{x}_2 \sim \mathcal{N}(\mu_2, 1)$, with (μ_1, μ_2) known to lie on a circle of radius 3 centered at the origin. We wish to estimate the angular location θ of (μ_1, μ_2) on the circle. The data vector $(\bar{x}_1, \bar{x}_2)$ is observed to have polar coordinates $(\hat{\theta}, r)$.

bivariate normal density, this follows from the circular symmetry of the distribution (6.5) of $(\bar{x}_1, \bar{x}_2)$ about (μ_1, μ_2).)

(ii) If r is small, then $\hat{\theta}$ has less accuracy than (6.8) indicates, while if r is large then $\hat{\theta}$ has greater accuracy that (6.8) indicates. Table 1 shows the conditional expected squared error $E\{(\hat{\theta} - \theta)^2 | r\}$ as a function of r.

In Fisher's terminology, r is an "ancillary" statistic. It doesn't directly contain information about θ, because of property (i), but its value determines the accuracy of $\hat{\theta}$. It now seems obvious that we should condition our assessment of the accuracy of $\hat{\theta}$ on the observed value of r. If $r = 2$, as in Figure 3, then $E\{(\hat{\theta} - \theta)^2 | r\} = .18$ is more relevant to the accuracy of $\hat{\theta}$ than is the unconditional expectation $E(\hat{\theta} - \theta)^2 = .12$.

r	1.5	2	2.5	3	3.5	4	4.5	5	Unconditional Value $E(\hat{\theta} - \theta)^2$
$E\{(\hat{\theta} - \theta)^2 \| r\}$	.26	.18	.14	.12	.10	.09	.08	.07	.12

TABLE 1. The conditional expected squared error of estimation in the circle problem, $E\{(\hat{\theta} - \theta)^2 | r\}$, as a function of the ancillary statistic $r = \sqrt{x_1^2 + x_2^2}$. The accuracy of $\hat{\theta}$ improves as r increases. Fisher argued that $E\{(\hat{\theta} - \theta)^2 | r\}$ is a more relevant measure of the accuracy of $\hat{\theta}$ than is the unconditional expectation $E(\hat{\theta} - \theta)^2$.

Many real statistical problems have the property that some data values are obviously more informative than others. Conditioning is the intuitively correct way to proceed, but few situations are as clearly structured as the circle problem. Sometimes more than one ancillary statistic exists, and the same data value will yield different accuracy estimates depending on which ancillary is conditioned upon. More often no ancillary exists, but various approximate ancillary statistics suggest themselves. What the circle example reveals is that frequentist statements like (6.8) may be true but irrelevant. Fisher's point was that the theoretical average of $(\hat{\theta} - \theta)^2$ should be taken not over all possible data values, but only over those containing the same amount of information for θ. So far it has proved impossible to codify this statement in a satisfactory way.

A Bayesian would agree that it is correct to condition one's opinion of the accuracy of $\hat{\theta}$ on the

observed value of r, but would ask why not go further and condition on the observed value of $(\bar{x}_1, \bar{x}_2)$ itself. This is impossible in the frequentist framework, since if we reduce our averaging set to one data point, there is nothing left to average over. Bayesian inferences are always conditional on the data point actually observed. In the circle problem the natural flat prior is a uniform distribution on $\theta \in [-\pi, \pi]$. With this prior distribution it turns out that $E\{(\theta - \hat{\theta})^2 \mid (\bar{x}_1, \bar{x}_2)\}$ equals $E\{(\hat{\theta} - \theta)^2 \mid r = \sqrt{\bar{x}_1^2 + \bar{x}_2^2}\}$ as given in Table 1, so in this particular case the objective Bayesian and conditional frequentist points of view agree. (Notice that in the first expectation "θ" is the random quantity, while in the second it is "$\hat{\theta}$" which varies.)

7. Randomization. Randomization is yet another form of inferential averaging introduced by R. A. Fisher. In order to discuss it simply we must change statistical problems, from estimation theory to "hypothesis testing." The data are now in the form of $2n$ independent normal observations $x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n$,

$$x_i \sim \mathcal{N}(\mu_1, \sigma^2), \qquad y_i \sim \mathcal{N}(\mu_2, \sigma^2) \qquad i = 1, 2, \ldots, n, \tag{7.1}$$

with σ known, μ_1 and μ_2 unknown. We wish to test the "null hypothesis" that $\mu_2 = \mu_1$ versus the "alternative hypothesis" that $\mu_2 > \mu_1$, often written

$$H: \mu_2 = \mu_1 \quad \text{versus} \quad A: \mu_2 > \mu_1. \tag{7.2}$$

(For our purposes, $\mu_2 < \mu_1$ is assumed impossible.)

In hypothesis testing the null hypothesis H usually plays the role of a devil's advocate which the experimenter is trying to disprove. For example, the x's may represent responses to an old drug and the y's responses to a new drug that the experimenter hopes is an improvement. Because there is a vested interest in discrediting H, conservative statistical methods have been developed which demand a rather stiff level of evidence before H is declared invalid. The frequentist theory, which is dominant in hypothesis testing, accomplishes this by requiring that the probability of falsely rejecting H in favor of A, when H is true, be held below a certain small level, usually .05. A test satisfying this criterion is said to be ".05 level" for testing H versus A.

With the data as in (7.1) it seems natural to compute $\bar{x} = \Sigma_1^n x_i / n$, $\bar{y} = \Sigma_1^n y_i / n$, and reject H in favor of A if

$$\bar{y} - \bar{x} > c. \tag{7.3}$$

The constant c is chosen so that if H is true then $\text{Prob}\{\bar{y} - \bar{x} > c\} = .05$. Standard probability calculations show that $c = 2.326 \cdot \sigma / \sqrt{n}$ is the correct choice. The theory of optimal testing developed by J. Neyman and E. Pearson around 1930 shows that (7.3) is actually the best .05 level test of H versus A, in the sense that if A is actually true then the probability of rejecting H in favor of A is maximized.

The x's and y's we observe are actually measurements on some sort of experimental units, perhaps college freshmen or white mice or headache victims. Let us denote these units by $U_1, U_2, U_3, \ldots, U_{2n}$. The opportunity for randomization arises when we have an experiment in which we can decide beforehand which n of the units are to be x's, and which n are to be y's. If we are lazy we can just give the first n units we happen to have at hand the x treatment and the last n the y treatment. This is begging for disaster! The first n headache victims may be those with the worst headaches, the first n mice those in the cage with the heavier animals, etc. An experiment done in the lazy way may have probability of falsely rejecting the null hypothesis much greater than .05 because of such uncontrolled factors.

In his vastly influential work on experimental design, Fisher argued that the choice of experimental units be done by randomization. That is, the assignment of the n units to the x treatment group and the n units to the y treatment group be done with equal probability for each of the $(2n!)/(n!)^2$ such assignments. A random number generating device is used to carry out the randomization process.

Fisher pointed out that randomized studies were likely to be free of the type of experimental biases discussed above. Suppose for example that there is some sort of "covariate" connected with the experimental units, by which we mean a quantity which is thought to affect the observation on that unit no matter which treatment is given. For example, weight might be an important covariate for the white mice. Heavy mice might respond less well to the stimulus than light mice. If n is reasonably large, say 10, it is very unlikely that the randomized experiment will have all the heavy mice in the x group and the light mice in the y group. This statement applies equally to every covariate, whether or not we know it affects the response, and even if we are unaware of its existence.

None of this has anything to do with averaging. The connection comes through Fisher's next suggestion: that we compute theoretical averages not over the hypothesized normal distributions, but instead over the randomization process itself. Suppose that if all $2n$ experimental units had received treatment x, the observations would have been $X_1, X_2, \ldots, X_{2n}$, X_i being the observation on unit U_i. The capital letters indicate that these are hypothetical observations and not necessarily the observed data. Under the null hypothesis H, treatment y is the same as treatment x, so we can indeed consider all $2n$ units to have received treatment x. In this case the observed data $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ coincide with the theoretical values $X_1, X_2, \ldots, X_{2n}$. Let $\mathcal{S}(x)$ be the indices of those units actually assigned to the x treatment and $\mathcal{S}(y)$ those assigned to the y treatment. Then, *if H is true*,

$$\bar{x} = \sum_{i \in \mathcal{S}(x)} X_i / n, \qquad \bar{y} = \sum_{i \in \mathcal{S}(y)} X_i / n. \tag{7.4}$$

If the study has been randomized then $\bar{x}$ is merely the average of n randomly selected X's and $\bar{y}$ the average of the remaining n X's.

The randomization (or "permutation") test of H analogous to (7.3) is constructed as follows:

(i) Given the observed data $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$, define $u_1 \equiv x_1$, $u_2 \equiv x_2, \ldots, u_{n+1} \equiv y_1, \ldots, u_{2n} \equiv y_n$. (Notice that, if H is true, the u's coincide with the X's of the previous paragraph.)

(ii) For each partition $\mathcal{P} \equiv \{\mathcal{S}_1, \mathcal{S}_2\}$ of $\{1, 2, \ldots, 2n\}$ into two disjoint subsets of size n, calculate

$$(\bar{y} - \bar{x})_{\mathcal{P}} \equiv \sum_{i \in \mathcal{S}_2} u_i / n - \sum_{i \in \mathcal{S}_1} u_i / n. \tag{7.5}$$

(iii) List all $(2n!)/(n!)^2$ values of $(\bar{y} - \bar{x})_{\mathcal{P}}$ in ascending order.

(iv) Reject H in favor of A if the value of $\bar{y} - \bar{x}$ actually observed is in the upper 5% of the list.

The randomization test has a .05 chance of falsely rejecting H, *where the probability .05 now refers to an average taken over all $(2n!)/(n!)^2$ random assignments of treatment types to experimental units.* The test is still of the form "reject H in favor of A if $\bar{y} - \bar{x} > c$," except that c no longer equals the constant $2.326 \cdot \sigma / \sqrt{n}$. Instead c is a function of the set of values $\{u_1, u_2, \ldots, u_{2n}\}$ constructed in (i). For each set $\{u_1, u_2, \ldots, u_{2n}\}$, c is selected to satisfy (iv).

The randomization test has one big advantage over test (7.3). Its .05 probability of falsely rejecting H remains valid under any null hypothesis that says the $2n$ x's and y's are generated by the same probability distribution, normal or otherwise. As a matter of fact, no randomness at all in the observations need be assumed. We can just take the null hypothesis to be that each unit U_i has a fixed response X_i connected with it, no matter whether it is given the x or y treatment. This last statement reemphasizes that the randomization test must involve a non-frequentist form of averaging.

Randomization, or at least inference based on randomization, appears heretical to a Bayesian statistician. The true Bayesian must condition on the assignment $\{\mathcal{S}(x), \mathcal{S}(y)\}$ of units to treatments actually used, since this is part of the available data, and not average over all possible partitions that might have been. (Fisher's arguments on ancillarity seem to point in exactly the same direction, which is to say directly opposite to randomization!)

One aspect of randomization makes both frequentists and Bayesians uneasy. Suppose, just by bad luck, that the randomization process does happen to assign all heavy mice to the x treatment and all light mice to the y treatment. Can we still use the .05 level randomization test to reject H in favor of

A? The answer seems clearly not, but it is difficult to codify a way of avoiding such traps. To put things the other way, suppose we know the weights $w_1, w_2, w_3, \ldots, w_{2n}$ of the mice before we begin the experiment. Under reasonable frequentist assumptions there will be a unique best way $\{\mathcal{S}(x), \mathcal{S}(y)\}$ of assigning the mice to the treatments for the purpose of testing treatment x versus treatment y, one that optimally equalizes the weight assignments to the two groups. Statisticians trained in the Fisherian tradition find it difficult to accept such "optimal experimental designs" because the element of randomization has been eliminated.

8. Stein's Phenomenon. The reader may have noticed that the controversies so far have been more academic than practical. All philosophical factions agree that in the absence of prior knowledge $[x - 2\cdot\sigma/\sqrt{n}, \bar{x} + 2\cdot\sigma/\sqrt{n}]$ is a 95% interval for μ, the disagreement being over what "95%" means. This situation changes, for the worse, when we consider the simultaneous estimation of many parameters.

Suppose then that we have several normal means $\mu_1, \mu_2, \ldots, \mu_k$ to estimate, for each one of which we observe an independent, unbiased normal estimate

$$\bar{x}_i \sim \mathcal{N}(\mu_i, 1) \quad \text{independently} \quad i = 1, 2, \ldots, k. \tag{8.1}$$

(Once again we have taken the variance σ^2/n equal to 1 for the sake of convenience.) The natural analogue of squared error loss when there are several parameters to estimate is Euclidean squared distance. To simplify notation, let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$ be the vector of observed averages, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$ the vector of true means, and $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_k)$ the vector of estimates. Then the squared error misestimation penalty is

$$\|\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\|^2 = \sum_{i=1}^{k} (\hat{\mu}_i - \mu_i)^2. \tag{8.2}$$

Before pursuing the problem of estimating $\boldsymbol{\mu}$ on the basis of $\mathbf{x}$, we note an elementary but important fact. This fact, which can be proved in one line by readers familiar with the multivariate normal distribution, is that for every parameter vector $\boldsymbol{\mu}$ we have

$$\text{Prob}\{\|\bar{\mathbf{x}}\| > \|\boldsymbol{\mu}\|\} > .50. \tag{8.3}$$

That is, the data vector $\bar{\mathbf{x}}$ tends to be farther away from the origin than does the parameter vector $\boldsymbol{\mu}$, no matter what $\boldsymbol{\mu}$ is. Table 2 shows that for $k = 10$ the probability is actually quite a bit greater than .50 for moderate values of $\|\boldsymbol{\mu}\|$.

Suppose that $k = 10$, and we observe a data vector $\bar{\mathbf{x}}$ with squared length $\|\bar{\mathbf{x}}\|^2 = 12$. Assume also that we have no prior knowledge about $\boldsymbol{\mu}$. Looking at Table 2, it seems to be a very good bet that $\|\boldsymbol{\mu}\|^2 < 12$. For $\|\boldsymbol{\mu}\|^2$ in the range $[0, 40]$, which is almost certainly the case if $\|\mathbf{x}\|^2 = 12$, more than 75% of the time we have $\|\mathbf{x}\| > \|\boldsymbol{\mu}\|$. However, this is a frequentist "75%," calculated with $\boldsymbol{\mu}$ fixed and $\bar{\mathbf{x}}$ varying randomly according to (8.1). The analogue of the objective Bayesian argument presented in Section 4 gives quite different results.

$\|\boldsymbol{\mu}\|^2$	0	6	12	18	24	30	40	60
$\text{Prob}\{\|\bar{\mathbf{x}}\| > \|\boldsymbol{\mu}\|\}$	1.00	.967	.904	.857	.822	.795	.762	.719

TABLE 2. The probability that $\|\bar{\mathbf{x}}\| \geqq \|\boldsymbol{\mu}\|$ is always greater than .5. For the case $k = 10$ the probabilities are much greater than .5 for moderate values of $\|\boldsymbol{\mu}\|$.

Given our complete prior ignorance about the parameter vector $\boldsymbol{\mu}$, it seems natural to use a flat prior of the form $\mu_i \sim \mathcal{N}(0, \infty)$ (that is, $\mu_i \sim \mathcal{N}(0, s^2)$ with $s^2 \to \infty$) independently for $i = 1, 2, \ldots, k$. This leads to the posterior distribution (4.8) for each parameter μ_i,

$$\mu_i \mid \bar{x}_i \sim \mathcal{N}(\bar{x}_i, 1) \tag{8.4}$$

independently for $i = 1, 2, \ldots, k$. This of course is a Bayesian statement, with the $\bar{x}_i$'s fixed at their observed values and the μ_i's varying randomly according to (8.4). Reversing the names of the fixed and random quantities in Table 2 gives

$$\text{Prob}\{\|\boldsymbol{\mu}\| > \|\bar{\mathbf{x}}\| \mid \|\bar{\mathbf{x}}\|^2 = 12\} = .904. \tag{8.5}$$

It now seems to be a very good bet that $\|\boldsymbol{\mu}\| > \|\mathbf{x}\|$. As a matter of fact,

$$\text{Prob}\{\|\boldsymbol{\mu}\| > \|\bar{\mathbf{x}}\| \mid \bar{\mathbf{x}}\} > .50 \tag{8.6}$$

for every observed data vector $\bar{\mathbf{x}}$! Fisher's fiducial argument of Section 5 also leads to (8.4)–(8.6).

Equations (8.3) and (8.6) show a clear contradiction between the frequentist and Bayesian points of view. Which is correct? There is a most surprising and persuasive argument in favor of the frequentist calculation (8.3). This was provided by Charles Stein in the mid 1950's and concerns the estimation of $\boldsymbol{\mu}$ on the basis of the data vector $\bar{\mathbf{x}}$ (or equivalently the estimation of the parameters $\mu_1, \mu_2, \ldots, \mu_k$ on the basis of $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$).

The obvious estimator is

$$\hat{\boldsymbol{\mu}}(\bar{\mathbf{x}}) = \bar{\mathbf{x}}, \tag{8.7}$$

which estimates each μ_i by $\bar{x}_i$, as at (3.1). This estimate has expected squared error loss

$$E\|\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\|^2 = k \tag{8.8}$$

for every parameter vector $\boldsymbol{\mu}$. What Stein showed is that if k, the number of means to be estimated, is $\geqq 3$, then the estimator

$$\tilde{\boldsymbol{\mu}}(\bar{\mathbf{x}}) = \left[1 - \frac{k-2}{\|\bar{\mathbf{x}}\|^2}\right]\bar{\mathbf{x}} \tag{8.9}$$

has

$$E\|\tilde{\boldsymbol{\mu}} - \boldsymbol{\mu}\|^2 < k \tag{8.10}$$

for every $\boldsymbol{\mu}$! (This particular form of $\tilde{\boldsymbol{\mu}}$ was developed jointly with W. James in 1960.) From a frequentist point of view, $\tilde{\boldsymbol{\mu}}$ estimates $\boldsymbol{\mu}$ uniformly better than does $\hat{\boldsymbol{\mu}}$. It is also better from a Bayesian point of view: given any prior distribution on $\boldsymbol{\mu}$, estimating by $\tilde{\boldsymbol{\mu}}$ rather than $\hat{\boldsymbol{\mu}}$ results in a lower overall expected squared error of estimation (averaging now over the randomness in $\boldsymbol{\mu}$ and the randomness in $\mathbf{x}$).

Stein's estimator is based on (8.3). Since $\|\hat{\boldsymbol{\mu}}\| = \|\bar{\mathbf{x}}\|$ tends to be greater than $\|\boldsymbol{\mu}\|$ with high probability, a shrinking factor $[1 - (k-2)/\|\bar{\mathbf{x}}\|^2]$ is used to give an estimate nearer $\boldsymbol{\mu}$. The shrinking factor is more drastic when $\|\bar{\mathbf{x}}\|^2$ is small. With $k = 10$, $\|\bar{\mathbf{x}}\|^2 = 12$, we have $\tilde{\boldsymbol{\mu}} = [.333]\bar{\mathbf{x}}$. If instead $\|\bar{\mathbf{x}}\|^2 = 800$ then $\tilde{\boldsymbol{\mu}} = [.99]\bar{\mathbf{x}}$. Figure 4 gives a schematic illustration.

Notice that the origin $\mathbf{O}$ plays a special role in the construction of $\tilde{\boldsymbol{\mu}}$, even though there is nothing in the statement of the estimation problem that favors $\mathbf{O}$. As a matter of fact, we can change the origin to any other point in k dimensional space, $\mathbf{O}'$ say, and obtain a different Stein estimate,

$$\tilde{\boldsymbol{\mu}}' = \mathbf{O}' + \left[1 - \frac{k-2}{\|\bar{\mathbf{x}} - \mathbf{O}'\|^2}\right](\bar{\mathbf{x}} - \mathbf{O}'), \tag{8.11}$$

which is also uniformly better than $\hat{\boldsymbol{\mu}}$.

Stein's result has created a host of difficulties for frequentists and Bayesians alike, which we can't pursue here. The implications for objective Bayesians and fiducialists have been especially disturbing. The seemingly flat prior distribution leading to (8.4) isn't flat at all: it forces the parameter vector to relatively far away from any prechosen origin $\mathbf{O}'$. If a satisfactory theory of objective Bayesian inference exists, Stein's estimator shows that it must be a great deal more subtle than previously expected.

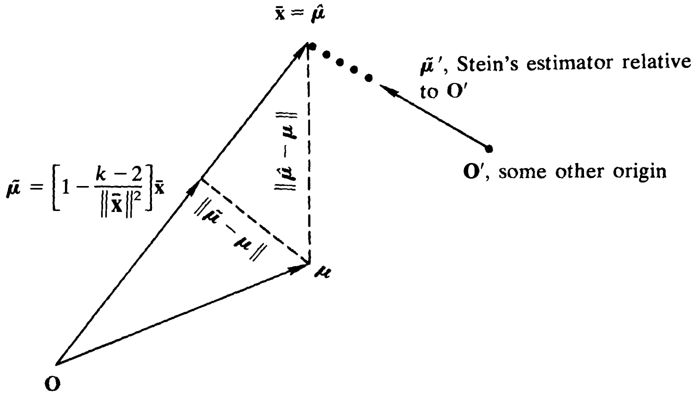

FIG. 4. Stein's estimate $\tilde{\boldsymbol{\mu}}$ is obtained by shrinking the obvious estimate $\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}}$ toward the origin **O**. The shrinking factor is more extreme the closer $\|\bar{\mathbf{x}}\|$ lies to **O**. Stein and James showed that $E\|\tilde{\boldsymbol{\mu}} - \boldsymbol{\mu}\|^2 < E\|\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\|^2$ for every $\boldsymbol{\mu}$. We can choose any other origin **O′** and obtain a different Stein estimate, $\boldsymbol{\mu}'$, which also dominates $\hat{\boldsymbol{\mu}}$.

The trouble with the multiparameter estimation problem is not that it is harder than estimating a single parameter. It is easier, in the sense that dealing with many problems simultaneously can give extra information not otherwise available. The trouble lies in finding and using the extra information. Consider the Bayesian model (4.1). With just a single μ to estimate this model must be taken on pure faith (or relevant experience). However, if we have several means to estimate, $\mu_1, \mu_2, \ldots, \mu_k$, each drawn independently from an $\mathcal{N}(m, s^2)$ population, the data $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ allows us to *estimate* m and s^2, instead of postulating their values. Plugging the estimated values into (4.6) gives an "empirical Bayes rule" very much like the Stein rule (8.11). Empirical Bayes theory, originally developed by Herbert Robbins in the early 1950's, offers some hope of a partial reconciliation between frequentists and Bayesians.

9. Some last comments. The field of statistics continues to flourish despite, and partly because of, its foundational controversies. Literally millions of statistical analyses have been performed in the past 50 years, certainly enough to make it abundantly clear that common statistical methods give trustworthy answers when used carefully. In my own consulting work I am constantly reminded of the power of the standard methods to dissect and explain formidable data sets from diverse scientific disciplines. In a way this is the most important belief of all, cutting across the frequentist-Bayesians divisions: that there do exist more or less universal techniques for extracting information from noisy data, adaptable to almost every field of inquiry. In other words, statisticians believe that statistics exists as a discipline in its own right, even if they can't agree on its exact nature.

What does the future hold? At a recent conference Dennis Lindley, of University College, London, gave a talk entitled, "The future of statistics—A Bayesian 21st century." My personal subjective probability is .15 on that eventuality. The big advantage of subjective Bayesianism, which is what Professor Lindley was referring to, is its logical consistency. Philosophers who investigate the foundations of scientific inference usually wind up being repelled by frequentism and attracted to the Bayesian argument.

But consistency isn't enough. Subjective Bayesianism must face the challenge of scientific objectivity. This is the ultimate stronghold of the frequentist viewpoint. If the 21st century is Bayesian, my guess is that it will be some combination of subjective, objective, and empirical Bayesian, not significantly less complicated and contradictory than the present situation. The complexity of the problems statisticians are asked to deal with is increasing at an alarming rate. It is not unusual these days to deal with data sets of a million numbers, and models with several thousand parameters. As Section 8 suggests, this trend is likely to exacerbate the difficulties of producing a logically consistent theory of statistics.

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WHEN IS A FUNCTION THAT SATISFIES THE CAUCHY–RIEMANN EQUATIONS ANALYTIC?

J. D. GRAY AND S. A. MORRIS

1. The Looman–Menchoff theorem—An extension of Goursat's theorem. It is well known[1] that a complex-valued function $f = u + iv$, defined and analytic on a domain D in the complex plane satisfies the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

throughout D. The standard textbooks, such as those authored by Ahlfors, Cartan, Churchill, Jameson, Knopp, Sansone and Gerretson, avoid answering the question as to whether or not the converse holds. Most instead offer the following partial converse due to Goursat **[13]**.

THEOREM 1. *If $f = u + iv$, defined on a domain D, is such that*

(i) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist everywhere in D,*

(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D, and if further*

(iii) *f is continuous in D,*

(iv) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are continuous in D,*

then f is analytic in D.

This is a substantially revised version of an article by the present authors and S. A. R. Disney that appeared in the Gazette of the Australian Mathematical Society 2 (3) (1975), 67–81. S. A. R. Disney's name does not appear above only because he preferred it that way.

[1] This is a rare instance of a well-known result that is indeed well known.

The remaining standard texts offer the stronger result:

THEOREM 2. *If $f = u + iv$, defined on a domain D, is such that*
(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ *exist everywhere in D,*
(ii) *u, v satisfy the Cauchy-Riemann equations everywhere in D, and if further*
(iii) *u, v, as functions of two real variables, are differentiable everywhere in D,*
then f is analytic in D.

Recently the authors began a search to discover precisely what is known regarding the converse. The only modern book we were able to find tnat addresses itself to this problem is Derrick **[8]**. He points out that far weaker conditions than those of Theorem 2 are known to imply analyticity but that the Cauchy–Riemann equations themselves do not imply analyticity! Indeed, the function f given by

$$f(z)=\begin{cases}\exp(-z^{-4}) & \text{if } z\neq 0\\ 0 & \text{if } z=0,\end{cases}$$

first noticed by Looman [**20**, 107], (see also [**39**, 70]), is readily seen to satisfy the Cauchy–Riemann equations everywhere, but, as $f(z)/z \to \infty$ as $z \to 0$ with $\arg z = \pi/4$, fails to be analytic at the origin. Observe that f must have an essential singularity at 0 otherwise $\partial f/\partial x$ could not exist there.

Derrick **[8]** suggests that the 'best' result in this direction appears to be

THEOREM 3. (Looman–Menchoff) *If $f = u + iv$, defined on a domain D, is such that*
(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ *exist everywhere in D,*
(ii) *u, v satisfy the Cauchy-Riemann equations everywhere in D, and if further*
(iii) *f is continuous in D,*
then f is analytic in D.

Menchoff's proof (see [**34**, 199] and [**24**, 9]), based on the concepts of Lebesgue integration and Baire category is, according to Saks **[36]** "... undoubtedly one of the most elegant and unexpected applications of the modern theory of real functions to the elementary problems of an entirely classical aspect." A proof of the theorem is given in the appendix.

A word of caution. The naive local version of the Looman-Menchoff theorem is: if a function is continuous at z_0 and satisfies the Cauchy-Riemann equations there, it is complex-differentiable at z_0. This assertion is false! For example [**8**, 15], the function

$$f(z)=\begin{cases}z^5/|z|^4 & \text{if } z\neq 0\\ 0 & \text{if } z=0,\end{cases}$$

is continuous everywhere, satisfies the Cauchy-Riemann equations at 0, but is not complex-differentiable at the origin. To the best of our knowledge the strongest result in this direction is the standard one: if $f = u + iv$ is such that (i) u, v are differentiable at z_0, (ii) u, v satisfy the Cauchy-Riemann equations at z_0, then f is (complex) differentiable at z_0. See [**17**, 35].

Although Looman and Menchoff clearly did improve on Goursat's theorem others have obtained still more subtle results.

2. Extensions of the theorems of Green, Morera and Goursat. The earliest contribution to the problem appears to be that of Paul Montel who, in a 1913 note in the *Comptes Rendus*, asserted the

THEOREM 4. *If $f = u + iv$, defined on a domain D, is such that*
(i) $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ *exist everywhere in D,*
(ii) *u, v satisfy the Cauchy-Riemann equations everywhere in D, and if further*
(iii) *f is bounded in D,*
then f is analytic in D.

Recall that a function f on D is said to be locally bounded if it is bounded in some neighbourhood

of each point of D. Now analyticity in D means analyticity in some neighbourhood of each point of D, whence condition (iii) can be replaced by

(iii)′ *f is locally bounded in D.*

As every continuous function is locally bounded it follows that Theorem 4 with (iii) replaced by (iii)′, implies Theorem 3. Although this result appears to be quite strong, observe that condition (i) implies the separate continuity of f which in turn implies its measurability, [26]. (In fact f is necessarily of Baire class I).

Montel neither proved this result in his note [27] nor did he publish a proof elsewhere. In spite of this, it was stated as a theorem in Menchoff's monograph [24]—one in a series edited by Montel. Montel did, however, indicate how the proof is an "immediate application" of a strengthened version of the following classical result on exact differentials. (Here, as elsewhere, the term integrable means Lebesgue integrable and all integrals are Lebesgue integrals. However, as every bounded Riemann integrable function is Lebesgue integrable, with the exception of Theorems 8, 9 and 10 and Question 1 at the end of §3, Lebesgue may be replaced by Riemann throughout.)

THEOREM 5. *Let C be a simple closed contour and K the closure of its interior. If P, Q are real-valued functions of two variables on K such that*

(i) *$\partial P/\partial y$, $\partial Q/\partial x$ exist everywhere in K,*

(ii) *$\partial P/\partial y = \partial Q/\partial x$ everywhere in K,*

and if further

(iii) *P, Q are continuous in K,*

(iv) *$\partial P/\partial y$, $\partial Q/\partial x$ are continuous in K,*

then

$$\int_C Pdx + Qdy = 0.$$

Of course this is a special case of the following version of Green's theorem—a proof of which may be constructed by making technical adjustments to that on page 289 of [1]. Parenthetically we remark that although condition (iv) below implies (ii), (ii) is included as it is clearly necessary.

THEOREM 6. *Let C be a simple closed contour and K the closure of its interior. If P, Q are real-valued functions of two variables on K such that*

(i) *$\partial P/\partial y$, $\partial Q/\partial x$ exist everywhere in K,*

(ii) *$\partial Q/\partial x - \partial P/\partial y$ is integrable in K,*

and if further

(iii) *P, Q are continuous in K,*

(iv) *$\partial P/\partial y$, $\partial Q/\partial x$ are continuous in K,*

then (Green's formula)

$$\int_C Pdx + Qdy = \int\int_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

In 1923 Looman [20], by weakening the hypotheses in Theorem 5, offered a proof of Theorem 3. Unfortunately the proof was found to contain a serious gap centring around a surprising fact. Even though $\partial P/\partial x$, $\partial Q/\partial y$ do not occur in Green's formula, an example of Fesq [11] reveals that some assumptions regarding them must be made if one wishes to relax condition (iii) and still have a valid statement. Indeed, even if the right-hand side of the formula is zero, the statement is false without such assumptions. It was Tolstoff [41] who first realized this. Unfortunately this error appears in the papers of both Montel and Looman (see also [46]), and was not corrected until Menchoff's paper [23, 9] appeared. (See also [34, 199].)

Tolstoff [40] was the first to prove Montel's theorem. Implicit in his work is the observation that

whenever one has a Green-type theorem (see Theorem 6) and a Morera-type theorem (see Theorem 7) one obtains a Goursat-type theorem (Theorem 1). For example, let us see how the classical Goursat theorem follows from the classical Green theorem (more accurately, from its corollary—Theorem 5—on exact differentials) and the classical Morera theorem.

THEOREM 7. (Morera, cf. [**35**, 120]) *If f, defined on a domain D, is such that*
(i) *f is continuous in D,*
(ii) $\int_{\partial R} f(z)dz = 0$ *for each rectangle R (*) in D,*
then f is analytic in D.

Proof (Of Goursat's theorem). For any rectangle R

$$\int_{\partial R} f(z)dz = \int_{\partial R} (u + iv)dz = \int_{\partial R} udx - vdy + i\int_{\partial R} vdx + udy = 0$$

by Theorem 5 and the Cauchy–Riemann equations. Hence by Morera's theorem, f is analytic in D.

The moral of this proof is clear. If one can reduce the conditions involved in Morera's theorem and those involved in Green's theorem one can obtain a strengthened version of Goursat's theorem. For instance, one can readily deduce the Looman–Menchoff theorem (Theorem 3) from the classical Morera theorem and the following extension of Green's theorem due to Paul J. Cohen [**7**]—the same Cohen of Continuum Hypothesis fame.

THEOREM 8. *Let R be a closed rectangle. If P, Q are real-valued functions of two variables on R such that*
(i) *$\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, $\partial Q/\partial y$ exist everywhere in R,*
(ii) *$\partial Q/\partial x - \partial P/\partial y$ is integrable in R,*
and if further
(iii) *P, Q are continuous in R,*
then

$$\int_{\partial R} Pdx + Qdy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

Such a proof would, however, reverse the chronological order of things. In fact Cohen's ideas were inspired by the proof of the Looman–Menchoff theorem.

It was via the implication Morera + Green $\Rightarrow$ Goursat that Tolstoff [**40**] proved Montel's theorem. First, he proved a strong version of Morera (with continuous replaced by integrable, locally bounded and separately continuous); next, he proved a strong version of Theorem 5, and finally he combined these as in the above proof of Goursat's theorem, to yield Montel's theorem.

3. More technical results. We saw in §1 that a function that satisfies the Cauchy–Riemann equations everywhere need not be analytic. With this in mind one cannot fail to be impressed by the distributional result.

THEOREM 9. *Suppose f is locally integrable on D and, as a distribution, satisfies the Cauchy–Riemann equations. Then f agrees almost everywhere with a function analytic in D.*

Proof. The proof proceeds by a smoothing argument. We present only an outline as full details may be readily found in [**49**, 117]. Let k be a 'bump' function, viz, k is a C^∞ function defined on the complex plane, $k \geqq 0$; $\int\int k(z)dxdy = 1$; and k vanishes outside the open unit disc. For $\varepsilon > 0$ put $k_\varepsilon(z) = \varepsilon^{-2}k(z/\varepsilon)$. Then the convolution $f_\varepsilon = f * k_\varepsilon$ given by

(*) Throughout, all rectangles are assumed to have their sides parallel to the co-ordinate axes.

$$f_\varepsilon(z) = \int\int f(z-\zeta)k_\varepsilon(\zeta)d\xi d\eta$$

is a C^∞ function of z for those z distant greater than ε from the boundary of D. Moreover [**47**, 118], $f_\varepsilon \to f$ a.e. in D as $\varepsilon \to 0$.

If g is a (differentiable) function the Cauchy–Riemann equations may be written as $\partial g/\partial \bar{z} = 0$, where $\partial/\partial\bar{z}$ is the formal partial differential operator

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Hence, to assert of f that as a distribution it satisfies these equations means that for each test function ϕ,

$$\int\int f(\zeta)\frac{\partial\phi}{\partial\bar{\zeta}}d\xi d\eta = 0.$$

Given this, the following equations are readily verified,

$$\frac{\partial f_\varepsilon}{\partial\bar{z}} = \frac{\partial}{\partial\bar{z}}\int\int f(\zeta)k_\varepsilon(z-\zeta)d\xi d\eta = \int\int f(\zeta)\frac{\partial}{\partial\bar{z}}k_\varepsilon(z-\zeta)d\xi d\eta$$
$$= -\int\int f(\zeta)\frac{\partial}{\partial\bar{\zeta}}k_\varepsilon(z-\zeta)d\xi d\eta = 0 \text{ as } k_\varepsilon \text{ is a test function.}$$

Thus f_ε is a C^∞ function satisfying the Cauchy–Riemann equations and hence, by Goursat's theorem, is analytic. But, by Cauchy's integral formula, for almost all $z \in D$, and for appropriate contours γ,

$$f(z) = \lim_{\varepsilon\to 0} f_\varepsilon(z) = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_\gamma \frac{f_\varepsilon(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta,$$

so that f agrees a.e. with a function analytic in D.

Theorem 9 is a particular instance of the general regularity theorem: any distributional solution of a hypo-elliptic system of partial differential equations (of which the Cauchy–Riemann system is a paradigm example) is in fact a C^∞ function. Moreover, the above proof-technique of "smoothing" is one of the key tools needed to prove the general result. See [**16**, 101].

From Theorem 9 the following result of Rademacher [**29**] may be deduced.

THEOREM 10. *If $f = u + iv$, defined on a domain D, is such that*
(i) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist a.e. in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations a.e. in D,*
and if further
(iii) *f is locally integrable in D,*
(iv) *f is separately absolutely continuous in D,*
(v) *$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are locally integrable in D,*
then f is analytic in D.

Proof. Note first the redundancy: (iv) implies (i). (iv) together with (v) suffice to ensure the validity of integration by parts:

$$\int\int f(z)\frac{\partial\phi}{\partial\bar{z}}dxdy = -\int\int\frac{\partial f}{\partial\bar{z}}\phi(z)dxdy,$$

for all text functions ϕ. This equation shows that the classical and distributional derivatives of f 'coincide'. Thus by Theorem 9, f agrees almost everywhere with a function g analytic in D. In fact, $f = g$ everywhere in D. To see this, suppose to the contrary that for some $z_0 = x_0 + iy_0 \in D$,

$f(z_0) \neq g(z_0)$. Then, as f is separately continuous, f and g must disagree at all points on some line-segment through z_0 parallel to the x-axis. For each $x_0 + iy$ on this segment they must similarly disagree at all points on some line-segment through $x_0 + iy$ parallel to the y-axis. The union of all these line segments is not of measure zero. Q.e.D.

Aside from its lack of elegance, due to (iii), (iv) and (v), Theorem 10 does not appear to be particularly strong. Nonetheless it is not contained in any of the others. On the credit side, however, the weakening of (i) and (ii) from 'everywhere' to 'almost everywhere' suggests the possibility of weakening Looman–Menchoff to, say,

If $f = u + iv$, defined on a domain D is such that

(i) $\partial u / \partial x$, $\partial u / \partial y$, $\partial v / \partial x$, $\partial v / \partial y$ exist a.e. in D,

(ii) u, v satisfy the Cauchy–Riemann equations a.e. in D,

and if further

(iii) f is continuous in D,

then f is analytic in D.

This conjecture is seriously false. For a counter-example, due to Urysohn, [**48**, 464], first construct a planar Cantor-set as follows. From the closed unit square delete $([0,1] \times (\frac{1}{3}, \frac{2}{3})) \cup ((\frac{1}{3}, \frac{2}{3}) \times [0,1])$ thus leaving a set K, consisting of four closed squares. Continue deleting in the usual way so that at the nth stage we are left with a set K_n consisting of 4^n closed squares whose centres we denote by $z_{n,k}(k = 1, 2, \cdots, 4^n)$. Then $K = \cap_n K_n$ is a (totally disconnected) set of planar measure zero. Next put

$$f(z) = \lim_{n \to \infty} 4^{-n} \sum_{k=1}^{4^n} (z - z_{n,k})^{-1};$$

then f is everywhere continuous, analytic off K (and thus satisfies, (i), (ii), (iii) above), yet fails to have an analytic continuation throughout K! The difficulty lies in weakening the condition that the partial derivatives exist everywhere. That some such weakening is possible is illustrated by the version of Looman–Menchoff appearing in Saks [**34**, 199]—a proof of which appears in the Appendix to this paper.

THEOREM 11. *If $f = u + iv$, defined on a domain D, is such that*

(i) *$\partial u / \partial x$, $\partial u / \partial y$, $\partial v / \partial x$, $\partial v / \partial y$ exist everywhere in D, except on a countable set,*

(ii) *u, v satisfy the Cauchy–Riemann equations a.e. in D,*

and if further

(iii) *f is continuous in D,*

then f is analytic in D.

One may weaken still further the conditions on the partial derivatives of f; results along these lines have been obtained by Cafiero [**6**] and Fesq [**11**], the latter deriving the following.

THEOREM 12. *If $f = u + iv$, defined on a domain D, is such that*

(i) *$\partial u / \partial x$, $\partial u / \partial y$, $\partial v / \partial x$, $\partial v / \partial y$ exist everywhere in D, except on a countable union of sets of finite one-dimensional Hausdorff measure,*

(ii) *u, v satisfy the Cauchy–Riemann equations a.e. in D,*

and if further

(iii) *f is locally bounded in D,*

(iv) *f is separately continuous in D,*

then f is analytic in D.

Recall ([**14**, 53] and [**12**]), that the one-dimensional Hausdorff measure of a set E in the plane is defined to be

$$\sup_{\varepsilon > 0} \inf \left\{ \sum_1^\infty \operatorname{diam}(E_n) \colon E = \bigcup_1^\infty E_n, \operatorname{diam}(E_n) < \varepsilon \right\}.$$

Any contour of finite length has finite one-dimensional Hausdorff measure, whereas a (non-trivial) rectangle has infinite one-dimensional Hausdorff measure.

Condition (i) dates back to Besicovitch **[3]** who proved that if a function defined on a simply-connected domain D is continuous everywhere and (complex) differentiable everywhere, except on a countable union of sets of finite linear measure, then in fact if is (complex) differentiable everywhere in D.

We conclude this section with two questions which, as far as we know, have not been answered.

QUESTION 1. *Suppose $f = u + iv$, defined on a domain D, is such that*
(i) $\partial u / \partial x$, $\partial u / \partial y$, $\partial v / \partial x$, $\partial v / \partial y$ *exist everywhere in D,*
(ii) *u, v satisfy the Cauchy–Riemann equations everywhere in D,*
and suppose further that
(iii) *f is locally integrable in D.*
Does it follow that f is analytic in D?

QUESTION 2. *Suppose $f = u + iv$, defined on a domain D, is such that u, v satisfy the Cauchy–Riemann equations everywhere in D. What is the structure of the set E of points of non-analyticity of f?*

Regarding Question 2, Trokhimchuk [**43**, 109] shows that E is (closed and) totally disconnected, so that f is necessarily analytic on a dense open subset of D. However, Pelling (private communication) has shown that E can have positive measure.

4. Odds and ends. The only 'odd' worth mentioning is the characterization (due to Heffter **[15]**) in terms of Cauchy–Riemann 'difference' equations, of those continuous functions that are analytic. His result is that a continuous function $u + iv$ on a domain D is analytic in D iff for each rectangle $R = [a, b] \times [c, d]$ in D there are points (x, y), $(x', y') \in R$ for which

$$\frac{u(b, y) - u(a, y)}{b - a} = \frac{v(x, d) - v(x, c)}{d - c},$$

$$\frac{u(x', d) - u(x', c)}{d - c} = -\frac{v(b, y') - v(a, y')}{b - a}.$$

For even odder odds see **[43]**.

As for 'ends', there are a number of papers dealing with the weakening of the conditions in Green's and Morera's theorems; see **[5]**, **[11]**, **[29]**, **[32]**, **[35]** and **[37]**. Further, the extension from rectangles to more general regions is dealt with in **[25]**, **[28]**, **[31]** and **[45]**. For a readable account of a fascinating and surprising extension of Morera's theorem consult Zalcman **[47]** and **[49]**. *Inter alia* he shows how relatively weak versions of Green lead to surprisingly strong versions of Morera. Whilst on the topic of Morera mention must be made of a remarkable example of Vitushkin [**12**, 95]. He constructs a compact, totally-disconnected planar set E of $(1 + \varepsilon)$-dimensional Hausdorff measure zero for each $\varepsilon > 0$, and a (non-constant) function f continuous on the extended complex plane, analytic on the complement of E (and thus without an analytic continuation to E), such that for any closed contour γ disjoint from E, $\int_\gamma f(z) dz = 0$!

Appendix. *A Proof of the Looman–Menchoff Theorem.* The proof rests heavily on two lemmas each of which is a variation on the 'fundamental theorem of calculus' theme. The first, due to Menchoff **[24]**, allows one to estimate certain naturally arising contour integrals. The proof of this lemma is entirely elementary and may be found in [**34**, 198]. The second lemma is well known in measure theory and may be found in [**33**, 166] or [**9**, 214].

LEMMA 1. *Let R be a closed rectangle and ϕ: $R \to \mathbf{R}$ a function both of whose partial derivatives $\partial \phi / \partial x$, $\partial \phi / \partial y$ exist everywhere in R. Let N be a constant and $E \subset R$ a closed non-empty set such that*

$$|\phi(\xi, y) - \phi(x, y)| \leqq N|\xi - x|,$$
$$|\phi(x, \eta) - \phi(x, y)| \leqq N|\eta - y|,$$

whenever $(x, y) \in E$, $(\xi, y) \in R$, $(x, \eta) \in R$. *Suppose* $[a, b] \times [\alpha, \beta]$ *is the smallest rectangle in* R *containing* E, *then*

$$\left| \int_a^b [\phi(x, \beta) - \phi(x, \alpha)]dx - \int\int_E \frac{\partial \phi}{\partial y} dxdy \right| \leqq 5N.m(R - E)$$
$$\left| \int_\alpha^\beta [\phi(b, y) - \phi(a, y)]dy - \int\int_E \frac{\partial \phi}{\partial x} dxdy \right| \leqq 5N.m(R - E),$$

where m *denotes (planar) Lebesgue measure.*

In order to state the second lemma we need the notion of the derivative of a set function. Suppose λ is a (complex) Borel measure on an open set K in $\boldsymbol{R}^2$, that $z \in K$ and c is a complex number. If for every sequence of Borel sets (B_n) that 'shrink nicely' to z (cf. [33, 163]),

$$\lambda(B_n)/m(B_n) \rightarrow c \quad \text{as } n \rightarrow \infty,$$

λ is said to be differentiable (with respect to m) at z. c is called the derivative of λ at z and is denoted by $(d\lambda/dm)(z)$.

LEMMA 2. *Let* λ *be a complex Borel measure on the open set* $K \subset \boldsymbol{R}^2$. *Then*

(i) $d\lambda/dm$ *exists a.e. in* K *and belongs to* $L^1(K)$.

If further λ *is absolutely continuous with respect to* m *then*

(ii) $\lambda(B) = \int_B \frac{d\lambda}{dm}(z)dm(z)$ *for every Borel set* B, *and*

(iii) $d\lambda/dm$ *coincides a.e. with the Radon–Nikodým derivative of* λ *with respect to* m.

We are now ready to present the

Proof. (Of Looman–Menchoff). By way of contradiction suppose that the (closed) set $E \subset D$ of points at which f fails to be analytic is non-empty. For each natural number n put

$$E_n = \{z \in D: |f(z+h) - f(z)|/|h| \leqq n \quad \text{and} \quad |f(z+ih) - f(z)|/|h| \leqq n,$$
$$\text{for real } h \text{ with } 0 < |h| \leqq 1/n\}.$$

As f is continuous, each E_n is closed and as both first-order partial derivatives of f exist throughout D, the (E_n) cover D. By the Baire category theorem [33, 103], applied to the complete metric space E, there are a natural number N and an open rectangle K whose closure lies in D, such that $\emptyset \neq E \cap K \subset E_N$.

For each (closed) rectangle R in K define the (complex-valued) set function λ by the contour integral

$$\lambda(R) = \int_{\partial R} f(z)dz.$$

If R, R' are rectangles in K put

$$\lambda(R \cup R') = \int_{\partial(R \cup R')} f(z)dz,$$

and if R is a rectangle in K put

$$\lambda(K - R) = \lambda(K) - \lambda(R).$$

λ so defined is an additive set function on the Boolean algebra $\mathcal{R}$ generated by the rectangles in K. By the Hahn extension theorem [9, 136, Corollary 9], λ has a (unique) extension to a measure (also denoted by λ) on the σ-algebra generated by $\mathcal{R}$. It is clear that this σ-algebra is precisely the collection of all Borel sets of K and hence λ is a (complex) Borel measure on K.

Our aim now is to show that λ is identically zero, as once this is established $\lambda(R)=0$ for each rectangle R in K, whence, by Morera's theorem f is analytic throughout K. As this conclusion contradicts the assumed non-emptiness of E (in K) the proof will then be complete. To show that λ is the zero measure it suffices to show that (a) λ is absolutely continuous with respect to m, and (b) the derivative $d\lambda/dm$ vanishes almost everywhere in K (cf. Lemma 2 (ii)).

The basic tool neeeded to prove both (a) and (b) is the estimate (*) below. To establish it let R be any rectangle in K meeting E and with side lengths $\leqq 1/N$. If J denotes the smallest rectangle containing $R \cap E$ it follows from the definition of E_N, and Lemma 1 when used in conjunction with the Cauchy–Riemann equations, that $|\lambda(J)| \leqq 20N.m(R-E)$. But by the Cauchy–Goursat theorem, λ vanishes on any rectangle not meeting E, so that this inequality may be written as

$$(*) \qquad |\lambda(R)| \leqq 20N.m(R-E).$$

Armed with (*) we can verify (a) above. Toward this end let F be a subset of K of (planar) Lebesgue measure zero. As such, given any $\varepsilon>0$ there is a sequence (R_n) of rectangles in K which cover F and which are such that $\Sigma_n m(R_n) < \varepsilon/20N$. By subdividing if necessary we may assume further that the side-lengths of all R_n are $\leqq 1/N$ and that all R_n meet E. Then it follows that $|\lambda(F)| \leqq \Sigma_n |\lambda(R_n)| \leqq 20N.\Sigma_n m(R_n - E) \leqq 20N.\Sigma_n m(R_n) < \varepsilon$ so that $\lambda(F)=0$ and (a) is established.

As for (b), if $z \in K-E$, by taking sufficiently small rectangles in the definition of the derivative of λ and invoking Cauchy–Goursat again, we see that $(d\lambda/dm)(z)=0$. As for points in E note first that for any $z \in K$, if $R \subset K$ is a rectangle containing z, meeting E, and of side length $\leqq 1/N$, by (*),

$$(**) \qquad \frac{|\lambda(R)|}{m(R)} \leqq 20N.\frac{m(R-E)}{m(R)} = 20N.\frac{m(CE \cap R)}{m(R)},$$

CE being the complement of E in K. Denote by m_{CE} the restriction of m to CE so that for any Borel set B,

$$m_{CE}(B) = m(CE \cap B) = \int_B \chi_{CE}(z)\,dxdy,$$

χ_{CE} being the characteristic function of CE. By Lemma 2 (iii) we have, for almost all $z \in K$

$$\frac{dm_{CE}}{dm}(z) = \chi_{CE}(z),$$

so that, for almost all $z \in E$, $(dm_{CE}/dm)(z)=0$. Thus if (R_n) is any sequence of rectangles as above that 'shrink nicely' to $z \in E$, for almost all such z

$$0 = \frac{dm_{CE}}{dm}(z) = \lim_{n\to\infty} \frac{m_{CE}(R_n)}{m(R_n)} = \lim_{n\to\infty} \frac{m(CE \cap R_n)}{m(R_n)},$$

so that by (**) $\lambda(R_n)/m(R_n) \to 0$. Hence, $d\lambda/dm(z)=0$ for a.a. $z \in E$, as was required.

As Lemma 1 holds if the partial derivatives are assumed to exist only on the complement of a countable set [34, 198], the above proof actually proves the slightly stronger Theorem 11.

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PROGRESS REPORTS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

SCHAUDER BASES

P. R. HALMOS

Euclidean spaces have three basic properties: they are (1) vector spaces, with (2) a concept of length, and (3) a concept of angle. Banach spaces, whose intensive study began in the 1930's, constitute a generalization that retains requirements (1) and (2) but does not insist on (3). (Precisely: a Banach space is a real or complex vector space endowed with a norm, with respect to which, as a metric space, it is complete.)

Generalizations can turn out to be too general (as wise hindsight sometimes shows); they can lead

to complicated inbred questions (do properties A, C, E, and G imply properties B, D, F, or H?), and to pathological counterexamples, that interest only the most dedicated taxonomist. Banach spaces are sometimes accused of such excessive generality. It is, however, undeniable that Banach space theory has illuminated the study of integration (via L^p spaces) and continuity (via spaces of continuous functions). Another sign that Banach spaces may not be too wildly general is that even the finite-dimensional ones are of interest, and, in fact, some experts feel that if we knew all about all finite-dimensional Banach spaces, we would know all about all of them. Many of the recent spectacular theorems in the field are based on deep finite-dimensional lemmas.

One of the earliest questions about Banach spaces was the basis problem, raised by Banach himself in his book (1932). A sequence $\{x_1, x_2, x_3, \dots\}$ of elements in a Banach space X is a (Schauder) *basis* for X in case every x in X has a unique(!) expression of the form $\sum_{n=1}^{\infty} \alpha_n x_n$. (That is, to each x there corresponds a unique sequence $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$ of scalars such that $\lim_n \|x - \sum_{j=1}^{n} \alpha_j x_j\| = 0$.) It is easy to see that if a Banach space has a basis, then it is separable. The basis problem, which was outstanding for 40 years, was the converse: does every separable Banach space have a basis? Each space that ever came up in analysis had one, and yet a proof that that had to be so remained elusive.

A classically important concept in the study of Banach spaces is that of a compact (completely continuous) operator, i.e., a linear transformation $T: Y \to X$ between Banach spaces, with the property that the closure of the image (under T) of the unit ball (in Y) is compact (in X). The easy compact operators are the ones of finite rank (i.e., the ones for which the range of T is finite-dimensional); the next easiest ones are the (uniform) limits of operators of finite rank. If X is a "good" Banach space, then every compact operator into X is such a limit (in technical language: "X has the approximation property"), and, in particular, if a Banach space has a basis, then it has the approximation property.

The basis problem was solved by Enflo (1973). The solution is negative: there exists a separable Banach space that does not have the approximation property. The technique is constructive; it is a combinatorial way of constructing and putting together infinitely many finite-dimensional Banach spaces.

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MATHEMATICAL NOTES

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TWO CONSTRUCTIONS OF LEBESGUE'S MEASURE

JAN MYCIELSKI

Introduction. My purpose is to point out that there exist two essentially different constructions of Lebesgue's measure. One of them generalizes into a vast family of so called Hausdorff measures (see [**4**, **6**]), the other into another family of so called Haar measures (the latter are usually done only on locally compact groups, but I shall write here about both constructions only for metric spaces; see [**3**, **4**]). The construction of Lebesgue's measure on the n-dimensional Euclidean space $\boldsymbol{R}^n$ depends on a sequence of elementary lemmas. Two of these lemmas are especially interesting since neither implies

the other, and we have a choice of developing the theory via any one of them. In the more general context of metric spaces those lemmas become postulates and are pointed out as (P) and (Q).

The Lebesgue measure in $\boldsymbol{R}^n$ can be defined as a countably additive function μ over the σ-algebra of Borel sets in $\boldsymbol{R}^n$ with nonnegative values (called a Borel measure) which satisfies the two additional requirements:

(1) $$\mu(C)=1, \text{ where } C \text{ is the unit } n\text{-cube in } \boldsymbol{R}^n.$$

(2) $$\mu(A)=\mu(B) \text{ if } A \text{ is isometric to } B.$$

We shall allow ourselves a natural generalization replacing $\boldsymbol{R}^n$ by any metric space M and C by any non-empty compact set in M.

Both constructions of μ utilize the following function defined for all compact sets $S \subseteq M$ and all $t>0$:

$$E(S,t)=\min\{\operatorname{card}(\mathscr{K}): \mathscr{K} \text{ is a covering of } S \text{ with sets of diameters } < t\}.$$

The Hausdorff way. For all Borel sets $A \subseteq M$ we put

$$m(A)=\lim_{\delta \downarrow 0}\left[\inf\left\{\sum_{i\in I}\frac{1}{E(C,t_i)}: t_i<\delta \text{ for all } i\in I, \text{ there exists a covering of } A \text{ with sets } A_i \text{ of diameters } < t_i \text{ respectively, and } |I|\leqq \aleph_0\right\}\right].$$

(We understand that $m(A)=\infty$ if for some δ the covering in question does not exist.) From this definition it follows that m has all the properties required of μ with the possible exception of (1). The proof of these facts is standard (for more details see **[4]**); the main point is countable additivity of m, which is derived by the method of Carathéodory from the property

(*) $$m(A\cup B)=m(A)+m(B),$$

whenever $\inf\{\text{distance.}(a,b): a\in A, b\in B\}>0$. Of course (*) follows directly from our definition of m.

Concerning (1), notice that $m(C)\leqq 1$ still easily follows from our definition of m. But the inequality

(P) $$m(C)>0$$

sometimes fails, e.g., if M is a countably infinite compact space and $C=M$. Thus (P) is an additional postulate necessary for the success of our construction. Having (P), we secure (1), putting $\mu(A)=m(A)/m(C)$. In case C is isometric to an n-dimensional parallelopiped, it is easy to prove (P) directly, and this is the special lemma of the "Hausdorff way".

The Haar way. We take a nonprincipal ultrafilter $\mathscr{F}$ of subsets of the set of natural numbers. Then for any compact $S\subseteq M$ we put

$$\lambda(S)=\lim_{n\to\mathscr{F}}\frac{E(S,1/n)}{E(C,1/n)}.$$

Here $\lim_{n\to\mathscr{F}}a_n$ is defined as λ where $\{n: |a_n-\lambda|<\varepsilon\}\in\mathscr{F}$ for all $\varepsilon>0$ if such a λ (necessarily unique) exists; otherwise $\lambda=\infty$. Then for every open set $V\subseteq M$ we put

$$m(V)=\sup\{\lambda(S): S\subseteq V,\ S \text{ compact}\},$$

and for any Borel set $A\subseteq M$ we put

$$m(A)=\inf\{m(V): A\subseteq V,\ V \text{ open}\}.$$

Now again we can check directly (*) and it follows by the method of Carathéodory that m is a Borel measure (for details see **[3]**).

Concerning (1) we still get $m(C) \geqq 1$ from our definition, but the inequality

$$m(C) < \infty \tag{Q}$$

sometimes fails, e.g., if $M = \boldsymbol{R}$ and $C = \{0\}$. Thus (Q) is an additional postulate necessary for the success of our construction. Having (Q) we secure (1) putting $\mu(A) = m(A)/m(C)$. To construct the Lebesgue measure, we must check (Q) for the n-dimensional cube in $\boldsymbol{R}^n$, which is easy (more generally, (Q) is also easy for a compact set C with non-empty interior in a topological group with a left (right) invariant metrization since a certain neighborhood of C can be covered by finitely many left (right) translates of C). This is the special lemma of the "Haar way". (Q) also holds whenever M is compact and $C = M$.

Concerning (2) it may fail too, e.g., if M is a countably infinite compact space and $C = M$. But the following weaker version of (2) follows, of course, from our definition of m:

(R) $m(A) = m(B)$ whenever A is isometric to B by an isometry which can be extended to some open sets including A and B respectively.

In A is isometric to B by an isometry which can be extended to some open sets including A and B respectively.

In particular (2) holds if A and B are both open in M. But for $\boldsymbol{R}^n$ the property (R) implies (2) since every isometry of two sets in $\boldsymbol{R}^n$ can be extended to an isometry of $\boldsymbol{R}^n$ onto itself. Also, in the case of a group, if A is isometric to B by a left (right) translation, then this isometry extends and (R) implies the left (right) invariance of the measure.

REMARKS. 1. Notice that (P) is a property of C itself, while (Q) may depend on the position of C in M. (P) is often difficult to check and, in fact, for many familiar compact spaces, we do not know if (P) is true (see [**3**, **4**]).

2. In the case when only a Haar measure μ is available, the following property of μ, which follows from our construction, may be useful (see [**4**]):

If S is a compact set in M, such that for every positive integer n there exists an open set V in M, such that $S \subseteq V$, and there are n disjoint sets isometric to V in C, then $\mu(S) = 0$.

3. The classical construction of the Haar measure in locally compact groups without left or right invariant metrizations is a natural analog of the above one via a suitable generalization of our definition of $\lambda(S)$ (see [**1**, **5**]). But I do not know any construction for separable metric spaces, analogous to the Cartan construction (see [**5**]), which would be free from the use of the axiom of choice for uncountable families of sets (the existence of $\mathscr{F}$).

4. The Lebesgue integral can be constructed directly, without prior development of the measure (see [**2**, **7**]). The content of this paper applies *mutatis mutandis* to such a construction.

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EXAMPLES OF FUNCTOR ADJUNCTIONS IN ELEMENTARY ANALYSIS

G. CICOGNA

Summary. Some basic concepts of real analysis, such as the "upper and lower limit", the "order of zero", the "upper and lower integral", are investigated by means of the language of categories and functors. It is shown that this approach provides a completely natural way for introducing these concepts and a convenient language for stating their properties. In this scheme, the most useful tools appear to be the notion of functor adjunctions and the technique of Kan extensions.

Prof. S. Mac Lane, in his book *Categories for the Working Mathematician*, has invited each reader "to find his own examples" of functor adjunctions, remembering also that "adjoint functors arise everywhere" **[1]**.

The principal and most attractive feature of the examples I am presenting here is that they are all concerned with elementary ideas and basic concepts of real analysis. In fact [**2**, **3**], the language of categories can be conveniently introduced even in the most elementary topics: the examples below will clearly confirm, in particular, that (i) this language provides a completely natural way for introducing some of the typical concepts of analysis, and (ii) the related results are often automatically self-contained in this approach and, in general, can be stated in a compact and expressive form which allows for a new insight—in a more general perspective—into these concepts.

For the convenience of the reader, I shall briefly recall here (in the form most suitable for our purposes; for details, see **[1]**) the two concepts of category theory which are being used in the following, namely the definition of functor adjunction and the construction of Kan extensions.

Given two categories A and B, the functor $F\colon B \to A$ is said to be a *left adjoint* for the functor $G\colon A \to B$ if for each object b in B, there is an arrow $\eta_b\colon b \to G \circ F(b)$ which is *universal* from b to G. This last statement means that for each arrow $\phi\colon b \to G(a)$, where a is an object of A, there exists one and only one arrow $\psi\colon F(b) \to a$, and ϕ factorizes $\phi = G(\psi) \circ \eta_b$.

Before giving the definition of Kan extension, we need the notion of ("projective") *limit* of a functor $F\colon J \to A$. It consists of an object r of A, together with a set ("cone") of arrows $\nu_j\colon r \to F(j)$, such that, for each object a which is vertex of another cone $\tau_j\colon a \to F(j)$, there is exactly one arrow $\phi\colon a \to r$ and $\tau_j = \nu_j \circ \phi$. Finally, instead of recalling its formal definition, we prefer to give a brief outline of the procedure used for constructing the Kan extensions. Let $F\colon A \to B$ and $G\colon A \to C$ be two given functors; for any fixed object c in C, consider the category of the arrows $\phi\colon c \to G(a)$, where a varies in (the class of objects of) A, and the functor sending each ϕ into $F(a)$. Call $R(c)$ the limit of this functor, if it exists (this is always true when B is complete). One then shows that the map $R\colon c \to R(c)$ of C in B defined in this way is really a functor, called the *right Kan extension* of F along G, and satisfying some peculiar "universal" properties.

Dualizing the above definitions, i.e., essentially reversing the arrows, one easily defines right adjoints, colimits, left Kan extensions.

A common feature of the examples we are considering below is that all the involved categories are preorders. A preorder in fact is simply a category in which there is at most one arrow between objects. We shall denote $\gtrsim$ the order relation, except for the standard relation $\geqq$ used between real quantities; we will also write $a \in P$ for "a is an object for the preorder (= category) P," and define the arrows $a_1 \to a_2$ in P with the meaning $a_1 \gtrsim a_2$. Note, of course, that the notion of limit and colimit of a functor whose codomain is a preorder ultimately coincides, in the above conventions, with that of least upper bound and of greatest lower bound, respectively.

In the following, R and N will denote respectively the set of real numbers (extended with $\pm\infty$) and of natural numbers, regarded as categories with the standard order relation $\geqq$.

Example 1 (limits of numerical sequences). Let S be the set of all real sequences $(a_n) \in S$; we shall define $(a_n) \gtrsim (b_n)$ when the following condition is satisfied: for each $c, c' \in R$ such that, for all

$\varepsilon > 0$, relations $b_n \leqq c - \varepsilon$ and $a_n \geqq c' + \varepsilon$ hold at most for a finite number of indices, then the same must be true for relations $a_n \leqq c - \varepsilon$ and $b_n \geqq c' + \varepsilon$.

Let K be the (trivial!) functor $K\colon R \to S$ defined by $K(a) = (a)$, where (a) stands for the constant sequence $a_n = a$ for all n; then the left adjoint and the right adjoint of K are defined on the whole S and are the functors l and $L\colon S \to R$ giving respectively the lower and the upper limit of sequences. To see this, observe e.g. that the universality of the arrow $(a_n) \gtrsim (\lambda)$, where $\lambda = l((a_n))$ = lower limit of (a_n), expresses the well-known property that if $(a_n) \gtrsim (\lambda')$, $\lambda' \in R$, then also $\lambda \geqq \lambda'$.

Example 2 (infinitesimal real functions). Let $\mathscr{I}$ be the set of all real continuous positive functions $f = f(x)$, defined on an interval $0 \leqq x \leqq b$ of the real axis, vanishing for x tending to zero. Let us introduce in $\mathscr{I}$ the relation $f_1 \gtrsim f_2$ if $\lim_{x\to 0}(f_1(x)/f_2(x))$ exists and is a finite number. Let R^+ be the subcategory of real positive numbers (included 0 and $+\infty$) and let us include in $\mathscr{I}$ also the constant functions 0 and 1. Define now the functor $E\colon R^+ \to \mathscr{I}$ by $E(a) = x^a$, with the obvious definition $E(0) = 1$ and $E(\infty) = 0$. The left and right Kan extensions of the identity $I\colon R^+ \to R^+$ along E define for all functions $f \in \mathscr{I}$ a sort of "upper" and "lower order of zero," as clearly shown by their construction. These functors, however, are *not* adjoint functors of E, due to the nonexistence of the characteristic universal arrow for some of the elements f.

Instead, let us consider the subcategory $\mathscr{I}_0$ of $\mathscr{I}$ whose objects are the functions f such that the above introduced upper and lower order coincide (call $O(f)$ this number) and, in addition, both relations $f \gtrsim x^{O(f)}$ and $x^{O(f)} \gtrsim f$ hold: then the functor $O\colon \mathscr{I}_0 \to R^+$ is simultaneously left and right adjoint of $E\colon R^+ \to \mathscr{I}_0$, as easily seen in a similar way as in Example 1, and so $\mathscr{I}_0$ is *equivalent* to R^+.

Finally, if one adopts in $\mathscr{I}_0$ a new definition of arrows (preorder relations) writing $f_1 \to f_2$ if $\lim_{x\to 0}(f_1(x)/f_2(x)) = l$ with $0 \leqq l \leqq 1$, one easily verifies the equivalence of this subcategory with $(R^+)_{\text{op}} \times R^+$, by means of the map sending f to the pair $(l(f), O(f))$ where $l(f) = \lim_{x\to 0}(f(x)/x^{O(f)})$. (In this case, exclude from $\mathscr{I}_0$ the functions 0 and 1, and from R^+ the elements 0 and $+\infty$. The subscript "op" stands for "opposite" category, i.e., with all the arrows reversed.) To see this result more clearly, one can observe that it ultimately expresses the standard computational rule of substituting each f in $\mathscr{I}_0$ with $l(f)x^{O(f)}$ in the evaluation of limits such as $\lim_{x\to 0}(f_1(x)/f_2(x))$.

Example 3 (limits of real functions and semicontinuous functions). Let E be a topological space, and $F = R^E$ the class of real functions f defined on E. For any fixed point x in E, let us define on F the following relation: we shall say $f_1 \gtrsim f_2$ in x if there is a neighbourhood of x where $f_1 \geqq f_2$. Let F_0 denote the subset of constant functions and $l_0\colon F_0 \to R$ the obvious isomorphism of F_0 with R; for any fixed point x, the left and the right Kan extensions of l_0 along the inclusion $K\colon F_0 \to F$ give respectively the upper limit and the lower limit of functions $f \in F$ at the point x. Alternatively, this scheme of functors can be viewed as an adjunction, exactly as in Example 1, but the above presentation may be preferable because of its constructive character, which clearly exhibits the peculiar property of, say, the lower limit of being the "greatest" of the real numbers λ' such that $f \gtrsim \lambda'$.

Another interesting feature can be viewed varying now the point x in E, but taking $f \in F$ fixed: thus the above left and right extensions describe respectively an upper semicontinuous function f^* and a lower semicontinuous function f_*, which are called the upper and lower envelopes of f **[4]**. The map sending each f to f^* is an example of the abstract situation called a *comonad* in category theory, this map being the right adjoint of the inclusion $K\colon F^* \to F$, where F^* is the coreflective subcategory of all upper semicontinuous functions defined on E. This inclusion does not possess a left adjoint: this fact depends essentially on the non-completeness of F^*. Similar (dual) results hold for the monad defined by the map sending each f to f_*.

Example 4 (integration of real functions). Let $\mathscr{R}$ be a Riesz space of real functions defined on an arbitrary set E and $\mu\colon \mathscr{R} \to R$ a Daniell measure on $\mathscr{R}$ **[4, 5]**. Let us consider the class $F = R^E$ of all real functions f defined on E as a partial order with the obvious definition $f_1 \geqq f_2$ if $f_1(x) \geqq f_2(x)$ for all

$x \in E$. Then μ is a functor $\mu : \mathcal{R} \to R$. Let us consider now the functor categories $\mathcal{R}^N$, R^N, F^N, as well as the functors $\mu^N: \mathcal{R}^N \to R^N$ and $K^N: \mathcal{R}^N \to F^N$ defined as follows

$$\mu^N(s) = \mu \circ s; \qquad K^N(s) = K \circ s,$$

where s is a functor in $\mathcal{R}^N$ (i.e., an increasing sequence of functions belonging to $\mathcal{R}$) and $K: \mathcal{R} \to F$ the inclusion of $\mathcal{R}$ in F. Finally, let us denote $L_F: F^N \to F$ and $L_R: R^N \to R$ the functors "limit" in F and respectively in R. Then one has: the left Kan extension of the functor $L_R \circ \mu^N: \mathcal{R}^N \to R$ along the functor $L_F \circ K^N: \mathcal{R}^N \to F$ defines the upper integral $\mu^*(f) = \int^* f d\mu$ of functions $f \in F$. In fact, for any fixed f, by the very construction of Kan extension, $\mu^*(f)$ is obtained as the greatest lower bound of the numbers of the type $\sup_n \mu(\phi_n)$ where (ϕ_n) is an increasing sequence of functions belonging to $\mathcal{R}$ and such that $\sup_n \phi_n(x) \geqq f(x)$ for all x, in agreement with the Daniell definition.

"Dualizing" this construction, i.e., taking the right Kan extension of $\tilde{L}_R \circ \mu^{N_{\text{op}}}: \mathcal{R}^{N_{\text{op}}} \to R$ along $\tilde{L}_F \circ K^{N_{\text{op}}}: \mathcal{R}^{N_{\text{op}}} \to F$ (where $\tilde{L}_R$, $\tilde{L}_F$ denote the functors "colimit" in R and F), we define the lower integral $\mu_*(f)$. The left extension of this second construction, and the right extension of the first one, define instead a different upper and lower integral $\mu^0(f)$ and $\mu_0(f)$ respectively; it can be shown that the natural transformations $\mu^0 \dot{\to} \mu^* \dot{\to} \mu_* \dot{\to} \mu_0$ hold; this means that for each f one has $\mu^0(f) \geqq \mu^*(f)$, etc. To understand more clearly this point, suppose that E is a locally compact topological space with a countable base and μ is a Radon measure on E, then, in the subset B of bounded functions f with compact support, one has that $\mu^0(f)$ is equal to $\mu^*(f^*) = \mu_*(f^*)$ and $\mu_0(f)$ to $\mu^*(f_*) = \mu_*(f_*)$, where f^* and f_* are the semicontinuous envelopes of f defined as in Ex. 3.

In the subset B just considered, one can, similarly, introduce very easily the usual concepts of upper and lower Riemann integral: they are simply the left and the right Kan extension of the obvious integral (i.e., the Euclidean measure) of "step functions" along the inclusion of these functions in B. Instead, finally, if one takes as Riesz space $\mathcal{R}$ the set of Riemann-integrable functions and as measure μ the Riemann integral, the above integrals μ^* and μ_* define the usual Lebesgue integrals.

Further details, especially about Examples 2 and 4, will be considered in subsequent papers **[3]** and **[6]**.

Acknowledgment. It is a pleasure to thank Prof. G. Letta for useful discussions and for his interest in the present approach. I want also to recall the late Prof. S. Ciampa, who initiated me in functor language.

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MISCELLANEA

7. The clouded crystal ball. "In the old days when people invented a new function they had some useful purpose in mind; now they invent them deliberately just to invalidate our ancestors' reasoning, and that is all they are ever going to get out of them."

H. Poincaré, *Science et Méthode*, 1909.

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

ADDENDA TO "MONTHLY RESEARCH PROBLEMS 1969–77"

RICHARD K. GUY

Printing and postal problems squeezed out a few items from the article in last December's issue.

There are further papers related to Ringel's tree-labelling and complete-graph-decomposition problems (Duke [1969, 1128]) by Bodendiek *et al.* (1977), Huang and Rosa (tbp) and Kotzig (1977).

The paper of Bloom and Golomb (1977) on the related problem of Golomb [1974, 499] was incompletely referenced and incorrectly titled.

Another paper on Lehmer's problem (Alter [1973, 192; 1975, 998]) is by Kishore (1977).

Joel Brenner writes that in the paper with Riddell [1977, 39] they should have mentioned the work of Fox (1952) and the Fox-Fenchel theorem, which is relevant, though different. What Fox proves is "every n-cycle" instead of "every permutation in S_n" and if n is even, but k, l both odd, this is false; you have to say "every product of two n-cycles." The Brenner-Riddell conjecture cannot by any means be deduced from the Fox-Fenchel theorem.

Jan Mycielski writes that the concept of decision procedure or algorithm applies only to properties of objects which can be represented by finite sequences of letters of a finite alphabet. Thus the problem on "Mortality of 2×2 matrices" [1977, 463] does not seem to make sense unless you replace the field of complex numbers by the field of algebraic numbers, say, or by $\mathbf{Q}(\sqrt{-1})$, for the elements of which such representations exist.

Mycielski has also found a negative solution for the case $n=2$ of problem (a) of [1977, 723]; his article is updated below.

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EQUATIONS UNSOLVABLE IN $GL_2(\mathbf{C})$ AND RELATED PROBLEMS

JAN MYCIELSKI

The case $n=2$ of problem (a) in [3] has a negative solution. In fact we can prove a little more. Let $GL_2(\mathbf{C})$ be the group of all nonsingular 2×2 matrices with complex entries. Let $[x,y]=x^{-1}y^{-1}xy$.

Examples. None of the equations

$$(1)\qquad [x^2, yxy^{-1}] = \begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix} \quad\text{and}\quad (2)\quad [x,y]^2 = \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$$

has solutions $x, y \in GL_2(\mathbf{C})$ unless $\alpha = 1$.

Proof. (1). Since $\det[u, v] = 1$ for all $u, v \in GL_2(\mathbf{C})$, equation (1) implies that $\alpha = \pm 1$. It remains to show that -1 is impossible. (1) with $\alpha = -1$ is equivalent to

$$yx^{-1}y^{-1}x^2yxy^{-1} = -x^2.$$

Since $\operatorname{trace}(uvu^{-1}) = \operatorname{trace}(v)$ and $\operatorname{trace}(-v) = -\operatorname{trace}(v)$ hence $\operatorname{trace}(x^2) = 0$. (1) with $\alpha = -1$ is also equivalent to

$$y^{-1}x^2yxy^{-1}x^{-2}y = -x$$

and hence $\operatorname{trace}(x) = 0$. But, in $GL_2(\mathbf{C})$,

$$\operatorname{trace}(x) = 0 \Rightarrow \operatorname{trace}(x^2) = -2\det(x).$$

Thus $\operatorname{trace}(x) = \operatorname{trace}(x^2) = 0$ is impossible.

(2). As in (1) we show that $\alpha = \pm 1$. Now, the only solutions of the equation

$$z^2 = \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$$

in $GL_2(\mathbf{C})$ are

$$\begin{pmatrix}i & -i/2\\ 0 & i\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}-i & i/2\\ 0 & -i\end{pmatrix},$$

where $i = \sqrt{-1}$. Both have determinant -1, and hence $z = [x, y]$ is impossible and, for $\alpha = -1$, (2) is impossible.

For $\alpha = 1$ equation (2) has solutions, e.g.,

$$x = \begin{pmatrix}2 & 0\\ 0 & 1/2\end{pmatrix}, \qquad y = \begin{pmatrix}3/2 & 1\\ 0 & 2/3\end{pmatrix}.$$

REMARKS. 1. By the Jordan normal form theorem every matrix in $GL_2(\mathbf{C})$ is conjugated to a matrix of one of the forms

$$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \quad \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}.$$

And so it is still an open question if there are equations of the form

$$x^{p_1}y^{q_1}\cdots x^{p_m}y^{q_m} = \begin{pmatrix}\alpha & 0\\ 0 & 1/\alpha\end{pmatrix}$$

with $0 \neq \alpha \neq \pm 1$, or of the form

$$x^{p_1}y^{q_1}\cdots x^{p_m}y^{q_m} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

where $m \geqq 1$ and p_i and q_i are integers all different from 0 except perhaps q_m, without any solutions $x, y \in GL_2(\mathbf{C})$.

2. It is easy to check that, for all $z \in SL_2(\mathbf{C})$ with $\det(z) = 1$, we have

$$\operatorname{trace}(z^2) = (\operatorname{trace}(z))^2 - 2.$$

Hence, for all $x, y \in GL_2(\mathbf{R})$, $\operatorname{trace}([x, y]^2) \geqq -2$. Is it true that for any integers $p_1, \ldots, p_m, q_1, \ldots, q_m$ as above the set

$$\operatorname{trace}(x^{p_1}y^{q_1} \cdots x^{p_m}y^{q_m}) : x, y \in GL_2(\mathbf{R})\}$$

includes the interval $[-2, \infty)$?

3. For extensive studies of $GL_2(\mathbf{C})$ and related groups see [1] and [2].

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CLASSROOM NOTES

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PICARD'S THEOREM WITHOUT TEARS

LAWRENCE ZALCMAN

1. Recently I noticed an approach to a restricted version of the Little Picard Theorem so short and simple that it can be presented to a class of undergraduates in a single lecture. This approach leads naturally to important generalizations of Liouville's Theorem (in which a one-sided bound on the real or imaginary part of a function replaces a bound on the modulus of the function); moreover, it exhibits Picard's theorem clearly as an analogue and extension of the Fundamental Theorem of Algebra. In fact, the proof is by an induction, the initial step of which is just the Fundamental Theorem of Algebra.

While it seems to me unlikely that this approach is really new, I have been unable to find it in the literature; and discussion with knowledgeable colleagues has failed to turn up a reference.

2. To begin with, we need some notation. Let $F(z) = U(z) + iV(z)$ be an entire function; it is understood that U and V are real functions. We set

$$M(r, F) = \max_{|z|=r} |F(z)| \qquad A(r, F) = \max_{|z|=r} U(z);$$

when no confusion can arise, we simply write $M(r)$, $A(r)$. The n-times iterated logarithm, $\log_n$, is defined by $\log_1 t = \log t$, $\log_n t = \log(\log_{n-1} t)$, where all logarithms are to the base e. Similarly, we denote by $\exp_n$ the n-times iterated exponential.

Our interest centers on the class of entire functions $F(z)$ for which

$$\limsup_{r\to\infty} \frac{\log_n M(r)}{\log r} < \infty \tag{1}$$

for some positive integer n. Among the functions satisfying (1) are the exponentials $\exp_{n-1} z^m$, m a

positive integer. On the other hand, not every entire function satisfies (1) for some n. Indeed, if the numbers $a_k > 0$ are chosen appropriately small, the series $\Sigma a_k \exp_k z$ converges uniformly on each disc about the origin and hence defines an entire function $F(z)$. Since $M(r) > a_n \exp_n r$ for each n, it is clear that F cannot satisfy (1).

While functions which satisfy (1) do not exhaust all entire functions, I know of no function or class of entire functions of any interest in complex analysis that does not already satisfy (1) for a small value of n. The case $n = 2$ is, of course, the much studied class of functions of *finite order.*

We shall prove the following result, apparently midway between the Big and Little Theorems of Picard. (Actually, it is an immediate consequence, hardly ever drawn, of the Little Theorem.)

THEOREM. *Let $F(z)$ be a nonconstant entire function satisfying* (1) *for some n. If F fails to take on some value $a \in \mathbf{C}$, it takes on every other value $b \in \mathbf{C}$ $(b \neq a)$ infinitely often.*

Thus, either F takes on *every* value in $\mathbf{C}$ or it omits a single value and takes on every other value infinitely often.

3. For the proof, we need the following result, which bounds $M(r)$ in terms of $A(R)$, $R > r$.

BOREL–CARATHÉODORY INEQUALITY. *Let $0 \leqq r < R$. Then*

$$M(r) \leqq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |F(0)|. \tag{2}$$

Proof. If $F(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = \alpha_n + i\beta_n$ (α_n, β_n real), we have

$$\begin{aligned} U(Re^{i\theta}) &= \operatorname{Re} \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) R^n (\cos n\theta + i \sin n\theta) \\ &= \sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) R^n, \end{aligned}$$

where the series converges uniformly in θ. For $n \geqq 1$ we have

$$\begin{aligned} \pi\alpha_n R^n &= \int_0^{2\pi} U(Re^{i\theta}) \cos n\theta \, d\theta \\ \pi\beta_n R^n &= -\int_0^{2\pi} U(Re^{i\theta}) \sin n\theta \, d\theta, \end{aligned}$$

so that

$$\pi a_n R^n = \int_0^{2\pi} U(Re^{i\theta}) e^{-in\theta} d\theta = \int_0^{2\pi} [U(Re^{i\theta}) - A(R)] e^{-in\theta} d\theta.$$

Thus

$$\pi |a_n| R^n \leqq \int_0^{2\pi} |U(Re^{i\theta}) - A(R)| \, d\theta = \int_0^{2\pi} [A(R) - U(Re^{i\theta})] \, d\theta = 2\pi[A(R) - \alpha_0],$$

so that

$$|a_n| R^n \leqq 2[A(R) + |F(0)|] \tag{3}$$

and $|a_n| r^n \leqq 2[A(R) + |F(0)|](r/R)^n$ for $n \geqq 1$. It follows that

$$\begin{aligned} |F(re^{i\theta}) - F(0)| &\leqq \sum_{n=1}^{\infty} |a_n| r^n \\ &\leqq 2[A(R) + |F(0)|] \sum_{n=1}^{\infty} (r/R)^n \end{aligned}$$

$$= \frac{2r}{R-r} A(R) + \frac{2r}{R-r} |F(0)|;$$

hence

$$|F(re^{i\theta})| \leqq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |F(0)|,$$

as required.

An immediate corollary is a version of Liouville's theorem which, though well known to experts, too often escapes mention in a first course in function theory.

LIOUVILLE'S THEOREM. *Let $F(z) = U(z) + iV(z)$ be entire and suppose that there exist positive constants C, K, and α such that $U(z) \leqq C|z|^\alpha$ whenever $|z| \geqq K$. Then $F(z)$ is a polynomial of degree no greater than α.*

Proof. The hypothesis implies that for each integer $n > \alpha$

$$\limsup_{R\to\infty} A(R)/R^n \leqq 0,$$

so by (3) $a_n = 0$ for $n > \alpha$.

Another consequence is the

FUNDAMENTAL THEOREM OF ALGEBRA. *Every nonconstant polynomial has a root in* **C**.

Proof. Suppose the polynomial $p(z)$ never vanishes. Then $p(z) = e^{F(z)}$ for some entire function F and $|p(z)| = \exp\{\operatorname{Re} F(z)\}$. Thus $M(r, p) = e^{A(r,F)}$, so that

$$\limsup_{r\to\infty} \frac{A(r,F)}{\log r} = \limsup_{r\to\infty} \frac{\log M(r,p)}{\log r} < \infty.$$

It follows that for large r we have

$$A(r,F) \leqq C(\log r) \leqq r^{1/2},$$

so by Liouville's Theorem F is constant. Hence $p = e^F$ is constant.

The preceding proof fits nicely into the present circle of ideas. A much simpler argument, which uses only a special case of the Cauchy integral formula, can also be given: if $p(z)$ never vanishes, $q(z) = 1/p(z)$ is entire and $q(0) = 1/p(0) \neq 0$. But

$$q(0) = \frac{1}{2\pi i} \int_{|z|=R} q(z)\,dz/z = \frac{1}{2\pi} \int_0^{2\pi} q(Re^{i\theta})\,d\theta$$

so that $|q(0)| \leqq M(R, q)$. It is easy to see that if p is nonconstant this last quantity tends to 0 as R tends to infinity; choosing R large enough gives the required contradiction.

We shall use the inequality of Borel and Carathéodory in the following form:

LEMMA. *Let F be a nonconstant entire function. Then $M(r) \leqq 3A(2r)$ for $r > r_0(F)$.*

Proof. Take $R = 2r$ in (2) to get $M(r) \leqq 2A(2r) + 3|F(0)|$. Since F is nonconstant, Liouville's theorem shows that $A(2r) \to \infty$. Done.

4. We now prove the theorem. For $n = 1$, condition (1) implies that $|F(z)| \leqq |z|^m$ for some integer $m > 0$ and all $|z| \geqq 2$. Thus F is a polynomial, so, by the Fundamental Theorem of Algebra (applied to $F(z) - a$), F takes on each value $a \in \mathbf{C}$.

Suppose the result has been proved for $n = k$ and let F satisfy (1) with $n = k+1$. If F fails to take on the value a, we have $F(z) - a = e^{G(z)}$, where $G(z)$ is again entire. Clearly $F(z) - a$ still satisfies (1)

with $n = k+1$, so the relation $M(r, e^G) = e^{A(r,G)}$ yields

$$\limsup_{r\to\infty} \frac{\log_k A(r, G)}{\log r} < \infty.$$

This inequality remains valid if $A(r, G)$ is replaced by $3A(2r, G)$. Thus, applying the lemma to G, we obtain

$$\limsup_{r\to\infty} \frac{\log_k M(r, G)}{\log r} \leqq \limsup_{r\to\infty} \frac{\log_k 3A(2r, G)}{\log r} < \infty.$$

By the induction hypothesis, G takes on every value in $\mathbf{C}$ with at most one exception. In particular, for each fixed value $w \in \mathbf{C}$, G takes on all values $w + 2\pi i n$, $n = 0, \pm 1, \pm 2, \ldots$, with at most one exception. It is now obvious that $F(z) = e^{G(z)} + a$ takes on each value in $\mathbf{C}\backslash\{a\}$ infinitely often.

5. The elegant proof of the Borel–Carathéodory inequality given above is taken almost verbatim from [**3**, p. 16]; it is a refinement of an argument which goes back at least to Hadamard [**4**]. Quite a different proof, involving Schwarz' lemma, can be found in [**5**, pp. 174–5]. Borel used his inequality to give the first "elementary" proof of the Little Picard Theorem [**1**]. His proof and the argument given here share only conventional elements: their central mechanisms seem quite different.

Finally, while we are very far from claiming that our result provides a satisfactory substitute for the full Picard theorem, we find it both amusing and instructive to consider Borel's own remarks [**2**, pp. 145–6] on generality in function theory with this (or, for that matter, any other) question in mind.

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A NOTE ON ASYMPTOTIC EXPANSIONS

KUSUM SONI

One of the simplest techniques, used in the asymptotic expansion of a function defined by a definite integral, is integration by parts. If this technique can be used, the successive terms in the expansion are obtained by repeated integration by parts. However, we obtain an asymptotic series only if the order of the remainder after n terms is lower than the order of the nth term for every positive integer n. In most of the examples we encounter in the literature, this condition is satisfied. Nevertheless, it should not be taken for granted. The following simple example is helpful in emphasizing this point. Let

$$I(x) = \int_0^1 t^\alpha J_0(xt)\, dt, \qquad -1 < \alpha < \tfrac{1}{2}.$$

$J_m(t)$ is the Bessel function of the first kind of order m. Since $(t^m J_m(t))' = t^m J_{m-1}(t)$, we can use integration by parts n times to obtain

$$I(x)=\sum_{k=1}^{n}2^{k-1}[\Gamma(k-\tfrac{1}{2}-\tfrac{1}{2}\alpha)/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)]\,x^{-k}J_k(x)$$
$$+2^n[\Gamma(n+\tfrac{1}{2}-\tfrac{1}{2}\alpha)/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)]\,x^{-n}\int_0^1 t^{\alpha-n}J_n(xt)\,dt. \tag{1}$$

It may appear reasonable to conclude that $I(x)$ has a generalized asymptotic series expansion as $x\to\infty$ and

$$I(x)\sim\sum_{k=1}^{\infty}2^{k-1}[\Gamma(k-\tfrac{1}{2}-\tfrac{1}{2}\alpha)/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)]\,x^{-k}J_k(x),\qquad x\to\infty. \tag{2}$$

However, (2) is false. By a simple change of the variable, we see that

$$x^{-n}\int_0^1 t^{\alpha-n}J_n(xt)\,dt=x^{-\alpha-1}\int_0^x u^{\alpha-n}J_n(u)\,du$$
$$\sim x^{-\alpha-1}2^{\alpha-n}[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)/\Gamma(n+\tfrac{1}{2}-\tfrac{1}{2}\alpha)],\qquad x\to\infty.$$

Thus, for every positive integer n, the remainder after n terms in (1) is of the same order as $x^{-\alpha-1}$ as $x\to\infty$. The correct asymptotic series expansion for $I(x)$ corresponding to (2) is the following:

$$I(x)\sim 2^{\alpha}[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)]\,x^{-\alpha-1}$$
$$+\sum_{k=1}^{\infty}2^{k-1}[\Gamma(k-\tfrac{1}{2}-\tfrac{1}{2}\alpha)/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)]\,x^{-k}J_k(x),\qquad x\to\infty. \tag{3}$$

The proof is quite simple. Note that

$$I(x)=\int_0^{\infty}t^{\alpha}J_0(xt)\,dt-\int_1^{\infty}t^{\alpha}J_0(xt)\,dt. \tag{4}$$

The first term in the expansion of $I(x)$ in (3) is the value of the first integral in (4), [**3**, pp. 391–392]. By repeated integration by parts,

$$\int_1^{\infty}t^{\alpha}J_0(xt)\,dt=-\sum_{k=1}^{n}2^{k-1}[\Gamma(k-\tfrac{1}{2}-\tfrac{1}{2}\alpha)/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)]\,x^{-k}J_k(x)+R_n,$$

where

$$R_n=2^n[\Gamma(n+\tfrac{1}{2}-\tfrac{1}{2}\alpha)/\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha)]\,x^{-n}\int_1^{\infty}t^{\alpha-n}J_n(xt)\,dt.$$

Since $J_n(t)=O(t^{-\frac{1}{2}})$, $t\to\infty$, and $-1<\alpha<\frac{1}{2}$, $R_n=O(x^{-n-\frac{1}{2}})$, $x\to\infty$.

References

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BATCH PROCESSING DIFFERENTIAL EQUATIONS ON A MINICOMPUTER

RICHARD BRONSON AND ALAN JONES

1. Introduction. Although numerical methods for solving differential equations are now a common part of many undergraduate courses in the subject, computer solutions *per se* are not. There are a number of reasons for this, not the least of which is the inordinate time required for students to successfully code even one numerical scheme. It is the purpose of this article to show how canned computer programs can be incorporated effectively into a sophomore course in differential equations, and how their use can have dramatic impact on the students' understanding of the subject matter.

2. Course description. During the 1973–74 academic year, the mathematics faculty at Fairleigh Dickinson University decided to emphasize numerical methods in its course MA210 Differential Equations which meets three hours a week in the spring semester and is required of all students in the College of Science and Engineering. The current text is *Introduction to Differential Equations* by Shepley L. Ross; prerequisites include a full calculus sequence and a two credit course in Fortran IV.

The new syllabus divides the course into three segments beginning with a traditional development of linear differential equations with constant coefficients. Variable coefficients are discussed briefly at the end of this segment — only first order equations are actually solved — leading naturally into the section on numerical methods. The course concludes with Laplace transforms, a topic required by the engineering departments whose majors constitute a majority of the student body. Topics (reluctantly) deleted from the old syllabus to accommodate the new material included methods for solving non-linear first order equations, series solutions of linear equations, and matrix methods.

3. Computer considerations. The canned program used for the course was written at Fairleigh Dickinson to run on its minicomputer, a Hewlett Packard HP-2116C with 32K words of memory. Input lines are prespecified — instruction sheets are distributed — with the student supplying his or her parameters in free-field format where appropriate, such as initial values, length of run, step size and so on. The differential equation itself is one input line, and the ability to code it is the only Fortran required of the student. The output can be specified as tabular or graphical or both, all emanating from a standard line printer. The mode of operation is batch.

Students can choose any one of ten different numerical methods ranging from Euler's method to Hamming's predictor-corrector method and a fourth order Runge-Kutta method. Only one differential equation at a time can be handled by the program, not systems, but the order can be as high as nine, and multiple runs are allowed so that the same equation with different parameters can be studied on the same job.

4. Pedagogy. The focal point for teaching numerical methods is a three part project, the first portion of which requires the students to solve via hand calculators a test differential equation of their own choosing over a small interval by each of the methods presented in class. Two important concepts are developed here: First, the students are forced to understand the algorithms, and second, they begin to appreciate the benefits of having a computer do the calculations. This latter point is hammered home convincingly by only a few iterations of a fourth order Runge-Kutta scheme.

Part two of the project requires the students to program Euler's method for a second order differential equation. The intent here is again twofold: First, it requires students to understand the reduction process of converting high order differential equations to a first order system, and then the extension of numerical methods to such systems. Second, the students understand the complexities involved in coding a numerical scheme, even one as simple as Euler's. For many, these complexities are so overwhelming that some instructors make this portion of the project optional, replacing it with solutions by hand calculators.

The third and most important part of the project is "turning the students loose" on the canned program and letting them explore differential equations for themselves. In general, the results have been enlightening and often unexpected.

5. Results. One result is that students cannot be "turned loose" on a minicomputer; their requests far exceeded the computer's ability to perform. By the end of the first week, student submissions of fifteen runs in succession, each over long intervals with small step sizes, had increased computer turn-around time by a factor of eight, resulting in unrest among all computer users. Normality returned only when limits were placed on the total number of runs per project, number of runs in succession, and total time per job.

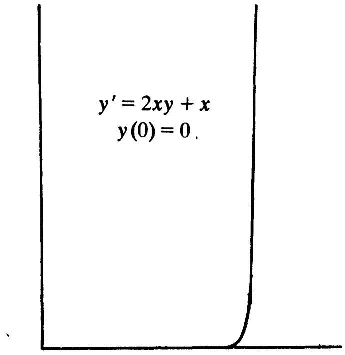

FIG. 1

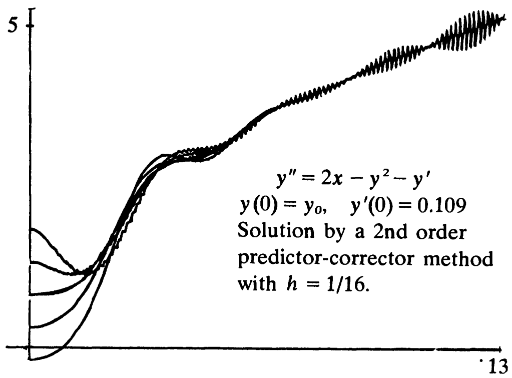

FIG. 2

Very early in the project the problem of scaling surfaced, with most students obtaining solutions similar to the one in Figure 1 which looked nothing like the exponential curves expected. A short lecture on the characteristics of the true solution, here

$$y = \frac{1}{2}(e^{x^2} - 1),$$

along with Table 1, soon cleared the mystery. Dividing 10^{38} (E38 in computer language) into ten, twenty, or even 100 equal parts still leaves the y-axis in units of E37 or E36. Almost all values of y for x less than 9.375 are necessarily plotted on the horizontal axis; the true nature of the curve is obscured by the rapid rise in y for each small change in x.

TABLE 1

x	y-computed	True solution $y = \frac{1}{2}(e^{x^2} - 1)$
7.5	9.38E21	1.34E24
8.0	1.43E25	3.12E27
8.5	3.67E28	1.19E31
9.375	1.21E35	7.40E37
9.4375	∞	2.40E38

Figure 2 was submitted by a student for the five values of y_0 of 1.8125, 1.3125, 0.8125, 0.3125, and -0.1875. The graph looked pretty, and he was pleased with the results. In later semesters, this example proved a good stepping stone for a short discussion on numerical instability. Figure 2, coupled with Figure 3 which is the solution to the same equation with a smaller step size h, vividly demonstrated how numerical instability often can be avoided by reducing the step size, and the risk involved in numerically solving a problem with only one run.

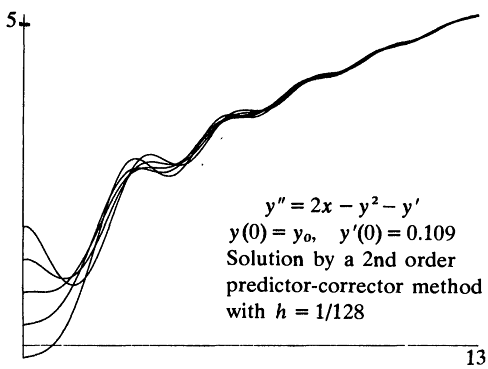

FIG. 3

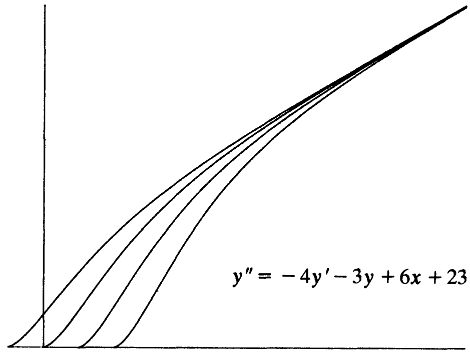

FIG. 4

Figure 4 was submitted by a student with the comment, "These curves behave very strange. They arrive at the same point." It seems that although most students could obtain the steady state solution and isolate the transient solution from the complete analytic solution, they had no insight into the geometric characteristics of such solutions. Figure 4, as well as Figure 3, drove home the concept of steady state solutions much quicker and with more authority than anything previously done in class. The fact that both figures were the result of student runs played no small part in capturing and holding student interest long enough to make the necessary points.

Figure 4 was interesting for another unexpected reason — it was submitted as an example of solution curves for varying initial conditions. Although such curves had been discussed analytically earlier in the semester, it was apparent that the students had no insight into the implications of such changes. A short lecture based on Figures 3, 4, and 5 convincingly accomplished what classroom lectures alone had not: the significance of changing the initial values of either the dependent or independent variable.

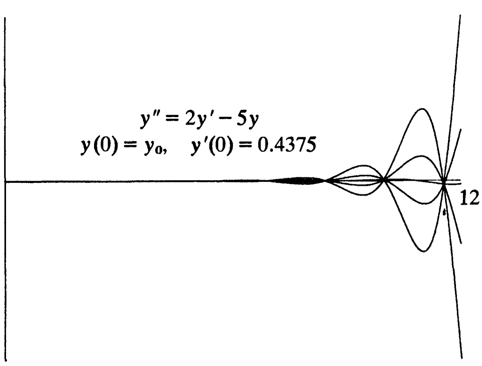

FIG. 5

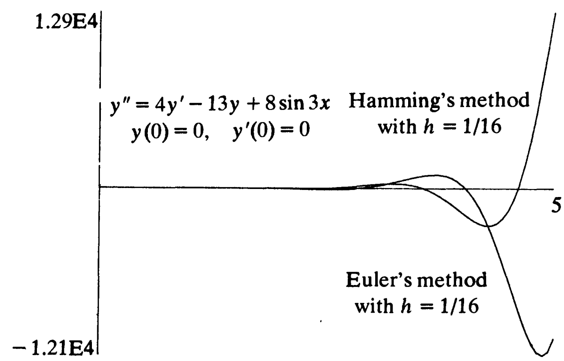

FIG. 6

The student who submitted Figure 5 wanted to attribute the curves to numerical instability but became confused when a higher order method yielded the same results. He was not convinced that he was being misled by scaling until after he reran the equation over a smaller range and derived the analytic solution

$$y = e^x \left[y_0 \cos 2x + \left(\frac{0.4375 - y_0}{2} \right) \sin 2x \right].$$

Finally, by the end of the project, all students were unanimous in their conclusion that fourth order methods were the most accurate and that lower order methods, especially Euler's, should be avoided.

These claims, based on the calculations performed in part one of the project and graphs similar to Figure 6 obtained in part three, were the springboard for a brief lecture on the compensating qualities of higher order methods with smaller step sizes, Figure 7, and the justification for using lower order methods — even Euler's as is done in Dynamo — when the differential equation itself is known only approximately, the usual case when modeling social or industrial systems.

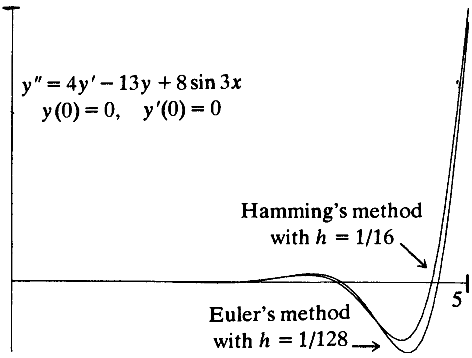

FIG. 7

6. Conclusions. Teaching numerical methods in differential equations without implementing those methods on a computer is much like serving a sandwich without the bread — the recipient will not go hungry but an important ingredient is missing. Without computer experience, the student cannot appreciate or recognize the computational difficulties inherent in all numerical procedures such as scaling and instability to name but two. Not having faced these difficulties in a classroom setting, he or she is likely to badly misinterpret computer solutions when first obtained in a vocational setting. As a practical matter, the numerical methods themselves become useless.

Computer solutions also have great pedagogical value in the traditional academic setting. Students can solve many equations quickly and, through graphical output, appreciate, often for the first time, the concepts of steady state solutions, initial conditions, families of curves, and, most importantly, the concept that a solution of a differential equation is geometrically a curve.

To do so, however, the student must have access to a canned program since coding one's own package is just too time consuming to be feasible. The package used at Fairleigh Dickinson can be implemented on most (mini) computers with a Fortran compiler, and it is available to interested academic departments upon request.

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MISCELLANEA

8. "It is a remarkable historical fact that there is a branch of science in which there has never been a prolonged dispute concerning the proper objects of that science. It is the mathematics. Mistakes in mathematics occur not infrequently, and not being detected give rise to false doctrine, which may continue a long time. Thus, a mistake in the evaluation of a definite integral by Laplace, in his *Mécanique céleste*, led to an erroneous doctrine about the motion of the moon which remained undetected for nearly half a century. But after the question had once been raised, all dispute was brought to a close within a year."

C. S. Peirce, *Collected Papers*, Paragraph 3.426 (Suggested by A. A. Mullin)

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before July 31, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2707. *Proposed by Leonard Shapiro, North Dakota State University*

Find $\sup \sigma(f)$ where

$$\sigma(f) = \inf_{x>0} \left\{ \frac{f(x)}{x} \int_0^x (1 - f(t))\, dt \right\}$$

and f ranges over continuous functions on $[0, \infty)$. For which f (if any) is this supremum achieved?

E 2708. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Ontario*

Find all n for which the symmetric group S_n has the following property: If $\alpha, \beta \in S_n$ are n-cycles, then either $\langle \alpha \rangle = \langle \beta \rangle$ or $\langle \alpha \rangle \cap \langle \beta \rangle = \{1\}$.

E 2709. *Proposed by R. M. Norton, The College of Charleston, South Carolina*

Let $A = (a_{ij})$, $0 \leqq i, j \leqq n$ be a Hankel matrix defined by

$$a_{ij} = \begin{cases} 0 & \text{if } i+j \text{ is odd} \\ \dbinom{2i+2j}{i+j} & \text{if } i+j \text{ is even.} \end{cases}$$

Compute $\det A$.

E 2710. *Proposed by J. A. Andrews*

Call two real numbers *equivalent* if their difference is rational. Call $S \subset \mathbb{R}$ a *choice set* if S is a set of representatives of the equivalence classes in $\mathbb{R}$. Let $\mathcal{F}$ be the family of all choice sets contained in $[0, 1]$. Show that the numbers $m^*(S)$ $(S \in \mathcal{F})$ are dense in $[0, 1]$. (m^* is the usual outer measure.)

E 2711. *Proposed by Frank Uhlig, Aachen, Germany*

Let A and B be $m \times m$ matrices over a field. If the characteristic polynomial of A is irreducible, show that $\operatorname{rank}(AB - BA) \neq 1$.

E 2712. *Proposed by A. Wilansky, Lehigh University*

Let A be a linear map from real bounded sequences to the real numbers, such that for each sequence x some subsequence of x converges to $A(x)$. Must $A(xy) = A(x)A(y)$?

SOLUTIONS OF ELEMENTARY PROBLEMS

Average Distance between Two Points in a Box

E 2629 [1977, 57]. *Proposed by David P. Robbins, Phillips Exeter Academy, New Hampshire*

Two points are chosen at random (uniform distribution) in the box $|x| \leqq a$, $|y| \leqq b$, $|z| \leqq c$ of $\mathbf{R}^3$. What is the expected distance between them?

(The cases when $\mathbf{R}^3$ is replaced by $\mathbf{R}$ or $\mathbf{R}^2$ are well known.)

Solution by Theodore S. Bolis, State University College at Oneonta. Let X_1, X_2; Y_1, Y_2; Z_1, Z_2 be independent random variables uniformly distributed in $[-a, a]$; $[-b, b]$; $[-c, c]$, respectively. Then the random variables $U = |X_1 - X_2|$, $V = |Y_1 - Y_2|$, $W = |Z_1 - Z_2|$ are also independent. It is easy to show that the density function of U, say, is

$$f(u) = (2a - u)/2a^2, \qquad 0 \leqq u \leqq 2a.$$

The sought expectation is given by

$$E = \frac{1}{8a^2b^2c^2} \int_0^{2c} \int_0^{2b} \int_0^{2a} \sqrt{u^2 + v^2 + w^2}\,(2a - u)(2b - v)(2c - w)\,du\,dv\,dw.$$

Let $P(a, b, c)$ be the pyramid determined by the planes $u = 2a$, $v = 0$, $w = 0$, $av = bu$, and $aw = cu$. Set

$$F(a, b, c) = \int\int\int_{P(a,b,c)} \sqrt{u^2 + v^2 + w^2}\,(2a - u)(2b - v)(2c - w)\,du\,dv\,dw.$$

Then it is clear that $F(a, b, c) = F(a, c, b)$ and that

$$E = \frac{1}{8a^2b^2c^2} \left(F(a, b, c) + F(c, a, b) + F(b, c, a)\right). \tag{1}$$

Thus it suffices to compute $F(a, b, c)$. By using spherical coordinates we obtain

$$F(a, b, c) = \int_0^{\tan^{-1}(b/a)} \int_{\cot^{-1}((c/a)\cos\theta)}^{\pi/2} \int_0^{2a \csc\phi \sec\theta} \Phi\,d\rho\,d\phi\,d\theta,$$

where

$$\Phi = \rho^3(2a - \rho \sin\phi \cos\theta)(2b - \rho \sin\phi \sin\theta)(2c - \rho\cos\phi)\sin\phi.$$

By tedious but routine successive integration one finds that

$$F(a, b, c) = \frac{64}{315} a^7 - \frac{8}{315} a^2(8a^4 - 19a^2b^2 - 6b^4)r_3 - \frac{8}{315} a^2(8a^4 - 19a^2c^2 - 6c^4)r_2$$

$$+ \frac{8}{315} a^2(8a^4 - 6b^4 - 6c^4 - 19a^2b^2 - 19a^2c^2 + 30b^2c^2)r$$

$$+\frac{8}{15}a^6b\sinh^{-1}\frac{b}{a}+\frac{8}{15}a^6c\sinh^{-1}\frac{c}{a}$$

$$-\frac{8}{15}a^2b(a^4-4a^2c^2-c^4)\sinh^{-1}\frac{b}{r_2}$$

$$-\frac{8}{15}a^2c(a^4-4a^2b^2-b^4)\sinh^{-1}\frac{c}{r_3}-\frac{32}{15}a^5bc\sin^{-1}\frac{bc}{r_2r_3},$$

where

$$r=\sqrt{a^2+b^2+c^2},\qquad r_1=\sqrt{b^2+c^2};$$

$$r_2=\sqrt{c^2+a^2},\qquad r_3=\sqrt{a^2+b^2}.$$

Substituting in (1) we obtain

$$E=\frac{2}{15}r-\frac{7}{45}\left[(r-r_1)\left(\frac{r_1}{a}\right)^2+(r-r_2)\left(\frac{r_2}{b}\right)^2+(r-r_3)\left(\frac{r_3}{c}\right)^2\right]$$

$$+\frac{8}{315a^2b^2c^2}(a^7+b^7+c^7-r_1^7-r_2^7-r_3^7+r^7)$$

$$+\frac{1}{15ab^2c^2}\left(b^6\sinh^{-1}\frac{a}{b}+c^6\sinh^{-1}\frac{a}{c}-r_1^2(r_1^4-8b^2c^2)\sinh^{-1}\frac{a}{r_1}\right)$$

$$+\frac{1}{15a^2bc^2}\left(c^6\sinh^{-1}\frac{b}{c}+a^6\sinh^{-1}\frac{b}{a}-r_2^2(r_2^4-8c^2a^2)\sinh^{-1}\frac{b}{r_2}\right)$$

$$+\frac{1}{15a^2b^2c}\left(a^6\sinh^{-1}\frac{c}{a}+b^6\sinh^{-1}\frac{c}{b}-r_3^2(r_3^4-8a^2b^2)\sinh^{-1}\frac{c}{r_3}\right)$$

$$-\frac{4}{15abc}\left(a^4\sin^{-1}\frac{bc}{r_2r_3}+b^4\sin^{-1}\frac{ca}{r_3r_1}+c^4\sin^{-1}\frac{ab}{r_1r_2}\right).$$

In the case of the unit cube, $a=b=c=\frac{1}{2}$, we obtain

$$E=\frac{1}{105}[4+17\sqrt{2}-6\sqrt{3}+21\log(1+\sqrt{2})+42\log(2+\sqrt{3})-7\pi], \tag{2}$$

i.e., $E\doteq 0.661707$.

Formula (2) was found also by Günter Bach & Frank Piefke (West Germany), and by the proposer.

Comments. Bach and Piefke use in their solution a formula due to J. F. C. Kingman (Journal of Applied Probability, 6 (1969), p. 668) for the expectation of the kth power of the distance between two random points of a convex body in $\mathbf{R}^n$, and a formula due to R. Coleman (ibid., p. 439) for the density function of secants of fixed length of the unit cube.

The solution of the analogous problem for an n-dimensional ball is known, see L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, 1976, p. 212.

Polyhedral Models

E 2630 [1976, 57]. *Proposed by Edward T. Ordman, University of Kentucky, Lexington*

Suppose that a polyhedral model (made, say, of cardboard) is slit along certain edges and unfolded to lie flat in the plane. The cuts may not be made so as to disconnect the figure. Now suppose that the resulting plane figure is again folded up to make a polyhedron (folding is allowed only on the original

lines). The new polyhedron is not necessarily congruent to the original one. Find some interesting examples.

Solution by the proposer. The model of a regular octahedron can be slit along five edges and unfolded as shown in Fig. 1.

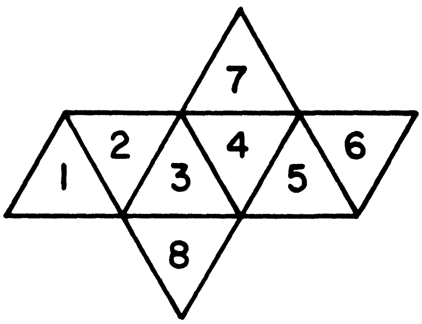

FIG. 1.

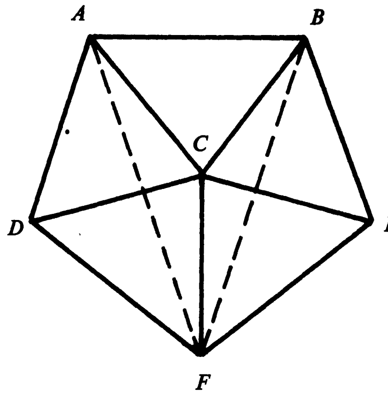

FIG. 2.

This can now be folded to obtain a new polyhedron in Fig. 2. This new polyhedron is built from three congruent regular tetrahedra, $ABCF$, $BCEF$, and $ACDF$. The triangles 1–8 from Fig. 1 are identified with the triangles in Fig. 2 as follows:

$$CDF=1,\quad ACD=2,\quad ABC=3,\quad ABF=4$$

$$ADF=7,\quad BEF=5,\quad CEF=6,\quad BCE=8.$$

Also solved by Michael Goldberg who, for additional examples, cites W. Wunderlich, *Starre, kippende, wackelige und bewegliche Achtflache*, Elemente der Mathematik 20(1965), 25–32.

Prime Satisfying Mirimanoff's Condition

E 2631 [1977, 57]. *Proposed by Barry Powell, Kirkland, Washington*

In 1910 Mirimanoff proved that if p is an odd prime and $3^p \not\equiv 3 \pmod{p^2}$ then the equation

$$x^p + y^p = z^p$$

has no solution in positive integers x, y, z not divisible by p. Show that the above condition is satisfied for all primes p having the form $p = \frac{1}{2}(3^{2k}+1)$ or $p = \frac{1}{2}(3^q - 1)$ with q also an odd prime.

Solution by M. J. DeLeon, Florida Atlantic University (revised by the editor). We shall prove a stronger result.

THEOREM. *Let a be a positive integer and p an odd prime. Assume that* $p \| (a^m+1)$ *or* $p \| (a^m - 1)$ *for some positive integer m. Then* $a^p \not\equiv a \pmod{p^2}$. [$p \| x$ *means* $p \mid x$ *and* $p^2 \nmid x$.]

Proof. If $p \| (a^m+1)$ then $p \| (a^{2m}-1)$, so we need consider only the case $p \| (a^m - 1)$. We have a short exact sequence of abelian groups

$$\{1\} \to K \to (Z/p^2Z)^* \to (Z/pZ)^* \to \{1\}$$

where f is the canonical map and $K = \ker f$. (For any ring R, R^* denotes its group of units.)

Considering a as an element of $(Z/p^2Z)^*$, we have $f(a^m)=1$ since $a^m \equiv 1 \pmod p$. Hence $a^m \in K$. Since K has order p and $a^m \not\equiv 1 \pmod{p^2}$; it follows that p divides the order of a in $(Z/p^2Z)^*$. Therefore $a^{p-1} \not\equiv 1 \pmod{p^2}$, i.e., $a^p \not\equiv a \pmod{p^2}$.

Also solved by Robert Breusch, Lorraine Foster, Irving Gerst, Wells Johnson, L. E. Mattics, Ernst Trost (Switzerland), and the proposer.

Minimizing Discrepancy

E 2632 [1977, 57]. *Proposed by Azriel Rosenfeld, College Park, Maryland*

Define the *discrepancy* $d(A,B)$, between two plane geometric figures to be the area of their symmetric difference. Let A be a circle of radius r. Determine the inradius of the regular n-gon B for which $d(A,B)$ is minimal.

Solution by John Oman, University of Wisconsin, Oshkosh. In fact A is a disk, and let O be its center. Let first B be of fixed size and denote by P the intersection of its axes of symmetry. It is easy to see that B can be positioned so that

(a) $d(A,B)$ is minimal;

(b) while subject to (a), $\overline{OP}$ is minimal.

We claim that then $O=P$. Assume that $O \neq P$ and let p be an axis of symmetry of B such that $O \notin p$. Let q be the line parallel to p such that $O \in q$. Pick a chord of B perpendicular to p and consider all translates of it lying on the same line. None of these translates can have greater length in A than the one whose midpoint lies on q. Thus if we translate B in the direction perpendicular to p so that $P \in q$ then $d(A,B)$ remains minimal while OP will decrease. This contradicts (b) and so we have proved that $O=P$.

Once we know that we may assume that $O=P$ the problem can be solved by elementary calculus. Let R be the inradius of B. Clearly

$$r\cos\frac{\pi}{n} \leqq R \leqq r$$

and

$$d(A,B)=nr^2\left[2\cos^{-1}\left(\frac{R}{r}\right)-2\frac{R}{r}\sqrt{1-\left(\frac{R}{r}\right)^2}+\left(\frac{R}{r}\right)^2\tan\frac{\pi}{n}-\frac{\pi}{n}\right].$$

Putting $x=R/r$ and equating first derivatives to zero we obtain

$$2\sqrt{1-x^2}=x\tan\frac{\pi}{n}.$$

Hence the minimum occurs for

$$R=r\left(1+\frac{1}{4}\tan^2\frac{\pi}{n}\right)^{-1/2}$$

This last condition means that half of the perimeter of B lies inside A and the other half lies outside A.

It is interesting to note that if B is now held fixed and r is varied then the given A does not minimize $d(A,B)$.

Partial solutions ($O=P$ was assumed) submitted by Marguerite Gerstell, Michael Goldberg, Michael Josephy (Costa Rica), Hans Kappus (Switzerland), Franklin Kemp, and the proposer.

Permutable Sets of Lattice Points

E 2633 [1977, 58]. *Proposed by Benjamin G. Klein, Davidson College, North Carolina*

Two points x and y in $\mathbf{Z}^n$ are said to be neighbors if $y - x = \pm e_i$ for some $i = 1, \ldots, n (e_1, \ldots, e_n$ is the canonical basis of $\mathbf{Z}^n$). A subset $S \subset \mathbf{Z}^n$ is said to be *permutable* if there is a bijection $T: S \to S$ such that for each $x \in S$, Tx and x are neighbors. Show that if a finite subset $S \subset \mathbf{Z}^n$ is permutable then $|S|$ is even.

Find necessary and sufficient conditions for a subset $S \subset \mathbf{Z}^2$ to be permutable.

Solution by Tom Moore and Emily Moore, Marietta College, Ohio (revised by the editor). We say that $(a_1, \ldots, a_n) \in \mathbf{Z}^n$ is *even* (*odd*) if $a_1 + \cdots + a_n$ is even (odd). For $X \subset \mathbf{Z}^n$ we let $X_e = \{a \in X \mid a$ is even$\}$ and $X_0 = X \backslash X_e$.

LEMMA. *If $S \subset \mathbf{Z}^n$ is permutable there exists an involution $f: S \to S$ such that $f(a)$ and a are neighbors for all $a \in S$.*

Proof. By hypothesis there exists a bijection $g: S \to S$ such that $g(a)$ and a are neighbors for all $a \in S$. An orbit T of g is a subset of S of the form $\{g^k(a) \mid k \in \mathbf{Z}\}$ where $a \in S$ is fixed. We shall construct the required involution f so that $f(T) = T$ for every orbit T of g.

Let T be an orbit of g, $a \in T$, and so $T = \{g^k(a) \mid k \in \mathbf{Z}\}$. Now we distinguish two cases.

CASE 1. T is infinite. Then the map $\mathbf{Z} \to T$ sending k to $g^k(a)$ is a bijection and we define f on T by

$$\left.\begin{aligned} f(g^{2k}(a)) &= g^{2k+1}(a) \\ f(g^{2k+1}(a)) &= g^{2k}(a) \end{aligned}\right\} \quad k \in \mathbf{Z}.$$

CASE 2. T is finite. If a is, say, even then $g(a)$ is odd, $g^2(a)$ is even, etc. Therefore $|T|$ is even, say $|T| = 2m$. Then we define f on T by

$$\left.\begin{aligned} f(g^{2k}(a)) &= g^{2k+1}(a) \\ f(g^{2k+1}(a)) &= g^{2k}(a) \end{aligned}\right\} \quad 0 \leqq k \leqq m-1.$$

Clearly this defines f on S and f has all the required properties.

Let G_S be the subgraph of the lattice graph $\mathbf{Z}^n$ spanned by S. Then clearly, G_S is a bipartite locally finite graph with (S_e, S_0) as the underlying partition of S. The lemma shows that $S \subset \mathbf{Z}^n$ is permutable iff there is a matching in G_S which matches S_e with S_0. The necessary and sufficient condition for the existence of such matchings are well known, see, e.g., R. A. Brualdi, *Transversal Theory and Graphs*, Theorems 4.2 and Corollary 4.5 (in D. R. Fulkerson, *Studies in Graph Theory*, Part 1, MAA Studies in Mathematics, vol. 11).

Using these theorems we can state that $S \subset \mathbf{Z}^n$ is permutable if and only if for each finite subset X of S_e or S_0 the set $S(X)$ of its neighbors in S satisfies $|S(X)| \geqq |X|$.

If $S \subset \mathbf{Z}^n$ is a finite permutable set then $|S_e| = |S_0|$ by the lemma and hence $|S|$ is even.

Also solved by D. M. Bloom, Michael Brozinsky, Thomas Elsner, Donald Fuller, Marguerite Gerstell, Richard Gibbs, Michael Josephy (Costa Rica), S. C. Locke (Canada), O. P. Lossers (Netherlands), Peter Pappas, Martin Schechter, James Walker, and the proposer.

A Triangle Inequality

E 2634 [1977, 58]. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, N.Y.*

Let A_i $(i = 0, 1, 2 \pmod 3)$ be the vertices of a triangle, Γ its inscribed circle with center O. Let B_i be the intersection of the segment A_iO with Γ and let C_i be the intersection of the line A_iO with the side $A_{i-1}A_{i+1}$.

Prove that $\Sigma A_iC_i \leqq 3\Sigma A_iB_i$.

Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands. Let r be the radius of Γ and $a_i = A_{i-1}A_{i+1}$. By the Sine Theorem we have

$$A_1C_0/a_2 = A_2C_0/a_1 = OC_0/OA_0$$

and so $a_0 = A_1C_0 + A_2C_0 = (a_1 + a_2)OC_0/OA_0$. Since $OC_0 \geqq r$ we have

$$\begin{aligned} 3A_0B_0 - A_0C_0 &= 3(OA_0 - r) - (OA_0 + OC_0) \\ &= OC_0\left(2\frac{a_1 + a_2}{a_0} - 1\right) - 3r \\ &\geqq 2r\left(\frac{a_1 + a_2}{a_0} - 2\right). \end{aligned}$$

Hence

$$3\sum A_iB_i - \sum A_iC_i \geqq 2r\sum \frac{(a_i - a_{i+1})^2}{a_ia_{i+1}} \geqq 0.$$

This proves the inequality and shows that $3\Sigma A_iB_i = \Sigma A_iC_i$ only if $a_0 = a_1 = a_2$.

Also solved by W. J. Blundon (Canada), Jordi Dou (Spain), Leonard Goldstone, M. G. Greening (Australia), J. M. Stark, and the proposer.

Comment. Goldstone proves that $\Sigma(3A_iB_i - A_iC_i) \geqq 2r(1 - 2(r/R))$ where R is the circumradius of the triangle. Since $R \geqq 2r$, this is also an improvement of the inequality in the problem.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before July 31, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6204*. *Proposed by F. David Hammer, Santa Cruz, California*

(1) If all proper subgroups of an infinite abelian group are free (as abelian groups), then the group is free.

(2) Find a weaker hypothesis for (1).

(3) Delete abelian in (1).

6205. *Proposed by Alan McConnell and Louis Shapiro, Howard University*

Let G be a group with no elements of finite order, and let H be a cyclic subgroup of finite order in G. Show that G is itself cyclic.

6206. *Proposed by Gérard Letac, Université Paul-Sabatier, Toulouse, France*

Prove that, if n is an integer $\neq 0$,

$$\int_{-\pi/2}^{+\pi/2} \exp[2in(x + \tan x)]dx = 0.$$

6207. *Proposed by Ignacy I. Kotlarski, Oklahoma State University*

Let X, Y be two independent $(2n + 2)$-dimensional normal random vectors with means 0 and positive definite variance covariance matrices C, C^{-1} respectively ($n = 0, 1, \ldots$). Find the distribution of their inner product $Z = X \cdot Y$.

6208. *Proposed by Gary Gunderson, University of New Orleans*

Let $p(z)$ and $q(z)$ be two polynomials with $\deg(q) \geqq \deg(p)$, and suppose there is a discrete real sequence $\{x_j\}_{j=1}^{\infty}$ with cluster points at $\pm\infty$. Prove: if $q(z) \in \mathbb{R}$ whenever $p(z) \in \{x_j\}_{j=1}^{\infty}$, then $q(z) = \sum_{i=0}^{n} c_i(p(z))^i$ where $c_i \in \mathbb{R}$ $(0 \leqq i \leqq n)$.

* Can the condition $\deg(p) \leqq \deg(q)$ be dropped?

6209. *Proposed by Marcel F. Neuts, Purdue University, Lafayette, Indiana*

Let A be a primitive nonnegative matrix of order m and let B be a finite real matrix of order m. Denote the spectral radius of A by ρ. Show that

$$\lim_{n\to\infty} \frac{1}{n} \rho^{-(n-1)} \sum_{\nu=0}^{n-1} A^{\nu} B A^{n-1-\nu}$$

exists and identify the limit.

SOLUTIONS OF ADVANCED PROBLEMS

Kuratowski Sets

5996 [1974, 1034]. *Proposed by Arthur Smith, Carleton University, Canada*

A well-known problem posed by Kuratowski concerns the maximum number of sets obtainable from a subset A of a topological space T by the operations of closure $(^-)$, taking interiors $(^0)$ and complementation. Suppose we do not allow complementation, but do allow arbitrary unions.

Show that at most 13 sets can be constructed from A and give an example of a set A for which 13 sets may be constructed, when T is the real line with the usual topology.

Solution by Chie Y. Yu, Wyomissing, Pennsylvania. Given a set A, at most 6 new sets can be constructed by taking closures and interiors; we list them together with A in the diagram below

$$\begin{array}{ccccc} A^{-} & \supset & A^{-0-} & \supset & A^{-0} \\ \cup & & \cup & & \cup \\ A & & A^{0-} & \supset & A^{0-0} \\ & \cup & & \cup & \\ & & A^{0} & & \end{array}$$

From the diagram we see that at most 6 more new sets can be constructed by taking unions of two sets at a time:

$$A \cup A^{-0-}, \quad A \cup A^{-0}, \quad A \cup A^{0-}, \quad A \cup A^{0-0}, \quad A \cup A^{0-} \cup A^{-0}, \quad A^{0-} \cup A^{-0}.$$

We now have altogether 13 sets.

It is easy to see that taking the union of any two of the 13 sets will yield no new set, and with the exception of $(A \cup A^{0-})^0 = A^{0-0}$, it is easy also to see that taking the closure or interior of any one of the 13 sets will yield no new set.

To show $(A \cup A^{0-})^0 = A^{0-0}$, we first show $(A \cup A^{0-})^0 \subset A^{0-}$. Suppose not, then there exists $x \in (A \cup A^{0-})^0$, and $x \notin A^{0-}$. It follows that there exist neighborhoods N_1 and N_2 of x such that $N_1 \subset A \cup A^{0-}$, and $N_2 \cap A^{0-} = \varnothing$. Let $N = N_1 \cap N_2$, then N is a neighborhood of x, and $N \subset A \cup A^{0-}$, $N \cap A^{0-} = \varnothing$. Thus $N \subset A$, which implies $x \in A^0 \subset A^{0-}$, a contradiction. We now have shown $(A \cup A^{0-})^0 \subset A^{0-}$, thus $(A \cup A^{0-})^0 \subset A^{0-0}$, but $(A \cup A^{0-})^0 \supset A^0 \cup A^{0-0} = A^{0-0}$, hence $(A \cup A^{0-})^0 = A^{0-0}$.

Thus we have shown at most 13 sets can be constructed.

The following example shows 13 sets can be constructed.

Let $A = D_1 \cup D_2 \cup (3,4) \cup (4,5)$, where

$$D_1 = \{1/n \mid n = 1,2,3,\cdots\}, \qquad D_2 = \{d \mid d \text{ irrational}, \quad 2 < d < 3\}.$$

$$\begin{array}{llll}
A & = D_1 \cup D_2 \cup (3,4) \cup (4,5), & A \cup A^{-0-} & = D_1 \cup [2,5], \\
A^{-} & = \{0\} \cup D_1 \cup [2,5], & A \cup A^{-0} & = D_1 \cup (2,5), \\
A^{-0-} & = [2,5], & A \cup A^{0-} & = D_1 \cup D_2 \cup [3,5], \\
A^{-0} & = (2,5), & A \cup A^{0-0} & = D_1 \cup D_2 \cup (3,5), \\
A^{0-} & = [3,5], & A \cup A^{0-} \cup A^{-0} & = D_1 \cup (2,5], \\
A^{0-0} & = (3,5), & A^{0-} \cup A^{-0} & = (2,5], \\
A^{0} & = (3,4) \cup (4,5). & &
\end{array}$$

Also solved by David Farnsworth, Carl Hurd, Eric Langford, Patrick McCray, Leroy Meyers, Louise Moser, David Neu, C. B. A. Peck, and the proposer.

Editor's comment. McCray's solution develops a consideration of an algebra of all functions which map the set of all subsets of some fixed set M into itself, and offers the reference: P. C. Hammer, *Kuratowski's closure theorem*, Nieuw Arch. Wisk. (3), 8, pp. 74–80.

For further results of this nature see Eric Langford, *Characterization of Kuratowski* 14*-sets*, this MONTHLY, v. 78, pp. 362, ff., and the references cited there.

Loxodromes on a Torus

6087 [1976, 293]. *Proposed by Nathaniel Grossman, University of California, Los Angeles*

A loxodrome on a Riemannian surface is a curve meeting members of a specified one-parameter family of curves at a constant angle. For example, the bagel (round torus or anchor-ring) has two special families, the meridians and the parallels, each defining the same family of loxodromes. Prove that a loxodrome on a bagel is either periodic or dense.

Solution by I. J. Schoenberg, University of Wisconsin, Madison. The line element of the torus

$$T: \begin{cases} x = (a + b\cos\theta)\cos\phi \\ y = (a + b\cos\theta)\sin\phi \\ z = b\sin\theta, \end{cases}$$

$-\pi \leqslant \phi \leqslant \pi, \ -\pi \leqslant \theta \leqslant \pi$, is found to be

$$ds^2 = dx^2 + dy^2 + dz^2 = (a + b\cos\theta)^2\left\{dy^2 + \left(bd\theta\,(a + b\cos\theta)^{-1}\right)^2\right\}.$$

It follows that if we pass from θ to the new variable ϑ by $bd\theta/(a + \cos\theta) = d\vartheta$, whence

$$\vartheta = b\int_0^\theta \frac{d\theta}{a + b\cos\theta} = \frac{2b}{\sqrt{a^2 - b^2}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\ \tan\frac{\theta}{2}\right),$$

then $ds^2 = (a + b\cos\theta)^2(d\phi^2 + d\vartheta^2)$, showing that the mapping $(\phi,\theta) \to (\phi,\vartheta)$ maps T *conformally* onto the rectangle

$$R = \left\{(\phi,\vartheta): -\pi \leqslant \phi \leqslant \pi, -b\pi/c \leqslant \vartheta \leqslant b\pi/c, c = \sqrt{a^2 - b^2}\ \right\},$$

in which we identify as usual "opposite" points of ∂R. This may be called the *Mercator map* of T. Images of loxodromes of T are the straight lines $\vartheta = \gamma\phi + C$, where the line, on meeting the boundary of R at (π,ϑ_0) say, is to be continued with the same slope γ, from the point $(-\pi,\vartheta_0)$ (similarly in the vertical direction; see Fig. 1).

These lines, reduced (mod 2π, mod $2b\pi/c$) are much like the path of a billiard ball moving in R (see Chap. 23 of G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th Ed.,

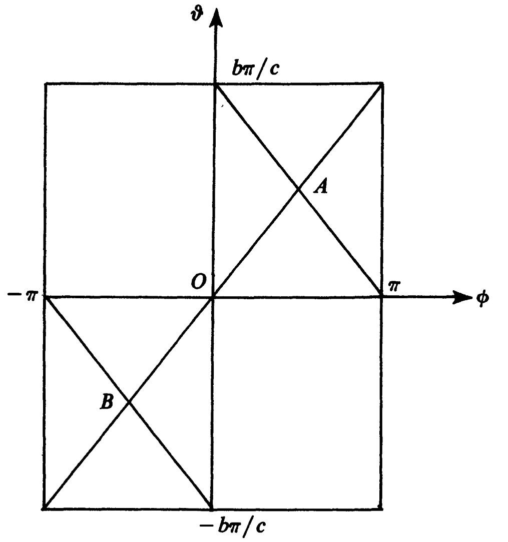

FIG. 1

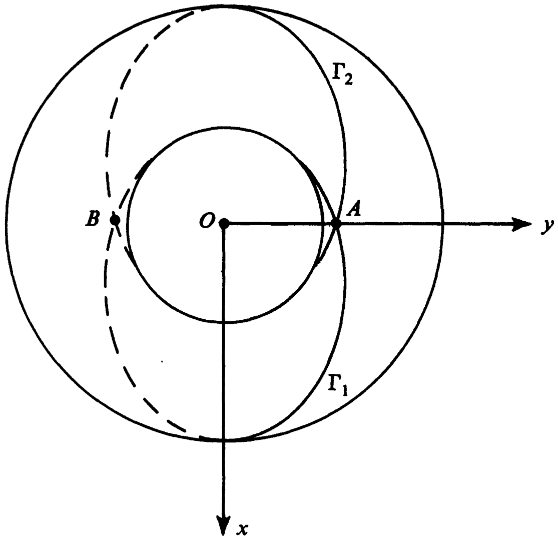

FIG. 2

Oxford 1970, or I. J. Schoenberg, *On the motion of a billiard ball in two dimensions*, Delta 5(1975), 1–18), except that the present problem is simpler. However, both problems depend on Kronecker's theorem. The result, solving our present problem, is as follows: *The loxodrome of slope γ is a closed curve if and only if the number $\gamma c/b$ is rational. If this number is irrational then the loxodrome is dense in T.*

In discussing this subject we should also mention the circles of Villarceau (a French astronomer of the last century). (1) The simplest non-trivial loxodromes of T correspond to the straight lines $\vartheta = \pm bc^{-1}\phi + C$ *and they are all circles.* (2) The two circles (See Fig. 1)

$$\Gamma_1 : \vartheta = bc^{-1}\phi, \Gamma_2 : \vartheta = \begin{cases} bc^{-1}(\pi-\phi) & \text{if } 0 \leqslant \phi \leqslant \pi, \\ -bc^{-1}(\pi+\phi) & \text{if } -\pi \leqslant \phi \leqslant 0, \end{cases}$$

are also obtained in a different way as stated by the remarkable THEOREM OF VILLARCEAU. *If the slanting plane Π is tangent to the torus T in the two points A and B, then the intersection of T and Π is identical with the union of the two circles $\Gamma_1 \cup \Gamma_2$.*

Fig. 2 represents the projection of T, Γ_1, Γ_2 onto the $x0y$ plane.

Also solved by the proposer.

Polynomial Algebra Generated by Symmetric Functions

6097 [1976, 489]. *Proposed by Glen E. Bredon, Rutgers University*

Consider the polynomial

$$P(t) = 2^{-n}(1+t^{a_1})(1+t^{a_2})\cdots(1+t^{a_n}).$$

The first k derivatives of $P(t)$ evaluated at $t=1$, that is

$$q_1 = P'(1),\ q_2 = P''(1), \ldots, q_k = P^{(k)}(1),$$

are symmetric functions of $a_1, a_2, \ldots, a_n$. Show that the polynomial algebra generated by these k symmetric functions coincides with that generated by σ_1 and the σ_{2j} for $2 \leqslant 2j \leqslant k$. Here σ_i is the sum of ith powers, $\sigma_i = a_1^i + a_2^i + \cdots + a_n^i$. (Note that it follows, in particular, that q_3 is a polynomial in q_1

and q_2. In fact $q_3=3q_1q_2-2q_1^3-3q_2+3q_1^2-q_1$. Similarly, q_5 is a polynomial in q_1, q_2, and q_4; and so on.)

Solution by J. A. Tyrrell, King's College, London, England. By Taylor's theorem we have

$$1+\sum_{r=1}^{\infty}\frac{q_r}{r!}(t-1)^r\equiv 2^{-n}\prod_{s=1}^{n}(1+t^{\alpha_s}). \tag{1}$$

If we write $t=\exp 2i\theta$, and use double angle formulas, the term $1+t^{\alpha_s}$ reduces to $2\exp i\alpha_s\theta\cos\alpha_s\theta$; the right-hand side of (1) becomes

$$(\exp i(\alpha_1+\cdots+\alpha_n)\theta)\prod_{s=1}^{n}\cos\alpha_s\theta.$$

We then get the following identity in θ:

$$1+\sum_{r=1}^{\infty}\frac{q_r}{r!}(\exp 2i\theta-1)^r=(\exp i\sigma_1\theta)\prod_{s=1}^{n}\cos\alpha_s\theta. \tag{2}$$

We now expand the various terms here in powers of θ and compare coefficients of $\theta^k (k>1)$. Noting that the expression of $(\exp 2i\theta-1)^r$ contains no terms of degree less than r, we see that the coefficient of θ^k on the left-hand side of (2) is

$$\frac{(2i)^k}{k!}q_k+b_{k-1}q_{k-1}+\cdots+b_1q_1, \tag{3}$$

where the b_i are constants whose precise values are not needed. On the other hand, remembering that the series for $\cos\alpha_s\theta$ contains only even powers of $\alpha_s\theta$, we find directly from the product that the coefficient of θ^k on the right-hand side of (2) is of the form

$$c_0\sigma_1^k+c_1\sigma_1^{k-1}+\cdots+c_{k-1}\sigma_1+c_k, \tag{4}$$

where, for even i, c_i is a symmetric polynomial of degree $i/2$ in $\alpha_1^2,\ldots,\alpha_n^2$ while, for odd i, c_i is zero. Therefore each c_i is expressible as a polynomial in those σ_{2j} for which $2\leqslant 2j\leqslant i$. Equating (3) and (4), we see that q_k belongs to the polynomial algebra generated by $q_1,\ldots,q_{k-1}$ together with σ_1 and the σ_{2j} for which $2\leqslant 2j\leqslant k$; and, by an induction, it follows that the polynomial algebra generated by $q_1,\ldots,q_k$ is contained in that generated by σ_1 and the σ_{2j} for which $2\leqslant 2j\leqslant k$.

To show that the two algebras are in fact the same, it is enough to establish the reverse inclusion for even k only. Now, if k is even, the last term c_k in (4), which is the coefficient of θ^k in

$$\prod_{s=1}^{n}\cos\alpha_s\theta,$$

is found, again from the product, to be of the form

$$\frac{(-1)^{k/2}}{k!}(\alpha_1^k+\cdots+\alpha_n^k)+A_k,$$

where A_k is a (symmetric) sum of terms of degree $k/2$ in $\alpha_1^2,\ldots,\alpha_n^2$, each of which involves at least two of the α_i^2; as such, A_k can be written as a polynomial in $\sigma_2, \sigma_4,\ldots,\sigma_{k-2}$. The coefficients $c_0,\ldots,c_{k-1}$ in (4) can also be so written, and (4) is therefore of the form

$$\frac{(-1)^{k/2}}{k!}\sigma_k+B_k,$$

where B_k is a polynomial in σ_1 and $\sigma_2, \sigma_4,\ldots,\sigma_{k-2}$. Equating this with (3) gives us σ_k as a polynomial in $q_1,\ldots,q_k,\sigma_1,\sigma_2,\sigma_4,\ldots,\sigma_{k-2}$; and thence (by an induction through even values of k) we prove that $\sigma_1,\sigma_2,\sigma_4,\ldots,\sigma_k$ all belong to the polynomial algebra generated by $q_1,\ldots,q_k$. This gives the reverse inclusion we wanted, and completes the proof.

Also solved by L. Carlitz, Andrew Gleason, Robert Strong, and by the proposer.

The Closure of $\sigma(n+1)/\sigma(n)$

6107 [1976, 573]. *Proposed by Roy E. DeMeo, Jr., Franklin Square, New York*

Let $\sigma(n)$ be the sum of all divisors of n. Let A be the set of all rational numbers $\sigma(n+1)/\sigma(n)$. Determine the closure of A in the set of real numbers.

Solution by Harold N. Shapiro, The Courant Institute, New York University. We will show that the closure of A consists of all the non-negative reals. Let B and Q be positive integers, $(B,Q)=1$, and set $n=BQ-1$. Then

$$\begin{aligned}\sigma(n+1)/\sigma(n)&=\sigma(BQ)/\sigma(BQ-1)\\&=(\sigma(B)/B)(\sigma(Q)/Q)(\sigma(BQ-1)/(BQ-1))^{-1}(BQ)/(BQ-1)\end{aligned}$$

and, for BQ large, this is close to

$$(\sigma(B)/B)(\sigma(BQ-1)/(BQ-1))^{-1}(\sigma(Q)/Q).$$

Given any real $a>1$, we will produce B and Q such that (i) $\sigma(Q)/Q$ is close to 1, and (ii) $(\sigma(BQ-1)/(BQ-1))(\sigma(B)/B)^{-1}$ is close to a. Then $\sigma(n+1)/\sigma(n)$ is close to $1/a$, and this suffices to show that the closed interval $[0,1]$ is in the closure under consideration. Similarly, for $n=BQ$ we have

$$\sigma(n+1)/\sigma(n)=(\sigma(BQ+1)/\sigma(BQ))=(\sigma(BQ+1)/(BQ+1))(\sigma(B)/B)^{-1}(\sigma(Q)/Q)^{-1}(BQ+1)/BQ$$

which for BQ large is close to

$$(\sigma(BQ+1)/(BQ+1))(\sigma(B)/B)^{-1}(\sigma(Q)/Q)^{-1}.$$

Again in this case we will find B and Q such that (i) $\sigma(Q)/Q$ is close to 1, and (ii) $(\sigma(BQ+1)/(BQ+1))(\sigma(B)/B)^{-1}$ is close to a. Then $\sigma(n+1)/\sigma(n)$ is close to a, and this suffices to put all real $a\geq 1$ in the closure.

We note first that for any given positive integer B, and real $b>1$, there exists an integer t_0 such that $(t_0,B)=1$ and $\sigma(t_0)/t_0$ is as close as we please to b. To see this, let q_{11} be a prime larger than all those dividing B, and such that $1+(1/q_{11})<b$. Then let $q_{11}, q_{12},\ldots,q_{1r_1}$ be consecutive primes such that

$$\left(1+\frac{1}{q_{11}}\right)\left(1+\frac{1}{q_{12}}\right)\cdots\left(1+\frac{1}{q_{1r_1}}\right)<b\leq\left(1+\frac{1}{q_{11}}\right)\cdots\left(1+\frac{1}{q_{1r_1}}\right)\left(1+\frac{1}{q_{1,r_1+1}}\right).$$

Such an r_1 exists, since the sum of the reciprocals of the primes diverges. Next choose q_{21} as a prime greater than q_{1r_1}, and large enough so that

$$\left(1+\frac{1}{q_{11}}\right)\cdots\left(1+\frac{1}{q_{1r_1}}\right)\left(1+\frac{1}{q_{21}}\right)<b.$$

Then let $q_{21},\ldots,q_{2r_2}$ be consecutive primes such that

$$\begin{aligned}&\left(1+\frac{1}{q_{11}}\right)\cdots\left(1+\frac{1}{q_{1r_1}}\right)\left(1+\frac{1}{q_{21}}\right)\cdots\left(1+\frac{1}{q_{2r_2}}\right)<b\\&\qquad\leq\left(1+\frac{1}{q_{11}}\right)\cdots\left(1+\frac{1}{q_{1r_1}}\right)\left(1+\frac{1}{q_{21}}\right)\cdots\left(1+\frac{1}{q_{2,r_2+1}}\right).\end{aligned}$$

Clearly, the left side above is of the form $\sigma(t_0)/t_0$, with t_0 square-free and prime to B; and this can be made as close as we like to b by continuing the process.

Fixing a large positive integer C, take $B=\Pi_{p\leq C}\,p$, and by the above choose t_0 prime to B such that $\sigma(t_0)/t_0$ is close to $b=a\sigma(B)/B$. Since $\sigma(B)/B>1$, it follows that $(\sigma(t_0)/t_0)(\sigma(B)/B)^{-1}$ is close to a. Since $(t_0,B)=1$, the Chinese Remainder theorem provides a solution k_0 of the simultaneous congruences

$$k_0t_0 \equiv e \pmod{B}, \qquad \text{and} \qquad k_0 \equiv 1 \pmod{t_0}$$

with $0<k_0<Bt_0$, where $e=-1$ or 1 in correspondence to the two cases described earlier. Further, we note then that $(k_0,B)=(k_0,t_0)=1$, and hence the congruence $t_0^2s_0 \equiv 1-((k_0t_0-e)/B) \pmod{B}$ has a solution for s_0, $0 \leqslant s_0 < B$. Finally, our candidates for Q are in the arithmetic progression

$$Q=t_0^2Bs+\left(t_0^2s_0+((k_0t_0-e)/B)\right), \tag{*}$$

and we will show that by taking C large, we can find many positive integers s such that the corresponding Q has the desired properties.

Note first that the above implies $Q \equiv 1 \pmod{B}$ so that $(Q,B)=1$. Also,

$$BQ+e=t_0\left(t_0B^2s+t_0s_0B+k_0\right),$$

and

$$k=k(Q)=t_0B^2s+t_0s_0B+k_0 \equiv k_0 \pmod{Bt_0}$$

implying that $(k(Q),Bt_0)=1$. Thus we have

$$(\sigma(BQ+e)/(BQ+e))(\sigma(B)/B)^{-1}=(\sigma(t_0)/t_0)(\sigma(B)/B)^{-1}(\sigma(k(Q)/k(Q))),$$

and the desired result follows if we can provide a Q in (*) such that both $\sigma(Q)/Q$ and $\sigma(k(Q))/k(Q)$ are close to 1. For this purpose consider

$$\mathbb{S}=\sum_{s<k}\left\{\sum_{q|Q}\frac{1}{q}+\sum_{p|k(Q)}\frac{1}{p}\right\},$$

where p, q denote primes, and $Q=Q(s)$ is given by (*). Then since both Q and $k(Q)$ are prime to B, we have

$$\begin{aligned}
\mathbb{S}&=\sum_{s\leqslant k}\left\{\sum_{\substack{q|Q\\ q>C}}\frac{1}{q}+\sum_{\substack{p|k(Q)\\ p>C}}\frac{1}{p}\right\}\\
&=\sum_{\substack{q\leqslant 3t_0^2Bx\\ q>C}}\frac{1}{q}\sum_{\substack{s\leqslant x\\ Q\equiv 0 \pmod{q}}}1+\sum_{\substack{p\leqslant 3t_0B^2x\\ p>C}}\frac{1}{p}\sum_{\substack{s\leqslant x\\ k(Q)\equiv 0 \pmod{p}}}1\\
&\leqslant x\sum_{q>C}\frac{1}{q^2}+\sum_{q\leqslant 3t_0^2Bx}\frac{1}{q}+x\sum_{p>C}\frac{1}{p^2}+\sum_{p\leqslant 3t_0B^2x}\frac{1}{p}\\
&\leqslant\frac{2}{C}x+c_1\log\log(3t_0^2B^2x).
\end{aligned}$$

Thus if N denotes the number of $s\leqslant x$ such that

$$\sum_{q|Q}\frac{1}{q}+\sum_{p|k(Q)}\frac{1}{p}>C^{-1/2}$$

we get

$$N\leqslant\frac{2}{\sqrt{C}}x+\left(\frac{c_1\log\log(3t_0^2B^2x)}{C^{-1/2}x}\right)x,$$

which implies that for any small $\eta>0$, by taking C large and then x large (in particular with respect to C), we obtain $N\leqslant\eta x$. Since the total number of $s\leqslant x$ is $[x]$, it follows that there are many values of s such that

$$\sum_{q|Q}\frac{1}{q}+\sum_{p|k(Q)}\frac{1}{p}\leqslant C^{-1/2}$$

and hence both

$$\sum_{q|Q} \frac{1}{q} \leqslant C^{-1/2} \qquad \text{and} \qquad \sum_{p|k(Q)} \frac{1}{p} \leqslant C^{-1/2}.$$

But this in turn yields

$$1 < \frac{\sigma(Q)}{(Q)} < \prod_{q|Q} \frac{1}{1-1/q} < e^{2\Sigma_{q|Q}\frac{1}{q}} < 1+4\sum_{q|Q}\frac{1}{q} < 1+4C^{-1/2},$$

and similarly

$$1 < \frac{\sigma(k(Q))}{k(Q)} < 1+4C^{-1/2},$$

so that both $\sigma(Q)/Q$ and $\sigma(k(Q))/k(Q)$ are forced close to 1 for C large.

REMARKS. (1) By using the Titchmarsh sieve inequality and the prime number theorem for arithmetic progressions the above argument may be structured more simply. Namely, Q may be taken to be a prime, and one has only to show that $\sigma(k(Q))/k(Q)$ can be made close to 1.

(2) This result is really a consequence of a kind of "statistical independence" between n and $n+1$. It easily generalizes to the same conclusion for $(\sigma(n+i_1)\cdots\sigma(n+i_r)/(\sigma(n+j_1)\cdots\sigma(n+j_r)$, where $i_1,\ldots,i_r,j_1,\ldots,j_r$ are distinct integers.

(3) Even more general results can be proved by the methods used above. For example, it can be shown that the closure of the set of points $\{\sigma(n)/n,\sigma(n+1)/(n+1)\}$ consists of all points (x,y) with $x \geqslant 1, y \geqslant 1$.

Also solved by by Douglas Hensley.

Editor's comments. (i) Hensley gives a shorter proof of the above result using a theorem (10.7) in Halberstam and Richert, *Sieve Methods*. The problem is stated as: Let f be multiplicative and $\lim_{p\to\infty} f(p^t)/p^t = 1$, p prime, $t \geqslant 1$, $\Sigma_p|1-f(p)/p| = \infty$, then $\{f(n)/n\}$ is dense in $\boldsymbol{R}^+$.

(ii) Bob Prielipp notes that a statement (without proof) of the result of the problem may be found in A. Schinzel, *Generalization of a theorem of B.S.K.R. Somayajulu on the Euler ϕ-function*, Ganita (Lucknow), v. 5 (1954), pp. 123–128.

(iii) This is also Problem 982 in the *Mathematics Magazine* (May, 1976); no solution is given but generalizations and further references are provided.

Multiplicative Identities for $\tau(n)$

6108 [1976, 661]. *Proposed by Aleksander Ivić, Novi Sad, Yugoslavia*

Find all multiplicative functions $f(n)$ such that:

$$(1) \qquad f(n^2) = \sum_{d|n} \mu^2(d) f\left(\frac{n}{d}\right) \qquad \text{and} \qquad (2) \quad f^2(n) = \sum_{d|n} f(d^2).$$

Solution by Al G. Braist, Courant Institute, New York University. $f(n)$ identically zero is one solution. For $f(n)$ not identically zero, multiplicativity implies $f(1)=1$; and $f(n)$ is determined by the values $f(p^m)$, p a prime. The solutions to (1) and (2) are those $f(n)$ such that for each fixed prime p, the sequence $f(p^m)$, $m=0,1,2,\ldots$ is *either* the sequence (A) or the sequence (B):

$$\text{(A)} \qquad f(p^m) = m+1 = \tau(m),$$

$$\text{(B)} \qquad f(p^m) = \begin{cases} 1 & \text{for } m \equiv 0 \pmod 3 \\ -1 & \text{for } m \equiv 1 \pmod 3 \\ 0 & \text{for } m \equiv 2 \pmod 3). \end{cases}$$

Because of the multiplicativity the conditions (1) and (2) may be replaced by the requirement that

for each fixed prime p

(3) $$f(p^{2m}) = f(p^m) + f(p^{m-1}) \quad \text{and} \quad (4)\ f^2(p^m) = \sum_{i=0}^{m} f(p^{2i}),$$

for all integers $m \geqq 1$. These imply that

$$f^2(p^m) = f^2(p^{m-1}) + f(p^{2m}) = f^2(p^{m-1}) + f(p^m) + f(p^{m-1}),$$

and solving this quadratic equation yields that

(5) $$f(p^m) = -f(p^{m-1}) \quad \text{or} \quad f(p^m) = f(p^{m-1}) + 1.$$

In fact, clearly, the conditions (3) and (5) are equivalent to (3) and (4). Both sequences (A) and (B) given above satisfy (5), and (3) is easily verified; so that they are indeed solutions.

Next we show that (A) and (B) are the only solutions to (3) and (5) with $f(1) = 1$. In fact, from (5), $f(p) = -1$ or 2; and the choice $f(p) = -1$ leads to sequence (B), $f(p) = 2$ leads to sequence (A).

For $f(p) = 2$, we proceed by induction and assume that $f(p^r) = r + 1$ for $0 \leqq r < m$. Then if m is even, $m = 2s$, and from (3),

$$f(p^m) = f(p^s) + f(p^{s-1}) = (s+1) + s = 2s + 1 = m + 1.$$

If m is odd, $m = 2s + 1$, $m + 1 = 2(s+1)$, and from (3),

$$f(p^{m+1}) = f(p^{s+1}) + f(p^s) = (s+2) + (s+1) = 2s + 3 = m + 2.$$

Then from (5), since $f(p^{m-1}) = m$, we have $f(p^m) = -m$ or $m + 1$ and then in turn $f(p^{m+1}) = m$ or $-m + 1$, or $-m - 1$ or $m + 2$, respectively. Since $f(p^{m+1})$ must be $m + 2$ by the induction hypothesis, therefore $f(p^m) = m + 1$, which completes the induction. Thus the case $f(p) = 2$ produces the sequence (A).

For $f(p) = -1$, it follows from (3) that $f(p^2) = f(p) + 1 = 0$ which establishes (B) for $m = 0, 1, 2$. Proceeding by induction, assume (B) to hold for all values of m of the form $3r$, $3r + 1$, $3r + 2$, with $0 \leqq r < k$. Consider then the triplet $3k$, $3k + 1$, $3k + 2$. If k is even, $k = 2s$, and (3) gives $f(p^{3k}) = f(p^{3s}) + f(p^{3s-1}) = 1 + 0 = 1$; and $f(p^{3k+2}) = f(p^{3s+1}) + f(p^{3s}) = -1 + 1 = 0$. Then from (5), $f(p^{3k+1}) = -1$ or 0 leading to $f(p^{3k}) = 1$ or -2, or 0 or -1, respectively. Since $f(p^{3k}) = 1$ we must have $f(p^{3k+1}) = -1$; and the induction is completed in case k is even.

A similar analysis takes care of the case k is odd. Now $f(n)$ is determined by writing $n = \Pi p_i^{e_i}$ and using for $f(p_i^{e_i})$ the value determined in either case (A) or case (B).

Also solved by Tom Apostol, Robert Breusch, Irving Gerst, Lael Kinch, Adnah Kostenbauder, O. P. Lossers (Netherlands), L. E. Mattics, Jerry Metzger, V. Sita Ramaiah (India), Bart Rice, Ernst Trost (Switzerland), Ken Yocom, and the proposer.

Editor's notes. (1) Gerst notes that if the primes p_i all are in case (A) then $f(n)$ is generated by the coefficients of $\zeta^2(s)$; and by $\zeta(3s)/\zeta(s)$ if p_i are all in case (B).

(2) Ramaiah comments: It may be of interest to note that there is no multiplicative function f satisfying

$$f(n^2) = \sum_{d \| n} (\mu^*)^2(d) f(n/d) \quad \text{and} \quad f^2(n) = \sum_{d \| n} f(d^2),$$

where the summations are over all divisors (positive) d of n, satisfying $(d, n/d) = 1$ and $\mu^*(n) = (-1)^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct prime factors of n.

Sylvester Series and Normal Families

6109 [1976, 661]. *Proposed by Stuart P. Lloyd, Bell Laboratories, Murray Hill, New Jersey*

Define functions $S_1(z), S_2(z), \ldots,$ by $S_1(z) = z$, $S_{n+1}(z) = \varphi(S_n(z))$, $n \geqq 1$, where $\varphi(s) = s + s^2$. When z is a positive integer, the series

$$(*)\qquad \frac{1}{z}=\sum_{n=1}^{\infty}\frac{1}{S_n(z)+1}$$

is the nonterminating Sylvester series for rational number $1/z$ [H. E. Salzer, *Further remarks on the approximation of numbers as sums of reciprocals*, this MONTHLY 55, (1948), 350–356]. Determine the region of convergence of (*) in the complex z-plane.

Solution by I. N. Baker, Imperial College, London, England. The solution is implicitly contained in the iteration theory of Fatou and Julia, see, e.g., P. Fatou, *Sur les équations fonctionelles*, Bull. Soc. Math. de France 47 (1919) 161–271 and 48 (1920) 33–94 and 208–314. The iteration of quadratic functions including $s+s^2$, has been treated explicitly by P. J. Myrberg, e.g., in Journ. de Math. 41 (1962) 339–351.

The function $S_{n+1}(z)$ is the nth iterate $\varphi_n(z)$ of $z+z^2$. The sequence $\{\varphi_n\}$ forms a normal family except on a set $\mathcal{F}$ which divides the plane into regions of normality. Both $\mathcal{F}$ and its complement are invariant both under $z\to\varphi(z)$ and $z\to\varphi^{-1}(z)$.

The disc D: $|z+\frac{1}{2}|<\frac{1}{2}$ is mapped so that $\varphi(D)\subset D$ and in fact $\varphi_n(z)\to 0$ locally uniformly in D, as can be seen, e.g., by considering the behavior for real negative $x>-1$. The component D_0 of $\mathcal{C}(\mathcal{F})$ which contains D therefore contains the only finite singularity of φ^{-1}, viz. $\frac{1}{4}$, and from this and invariance of $\mathcal{C}(\mathcal{F})$ under $z\to\varphi^{-1}$ it follows that D_0 is invariant under $z\to\varphi^{-1}$. Also $\varphi_n(z)\to 0$ in the whole of D_0 and by Weierstrass' theorem D_0 is therefore simply-connected. From further results of the theory it follows that $\mathcal{F}$ is identical with the boundary of D_0 and that this boundary is in fact a Jordan curve Γ.

For real $x>0$ we have $\varphi_n(x)\to\infty$, so $z=0$ is a boundary point of D_0 and so therefore is $\varphi^{-1}(0)=-1$. Γ is a curve symmetric to the real axis, which it cuts at $0,-1$. D_0 contains sets of the form $\{z: |\arg z-\pi|<\pi-\varepsilon, |z|<\sigma(\varepsilon)\}$ for any $\varepsilon>0$, so Γ is not differentiable at 0, nor, as it turns out, at any other point.

The exterior of Γ is a domain D_1 in which $\varphi_n(z)\to\infty$, and this convergence will be uniform in D_1 outside an arbitrary neighborhood N of Γ. We may choose $A>2$ so large that $|z|\geqq A$ implies $|z^2+z|>\frac{1}{2}|z|^2$. Then there exists n_0 so that $\varphi_{n_0}(D_1-N)\subset\{|z|>A\}$, and so for $n>n_0$,

$$|\varphi_n(z)|>A^{2n-n_0}/2\cdot 2^{2(n-n_0-1)}.$$

This implies that (*) converges uniformly in D_1-N to a function ψ which is regular there (including at ∞). Thus in fact the right hand side of (*) converges exactly in D_1 with the regular sum $\psi(z)$. Since $\psi(z)=1/z$ for $z=n\to\infty$ and is regular at ∞ it follows that $\psi(z)\equiv 1/z$ in D_1.

Also solved partially by the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
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We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Mathematical Methods in Science. By George Pólya. Mathematical Association of America, Washington, D.C. 1977. xi + 234 pp. \$4.50 (paper back). (Telegraphic Review, June-July 1977.)

After reading this thought-provoking slender introductory "physics" book, I asked myself how it had come to be written by a distinguished mathematician and published by the MAA as part of its "New Mathematical Library." My conclusion was that it represented a missionary effort to break down the isolation of mathematics teaching, quite different in orientation from what the titles of the first twenty-five volumes in the "New Mathematical Library" communicated to me. Pólya's stature qualified the book as "mathematics" and the MAA thought it worthwhile to call to the attention of its members, other teachers, and students, a selection of physical applications of mathematics with historical and heuristic interest.

Reading the book also led me to examine what I consider to be mathematics. I found that I view the field as consisting of two quite distinct components. One of these includes the algorithms and other procedures of algebra, geometry, and analysis that I had learned as a student and have come to use in a more or less routine way in my professional work. The other component includes the mysterious and largely useless (to me) pursuits engaged in by mathematicians when they are not teaching introductory or service courses: topology, formal logic, set theory, and so on.

These statements may strike you as a confession of ignorance, yet they indicate the opinions of a diligent but not professionally oriented former student—probably reflecting the image of mathematics communicated by his high school and undergraduate education. *Mathematical Methods in Science* may have the secondary purpose of overcoming prejudices about mathematics like mine, but with me it did not succeed: in spite of its authorship, I classified it as a *physics* book explaining and using familiar mathematical processes. I did not become clearly aware of the examples' significance for mathematics.

The four substantial chapters in the book deal with measurement in the history of astronomy, statics, dynamics, and application of differential equations. A fifth chapter, on "Physical Reasoning in Mathematics," had aroused my particular curiosity, but is not included in the book; the reader is directed to Pólya's *Mathematics and Plausible Reasoning*, Vol. 1, for this material.

The examples from the history of astronomy in the first chapter are very appropriate for illustrating triangulation, calculation of the earth's radius, and the geometrical relations associated with planetary motion around the sun. Because direct measurements are so obviously out of the question in these situations, the mathematical approach is clearly necessary. Since it is also relatively simple, some of the same topics have even been recently included in new science curricula for junior high schools.

Chapter Two is concerned with equilibrium on an inclined plane, levers, and the algebra of vectors that occurs in composition of forces and velocities. Here physical examples and thought experiments—another important scientific concept— are used copiously and successfully.

The third chapter presents examples from the history of dynamics. Motion in free fall and on an inclined plane, uniform circular motion and the law of universal gravitation, oscillation of the pendulum, and orbital/escape velocities of earth satellites are taken up in order. Here again, thought experiments figure importantly. Another important idea introduced is that of velocity space: a space in which points represent velocity rather than position vectors.

A pedagogical point raised by Pólya is the translation of verbally stated problems into mathematical relationships and formulae. My experience with my college physics students, many of whom would have difficulty with the material of Chapter Three, is that this goal is not achieved satisfactorily at present. Perhaps one cause is the poor quality of word problems in mathematics texts, as asserted by Pólya. Another may be inadequate connections between science and mathematics courses.

The last substantial chapter, concerned with the application of differential equations, requires much more sophisticated techniques than did the first three. Accordingly, the analysis of problems is divided into a physical, a transitional, and a mathematical phase. This approach further illustrates that the solution of word problems requires qualitative considerations before computational techniques are applied.

I invited a gifted high school senior to read the book, since the editors of the New Mathematical Library identify this audience in their "Note to the Reader." He only completed Chapter One, undoubtedly without the paper-and-pencil approach recommended by the editors in their "Note." He reported that the chapter had been interesting, but that the *tours de force* used to solve many of the tasks had awed him without raising the challenge of how he himself might invent a successful procedure.

In summary, I would recommend the book for occasional reading to gifted high school and beginning college students in just the way suggested in the editors' "Note." Mathematics and physics teachers should suggest it to their students and might well browse in it themselves. Teachers of chemistry, biology, and other sciences will find it much less relevant to their fields. Further, I fear that the model provided by Pólya himself as he approaches a physics problem, turns it this way and that, attacks it from an unexpected angle, and then solves it elegantly, is likely to be admired more than it is emulated by the reader.

ROBERT KARPLUS, University of California—Berkeley

Since I am a mathematician, I view Pólya's book quite differently from Karplus. My own ignorance, like that of many persons trained in mathematics, is of the world of science surrounding mathematics. *Mathematical Methods in Science* serves as an introduction to an alien universe, one to which Karplus needs no introduction. The "mathematics" of the book resides, I think, in its organization and perspective: the *themes* are mathematical, and the science is approached through these themes. Instead of "Consequences of the law $F = ma$," for example, we have instances of "measurement and successive approximation." The starting ground is at least familiar to the xenophobic mathematician, and less frightening when seen from that perspective.

Therein, I believe, resides the primary virtue of the book. Karplus' "outsider" perspective of mathematics is unfortunately all too common, and he is quite right in laying much of the blame at our own doorstep. The interplay of science and mathematics throughout history has been enormous and beneficial to both. An artificial compartmentalization of mathematics and the introduction of techniques and tools without demonstrating their value and fertility has unquestionably been to our disadvantage. Let the mathematics teacher occasionally peek outside his domain; let him demonstrate to his students that mathematics can be useful, fruitful, and aesthetically gratifying, and perhaps future generations of students will not despise the subject as so many of our students do now. This is indeed, as Karplus suggests, "missionary work," and we would do well to consider it.

ALAN H. SCHOENFELD, University of California—Berkeley

Mathematical Gems. By Ross Honsberger. Mathematical Association of America. Vol. I, 1973, xi + 176 pp. $11.00; Vol. II, 1976, ix + 182 pp. $11.00. (TR, V. I: June–July 1974; V. II: December 1976.)

If p is a prime, then p divides $a^p - a$. Is it true, conversely, that if an integer n (>1) divides $a^n - a$ for every integer a, then n is a prime? What if n is known to divide $2^n - 2$—is that by itself enough to imply that n is a prime? If you think that the answer to the latter question is yes, you're in trouble. The trouble is with the famous Fermat numbers $2^{2^n} + 1$ ($n = 0, 1, 2, \ldots$). If F is any one of them, then F divides $2^F - 2$, and, although 3, 5, 17, 257, and 65537 are primes, the next Fermat number ($2^{2^5} + 1 = 4294967297$) is equal to 641×6700417. By the way, the answer to the first question is no also; 561 ($= 3 \times 11 \times 17$) is an "absolute pseudoprime", i.e., it divides $a^{561} - a$ for every a.

Does anyone think that such facts are interesting? Yes, I do, and many people I know do. Why? Is it partly because of the history of the subject, because Fermat guessed wrong (he said $2^{2^n} + 1$ is always

a prime), and because of the relation between Fermat primes and the ruler-and-compass constructibility of regular polygons?

Here is another "gem". A simple graph is a finite set of objects (each of which is called a vertex), and a finite set of unordered pairs $\{a, b\}$ of vertices (where $a \neq b$). In common parlance $\{a, b\}$ is called an "edge" that "joins" a and b. If the number n of vertices is at least 3, and if each vertex occurs in at least $n/2$ edges, then there exists an ordering of the vertices, $v_1, v_2, \ldots, v_n$, such that each of $\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$ is an edge—there exists a "Hamiltonian circuit". Does anyone think that such facts are interesting? I don't, but many people I know do. You can't win them all.

The two volumes under review consist of a total of 27 essays about the kinds of subjects illustrated by these examples, the kind that mathematicians, on a busman's holiday, like to talk about.

Did you know that if equilateral triangles are erected outward on the sides of an arbitrary triangle, then their centers form an equilateral triangle? And did you know that that's called Napoleon's theorem? That's another sample of the gems and their facets that can be found in the book; let me give you a few more, non-randomly sprinkled through the two volumes.

A plane region of area 8.39 can always be translated to cover at least nine lattice points. (This is called Blichfeldt's lemma and I found its geometric, measure-theoretic proof a beautiful piece of pure mathematical thought. Needless to say, 8.39 and 9 are irrelevant; an area greater than n implies a translate with more than n lattice points.)

It is trite that if two diagonally opposite corner squares are removed from an ordinary 8×8 chessboard, then the remainder cannot be covered by 31 dominoes of size 1×2; the reason is that 31 dominoes cover 31 black squares and 31 white ones, whereas the removal of opposite corners leaves 32 squares of one color and 30 of the other. What if two squares of different colors are removed from two possibly distant parts of a chessboard—can the remainder always be covered by 31 dominoes? The answer is not immediately obvious, but a satisfying combinatoric argument (Gomory's theorem) says that it's yes.

Is it obvious that the maximum number of regions into which a convex hexagon can be divided by its diagonals is 25? (Caution: the diagonals of a regular hexagon divide it into only 24 pieces.) What about a convex n-gon?

Can you find a circle that contains exactly 19 lattice points in its interior? How about a circle that contains exactly 19 lattice points on its perimeter? What if squares are wanted instead of circles?

Does it please you to know that for each of the 80 integral values of x between 1460 and 1539 inclusive the value of the polynomial $x^2 - 2999x + 2248541$ is a prime? If it does, you might be interested in the function f defined (for positive integers x and y) as follows:

$$f(x, y) = 2 + (y - 1)\operatorname{neg}(x(y + 1) - (y! + 1)).$$

Here neg is the "negative part" function, which assigns to each number Z the value 0 if $Z \geqq 0$ and the value $-Z$ if $Z < 0$; explicitly

$$\operatorname{neg}(z) = \tfrac{1}{2}(|z| - z).$$

What's good about the function f is that its range is the set of all prime numbers, and, moreover, that it takes each odd prime value just once. Is that good?

Here is a bit of geometry that probably none of us learned in high school: if all the faces of a tetrahedron have the same area, then they are congruent. Here is another: 14 points in the plane determine $\binom{14}{2} = 91$ not necessarily distinct distances, but there must be at least 4 distinct ones among them, and no distance can ever occur more than 40 times.

That should give you the idea. Honsberger's two volumes are recreational mathematics. There is already a large library of such books, ranging from "How many 3-cent stamps in a dozen?" by

Herman Hover (Price–Stern–Sloan, 1976—the answer is 12), good for winning bar bets and amusing ten-year olds, to "Famous problems of mathematics" by Heinrich Tietze (Graylock, 1976), good for introducing serious students to the joy of mathematics, to some of its history, and to its immense extent. The recent "Annotated bibliography" by M. P. Gaffney and L. A. Steen (MAA, 1976) was designed for other purposes but lists many of the best recreational books also. At least two others of my favorites deserve explicit mention: "Mathematical snapshots" by Hugo Steinhaus (Oxford, 1969), and "The enjoyment of mathematics" by Hans Rademacher and Otto Toeplitz (Princeton, 1957). And do I dare stop without mentioning at least the names of W. W. Rouse Ball, Lewis Carroll, George Pólya, and Martin Gardner?

Why is the recreational library growing? Why do authors write such books? I can think of two answers: to attract the amateur and introduce him to new subjects, and to amuse the professional and "remind" him of possibly never learned ones. The first of these is surely more important. It's not that I advocate drumming up business for mathematics—in fact I believe that that's bad, and that no one should become a mathematician unless he is certain that that's the only profession he is willing to spend his life professing—but it is important to show the potential Archimedeses, Gausses, Ramanujans, and more nearly ordinary mortals of the world that they have options that their home and their school may not have made accessible.

Even if the amateur (whether or not he is a potential future mathematician) is the reader for whom recreational books are written, the writing is very strongly influenced by the knowledge that the books will be read by the professional also. Knowing that, authors hunt far and wide for new material, go out of their way to reformulate the old, and, often, lean over backward much too far to avoid boring the blasé. The result is likely to be a book of uneven quality—some bad stuff is included just because the experts would sneer at the good stuff for being familiar.

Another obstacle that faces the recreational author is that of prerequisites. Almost all of Honsberger's gems are from number theory, combinatorics, and Euclidean geometry—subjects that can often state surprising results without technical terms, and can sometimes prove them without sophisticated tools. What else could he have done? What else is accessible? Probability theory is one answer (there are two pages on the gambler's ruin), and some of the pleasant paradoxes about infinite series offer another (there is one essay on that). We all face the problem when we lecture to lay audiences, and, with only occasional exceptions, we all solve it the same way. Some Boolean algebra can be used (Smith is the pilot, Jones beats the copilot at billiards, and Robinson is married to the engineer's sister—who lands the plane?), some logic (how to get reliable answers by questioning an unpredictable liar?), a little game theory (is it obvious that someone must always win at chess?, can poker be played without bluffing?), the topology of two-manifolds (Möbius strips, the Euler formula), and, in moderation, some calculus (Zeno's paradox) and some set theory (there are just as many rational numbers as integers). Honsberger's choices are more limited than they need to be, but they are legitimate choices, and authors should not be faulted for not doing what they never promised to do.

How well did he do what he did? Answer: remarkably well. The quality of the "gems" is not uniformly high (and seems uniformly lower in Volume 2)—Kool-Aid is indiscriminately mixed with Liebfraumilch. Definitions are quick and clear—no nonsense about them. Proofs are short and usually elegant. There is some hard sell ("the present work seeks to charm you with some wonderful pieces of mathematics", "...an intriguing problem and the beautiful mathematics involved in its solution", "...another delightful proof"). The two volumes overlap a little. There are conspicuously many references to the students, the faculty, the visitors, and the neighbors of the University of Waterloo.

These are minor cavils withal. I'd be glad to have written this book, and the next few times I talk to lay audiences I shall happily steal material from it.

P. R. HALMOS, University of California—Santa Barbara

TELEGRAPHIC REVIEWS

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T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P*, L*. *Why the Professor Can't Teach: Mathematics and the Dilemma of University Education.* Morris Kline. St. Martin's Pr, 1977, 288 pp, $10. [ISBN: 0-312-87867-2] A lively, deliberately exaggerated polemic: "Mathematics education has been a debacle...[for] in today's world, research and undergraduate education are in direct conflict." "The creative researcher is most likely to be no more than a proficient but limited technician..." "University administrations are staffed largely with people who strive hard to perpetuate what they do not understand." "...[P]rofessors' presentations promote not the emancipation but the enslavement of minds." "The elitist, narcissistic mathematician...is totally unfit to be a teacher in any course." Undergraduate mathematics texts are, in Kline's view, "unintelligible," "challenges to clairvoyance," "puerile," "cold, monotonous, dry, dull, and even ungrammatical," "pedagogically disastrous," and "deliberately fraudulent." Dedicated to the American undergraduate and addressed to administrators, parents, teachers, legislators and ordinary citizens concerned with the effectiveness of education, this latest of Kline's fusillades on the mathematical establishment will surely reverberate in university halls for years to come. LAS

GENERAL, T(13: 1). *Unifying Concepts and Processes in Elementary Mathematics.* U. of Maryland Math. Project. Allyn, 1978, x + 323 pp, $18.95. [ISBN: 0-205-05844-2]; *Instructor's Supplement,* 268 pp. [ISBN: 0-205-05845-0] This could be a text for elementary school teachers or a text for a basic liberal arts course. Topics include probability and statistics, geometric transformations, binary operations and groups. Instructor's supplement includes suggestions for use with small groups, answers to exercises, and additional problems. LLK

GENERAL, L. *Mathematical Circles Adieu.* Howard W. Eves. Prindle, 1977, xxiii + 181 pp, $10. [ISBN: 0-87150-240-2] An unintended sequel to the author's four well-known previous *Circles,* motivated and fueled by readers' contributions. This is the *final* installment. LAS

GENERAL, S(13-14). *Problem-mathics: Mathematical Challenge Problems with Solution Strategies.* Carole E. Greenes, Rika Spungin, Justine M. Dombrowski. Creative Pub, 1977, 141 pp, $5 (P). [ISBN: 0-88488-085-0] Presents 25 problems in a non-mathematical setting. Discussions of the solutions illustrate the basic techniques of problem solving, while presenting simple ideas from abstract algebra, number theory, geometry, and logic. Done in a clever, interesting way. TLS

GENERAL, S, L. *Euclidians.* Guillevic. Trans: Teo Savory. Unicorn Pr, 1977, 63 pp, $4 (P); $10. [ISBN: 0-87775-079-3; 0-87775-053-X] A collection of simple, wry poems about Euclidean things (e.g., cone, rhomboid, parallels) that, in the words of one reviewer, "offers not the words *for* things, but the words *of* things." One of Europe's leading contemporary poets, Guillevic was recently awarded the *Grand Prix de la Poésie* by the Académie Francaise. *Euclidians* is an English translation of the 1967 *Euclidiennes.* LAS

GENERAL, S(13), L. *Mind Tickling Brainteasers.* E.R. Emmet. Emerson, 1978, 255 pp, $7.95. [ISBN: 87523-192-6] 127 puzzles from very elementary to moderately advanced, graded accorded to difficulty, requiring no special knowledge of mathematics. Includes hints, as well as complete explanations. LCL

PRECALCULUS, T(13: 1, 2). *College Algebra, Fourth Edition.* Edwin F. Beckenbach, Irving Drooyan, William Wooton. Wadsworth, 1978, x + 411 pp, $14.95. [ISBN: 0-534-00536-5] In addition to minor revisions and organizational changes, a section on complex numbers has been added. (*Second Edition,* TR, February 1969.) JRG

PRECALCULUS, T(13). *An Elementary Approach to Functions, Second Edition.* Henry R. Korn, Albert W. Liberi. McGraw, 1978, xii + 459 pp, $14.50. [ISBN: 0-07-035401-4] A clearly and simply written precalculus text. Separate chapters on quadratic, polynomial, rational, trigonometric, and exponential functions. Also chapters on inequalities, straight lines, and conic sections. Many word problems. SG

PRECALCULUS, T(13: 1). *A Primer for Calculus.* Leonard I. Holder. Wadsworth, 1978, xi + 493 pp, $13.95. [ISBN: 0-534-00554-3] The author has written a precalculus text in which he has tried to provide "a middle ground between texts which are either too elementary or too sophisticated." It is a good review of topics needed for calculus. LLK

PRECALCULUS, T(13: 1). *Plane Trigonometry with Tables, Fifth Edition.* Gordon Fuller. McGraw, 1978, xi + 276 pp, $12.95. [ISBN: 0-07-022612-1] Same topics as *Fourth Edition* (TR, June/July 1972). Almost all problems are new. LCL

EDUCATION, P. *The Gifted and the Creative: A Fifty-Year Perspective.* Ed: Julian C. Stanley, William C. George, Cecilia H. Solano. Johns Hopkins U Pr, 1977, xiv + 284 pp, $17.50; $4.95 (P). [ISBN: 0-8018-1974-1; 0-8018-1975-X] Third volume in the Studies of Intellectual Precocity based on symposia at Johns Hopkins held in conjunction with their Study of Mathematically Precocious Youth (SMPY). Includes longitudinal studies based on SMPY, and various reports on the effects of sex differences. LAS

EDUCATION, T*(14-17: 1), S, P, L*. *Thinking and Problem Solving: An Introduction to Human Cognition and Learning.* Richard E. Mayer. Scott, Foresman, 1977, 214 pp, $6.95 (P). [ISBN: 0-673-15055-0] Why is it that some people, when they are faced with problems, get clever ideas, make inventions and discoveries? This book constitutes a highly readable summary of how psychologists respond to this question. LCL

HISTORY, L. *The Crime of Claudius Ptolemy.* Robert R. Newton. Johns Hopkins U Pr, 1977, xiv + 411 pp, $22.50. [ISBN: 0-8018-1990-3] A controversial analysis of Ptolemy's astronomical data, leading the author to the conclusion that it was "dry-labbed:" the reported "observations" frequently agree to every significant digit with what would have been inferred by known calculation from existing Hellenic data. For contrasting view, see review in *Science* 199 (1978) 872. LAS

HISTORY, P, L*. *Dear Russell-Dear Jourdain.* I. Grattan-Guinness. Columbia U Pr, 1977, 234 pp, $20. [ISBN: 0-213-04460-7] Previously unpublished correspondence between Bertrand Russell and Philip Jourdain from 1902 until Jourdain's death in 1919, supplemented by a few related manuscripts and notes, and unified by extensive interpretive commentary by Grattan-Guinness. A new, uninhibited view of the vigor with which the antinomies and axioms of set theory were debated. LAS

FOUNDATIONS, S, P. *Chance, Cause, Reason: An Inquiry Into the Nature of Scientific Evidence.* Arthur W. Burks. U of Chicago Pr, 1977, xvi + 694 pp, $27.50. [ISBN: 0-226-08087-0; 0-226-08088-9] Attempts a critique and reformulation of inductive logic and the logic of probability that culminates in a unified theory of probability, causality, and induction. FLW

FOUNDATIONS, S(17-18), P, L*. *Lecture Notes in Mathematics-617: The Axiom of Constructibility: A Guide for the Mathematician.* Keith J. Devlin. Springer-Verlag, 1977, viii + 95 pp, $8.30 (P). [ISBN: 0-387-08520-3; 3-540-08520-3] A careful description of V = L with applications to resolving problems in set theory, algebra, topology and measure theory that otherwise (e.g., with only AC) are not resolvable. Well motivated, and clearly written for the mathematician who knows no formal logic. LAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-619: Set Theory and Hierarchy Theory V.* Ed: A. Lachlan, M. Srebrny, A. Zarach. Springer-Verlag, 1977, viii + 358 pp, $14.30 (P). [ISBN: 0-387-08521-1; 3-540-08521-1] Proceedings of the third conference held at Bierutowice, Poland, September 17-24, 1976. JAS

COMBINATORICS, T*(16-18), P*, L**. *The Theory of Error-Correcting Codes.* F.J. MacWilliams, N.J.A. Sloane. Math. Lib., V. 16. North-Holland, 1977. *Part I,* xvi + 369 pp, $24.50 [ISBN: 0-444-85009-0]; *Part II,* ix + 392 pp, $32.75. [ISBN: 0-444-85010-4] A broad and readable introduction to the algebraic theory of block codes. Contains introductory chapters in linear, nonlinear, and cyclic codes, finite fields, weight distribution, and entire chapters to each of several important classes of codes: BCH, Reed-Solomon, MDS, Reed-Muller quadratic residues, and Golay. Suitable as a text in an elementary or advanced course in coding theory. Almost 1500 bibliography entries. May well become the bible in the field. SG

COMBINATORICS, P, L. *Interaction Models.* Norman Biggs. London Math. Soc. Lect. Notes, No. 30. Cambridge U Pr, 1977, 101 pp, $7.95 (P). [ISBN: 0-521-21770-9] An interaction function measures the "strength of interaction" between two vertices of a graph. Applications to physics. JEG

COMBINATORICS, P. *Proceedings of the Second Caribbean Conference in Combinatorics and Computing.* Ed: R.C. Read, C.C. Cadogan. U of West Indies, 1977, 223 pp, $12.50 (P). Texts or abstracts of seventeen contributed papers along with the texts of invited addresses by R.F. Churchhouse, R.K. Guy, F. Harary, and R.C. Read. CEC

ALGEBRA, T*(16-17: 1, 2), S*, P*, L*. *Application-Oriented Algebra, An Introduction to Discrete Mathematics.* James L. Fisher. Dun-Donnelley, 1977, xviii + 362 pp, $16.50. [ISBN: 0-7002-2504-8] Suitable as a one-term course in discrete mathematics (partially ordered sets, graph theory, Boolean algebra with applications to theories of social choice, networks, languages, finite state machines) and as a one-term course in applied algebra (semigroups and groups, modules, fields and universal algebras with applications to coding, machines, and cryptography). Similar in style and sophistication to Stone's *Discrete Mathematical Structures* (TR, February 1974; ER, March 1976) but not quite as engineering oriented. Matrix algebra prerequisite. LCL

ALGEBRA, P, L. *Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin.* Ed: Hyman Bass, Phyllis J. Cassidy, Jerald Kovacic. Acad Pr, 1977, xxii + 424 pp, $39.50. [ISBN: 0-12-080550-2] A collection of 29 papers celebrating "the influence that Kolchin's work on the Galois theory of differential fields has had on the development of differential algebra and linear algebraic group theory." JRG

ALGEBRA, T(18), P. *The Algebraic Structure of Group Rings.* Donald S. Passman. Wiley, 1977, xiv + 720 pp, $34.95. [ISBN: 0-471-02272-1] A comprehensive introduction to the subject. Includes extensive discussions of the trace map, augmentation ideal, polynomial identities, the semisimplicity problem, primitive rings, the zero divisor problem, and the isomorphism problem. Many exercises are included. SG

ALGEBRA, T(17-18), S, P, L. *Boolean Rings.* Alexander Abian. Branden Pr, 1976, ix + 394 pp, $12.50 (P). [ISBN: 0-8283-1687-3] A self-contained and well-written exposition of boolean rings. The development is based on algebraic, lattice-theoretic and topological approaches, so that it is accessible to students of algebra, foundations, logic, computer science and physics. JEG

ALGEBRA, P. *Lecture Notes in Mathematics-614: Theory of Hopf Algebras Attached to Group Schemes.* Hiroshi Yanagihara. Springer-Verlag, 1977, vii + 308 pp, $14.30 (P). [ISBN: 0-387-08444-4; 3-540-08444-4]

ALGEBRA, T(16). *Tratat de Algebră Modernă, V. I.* Ioan Purdea, Gheorghe Pic. Editura Academiei (Romania), 1977, 344 pp, Lei 27. A senior-level algebra text written in Romanian. Chapter titles: relations, universal algebras, groups, rings. A second volume is to follow. SG

CALCULUS, S(13-14), L. *If And Only If In Analysis.* Robert R. Dobbins. U Pr of America, 1977, x + 215 pp, $8.65 (P). [ISBN: 0-8191-0344-6] A "Gelbaum and Olmsted" at the freshman-sophomore level, providing a commentary (via examples and counterexamples) for the major theorems in one-variable calculus. Will serve as a useful student supplement to most standard texts. LCL

REAL ANALYSIS, T(16-17: 1), S. *Integral, Measure and Derivative: A Unified Approach, Revised English Edition.* G.E. Shilov, B.L. Gurevich. Trans: Richard Silverman. Dover, 1977, xiv + 233 pp, $4.50 (P). [ISBN: 0-486-63519-8] A corrected republication of the 1966 Prentice-Hall edition (TR, April 1967; ER, August-September 1968). Features the Daniell approach to the Lebesgue integral, beginning with an elementary integral defined on a class of elementary functions. LAS

REAL ANALYSIS, T(16-17: 1, 2). *Measure and Integral, An Introduction to Real Analysis.* Richard L. Wheeden, Antoni Zygmund. Pure and Appl. Math., V. 43. Dekker, 1977, x + 274 pp, $16.75. [ISBN: 0-8247-6499-4] A straight-forward, thinly motivated theorem-proof presentation of the Lebesgue theory, from classical integration in Euclidean space to abstract integration and harmonic analysis. Each of the twelve chapters concludes with approximately 20 exercises, many of which extend the theory. Covers a lot of mathematics in a small number of pages. LAS

COMPLEX ANALYSIS, P. *Infinite Dimensional Holomorphy and Applications.* Ed: Mario C. Matos. Math. Stud., V. 12. North-Holland, 1977, viii + 443 pp, $30.75 (P). [ISBN: 0-444-85084-8] Proceedings of the International Symposium on Infinite Dimensional Holomorphy, held at the Universidad Estatual de Campinas, Brazil. Contains twenty-five original research articles covering virtually the full range of infinite dimensional homomorphy topics currently the subject of active research, as well as numerous applications. MU

COMPLEX ANALYSIS, P. *Méthodes algébriques dans la théorie globale des espaces complexes, V. 2.* C. Bănică, O. Stănăşilă. Gauthier-Villars (US Distr: SMPF, 111 W. 57th St., NY 10019), 1977, 183 pp, 68F (P). [ISBN: 2-04-009793-2] Second half of the revised and augmented third edition (TR, April 1975) of a 1974 Romanian monograph. An English translation of the original edition was published by Wiley in 1976 (TR, June 1977). LAS

COMPLEX ANALYSIS, T(16-18: 2, 3), L. *Theory of Functions of a Complex Variable, Second Edition.* A.I. Markushevich. Trans: Richard A. Silverman. Chelsea, 1977, xxiii + 1138 pp, $29.50. [ISBN: 0-8284-0296-5] A revision and combining of the previously published three-volume set in one volume. Revisions in the text are minor but there are combined indices and an updated bibliography. JAS

COMPLEX ANALYSIS, P. *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces.* Richard Courant. Springer-Verlag, 1977, xi + 332 pp, $19.80. [ISBN: 0-387-90246-5; 3-540-90246-5] Reprint of the 1950 edition, supplemented with two pages of notes and bibliography providing references to further work on some of the open problems mentioned in the text. LAS

DIFFERENTIAL EQUATIONS, T(17-18). *Partial Differential Equations.* Günter Hellwig. Teubner, Stuttgart, 1977, xi + 259 pp, (P). [ISBN: 3-519-12213-8] An expanded edition of the 1964 translation. Assumes knowledge of ordinary differential equations, some complex variable theory and functional analysis. Covers wave, potential, and heat equations; existence and uniqueness for elliptic, parabolic, hyperbolic, and mixed type, and other topics. Some exercises, with solutions, are included. SG

DIFFERENTIAL EQUATIONS, T(18: 2), S, P. *Fourier Transformation and Linear Differential Equations.* Zofia Szmydt. Trans: Marcin E. Kuczma. Reidel, 1977, xix + 503 pp, $34. [ISBN: 90-277-0622-0] This self-contained modern treatment of the theory of partial differential equations is an outgrowth of lectures given by the author at Jagiellonian University (Cracow, 1966-74). The main purpose of the book is a presentation, in a distributional setting, of the simplest limit problems for the basic operators of mathematical physics. MU

DIFFERENTIAL EQUATIONS, S(16-18), P, L. *Studies in Ordinary Differential Equations.* Ed: Jack Hale. Stud. in Math., V. 14. MAA, 1977, ix + 278 pp, $11. [ISBN: 0-88385-114-8] A presentation in eight essays of "some of the areas of current research in such a way as to be accessible to specialists." Topics covered include: dynamical systems and stability, the role of functional analysis, and a modern view of asymptotics. JAS

DIFFERENTIAL EQUATIONS, P. *Nonlinear Systems and Applications, An International Conference.* Ed: V. Lakshmikantham. Acad Pr, 1977, xvi + 700 pp, $25. [ISBN: 0-12-434150-0] Proceedings of the conference held at the University of Texas at Arlington, July 19-23, 1976. JAS

NUMERICAL ANALYSIS, T(16-17: 2), S, P. *Numerik gewöhnlicher Differentialgleichungen.* R.D. Grigorieff. Teubner, Stuttgart, 1977, 411 pp, (P). [ISBN: 3-519-02045-9] The second of two volumes on numerical solutions of ordinary differential equations. Devoted to multi-step methods. Treats the methods and their theoretical justification separately. Many examples worked out, but no exercises. Extensive bibliography. JD-B

NUMERICAL ANALYSIS, P. *Numerik und Anwendungen von Eigenwertaufgaben und Verzweigungsproblemen.* E. Bohl, L. Collatz, K.P. Hadeler. Int. Ser. Num. Math., V. 38. Birkhäuser, 1977, 218 pp, sFr. 42 (P). [ISBN: 3-7643-0938-5] Papers from the conference held at Oberwolfach, November 14-20, 1976. JAS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-623: Spectral Decompositions on Banach Spaces.* Ivan Erdelyi, Ridgley Lange. Springer-Verlag, 1977, vii + 122 pp, $8.30 (P). [ISBN: 0-387-098525-4; 3-540-08525-4] Exposition of an intrinsic axiomatic approach to spectral theory based on invariant subspaces and the so-called single-valued extension property (SVEP): the authors develop a unified spectral theory by characterizing operators that admit a spectral decomposition as those that satisfy SVEP. LAS

FUNCTIONAL ANALYSIS, P. *Functional Analysis: Surveys and Recent Results.* Ed: Klaus-Dieter Bierstedt, Benno Fuchssteiner. Math. Stud., V. 27. North-Holland, 1977, xi + 290 pp, $28.75 (P). [ISBN: 0-444-85057-0] Proceedings of the conference on functional analysis at Paderborn, Germany, November 17-21, 1976. JAS

FUNCTIONAL ANALYSIS, T(17-18: 1, 2), S, P. *Introduction to Riesz Spaces.* E. De Jonge, A.C.M. Van Rooij. Math. Centre Tracts, No. 78. Math Centrum, 1977, ix + 229 pp, Dfl. 24 (P). [ISBN: 90-6196-133-5] A readable survey of the theory of Riesz spaces containing a generous sprinkling of exercises and examples. MU

OPTIMIZATION, T(17-18: 1), S, P. *Stochastische Methoden des Operations Research.* J. Kohlas. Teubner, Stuttgart, 1977, 192 pp, (P). [ISBN: 3-519-02342-3] An introduction to stochastic methods in operations research. Chapters on probability (largely without proofs), renewal theory, Markov chains, queueing theory, dynamic programming, and simulation and Monte-Carlo methods. Selective bibliography, largely restricted to monographs. JD-B

OPTIMIZATION, S(17-18), P. *Lecture Notes in Economics and Mathematical Systems-122: Foundations of Optimization.* M.S. Bazaraa, C.M. Shetty. Springer-Verlag, 1976, vi + 193 pp, $8.20 (P). [ISBN: 0-387-07680-8; 3-540-07680-8] The first part of this monograph establishes the foundations of nonlinear optimization in convex analysis using convex sets, cones and functions. The second part is concerned with optimality conditions with and without differentiability and with Lagrangian and conjugate duality. RWN

OPTIMIZATION, S(17-18), P, L. *Topics in Combinatorial Optimization.* Ed: S. Rinaldi. Springer-Verlag, 1975, 186 pp, $15.20 (P). [ISBN: 0-387-81339-X; 3-211-81339-X] Seven papers surveying the present state of the art. LCL

ANALYSIS, T***(15-17: 1, 2), S*, P*, L***. *Advanced Calculus, Third Edition.* R. Creighton Buck. McGraw, 1978, xii + 622 pp, $18.95. [ISBN: 0-07-008728-8] A substantial reworking of one of the classics. Essentially the same topics, with some rearrangement to allow for more flexible use. Earlier introduction to topological concepts (Chapter 1) and differentiation of functions of several variables (Chapter 3). A section on Fourier series added. Improved mix of easy and challenging exercises. Increased attention to numerical techniques throughout (including a chapter on numerical methods). LCL

ANALYSIS, T(16-17). *Applied Abstract Analysis.* Jean-Pierre Aubin. Trans: Carole Labrousse. Wiley, 1977, xi + 263 pp, $21.95. [ISBN: 0-471-02146-6] A clearly written introduction to metric topology and analysis. The applications are few in number and are concentrated in optimization and game theory. Among the mathematical topics: elementary functional analysis, convex functions, Stone-Weierstrass. Includes a résumé of results and a number of exercises. SG

ANALYSIS, T(16-18: 1), S, L. *Differential Forms, A Heuristic Introduction.* M. Schreiber. Springer-Verlag, 1977, xi + 147 pp, $9.80 (P). [ISBN: 0-387-90287-2; 3-540-90287-2] Lecture notes from a course which attempted to present the technique of differential forms "with minimal apparatus and very few prerequisites." Lack of index, except of notation, and total lack of problems would hinder use as a text, but there is here an unusually accessible presentation of some integral geometry and other possible extensions of more standard "advanced calculus" material. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-576: Harmonic Analysis of Real Reductive Groups.* V.S. Varadarajan. Springer-Verlag, 1977, 521 pp, $17.20 (P). [ISBN: 0-387-08135-6; 3-540-08135-6] Essentially notes from the author's seminar at the University of California at Los Angeles during 1969-1973. "A self-contained exposition of Harish-Chandra's work on harmonic analysis on real reductive groups, leading to the complete determination of the discrete series." JAS

ALGEBRAIC GEOMETRY, P. *Algebraic Geometry, The Johns Hopkins Centennial Lectures.* Ed: Jun-Ichi Igusa. Johns Hopkins U Pr, 1977, 125 pp, $12. [ISBN: 0-8018-2021-9] A collection of six papers (one each by Kempf, Mumford, Zariski, Dwork, Griffiths, and Hironaka) on a variety of topics. SG

ALGEBRAIC GEOMETRY, P. *Stability of Projective Varieties.* David Mumford. L'Enseignement Math, 1977, 74 pp, sFr. 21 (P). A treatment of and extension of Gieseker's work on the construction of moduli spaces of algebraic varieties. SG

ALGEBRAIC GEOMETRY, P. *Complex Analysis and Algebraic Geometry: A Collection of Papers Dedicated to K. Kodaira.* Ed: W.L. Baily, Jr., T. Shioda. Cambridge U Pr, 1977, xii + 401 pp, $54.50. [ISBN: 0-521-21777-6] The collection is divided into three parts. Part I includes topics in the theory of algebraic surfaces; Part II includes topics in moduli and classification problems; and Part III includes topics in algebraic geometry, analysis and arithmetic. JEG

DIFFERENTIAL GEOMETRY, T(18: 1), P. *Lectures on Symplectic Manifolds.* Alan Weinstein. CBMS Reg. Conf. in Math., No. 29. AMS, 1977, v + 48 pp, $7.60 (P). [ISBN: 0-8219-1679-9] A concise introduction to the subject. Lecture form; the first six give the basic theory, while the last four give some indication of where research is headed. Particularly suited for a short introduction for someone grounded in Riemannian geometry. TLS

DIFFERENTIAL GEOMETRY, P. *Lecture Notes in Mathematics-605: Classification Theory of Riemannian Manifolds.* Leo Sario, *et al.* Springer-Verlag, 1977, xx + 498 pp, $17.10 (P). [ISBN: 0-387-08358-8; 3-540-08358-8] A systematic and newly developed classification theory of Riemannian manifolds based on the existence or non-existence of harmonic, quasiharmonic, and biharmonic functions with various boundedness properties. JAS

DIFFERENTIAL GEOMETRY, S(15-16). *Géométrie des Courbes et des Surfaces.* Jean-Marc Braemer, Yvan Kerbrat. Hermann, 1976, 198 pp, 38F (P). [ISBN: 2-7056-5841-6] Introduction to differential geometry consisting of an extensive review of the necessary calculus and linear algebra, followed by

thorough but elementary treatments of parametrized curves and surfaces. The final chapter, an introduction to subvarieties, is at a more advanced level. JRG

TOPOLOGY, P. *Supercompactness and Wallman Spaces*. J. Van Mill. Math. Centre Tracts, No. 85. Math Centrum, 1977, iv + 238 pp, Dfl. 29 (P). [ISBN: 90-6196-151-3] Supercompactness and superextension are discussed in detail. One main result: the superextension of the closed unit interval is homeomorphic to the Hilbert cube. A survey of recent results and an extensive bibliography is included. JEG

TOPOLOGY, T*(15-18), S, L*. *Algebraic Topology: An Introduction*. William S. Massey. Grad. Texts in Math., V. 56. Springer-Verlag, 1977, xxi + 261 pp, $14.80. [ISBN: 0-387-90271-6; 3-540-90271-6] A text for advanced undergraduates covering 2-dimensional manifolds, the fundamental group and covering spaces. The topics have been chosen to provide interesting applications, while keeping complicated machinery to a minimum. Many examples and exercises are included and references have been updated from the first edition (TR, November 1967). JEG

TOPOLOGY, S(16-17). *Spaces of Functions and Sets*. R.B. Sher. J. of Undergrad. Math. (Dept. of Math., Guilford Coll., Greensboro, NC 27410), 1976, 40 pp, $5 (P). This book attempts to give practical experience in topology by exploring the topological properties of function spaces. The style is very concise, much in the Moore method. Might be well used as a supplement to a point set course. TLS

TOPOLOGY, T*(16-17: 2, 3), L*. *Aspects of Topology*. Charles O. Christenson, William L. Voxman. Pure and Appl. Math., V. 39. Dekker, 1977, xi + 517 pp, $19.75. [ISBN: 0-8247-6331-9] A text for a solid year's course with many good problems, a useful index, and an unusually broad range of topics. The material found in most beginning texts at this level is enriched by coverage of homotopy theory, inverse systems, covering spaces, triangulation and the classification of 2-manifolds, and the topology of n-manifolds. The presentation is clean, organized, and moderately formal. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-577: E_∞ Ring Spaces and E_∞ Ring Spectra*. J. Peter May. Springer-Verlag, 1977, 268 pp, $11 (P). [ISBN: 0-387-08136-4; 3-540-08136-4]

TOPOLOGY, P. *Lecture Notes in Mathematics-609: General Topology and Its Relations to Modern Analysis and Algebra IV*. Ed: J. Novák. Springer-Verlag, 1977, xvii + 225 pp, $11.50 (P). [ISBN: 0-387-08437-1; 3-540-08437-1] The 19 invited papers (with a listing of submitted papers) from the Fourth Prague Topological Symposium 1976. The remaining papers (105 in all) have been published by the Association of Czechoslovak Mathematicians and Physicists. JAS

TOPOLOGY, P. *Local Surgery and the Exact Sequences of a Localization for Wall Groups*. William Pardon. Memoirs No. 196. AMS, 1977, xii + 171 pp, $8.40 (P). [ISBN: 0-8218-2196-2] Information about the Wall groups of an integral group ring is obtained via the localization sequence for the rational group ring. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-591: Surgery with Coefficients*. G.A. Anderson. Springer-Verlag, 1977, 157 pp, $8 (P). [ISBN: 0-387-08250-6; 3-540-08250-6] A study, extending the author's thesis and a 1973 seminar at the University of Michigan, which shows that in a reasonably general case (rings of the form $R\pi$) the obstruction to finding a homotopy equivalence (over a ring with given torsion) cobordant to a given map lies in a Wall group of a localized group ring. JAS

PROBABILITY, P. *Lecture Notes in Mathematics-607: Reelle und Vektorwertige Quasimartingale und die Theorie der Stochastischen Integration*. Michel Métivier. Springer-Verlag, 1977, ix + 310 pp, $14.30 (P). [ISBN: 0-387-08434-7; 3-540-08434-7]

PROBABILITY, T(17-18: 2), S, P. *Wahrscheinlichkeitstheorie*. P. Gänssler, W. Stute. Springer-Verlag, 1977, xii + 418 pp, $16.60 (P). [ISBN: 0-387-08418-5; 3-540-08418-5] A sophisticated and compactly written text on probability. Prerequisites: an introduction to the subject and a considerable knowledge of analysis. JD-B

PROBABILITY, T(16-18: 1), S, P, L. *Combinatorial Methods in the Theory of Stochastic Processes*. Lajos Takács. Krieger, 1977, xi + 262 pp, $15. [ISBN: 0-88275-491-2] A generalization of the classical ballot theorem is used to find the distribution of the maximum of random variables or the supremum of stochastic processes. Applications to random walks, queueing, storage processes, insurance, and other statistics. FLW

PROBABILITY, P. *Entropy and Ergodic Theory*. J. Aczél, *et al*. U Pr of Canada, 1975, 113 pp. A collection of papers on the title topic. JAS

PROBABILITY, P. *Statistics of Random Processes I, General Theory*. R.S. Liptser, A.N. Shiryayev. Trans: A.B. Aries. Appl. of Math., No. 5. Springer-Verlag, 1977, x + 394 pp, $29.80. [ISBN: 0-387-90226-0; 3-540-90226-0]

PROBABILITY, P. *Lecture Notes in Mathematics-598: Ecole d'Eté de Probabilités de Saint-Flour VI-1976*. J. Hoffmann-Jørgensen, T.M. Liggett, J. Neveu. Springer-Verlag, 1977, xii + 447 pp, $17.10 (P). [ISBN: 0-387-08340-5; 3-540-08340-5] The three principal presentations appear in this volume: Probability in Banach spaces by J. Hoffmann-Jørgensen; The stochastic evolution of infinite systems of interacting particles by T.M. Liggett; Processus ponctuels by J. Neveu. Most of the other presentations appear in Number 61 of the Annales Scientifiques de l'Université de Clermont. JAS

PROBABILITY, P. *Lecture Notes in Mathematics-581: Séminaire de Probabilités XI*. Ed: C. Dellacherie, P.A. Meyer, M. Weil. Springer-Verlag, 1977, v + 573 pp, $18.50 (P). [ISBN: 0-387-08145-3; 3-540-08145-3]

STATISTICS, S(13-16), L. *Sampling Inspection and Quality Control, Second Edition*. G. Barrie Wetherill. Chapman and Hall, 1977, viii + 146 pp, $7.50 (P). [ISBN: 0-470-98993-9] Revision of a 1969 Methuen monograph (TR, October 1970), it attempts broad coverage and a simple treatment of the background theory. This new edition has some new tables and some revised sections. FLW

STATISTICS, T(13: 1). *Essentials of Statistics*. Stephen A. Book. McGraw, 1978, xii + 315 pp, $12.95. [ISBN: 0-07-006464-4] Presupposes only high school algebra. The usual topics. FLW

STATISTICS, T(16-18: 1), P. *Linear Estimation and Stochastic Control*. M.H.A. Davis. Chapman and Hall, 1977, xii + 224 pp, $14.95. [ISBN: 0-470-99215-8] Stochastic processes, orthogonal increments processes, estimation in dynamical systems, linear stochastic control. Most of the material can be read without a measure theoretic background. FLW

STATISTICS, P. *Tratat de Statistică Matematică, Volumul II: Verificarea Ipotezelor Statistice*. Gheorghe Mihoc, Virgil Craiu. Editura Academiei (Romania), 1977, 405 pp, Lei 30. A reference work without problems or index but with a very extensive bibliography. JAS

STATISTICS, P. *Proceedings of the Twenty-Second Conference on the Design of Experiments*. US Army Research Office (Research Triangle Park, NC 27709), 1977, xiv + 344 pp, (P).

STATISTICS, T(16-17: 1, 2), S, P, L. *Sampling Techniques, Third Edition*. William G. Cochran. Wiley, 1977, xvi + 428 pp, $17.95. [ISBN: 0-471-16240-X] A comprehensive treatment of sampling theory. This new edition contains much new material including a chapter on sampling with unequal probabilities of selection. FLW

STATISTICS, T*(13: 1). *Elementary Statistics, An Applied Approach*. Neil R. Ullman. Wiley, 1978, xi + 372 pp, $15.25. [ISBN: 0-471-02105-9] Simplifed version of the author's *Statistics: An Applied Approach* (TR, February 1973). Includes estimation, hypothesis testing, regression, ANOVA, nonparametrics. Well-motivated; easy-to-read; mathematical skills mainly involve arithmetic. LCL

STATISTICS, T(13: 1), S. *Descriptive and Inferential Statistics*. N.M. Downie, A.R. Starry. Har-Row, 1977, ix + 362 pp, $11.50; $5.25 (P). [ISBN: 0-06-041721-8] Distinctive treatment features early introduction to inference, chi-squared tests before sampling distributions, very limited attention to probability, expanded discussions concerning intelligent use of statistics. Includes linear regression, ANOVA, nonparametrics. LCL

STATISTICS, S(14-17), P, L. *Statistical Forecasting*. Warren Gilchrist. Wiley, 1976, xiii + 308 pp, $21. [ISBN: 0-471-99402-2; 0-471-99403-0] "A systematic account of the methods of statistical forecasting in common use and of the practical aspects of their use." Relatively non-mathematical. Presupposes only elementary statistics. FLW

STATISTICS, T*(14-17: 1, 2), S*, L*. *Selecting and Ordering Populations: A New Statistical Methodology*. Jean Dickinson Gibbons, Ingram Olkin, Milton Sobel. Wiley, 1977, xxi + 569 pp, $24.95. [ISBN: 0-471-02670-0] Techniques for ranking populations. Presupposes some statistics but no calculus. Pulls together a large body of results. Many examples and useful tables. FLW

STATISTICS, T(13: 1). *Introduction to Probability and Statistics, Sixth Edition*. Henry L. Alder, Edward B. Roessler. Freeman, 1977, xii + 426 pp, $12.95. [ISBN: 0-7167-0467-6] Revision of the 1972 *Fifth Edition* (TR, March 1973). Main changes include a set of over 100 new exercises in the appendix to review the basic statistical procedures developed in the text, some expansion of material, and some bringing up to date of data and terminology. RSK

STATISTICS, P*. *Robust Statistical Procedures*. Peter J. Huber. CBMS Reg. Conf. in Appl. Math., No. 27. SIAM, 1977, v + 56 pp, $6.75 (P). Material from a series of lectures given in Iowa City in July 1976. Theoretical discussion of procedures which are insensitive to small changes in the shape of the underlying distribution. RSK

COMPUTER SCIENCE, T(16-17: 1), S. *Effiziente Algorithmen*. K. Mehlhorn. Teubner, Stuttgart, 1977, 240 pp, (P). [ISBN: 3-519-02343-1]

COMPUTER SCIENCE, T(13: 1), S. *Using BASIC: An Introduction to Computer Programming*. Julien Hennefeld. Prindle, 1978, xi + 208 pp, $8.95 (P). [ISBN: 0-87150-248-8] Each feature of Basic is presented in a relaxed manner with many examples of programming code to illustrate the topic. Many exercises ask the student to predict what a segment of code will do. Includes several longer examples of complete programs on game playing, learning programs, and sorting. Five appendices, one on statistics and another on using computing in a calculus course. Can be used as a supplemental text in calculus or in computers-in-society courses, or as a text in a short introduction to programming courses. Selected answers to exercises. Index. RJA

COMPUTER SCIENCE, S(15-17), L. *Introduction to Microprocessors*. Ed: D. Aspinall, E.L. Dagless. Pitman, 1977, v + 162 pp, $9.50 (P). [ISBN: 9-12-064550-5] Based on a workshop course (for practicing engineers), this book presents a coordinated set of technical essays about hardware, software, and their inter-relation frequently supported by detailed examples. Knowledge of basic logic circuits is assumed. JAS

COMPUTER SCIENCE, S, L. *Artist and Computer*. Ed: Ruth Leavitt. Creative Computing Pr, 1976, ix + 121 pp, $4.95 (P). [ISBN: 0-517-52735-9; 0-517-52787-1] 35 statements by computer artists on the role of computers in their artistic activity, illustrated with samples of their work. An intriguing sample of current computer art. LAS

APPLICATIONS (ARCHAEOLOGY), S*(13-18), P, L**. *Mathematics and Computers in Archaeology*. J.E. Doran, F.R. Hodson. Harvard U Pr, 1975, xi + 381 pp, $18. Probability, statistics, computation, quantification of evidence, measures of similarity, clustering, multivariate analysis (briefly), automatic seriation, modelling, simulation, and data banks. Many examples. FLW

APPLICATIONS (BIOLOGY), P. *Geographic Variation, Speciation, and Clines*. John A. Endler. Mono. in Population Bio., No. 10. Princeton U Pr, 1977, ix + 246 pp, $16. Investigation of the effects of boundaries and geographic character gradients on the development of geographic variation among species.

Employs modest amounts of calculus and statistics, and extensive vocabulary from population genetics. Includes a 50-page bibliography. LAS

APPLICATIONS (CODING), S(17-18), P, L. *Coding and Complexity*. Ed: G. Longo. Springer-Verlag, 1975, vii + 334 pp, $24.40 (P). [ISBN: 0-387-81341-1; 3-211-81341-1] Proceedings (nine papers) of the two week summer workshop on coding held at Udine, Italy, July 1974. LCL

APPLICATIONS (COMMUNICATION), T(16-17), P. *The Theory of Information and Coding, A Mathematical Framework for Communication*. Robert J. McEliece. A-W (Adv. Bk. Prog.), 1977, xvi + 302 pp, $21.50. [ISBN: 0-201-13502-7] Part I covers information theory: entropy, discrete memoryless channels and sources, the Gaussian channel and source, the source-channel coding theorem. Part II is devoted to coding theory: linear codes, BCH, Goppa, Reed-Solomon and convolutional codes, variable-length source coding. The flavor is probabilistic rather than algebraic or combinatorial. Many examples and exercises. A useful addition to the literature. SG

APPLICATIONS (CONTROL THEORY), P. *Differential Games and Control Theory II*. Ed: Emilio O. Roxin, Pan-Tai Liu, Robert L. Sternberg. Lect. Notes in Pure and Appl. Math., V. 30. Dekker, 1977, xii + 485 pp, $35 (P). [ISBN: 0-8247-6549-4] Twenty-four of the thirty-six papers presented at the Second Kingston (Rhode Island) Conference on Differential Games and Control Theory, June 7-10, 1976. JAS

APPLICATIONS (ECONOMICS), T(15-17: 1, 2), S, L. *Game Theory: Lectures for Economists and Systems Scientists*. N.N. Vorob'ev. Trans: S. Kotz. Appl. of Math., No. 7. Springer-Verlag, 1977, xi + 178 pp, $16.80. [ISBN: 0-387-90238-4; 3-540-90238-4] Presupposes calculus and elementary linear algebra. Keeps the mathematics relatively simple. Matrix games, infinite antagonistic games, non-cooperative games, games in characteristic function form. FLW

APPLICATIONS (ECONOMICS), P. *Economics of Space and Time, The Measure-Theoretic Foundations of Social Science*. Arnold M. Faden. Iowa St U Pr, 1977, xiii + 703 pp, $39.95. [ISBN: 0-8138-0500-7] "It is the thesis of this book that measure theory is the natural language for spatial economics and, indeed, for all social science." A truly impressive reconceptualization of diverse classical economic theories under the common language of measure theory, including extensions to pseudomeasures (to permit analysis of net values even when gross values are possibly infinite). To the practical man who may wonder "But how does it help me meet a payroll?", economist Faden offers a stunning, uncompromising mathematical world-view with a "certain grandeur. LAS

APPLICATIONS (ENGINEERING), S(16-17), P. *Computation of Power System Transients*. J.P. Bickford, N. Mullineux, J.R. Reed. Peter Peregrinus (US Distr: ISBS, Inc., P.O. Box 555, Forest Grove, OR 97116), 1976, ix + 176 pp. [ISBN: 0-901223-85-9] An up-to-date analysis of problems involved in circuit-breaker design. A detailed description of the mathematical methods applied to the calculation of fast transient overvoltages in power-system networks. Eminent engineers have written chapters in areas of special interest to them. Good list of references, no exercises. CEC

APPLICATIONS (ENGINEERING), T(17-18: 1), S, P. *Discrete Field Analysis of Structural Systems*. Donald L. Dean. Springer-Verlag, 1976, 166 pp, $14.80 (P). [ISBN: 0-387-81377-2; 3-211-81377-2] Lecture notes, prepared before the fact, for a course in discrete field mechanics and applications. A background in differential equations with an introduction to partial differential equations is assumed, but review and development of difference equations, Fourier series and the inclusion of tables of difference equation operators makes the presentation reasonably self-contained. No index. JAS

APPLICATIONS (ENGINEERING), P. *Nuclear Systems Reliability, Engineering and Risk Assessment*. Ed: J.B. Fussell, G.R. Burdick. SIAM, 1977, xi + 849 pp, $38.50. Papers from the conference held at Gatlinburg, Tennessee, June 20-24, 1977. JAS

APPLICATIONS (ENGINEERING), P. *Dynamics of Systems of Rigid Bodies*. Jens Wittenburg. Teubner, Stuttgart, 1977, 224 pp. [ISBN: 3-519-02337-7] A development, with background material so as to make the book relatively self-contained, of a general formalism for the dynamics of systems of rigid bodies. This formalism is among the earliest applications in mechanics of graph-theoretic ideas. JAS

APPLICATIONS (ENGINEERING), P. *Nonlinear Networks*. Vaclav Dolezal. Elsevier Sci Pub, 1977, ix + 156 pp, $29.95. [ISBN: 0-444-41571-8] This book gives an analytic setting for the study of electrical networks. The author claims to generalize the subject which heretofore has been focused on specific types of networks. Requires functional analysis and Hilbert spaces. TLS

APPLICATIONS (FLUID DYNAMICS), P. *Fluid Dynamics*. Ed: R. Balian, J.-L. Peube. Gordon, 1977, xiv + 677 pp, $69. [ISBN: 0-677-10170-8] Material developed from the course material and special seminars from the summer school of theoretical physics at Les Houches in July 1973. JAS

APPLICATIONS (GEOGRAPHY), T(15-18: 1), S, P, L. *Models of Spatial Processes, An Approach to the Study of Point, Line and Area Patterns*. Arthur Getis, Barry Boots. Cambridge U Pr, 1978, xvi + 198 pp, $19.95. [ISBN: 0-521-20983-8] A basic problem of interest to geographers is that of explaining observed map patterns (e.g., of settlements). This book draws together, in a relatively simple verbal form, a description of several mathematical models for map patterns based on different assumptions about the two opposing tendencies toward agglomeration and diffusion. Requires only a minimal background in statistics. LCL

APPLICATIONS (INFORMATION THEORY), P. *Topics in Information Theory*. Ed: I. Csiszár, P. Elias. North-Holland, 1977, 592 pp, $81.75. [ISBN: 0-7204-0699-4; 963-8021-20-9] Selected papers from the conference held at Keszthely (Hungary), August 25-29, 1975. Main subject areas are: Shannon theory, algebraic coding theory, interplay of information theory and statistics. JAS

APPLICATIONS (OCEANOGRAPHY), P. *Hydrologic Optics, V. VI, Surfaces.* R.W. Preisendorfer. Nat. Oceanic and Atmos. Ad., 1976, xii + 390 pp, $13.25 (P). A study of the reflected radiance distribution from a random sea surface, and the transmitted radiance distribution entering the sea below the surface. JAS

APPLICATIONS (PHYSICS), S(13-16), L*. *Space and Time in the Modern Universe.* P.C.W. Davies. Cambridge U Pr, 1977, viii + 232 pp, $13.95; $5.95 (P). [ISBN: 0-521-21445-9; 0-521-29151-8] A lucid narrative tracing the evolution of our concept of space-time, from Newton to Penrose, with numerous philosophical digressions on the implications of these views for our understanding of man's place in the universe. Intended to entertain and excite, as well as to educate, this rather personal monograph mixes speculation with science in a manner that the layman may find hard to distinguish. It is seductive science at its best. LAS

APPLICATIONS (PHYSICS), P. *The Physics of Time Asymmetry.* P.C.W. Davies. U of Calif Pr, 1977, xviii + 214 pp, $3.45 (P). A revision of the 1974 edition (TR, March 1975) including recent work of Bekenstein and Hawking on the quantum mechanics of black holes. JAS

APPLICATIONS (PHYSICS), S, P, L. *Crystals and Light: An Introduction to Optical Crystallography, Second Revised Edition.* Elizabeth A. Wood. Dover, 1977, iv + 156 pp, $2.75 (P). [ISBN: 0-486-23431-2] A clear, concise, and carefully illustrated study of the behavior of light in crystals with color photographs of interference figures and particles between crossed polarizers. Presumes no background in either crystallography or optics. Provides an example of the use of group theory. JNC

APPLICATIONS (PHYSICS), T*(16-17: 1, 2), S, L. *Essential Relativity: Special, General, and Cosmological, Second Edition.* Wolfgang Rindler. Springer-Verlag, 1977, xiv + 284 pp, $29.80. [ISBN: 0-387-07970-x; 3-540-07970-x] The first edition (TR, February 1970; ER, October 1970) of this conceptually-oriented physics text has been revised throughout to take account of the implications of recent advances. The author comments that, to maintain his non-encyclopaedic conceptual approach, "ultimately, the new material was determined by the enthusiasm with which I felt I could present it." JAS

APPLICATIONS (PHYSICS), T*(15-16: 1), S, L*. *Mathematical Cosmology, An Introduction.* Peter L. Landsberg, David A. Evans. Clarendon Pr, 1977, x + 309 pp, $15.95. [ISBN: 0-19-851136-1] An inviting exposition of modern cosmology making extensive use of elementary undergraduate mathematics. It traverses a useful middle ground between popular, nonmathematical exposition and research-level monographs. Includes problems (with full solutions at the back of the book) and extensive references to collateral reading. LAS

APPLICATIONS (PHYSICS), S*(13-16), L*. *What is the World Made Of? Atoms, Leptons, Quarks, and Other Tantalizing Particles.* Gerald Feinberg. Anchor Pr, 1977, xvii + 290 pp, $10. [ISBN: 0-385-07694-0] A thorough yet non-mathematical tour of contemporary particle physics, explaining current theories of the bizarre ephemeral world of the subatomic particles. Uncompromising in vocabulary and physical detail, Feinberg explains very clearly about as much as can be expected in a non-mathematical exposition. Requires and rewards a careful reading. LAS

APPLICATIONS (PHYSICS), P. *Group Theoretical Methods in Physics.* Ed: Robert T. Sharp, Bernard Kolman. Acad Pr, 1977, xvi + 668 pp, $29.50. [ISBN: 0-12-637650-6] Proceedings of the Fifth International Colloquium at the Université de Montréal, July 5-9, 1976. The papers are grouped in six principal areas: nuclei, atoms, solids; coherent states, supersymmetry, gauge fields, relativity; classical and quantum mechanics; relativistic quantum physics; mathematical physics; and representation theory. JAS

APPLICATIONS (PHYSICS), T*(18: 1, 2), S, L. *General Relativity for Mathematicians.* R.K. Sachs, H. Wu. Grad. Texts in Math., V. 48. Springer-Verlag, 1977, xii + 291 pp, $19.80. [ISBN: 0-387-90218-X; 3-540-90218-X] A mathematically clear presentation of the core ideas of general relativity aimed at mathematicians with some knowledge of global differential geometry and elementary physics. This is not a differential geometry text for physicists. The book is nicely written, has good indices and a bibliography, and lots of relevant and meaningful exercises. JAS

APPLICATIONS (PHYSICS), T(16-18), P. *Scattering Theory in Quantum Mechanics.* Werner O. Amrein, Josef M. Jauch, Kalyan B. Sinha. Benjamin (Adv. Bk. Prog.), 1977, xxiii + 691 pp, $17.50 (P); $29.50. [ISBN: 0-8053-0203-4; 0-8053-0202-6] A rigorous exploration, using the spectral theory of self-adjoint operators, of the laws of interaction between elementary particles. Prerequisite: the Lebesgue integral on R^n. LAS

APPLICATIONS (SYNERGETICS), P. *Synergetics, A Workshop.* Ed: H. Haken. Springer-Verlag, 1977, x + 274 pp, $26.70. [ISBN: 0-387-08483-5; 3-540-08483-5] Proceedings of the workshop held at Schloss Elmau, Bavaria, May 2-7, 1977, wherein the general structure of diverse systems were studied and analogies developed--e.g., catastrophe theory and morphogenesis. JAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jennifer R. Galovich, St. Olaf; Steven Galovich, Carleton; Jay E. Goldfeather, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D.C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

University of California, Berkeley: Professor P. Emery Thomas has accepted the position of Vice-Chairperson for Faculty Appointments of the Department of Mathematics for 1977-78. Professor Calvin C. Moore has accepted the position of Director of the Center for Pure and Applied Mathematics for 1977-78. Professor Isadore M. Singer, Norbert Weiner Professor of Mathematics at M.I.T., has accepted a visiting appointment as Professor for two years. Professor Angus Taylor has been appointed Professor (now Emeritus). Dr. Steven Shreve, Assistant Professor at the University of Delaware, is a visiting Assistant Professor for 1977-78. Professor Hung-Hsi Wu is a Miller Research Professor in 1977-78 and Professor Calvin C. Moore will be a Miller Research Professor in 1978-79.

University of Texas, Austin: Professor R. H. Bing, president of the American Mathematical Society, asked to be relieved of his duties as Chairman of the Department of Mathematics, and Professor James W. Daniel has been appointed Chairman. Associate Professor William T. Eaton has been promoted to Professor. Dr. Gregory T. Bachelis, Wayne State University, Detroit, is a visiting Associate Professor. Dr. J. H. B. Kemperman, University of Rochester, is a visiting Professor. Dr. Patrick Lee Brockett, Tulane University, and Dr. Murray Cantor, Duke University, have been appointed Assistant Professors. Dr. Ronald M. Dotzell, Rutgers University, has been appointed Instructor. Former staff member Dr. Richard Stark is at San Jose State University, San Jose, California, and Dr. Gerard Venema is at the Institute for Advanced Study, Princeton, New Jersey.

Olive-Harvey College: Assistant Professor John Fenley has been appointed Chairperson of the Department of Mathematics. Associate Professor Thomas McGannon has been promoted to Professor. Assistant Professor Joseph A. Smith has been promoted to Associate Professor.

Tuskegee Institute: Assistant Professor Herman Windham has been appointed Chairman of the Department of Mathematics. Assistant Professor William Lester, former Chairman of the Mathematics Department, has been promoted to Assistant Provost.

Clemson University: Associate Professors Renu C. Laskar and Robert F. Ling have been promoted to Professors.

University of Maine: Associate Professor Eric L. Langford has been promoted to Professor. Associate Professor Gary Haggard has been promoted to Associate Professor and Chairman.

Indiana Vocational Technical College — Region 2: Dr. Darleen Pigford, Eastern Montana College, has been appointed Instructor. Instructor Ray E. Collings has been promoted to Assistant Professor.

Illinois State University, Normal: Assistant Professor Robert Speiser has been promoted to Associate Professor. Professor Albert D. Otto has been appointed Chairman of the Mathematics Department.

University of Michigan: Associate Professor Eugene F. Krause has been promoted to Professor. Professor Frederick W. Gehring received an Honorary Doctorate from the University of Helsinki in May, 1977.

Eastern Washington University, Cheney: Assistant Professors A. George Dors and Kit Hanes were promoted to Associate Professors.

Dr. Richard Brauer, Harvard University, died on April 17, 1977, at the age of 76. He was a member of the Association for forty-three years.

Mrs. Eleanor C. Guentert, Cuyahogu Community College, Parma, Ohio, died on April 27, 1977. She was a member of the Association for ten years.

Associate Professor Gerald A. Hutchison, Dean of the School of Natural Sciences and Mathematics, University of Alabama, died on June 11, 1977, at the age of 39. He was a member of the Association for seventeen years.

Professor Emeritus Sigurd Mundhjeld, Concordia College, Moorhead, Minnesota, died on July 4, 1977. He was a member of the Association for forty-eight years.

Dr. Merry L. Morgan, Socorro, New Mexico, died on July 9, 1977. She was a member of the Association for seventeen years.

Professor Emeritus Charles H. Rawlins, U. S. Naval Postgraduate School, died on August 20, 1977. He was a charter member of the Association since 1915.

Professor David L. Howard, Miami Dade Community College, Miami, Florida, died on August 27, 1977. He was a member of the Association for fourteen years.

Dr. Wendell A. Dwyer, Mercer Island, Washington, died on August 15, 1977. He was a member of the Association for forty-one years.

Dr. Robert Davies, Bloomfield Hill, Michigan, died on September 7, 1977. He was a member of the Association for thirty years.

Retired Professor H. Earl Spencer, Virginia Polytechnic Institute, died on October 11, 1977, at the age of 79. He was a member of the Association for forty years.

Professor William T. Reid, University of Texas, died on October 14, 1977. He was a member of the Association for fifty years.

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

The Undergraduate Mathematics Applications Project (UMAP) has published a catalog of available materials. The catalog contains information about content, mathematics applications fields, prerequisites, post-options, and cost of the modules currently available for classroom use and field testing. To receive a copy write to E.D.C-UMAP, 55 Chapel Street, Newton, Massachusetts 02160.

NSF ROTATOR PROGRAM

The Office of Equal Employment Opportunity is accepting applications year-round from individuals who desire to participate in the Rotator Program. Applications received too late for this year's program, for which duty assignments will begin in September 1978, will be held over for next year's competition.

Under this program NSF augments its permanent staff of scientists and other professional employees with qualified individuals from the faculties of colleges and universities across the country who serve in non-career positions for periods of one or two years. Those selected gain a rich developmental experience and additional insight into Federal support of scientific research, the improvement of science education and the dissemination of science information.

Interested faculties may wish to bring this to the attention of qualified women, minorities, and handicapped individuals. It is desirable that the applicant have a Ph.D. and six years of successful scientific research experience, but these are not rigid rules. Applicants should forward their vitae and statements of interest to Herbert Harrington, Jr., National Science Foundation, 1800 G Street, N.W., Washington, D.C. 20550.

SUMMER SHORT COURSE IN APPLICATIONS FROM CONTROL THEORY

The Ohio and Allegheny Mountain Sections of the Mathematical Association of America are the sponsors of a short course to be held June 13-17, 1978 at Allegheny College, Meadville, Pennsylvania. There will be presentations of control problems with applications to business, engineering, life sciences, and others suitable for the college curriculum. The principal speaker will be Donald O. Norris, Ohio University. For details contact Charles Cable, Department of Mathematics, Allegheny College, Meadville, Pennsylvania 16335.

MATHEMATICS AND STATISTICS CONFERENCE

The sixth annual Mathematics and Statistics Conference will be held at Miami University, Oxford, Ohio, September 29 and 30, 1978. Featured speakers will be George Carrier, Harvard; Victor Klee, University of Washington; Frederick Mosteller, Harvard. For information write to Charles Holmes or Elwood Bohn, Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The fifty-second meeting of the Philadelphia Section of the MAA was held November 19, 1977, at Moravian College, Bethlehem, Pennsylvania. Section Chairman, Professor Eugene Klotz, and Program Chairman, Professor Gerald Porter, presided. A total of 170 persons attended.

The following papers were presented:

Tiling the plane with Pentagons: A perplexing problem, Doris Schattschneider, Moravian College.

Algebraic curves: Confluence of Algebra, Geometry and Analysis, Stephen S. Shatz, University of Pennsylvania.

Some mathematical aspects in the design of automotive components, Steve M. Rohde, General Motors Research Laboratories.

Symmetry, William Thurston, Princeton University.

A special session for students heard the following papers:

Compiling techniques of a mini-language (PLCP), Mohsen Bonakdarpour, Temple University.

Hyperbolic Geometry: Effects and defects, Nora Wunder, Muhlenberg College.

Piecewise curve-fitting in an engineering application, Michelle Hartman, Shippensburg College.

Think-a-dot: An algebraic game analysis, Jordan Levy, University of Delaware.

Section officers elected for 1978 are: D. W. Schattschneider, Moravian College, Chair; H. Anton, Drexel University, Vice-Chair. Newly elected members of the Executive Committee are: S. Schwartz, Philadelphia Board of Education; L. C. Leinbach, Gettysburg College; and B. Babcock, Penn State University, York Campus.

W. E. BAXTER, *Secretary-Treasurer*

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

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Volumes 1 through 80 (1894–1973)

Edited by KENNETH O. MAY

This is a conventional index to articles and a selection of other items in the first eighty volumes of the American Mathematical Monthly (1894–1973). It consists of a chronological table of contents of articles indexed, an author index to these articles, and a subject index to them and a few items not listed in the Contents. The indexes are like those at the backs of books and should be used in the same way.

Table of Contents, Author Index, Subject Index: vi + 269 pages.

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Fifty-eighth Summer Meeting, Brown University, Providence, August 8–10, 1978.

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, University of Pittsburgh, Pennsylvania, April 14–15, 1978.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, Western Illinois University, Macomb, May 5–6, 1978.

INDIANA, Earlham College, Richmond, April 22, 1978.

INTERMOUNTAIN

IOWA, University of Northern Iowa, Iowa Falls, April 22, 1978.

KANSAS, Wichita State University, Wichita, late March —early April, 1978.

KENTUCKY, Northern Kentucky University, Highland Heights, April 7–8, 1978.

LOUISIANA-MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Queensborough Community College, May 7, 1978.

MICHIGAN, Michigan State University, East Lansing, May 5–6, 1978.

MISSOURI, Central Missouri State University, Warrensburg, April 7–8, 1978.

NEBRASKA, University of Nebraska at Omaha, April 14–15, 1978.

NEW JERSEY, Steinhart High School, Trenton, April 28, 1978.

NORTH CENTRAL, College of St. Thomas, St. Paul, Minnesota, April 21–22, 1978.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, The University of Akron, Akron, April 28–29, 1978.

OKLAHOMA-ARKANSAS, Henderson State University, Arkadelphia, Arkansas, March 31–April 1, 1978.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 16–17, 1978.

PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.

ROCKY MOUNTAIN, South Dakota School of Mines and Technology, Rapid City, April 28–29, 1978.

SEAWAY, Brock University, St. Catharines, Ontario, Canada, May 5–6, 1978.

SOUTHEASTERN, Clemson University, Clemson, South Carolina, March 31–April 1, 1978.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, New Mexico Institute of Mining and Technology, Socorro, spring 1978.

TEXAS, Stephen F. Austin State University, Nacogdoches, March 31–April 1, 1978.

WISCONSIN, University of Wisconsin, Whitewater, late April 1978.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Chicago, January 3–8, 1979.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Stouffer's Greenway Plaza Hotel, Houston, Texas, fall 1978.

AMERICAN MATHEMATICAL SOCIETY, Brown University, Providence, Rhode Island, August 9–12, 1978.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of British Columbia, Vancouver, June 19–22, 1978.

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton Park Hotel, Washington, D.C., December 4–6, 1978.

ASSOCIATION FOR SYMBOLIC LOGIC, Madison, Wisconsin, June 18–24, 1978.

ASSOCIATION FOR WOMEN IN MATHEMATICS, Brown University, Providence, Rhode Island, August 8–12, 1978.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHEMATIQUES, University of Western Ontario, London, Ontario, June 1–2, 1978.

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, San Diego, California, April 12–15, 1978.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, New York City, May 1–3, 1978 (Joint Meeting with the Institute of Management Sciences).

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Hotel Fort Des Moines, Des Moines, Iowa, November 2–4, 1978.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, University of Wisconsin, Madison, May 24–26, 1978.

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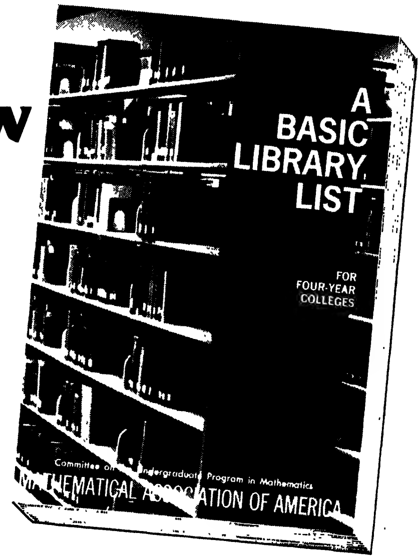

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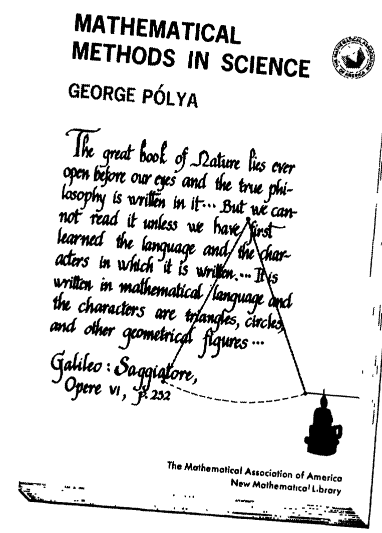

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Volume 85, Number 5

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Hilbert's 14th Problem

Sixth U.S.A. Mathematical Olympiad

The Serre Conjecture

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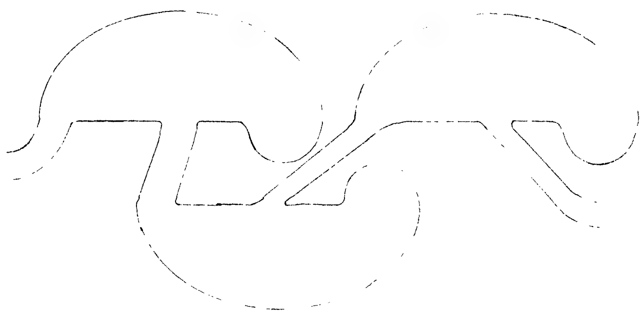

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1978

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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THE AMERICAN ASSOCIATION OF PHYSICS TEACHERS

ROBERT KARPLUS

Consider these problems taken from widely-used first-year college texts, one in physics, the other in calculus:

PROBLEM 1. A cable 100 meters long and having the mass density 5 kg/m is hanging from a winch. Find the work done in winding it up. (Ans. 25000g, g is the acceleration of gravity)

PROBLEM 2. A cord is used to lower vertically a block of mass M a distance d at a constant downward acceleration of g/4. Find the work done by the cord on the block. (Ans. $-3Mgd/4$)

Can you tell which problem came from what text?

Calculus and first-year college physics seem to have a great deal in common. A large number of students take both courses, and their success in physics depends in an important way on their mathematical proficiency. Many of the physics faculty concerned with these and also more advanced courses are among the approximately 9000 members of the American Association of Physics Teachers, AAPT for short.

The AAPT was founded in 1930 by a group of physicists who dedicated the Association to the advancement of the teaching of physics and the furtherance of the role of physics in our culture. A majority of the members are college or university faculty, while a minority teach in secondary schools. The AAPT therefore resembles the Mathematical Association of America in many ways.

Three periodicals are published by the Association. *The American Journal of Physics*, with twelve issues totaling more than one thousand pages annually, focuses on instructional and cultural aspects of the physical sciences. It includes expository articles on physics as well as reports of new teaching approaches, physics instructional apparatus, and reviews of books and films. The *Journal* is the oldest publication of AAPT and corresponds to the *American Mathematical Monthly*. *The Physics Teacher* comes out nine times annually and is primarily concerned with the teaching of introductory physics at all levels. The *AAPT Announcer* is a bulletin with four issues a year, carrying information about AAPT's annual meetings, news about the more than thirty Regional Sections, progress reports on AAPT's ongoing projects, and other items of general interest to physics teachers.

The Association's winter meeting in late January is held jointly with the American Physical Society and alternates between East Coast, Midwest, and West Coast metropolitan centers. The summer meeting, which attracts about 350 participants, takes place at the end of June on a college campus in a more scenic environment. The meeting programs include invited and contributed papers as well as poster sessions on areas in physics, laboratory equipment, educational approaches, and possible new course topics. During recent years the meeting programs have also come to include interactive workshops in which the participants use programmable calculators, make single concept films, study developmental psychology (see below), design and assemble integrated circuit electronic devices, investigate the use of model rockets for physics instruction, or complete other short projects.

The educational products sponsored and/or distributed by AAPT include more than 70 single-concept physics films, more than thirty sets of slides illustrating physical phenomena, and *Women in Science*, a multi-media package presenting information about six women scientists to help girls inform themselves about scientific careers. Among the most popular film titles have been twelve made with NASA cooperation in the zero-gravity environment of Skylab.

Annotated bibliographies of special interest to physics teachers are published in the *American Journal of Physics* as "Resource Letters" and are often accompanied by reprint books that include many of the references recommended. Sample titles from this list are "Kinematics and Dynamics of Satellite Orbits," "Quantum and Statistical Aspects of Light," and "Achievement Testing in Physics."

A recently introduced product is the *Workshop on Physics Teaching and the Development of*

Reasoning. It explains key aspects of developmental psychology and Jean Piaget's theories in a way that physics teachers can apply directly to their selection of textbooks, assignment of problems, preparation of tests, and organization of laboratories. This self-paced, interactive workshop with films, videotapes, and lab activities, has been used by more than 1000 physics teachers during the last thirty months. Because of the many similarities between the two fields, the Workshop has obvious implications for mathematics instruction.

Do inquire at the AAPT Executive Office, Graduate Physics Building, State University of New York, Stony Brook, N.Y. 11794 for more information about AAPT's activities and products. If it is convenient, you might attend one of our upcoming meetings: at the San Francisco Hilton, January 23–26, 1978; at the University of Western Ontario, London, Ontario, June 14–16, 1978; at the New York Hilton, January 29–February 1, 1979, and at New Mexico State University, Las Cruces, New Mexico, June 21–24, 1979.[1] And still better, contact physics instructors at your institution to invite their collaboration on educational projects you have in mind—even if your current project merely means choosing a new calculus text or looking for a wider variety of problems for your students.

LAWRENCE HALL OF SCIENCE, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720.

[1]Meeting programs are available at the registration desk or may be obtained from the Executive Office about a month in advance.

QUANTITATIVE APPROXIMATION THEORY

STEPHEN D. FISHER

Introduction. Approximation theory deals with the problem of whether a given function can be accurately approximated in some given topology or norm by an element from a prescribed set of functions, which generally possess some noteworthy property. Quantitative approximation theory attempts to determine as precisely as possible the size of the error in this approximation given specific information about the function to be approximated and the set of functions from which the approximant is to be taken. Thus, Weierstrass' famous theorem of 1883 which asserts that each continuous function on a closed interval in the real line can be approximated to within any specified tolerance by a polynomial is the prototype of the former while Jackson's Theorems (see Section 1 below) are prototypes of the latter. I shall sketch in what follows some of what are for me the high points in this and some closely related fields. Many of the results are 30, 40 or even 60 or more years old and are considered to be classics of their types. Others of the results are newer and a few are even so current as not to be yet in print. The theorems themselves are easy to state and understand but their proofs are frequently involved and make use of a good deal of computation and secondary notation. For this reason I shall give only part of the proof of some of the theorems, no proof completely, and frequently none at all. I shall, however, cite references, both primary and secondary, to which the reader may refer if he wishes to pursue the details. In this essay I discuss only uniform approximation—that is, approximation where the distance between two functions f and g on some interval $[a, b]$ is given by

$$\|f-g\|_{L^\infty(a,b)} = \sup\{|f(x)-g(x)|: a \leqq x \leqq b\}.$$

The related theorems for approximation in other norms, mainly the L^p norms, follow paths parallel to

Stephen Fisher received his Ph.D. at the University of Wisconsin under the direction of Frank Forelli. He was an instructor at MIT during 1967–69 and since 1969 has been at Northwestern University. His research interests are complex analysis, variational problems, and approximation theory.

those elucidated here. Further, I cite only theorems that concern approximation of functions of a single real variable, mainly because the several variable analogues are either not as sharp or not even yet worked out!

1. Direct theorems of D. Jackson. Our story of quantitative approximation theory begins with Dunham Jackson's dissertation, written in 1911 at Göttingen under the direction of the prominent German mathematician Edmund Landau. (Landau will reappear in a later chapter of this story.) As Jackson recounts **[13]**

> One day... I was admitted to the study of Professor Landau, seeking advice as to a subject for a thesis. After some preliminary inquiries as to my experience and preferences, he handed me a long sheet of paper, and directed me to take notes as he enumerated some dozen or fifteen topics in various fields of analysis and number theory, with a few words of explanation of each. He told me to think about them for a few days, and to select one of them, or any other problem of my own choosing... . Guided partly by natural inclination, perhaps, and partly by recollection of a course on methods of approximation which I had taken with Professor Bôcher a few years earlier I committed myself to one of the topics which Landau had proposed, an investigation of the degree of approximation with which a given continuous function can be represented by a polynomial of given degree. When I reported my choice, he said meditatively, in words which I remember vividly in substance, if not perfectly as to idiom: "Das ist ein schönes Thema, ich beneide Sie um das Thema... Nein, ich beneide Sie nicht, aber es ist ein wunderschönes Thema!"*

The simplest of Jackson's theorem is the following.

THEOREM 1. *Let f be a continuous 2π-periodic function satisfying*

$$|f(x_2)-f(x_1)|\leqq M|x_2-x_1|, \quad \text{all } x_1, x_2. \tag{1}$$

Then there is, for each positive integer n, a trigonometric polynomial

$$T(x)=\sum_{j=0}^{n}(a_j\cos jx+b_j\sin jx) \quad \text{with} \quad |f(x)-T(x)|\leqq KM/n, \qquad -\pi\leqq x\leqq\pi,$$

where K is a constant which does not depend on x, n, M, or f.

In other words, f can be uniformly approximated on $[-\pi,\pi]$ to within order $1/n$ by a trigonometric polynomial of degree n provided f satisfies a Lipschitz condition of order 1. Let us set the notations

$\mathcal{T}_n$ for all trigonometric polynomials of degree n or less

and

$\mathrm{Lip}_M(\alpha)$ for all continuous functions h on $[a,b]$ which satisfy

$$|h(x)-h(y)|\leqq M|x-y|^{\alpha}, \qquad x,y\in[a,b]. \tag{2}$$

I will use the superscript $*$ consistently to denote 2π-periodic functions; hence, $\mathrm{Lip}_M^*(\alpha)$ will mean that $a=-\pi$, $b=\pi$ and that the functions in the indicated class are not only Lipschitz of order α but are also 2π-periodic. With these notations we can phrase Theorem 1 in this concise way:

$$\sup_{f\in\mathrm{Lip}_1^*(\alpha)}\ \inf_{T\in\mathcal{T}_n}\|f-T\|_{L^\infty(-\pi,\pi)}\leqq K/n \tag{3}$$

for some constant K. Of course, the Lipschitz condition (1) is a restriction on the *modulus of continuity* of f. The modulus of continuity is defined for a continuous function F on $[a,b]$ by

$$\omega(F;\delta)=\sup\{|F(x)-F(y)|:|x-y|\leqq\delta,\ x,y\in[a,b]\}. \tag{4}$$

* This is a beautiful subject, I envy you this subject... No, I do not envy you, but it is a wonderful subject.

With this additional notation we have a more general form of Theorem 1.

THEOREM 2. *Let f be continuous and 2π-periodic. For each positive integer n there is an element T of $\mathcal{T}_n$ with*

$$\|f-T\|_{L^\infty(-\pi,\pi)} \leqq K'\omega(f;1/n),$$

where K' is a constant, independent of f and n.

Theorem 2, and hence Theorem 1, is proved in a pattern familiar to approximation theory: for each integer n a trigonometric polynomial K_n of degree n is given with the conventional properties that

$$K_n \geqq 0, \qquad \int_{-\pi}^{\pi} K_n(t)dt = 1$$

and the less conventional property that the numbers

$$n^{-k}\int_0^{\pi} K_n(t)t^k\,dt, \qquad k=1,2 \quad \text{and} \quad n=1,2,3,\ldots$$

are bounded away from zero and also above as $n\to\infty$. The trigonometric polynomial of degree n given by

$$J_n(x)=\int_{-\pi}^{\pi} f(t)K_n(t-x)dt \tag{5}$$

is then the appropriate approximant to f. Of course, the trick is to find such a trigonometric polynomial! See [**19**; p. 55] for the details of the definition of K_n and how the computation is carried out.

Let us set another notation:

$$E_n^*(f) = \text{distance of the } 2\pi\text{-periodic function } f \text{ to } \mathcal{T}_n$$
$$= \inf_{T\in\mathcal{T}_n}\|f-T\|_{L^\infty(-\pi,\pi)}.$$

From Theorem 2 an induction argument yields the following result.

THEOREM 3. *Let f be a continuous 2π-periodic function with continuous 2π-periodic derivatives of order $1,2,\ldots,r$. Then*

$$E_n^*(f)\leqq K''n^{-r}\omega(f^{(r)};1/n), \tag{6}$$

where K'' is a constant independent of f and n but not necessarily of r.

We shall return in Section 4 to discuss the size, indeed the exact value, of the constants K, K', and K'' which appear in Theorems 1, 2, and 3 under certain restrictions on the modulus of continuity of $f^{(r)}$.

There is a non-periodic version of Theorem 3 (which, of course, includes Theorems 1 and 2). We define π_n to be the ($n+1$ dimensional) vector space of all algebraic polynomials of degree n or less. That is, π_n consists of all functions of the form

$$p(x)=\sum_{j=0}^{n} a_j x^j, \qquad a_j \text{ a real number for all } j.$$

We further define $E_n(f)$ to be the distance from the continuous function f to π_n:

$$E_n(f) = \inf_{p\in\pi_n}\|p-f\|_{L^\infty(a,b)}. \tag{7}$$

It is worth noting that we can use the simple linear transformation

$$x' = (b-a)^{-1}(2x-a-b)$$

to convert $[a, b]$ to $[-1, 1]$; this transformation and its inverse preserve π_n. With this in mind we shall henceforth assume with no further comment that all periodic problems are considered on $[-\pi, \pi]$ and all non-periodic problems on the interval $[-1, 1]$, which we denote by I. Now to the non-periodic version of Theorem 3.

THEOREM 4. *Let f be a continuous function on I which has a continuous r-th derivative $f^{(r)}$ on I. Then*

$$E_n(f) \leqq Ln^{-r}\omega(f^{(r)}; 1/n), \tag{8}$$

where L is a constant independent of f and n (but not of r).

This theorem can be proved by induction from the case $r = 0$, which in turn follows directly from Theorem 2 by the device of transforming f to be even on the interval $[-1, 1]$ and then applying the transformation $x = \cos\theta$, $-\pi \leqq \theta \leqq \pi$. In Section 4 we shall give some information on the constant L appearing in Theorem 4.

The interest and beauty in Theorems 1–4 is that they relate an internal characteristic of a continuous function f, namely its degree of differentiability or smoothness, to an external characteristic, the rapidity with which f can be uniformly approximated on the entire interval I or $[-\pi, \pi]$ by a sequence of algebraic or trigonometric polynomials. As we shall see in the next section, this relation has a valid converse: the (small) size of $E_n(f)$ or $E_n^*(f)$ implies that f has a certain amount of smoothness.

One more comment is pertinent here. Let us denote by W_r those functions f on $I = [-1, 1]$ which possess $r-1$ continuous derivatives with $f^{(r-1)}$ lying in $\mathrm{Lip}_1(1)$; that is,

$$|f^{(r-1)}(x) - f^{(r-1)}(y)| \leqq |x-y|, \qquad x, y \in I.$$

Here r is a positive integer. Let W_r^* denote the similarly defined class of functions on $[-\pi, \pi]$ for which $f^{(k)}$ is 2π-periodic for $k = 0, \ldots, r-1$. Theorem 3 gives an upper bound on the distance of the set W_r^* from $\mathcal{T}_n$:

$$d(W_r^*, \mathcal{T}_n) = \sup_{f \in W_r^*} E_n^*(f) \leqq K'' n^{-r}, \qquad n = 1, 2, \ldots \tag{9}$$

and Theorem 4 gives an upper bound on the distance of the set W_r from π_n:

$$d(W_r, \pi_n) = \sup_{f \in W_r} E_n(f) \leqq Ln^{-r}, \qquad n = 1, 2, \ldots. \tag{10}$$

Let us concentrate for a moment on the latter. π_n is only one of many subspaces of $C(I)$ of dimension $n+1$ with which we can attempt to approximate the elements of W_r. Is there perhaps some sequence $\{E_m\}$ of subspaces of $C(I)$ of respective dimensions m, $m = 1, 2, \ldots$, for which the distance of W_r to E_m, given by

$$d(W_r, E_m) = \sup_{f \in W_r} \inf_{g \in E_m} \|f - g\|_{L^\infty(I)},$$

decreases to zero appreciably faster than m^{-r}? This question is the topic of Section 6 and there we shall see that the answer is basically no. That is, π_n is about the best subspace of dimension $n+1$ and the estimate (10) is the best possible, at least in terms of its order of magnitude.

2. Converse theorems of S. N. Bernstein and others. In 1912 there appeared a long and significant paper [2] by the Russian mathematician Serge Bernstein. Bernstein, who was to make original and

profound contributions to approximation theory for more than 40 years, proved among other things in this paper that Jackson's theorems have a valid converse; that is, it is possible to deduce a certain degree of smoothness for a function f given the rapidity with which $E_n(f)$ goes to zero. For example, Bernstein showed that the following holds.

THEOREM 5. *A continuous 2π-periodic function f has a continuous r-th derivative $f^{(r)}$ which lies in* $\mathrm{Lip}_M(\alpha)$ *for some α, $0<\alpha<1$, $M>0$, if and only if*

$$E_n^*(f) \leqq Cn^{-r-\alpha}, \qquad n = 1,2,\ldots, \tag{11}$$

for some constant C. Here r is a non-negative integer.

Again we note that the "only if" is just a special case of Theorem 3; the main point of Theorem 5 is the "if" statement. There is a non-periodic analogue of Theorem 5 but it is complicated by the fact that $E_n(f)$ is not a fine enough measure of the approximation needed *at the end points* of the interval. Let me be more precise. We assume as usual that the interval on which we are working is $I = [-1,1]$. Set

$$\Delta_n(x) = n^{-1}\max\{\sqrt{1-x^2}, 1/n\}, \quad n = 1,2,\ldots$$

so that Δ_n has height n^{-2} near ± 1 and is everywhere less than or equal to n^{-1}. Jackson's Theorem 3 has been sharpened by A. F. Timan [**34**] and its converse obtained (in special cases) by V. K. Dzjadyk [**8**]; one such case is this.

THEOREM 6. *Let f be a continuous function on I, let $0<\alpha<1$, and let r be a non-negative integer. Then f has a continuous r-th derivative which lies in* $\mathrm{Lip}_M(\alpha)$ *for some M if and only if there is a sequence $\{q_n\}$ of polynomials of respective degree n such that*

$$|f(x) - q_n(x)| \leqq C(\Delta_n(x))^{r+\alpha}, \qquad n = 1,2,\ldots, |x| \leqq 1, \tag{12}$$

where C is some constant.

Notice that in order to assure that $f^{(r)}$ lies in $\mathrm{Lip}(\alpha)$ on *all* of I we must have approximation to the order $n^{-2(r+\alpha)}$ at the end points of I. If we stay away from the endpoints of I we can state this:

> *if $E_n(f) \leqq Cn^{-r-\alpha}$ for some constant C, non-negative integer r and $\alpha \in (0,1)$ then $f^{(r)}$ exists and satisfies a Lipschitz condition of order α on $[-1+\varepsilon, 1-\varepsilon]$ for each $\varepsilon>0$.*

Of course, we find for such a function f that the Lipschitz constant $M = M_\varepsilon$ in (2) for the interval $[-1+\varepsilon, 1-\varepsilon]$ may become unbounded as $\varepsilon \to 0$. The difference between the estimate in (12) and that in (11) points up the substantial difference between approximation of non-periodic and periodic functions: with periodic functions there are no endpoint conditions to worry about.

There are converse theorems for moduli of continuity other than h^α but they are not as succinct as Theorems 5 and 6; see [**19**; p. 58, 73].

3. Geometric convergence and analytic extension. Suppose that f is a continuous function on the closed interval $[a,b]$ which we may as well assume to be $[-1,1]$. The theorems in Sections 1 and 2 show that the quantity

$$E_n(f) = \inf_{p\in\pi_n} \|f-p\| \qquad n = 1,2,\ldots$$

decreases to zero like some power of n if (and only if) f possesses a certain number of continuous derivatives. Suppose, however, that $E_n(f)$ decreases geometrically:

$$E_n(f) \leqq CR^{-n}, \qquad n = 1,2,\ldots,$$

where C is a constant and $1 < R$; or even

$$\limsup_{n\to\infty} (E_n(f))^{1/n} \leqq R^{-1},$$

where $1 < R$. Of course, it follows directly from Theorem 6 that f is C^∞ on $[-1, 1]$ but, in fact, a great deal more is true: f has an analytic extension to an open set containing $I = [-1, 1]$, the exact size of which is determined by R. More precisely, if $R > 1$, let E_R be the ellipse given by the equations

$$2x = (R + 1/R)\cos\theta$$

$$2y = (R - 1/R)\sin\theta, \qquad 0 \leqq \theta \leqq 2\pi.$$

Thus, E_R contains I in its interior and has foci at ± 1. The formulation of the theorem, which is due to S. N. Bernstein [**2**], is this:

THEOREM 7. *Let R_0 be the supremum of all those values of R such that f extends to be analytic within the ellipse E_R. (It is possible that $R_0 = \infty$.) Then*

$$\limsup_{n\to\infty} (E_n(f))^{1/n} = 1/R_0. \tag{13}$$

Conversely, if R_0 is defined by equation (13), *then f extends to be analytic within the ellipse E_{R_0}.*

As a corollary of Theorem 7 we obtain the result that a function f on I is the restriction of a function analytic in the whole complex plane if and only if $(E_n(f))^{1/n} \to 0$ as $n \to \infty$. Note again that like Theorems 1, 2, and 3, Theorem 7 relates an internal property of f, namely the growth of its derivatives, to the external property of its degree of approximation by algebraic polynomials.

There is another theorem in the spirit of Theorem 7 which involves approximation of a bounded function on the whole real line $\mathbb{R}$. Polynomials are not good approximants on $\mathbb{R}$ since they are inevitably unbounded if not constant. We choose instead for our approximants a certain family of entire functions, namely for $\sigma > 0$, let $\mathscr{E}_\sigma$ denote the set of entire functions F which are bounded on $\mathbb{R}$ and which are of exponential type *less than* σ. That is, F lies in $\mathscr{E}_\sigma$ if and only if F is analytic on the whole complex plane, bounded on $\mathbb{R}$, and satisfies the growth condition

$$|F(z)| \leqq \exp(\rho|z|)$$

for some $\rho < \sigma$ and all z with $|z|$ large. An example of such a function is $F(z) = \sin(\rho z)$ for any ρ, $0 < \rho < \sigma$.

Let V_r consist of those functions on $\mathbb{R}$ which are continuous and bounded and which possess continuous derivatives through order $r-1$ with the $r-1$st derivative satisfying

$$|f^{(r-1)}(x_2) - f^{(r-1)}(x_1)| \leqq |x_2 - x_1|, \qquad x_1, x_2 \in \mathbb{R};$$

that is, $f^{(r-1)}$ lies in $\mathrm{Lip}_1(1)$ on all of $\mathbb{R}$. For f in V_r we define its distance to the set $\mathscr{E}_\sigma$ by

$$D_\sigma(f) = \inf_{F\in\mathscr{E}_\sigma} \sup_{-\infty<x<\infty} |f(x) - F(x)| = \inf_{F\in\mathscr{E}_\sigma} \|f - F\|_{L^\infty(\mathbb{R})}.$$

The theorem we are interested in is this:

THEOREM 8. *Let $f \in V_r$. In order that f have an analytic extension to the strip $\{x + iy : |y| < \delta\}$ it is necessary and sufficient that*

$$\limsup_{\sigma\to\infty} (D_\sigma(f))^{1/\sigma} \leqq e^{-\delta}. \tag{14}$$

The sufficiency of (14) is due to S. N. Bernstein while its necessity is due to N. I. Achieser; see [**33**;

Theorem 6.5.3 and 5.7.22]. In Section 4 we will return to the classes $\mathscr{E}_\sigma$ and give a theorem in the form of Theorem 4 for $f \in V_r$ with $\mathscr{E}_\sigma$ replacing π_n.

4. Best constants. We return to the constant K'' which appears in Theorem 3. In fact, let us first consider the case when f lies in W_r^*; that is,

$$|f^{(r-1)}(x) - f^{(r-1)}(y)| \leqq |x-y|, \qquad x, y \in [-\pi, \pi]$$

and $f, \ldots, f^{(r-1)}$ are all 2π-periodic. We are interested in the size of K''—how small (or large) is it? If we set

$$\beta_{rn} = \sup_{f \in W_r^*} \inf_{T \in \mathscr{T}_n} \|f - T\|, \tag{15}$$

then Theorem 3 asserts that β_{rn} is no more than a constant times n^{-r}. But much more is known. In a remarkable case of simultaneous discovery the exact value of β_{rn} was found by the French mathematician J. Favard [9] and the Russian mathematicians N. I. Achieser and M. G. Krein [1] in 1937. This is their result.

THEOREM 9. *Let β_{rn} be given by* (15). *Then*

$$\beta_{rn} = K_r(n+1)^{-r}, \qquad n = 1, 2, \ldots, \qquad r = 1, 2, \ldots, \tag{16}$$

where

$$K_r = \frac{4}{\pi} \sum_{j=0}^{\infty} (-1)^{j(r+1)}/(2j+1)^{r+1}. \tag{17}$$

The numbers K_r satisfy

$$\pi^2/8 = K_2 < K_4 < \cdots < 4/\pi < \cdots < K_3 < K_1 = \pi/2$$

and clearly $K_r \to 4/\pi$ as $r \to \infty$. These numbers will reappear with some frequency in the succeeding sections.

It is worthwhile to pause here to give an outline of the proof of Theorem 9. Suppose $f \in W_r^*$; then f can be represented by its Fourier series

$$f(x) = \sum_{j=0}^{\infty} (a_j \cos jx + b_j \sin jx), \tag{18}$$

where

$$a_j \cos jx + b_j \sin jx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos j(x-t)\, dt. \tag{19}$$

There is nothing lost if we assume that $a_0 = 0$ since the constants lie in $\mathscr{T}_n$ for all n. Repeated integration by parts and some trigonometric identities convert (19) to

$$a_j \cos jx + b_j \sin jx = \frac{1}{\pi} \int_{-\pi}^{\pi} f^{(r)}(t) \frac{\cos(j(x-t) - r\pi/2)}{j^r}\, dt \tag{20}$$

for $j = 1, 2, \ldots$. We now insert (20) into (18) and interchange summation and integration which is permissible since the series is uniformly convergent if $r \geqq 2$ and has bounded partial sums and converges uniformly on $[\varepsilon, \pi - \varepsilon]$ for all $\varepsilon > 0$ if $r = 1$. We thus find that

$$f(x) = \int_{-\pi}^{\pi} u(t) D_r(x-t)\, dt,$$

where u is bounded by 1 and has mean-value 0 and

$$D_r(\theta)=\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{\cos(j\theta-r\pi/2)}{j^r}.$$

This function D_r has a very special and delicate property, namely: for r even, any even trigonometric polynomial of degree $n-1$ can equal D_r at no more than n places in $(0,\pi)$; for r odd, any odd trigonometric polynomial of degree $n-1$ can equal D_r at no more than $n-1$ places in $(0,\pi)$. This special property implies that the best L^1 approximation to D_r from $\mathcal{T}_n$, call it S_n, has the very regular behavior that

$$\operatorname{sign}(D_r-S_n)=\begin{cases}\operatorname{sign}(\cos nx) & \text{if } r \text{ is even}\\ \operatorname{sign}(\sin nx) & \text{if } r \text{ is odd.}\end{cases}\tag{21}$$

This is the major point in the proof and the remainder now follows directly. The integral

$$\int_{-\pi}^{\pi} u(t)S_n(x-t)dt=Lu(x)\tag{22}$$

is a trigonometric polynomial of degree n and

$$f(x)-Lu(x)=\int_{-\pi}^{\pi} u(t)(D_r(x-t)-S_n(x-t))dt$$

so that an application of Hölder's inequality implies that the uniform norm of $f-Lu$ is at most the L^1 norm of D_r-S_n. A computation, making use of (21), shows that this number is $K_r(n+1)^{-r}$. On the other hand, if F_{nr} is the unique function in W_r^* whose rth derivative is $\operatorname{sign}(D_r-S_n)$, then it turns out that F_{nr} has enough oscillation that its distance to W_r^* is just the same as its sup norm which is also its absolute value at $x=0$ and this number is precisely $K_r(n+1)^{-r}$. Full details of this proof may be found in [**19**; p. 115].

The major point to be made about the proof is that the class of functions we wish to approximate, W_r^*, can be represented as the integral transform of the unit ball of L^∞ by a certain kernel. The special properties of this kernel then allow us to solve the problem, and to find those functions, F_{nr} in the case above, which are at a maximum distance from the approximating subspace. There are a number of other theorems of this type in the non-periodic case which we will meet in Sections 6 and 7.

The function F_{nr} in W_r^* whose rth derivative is sign $(\cos nx)$ for r even, or sign $(\sin nx)$ for r odd, is important; let us spend a moment examining it. We have, except possibly for a sign,

$$F_{nr}(x)=\begin{cases}\dfrac{4}{\pi}(n+1)^{-r}\displaystyle\sum_{j=0}^{\infty}\frac{\cos((n+1)(2j+1)x)}{(2j+1)^{r+1}}, & r \text{ odd}\\[2ex] \dfrac{4}{\pi}(n+1)^{-r}\displaystyle\sum_{j=0}^{\infty}(-1)^j\frac{\cos((n+1)(2j+1)x)}{(2j+1)^{r+1}}, & r \text{ even.}\end{cases}\tag{23}$$

Hence, the maximum of the absolute value of F_{nr} is

$$\|F_{nr}\|=|F_{nr}(0)|=K_r/(n+1)^r$$

and F_{nr} equi-oscillates at the $2n+3$ places $x_j=-\pi+j(\pi/(n+1))$, $j=0,\dots,2n+2$; that is:

$$F_{nr}(x_j)=(-1)^j\|F_{nr}\|,\qquad j=0,1,\dots,2n+2\tag{24}$$

and $|F_{nr}(x)|<\|F_{nr}\|$ elsewhere. Note also that

$$F'_{nr}(x)=F_{n,r-1}(x),\qquad r=2,3,\dots.$$

This function F_{nr} or various scalings of it will reappear in later sections. Note that F_{nr} is a *spline* function of degree r; that is, there are points $x_0<x_1<\cdots<x_{2n+2}$ with the property that in each open interval (x_{j-1},x_j), $j=1,\dots,2n+2$, F_{nr} is an algebraic polynomial of degree r (since $F_{nr}^{(r)}$ is constant in this interval) and F_{nr} is in continuity class C^{r-1} on all of $[-\pi,\pi]$.

Now we continue our discussion of best constants. Once again we note that the condition $f \in W_r^*$ is merely the restriction that $f^{(r-1)}$ lie in $\mathrm{Lip}_1(1)$. The question of best constants for other moduli of continuity has been investigated and answered by N. P. Korneichuk, at least when the modulus in question is concave. Specifically, let $\omega(t)$ be a continuous function on $[0, \pi]$ with these properties

$$
\text{(25)}\quad
\begin{array}{rl}
\text{(i)} & \omega \text{ is concave, increasing} \\
\text{(ii)} & \omega(t_1 + t_2) \leqq \omega(t_1) + \omega(t_2) \\
\text{(iii)} & \omega(0) = 0,\ \omega(t) > 0 \text{ if } t > 0
\end{array}
$$

and let $\Lambda_r^*(\omega)$ denote those continuous functions which possess r continuous 2π-periodic derivatives with

$$\omega(f^{(r)}; t) \leqq \omega(t), \qquad 0 < t < \pi. \tag{26}$$

For example the choice $\omega(t) = t^\alpha$ gives the class of funtions whose rth derivative lies in $\mathrm{Lip}_1(\alpha)$. We set

$$\beta_r(\omega) = \sup_{f \in \Lambda_r^*(\omega)} E_n^*(f). \tag{27}$$

Define a sequence of functions φ_k by

$$\varphi_0(x) = 1/2, \qquad 0 \leqq x \leqq \pi$$

and for $k = 1, 2, \ldots,$

$$\varphi_k(x) = \tfrac{1}{2} \int_0^{\pi - x} \varphi_{k-1}(t)\,dt, \qquad 0 \leqq x \leqq \pi.$$

Then the following theorem holds; see [**16**].

THEOREM 10. *Let ω satisfy* (25); *then $\beta_r(\omega)$ defined by* (27) *is given by*

$$\beta_r(\omega) = \tfrac{1}{2}(n+1)^{-r} \int_0^\pi \varphi_{r-1}(\pi - t)\,\omega(t/n+1)\,dt, \qquad r = 1, 2, \ldots \tag{28}$$

and

$$\beta_0(\omega) = \tfrac{1}{2}\omega(\pi/n+1). \tag{29}$$

Some special cases of Theorem 10 are worth pointing out explicitly. Let $\omega_0(t) = t^\alpha$ for $0 < \alpha \leqq 1$. Then

$$\beta_0(\omega_0) = \tfrac{1}{2}(\pi/n+1)^\alpha$$

$$\beta_1(\omega_0) = \left(\frac{1}{4}\right)\left(\frac{1}{\alpha+1}\right)\left(\frac{\pi}{n+1}\right)^{\alpha+1}$$

$$\beta_2(\omega_0) = \left(\frac{1}{8}\right)\left(\frac{1}{\alpha+2}\right)\left(\frac{\pi}{n+1}\right)^{\alpha+2}.$$

Note that, in general, $\phi_k(x) \leqq (\pi/2)^k$ and hence

$$\beta_r(\omega) \leqq (n+1)^{-r}(\pi/2)^r \omega(\pi/n+1).$$

This shows, at least for a concave modulus of continuity, that the constant in Theorem 3 may be chosen to be $(\pi/2)^r$.

We turn to an approximation question for the special case when $f^{(r-1)}$ satisfies a Lipschitz condition of order 1 on all of $\mathbb{R}$. The following is an analogue on $\mathbb{R}$ of Theorem 9. The definitions of $\mathscr{E}_\sigma$ and V_r are in the second part of Section 3.

THEOREM 11. *The following holds*

$$\sup_{f\in V_r}\inf_{F\in\mathscr{E}_\sigma}\|f-F\|_{L^\infty(\mathbb{R})}=K_r/\sigma^r \tag{30}$$

where K_r is given by (17).

This theorem which is due to M. G. Krein **[17]** was proved in 1938 and implies Theorem 9 as Krein points out, basically because a periodic element of $\mathscr{E}_\sigma$ must be a trigonometric polynomial of degree less than σ; see **[10]** for the details.

Finally, we can obtain some information on the constant L which appears in Theorem 4, at least in the case W_r. This is contained in the following theorem, due again to S. N. Bernstein and published in 1947.

THEOREM 12. *Let*

$$\alpha_{rn}=\sup_{f\in W_r}E_n(f). \tag{31}$$

Then

$$\lim_{n\to\infty}n^r\alpha_{rn}=K_r \tag{32}$$

where K_r is the constant given in (17).

It is a remarkable fact that the limit of the left hand side of (32) is the number K_r which originally appeared as a constant involved with approximation of 2π-periodic functions by trigonometric polynomials. It is also noteworthy that Theorems 9, 11, and 12 are equivalent; that is, each implies the others. This is particularly interesting because Theorem 9 deals with periodic functions, Theorem 11 with approximation on $\mathbb{R}$ and Theorem 12 with the asymptotic behavior of a constant related to non-periodic approximation on $I=[-1,1]$. See **[10]** for the way in which each of the theorems implies the others. One further comment: any function f for which equality holds in (31) is a spline function with exactly $n-r+1$ knots and $|f^{(r)}|=1$ a.e. That is, there are points $-1=x_0<x_1<\cdots<x_{n-r+2}=1$ and $f^{(r)}(x)=(-1)^j$ for $x\in(x_{j-1},x_j)$, $j=1,2,\ldots,n-r+2$; this is proved in **[10]**. Note the similarity to the functions F_{nr} which are the extremals in Theorem 9.

It is worth pointing out that Jackson obtained upper estimates of 3 and $5\cdot3^r$ for the constants K and K'' in Theorems 1 and 3, respectively, and the estimate

$$L\leqq4(r+1)^{r-1}3^r/r!$$

for the constant in Theorem 4. (He actually has somewhat sharper results; see **[13]**.)

5. Extremal problems. It should come as no surprise that there are a number of (pointwise) extremal problems which are intimately connected with theorems on quantitative approximation theory. The entire problem of proving inverse theorems is tied up with estimating how large the derivative of a polynomial (algebraic or trigonometric) of degree n can be given a bound on the polynomial itself. For this we have results like these:

$$\|T'\|_{L^\infty(-\pi,\pi)}\leqq n\|T\|_{L^\infty(-\pi,\pi)}\qquad\text{if } T\in\mathscr{T}_n \tag{33}$$

$$\|p'\|_{L^\infty(I)}\leqq n^2\|p\|_{L^\infty(I)}\qquad\text{if } p\in\pi_n \tag{34}$$

$$|p'(x)|\leqq n(1-x^2)^{-1/2}\|p\|_{L^\infty(I)}\qquad\text{if } p\in\pi_n \text{ and } -1\leqq x\leqq1 \tag{35}$$

$$|p^{(k)}(0)|\leqq n^k\|p\|_{L^\infty(I)}\qquad\text{if } p\in\pi_n,\ k=0,\ldots,n \tag{36}$$

$$\|F'\|_{L^\infty(\mathbb{R})}\leqq\sigma\|F\|_{L^\infty(\mathbb{R})}\qquad\text{if } F\in\mathscr{E}_\sigma. \tag{37}$$

Inequality (33) is due to S. N. Bernstein as is (35) and also (37), which implies (33). (34) is due to A. A. Markov and (36) to V. A. Markov, brother of A. A. Markov. For example, (36) while crude, is essential in showing that Theorem 12 implies Theorem 11. In the same spirit we note that the classical Chebyshev polynomial C_n (the monic polynomial of degree n which has the smallest sup norm on I) satisfies the extremal property that

$$|p(x)| \leqq |C_n(x)|$$

for each x *outside* I if $p \in \pi_n$ and $\|p\|_{L^\infty(I)} \leqq \|C_n\|_{L^\infty(I)}$; see [**27**; p. 48]. Of course C_n is extremal in (34) and (35).

One of the more interesting extremal problems connected with our topic is this. (Recall that V_r consists of those bounded functions on $\mathbb{R}$ whose rth derivative is bounded by 1.)

If $x \in \mathbb{R}$, $f \in V_r$ and $1 \leqq s \leqq r-1$, how big is $|f^{(s)}(x)|$ in terms of $\|f\|_{L^\infty(\mathbb{R})}$?
The change of scale $x \to ax + b$ shows that no one point is really any different from any other nor is it important that $f^{(r)}$ be bounded by one so that we consider instead this problem:

Find the smallest constant C_{sr} such that

$$\|f^{(s)}\|_{L^\infty(\mathbb{R})} \leqq C_{sr} \|f\|_{L^\infty(\mathbb{R})}^{1-s/r} \|f^{(r)}\|_{L^\infty(\mathbb{R})}^{s/r}. \tag{38}$$

(The exponents are chosen to make the problem invariant under the changes of scale $f \to bf(ax)$.) This is known as the Landau problem, after Edmund Landau who solved it in 1913 in the case $r = 2$, $s = 1$; see [**18**]. The general case was solved in 1939 by Kolmogoroff [**15**] who proved this.

THEOREM 13. *The constant C_{sr} is given*

$$C_{sr} = K_{r-s} K_r^{(s/r)-1}, \tag{39}$$

where K_{r-s}, K_r are given in (17). *Equality holds in* (38) *if and only if f is a scaling of the function F_{nr} given in* (23).

An elementary proof of Theorem 13 can be found in [**6**]. The Landau problem for the semi-infinite line $[0, \infty)$ was solved in [**29**]; the solution for this case is intimately tied up with a certain spline function that we shall discuss in Section 6, Theorem 15(bis). The seemingly more elementary Landau problem for the finite interval $[a, b]$ is still open. Its solution is known to be connected with certain special spline functions; see [**28**].

6. *n* width. We turn here to the question raised at the end of Section 1. By way of review, there we noted that the distance from the set W_r to the $n+1$ dimensional subspace π_n was on the order of n^{-r} and we asked whether there is some other subspace of the same dimension which is "closer" to W_r in the sense that its distance to W_r is appreciably smaller than n^{-r}. We can make this notion very precise.

Let A be a subset of a Banach space X (think of X as being $C(I)$ or $C^*(-\pi, \pi)$) and consider the number $d_n(A)$ given by

$$d_n(A) = \inf_{X_n} \sup_{f \in A} \inf_{g \in X_n} \|f - g\|_X, \tag{40}$$

where X_n runs over *all* subspaces of X of dimension n. This quantity is called the *n-width* of A in X. This concept of n-width was introduced in 1937 by Kolmogoroff [**14**] with $X = \ell^2$, a Hilbert space, and A a particular subset determined by a certain linear differential operator. Our concern will be to estimate the n-widths of the sets W_r and W_r^* (and certain of their close relatives) in the spaces $C(I)$ and $C^*(-\pi, \pi)$, respectively, and to compare these numbers to the numbers β_{rn} and α_{rn} given in (15) and (31) which we have already discussed in Section 3. Let us begin with the periodic case since the answer is sharp there.

THEOREM 14. *The $2n+1$-width of W_r^* in $C^*(-\pi,\pi)$ is $K_r(n+1)^{-r}$ and thus $\mathcal{T}_n$ is an optimal approximating subspace to W_r^*.*

The answer for W_r is not quite as sharp.

THEOREM 15. *The n-width of W_r in $C(-1,1)$ is asymptotic to $K_r(2/\pi)^r n^{-r}$.*

Thus, Theorems 15 and 12 imply that π_n is not an optimal approximating subspace. On the other hand, the order of magnitude of the distance of π_n to W_r is the same as the $(n+1)$-width of W_r for n large so that for purposes of approximation π_n is a good choice. Theorems 14 and 15 are due to V. M. Tikhomirov **[31]**, **[32]** who pioneered in finding n-widths for a number of classes.

There are a number of other classes of functions whose n-width can be estimated. For example,

THEOREM 16. *Let A_R consist of the functions analytic and bounded by* 1 *within the ellipse E_R* (see Section 3). *Then the n-width of A_R in $C(I)$ satisfies*

$$R^{-n} \leqq d_n(A_R) \leqq 2(R-1)^{-1}R^{-n}.$$

Another case which is known is this; see **[16]**.

THEOREM 17. *The $2n+1$-width of the class $\Lambda_r^*(\omega)$ is given by the right-hand side of* (28); *hence, $\mathcal{T}_n$ is an optimal approximating subspace.*

It is interesting to note that the sharp theorems are generally for periodic functions and $\mathcal{T}_n$ is an optimal approximating subspace whereas for non-periodic functions generally only asymptotic behavior is known and π_n is *not* an optimal approximating subspace.

Let us return to Theorem 15 for a moment and go into a little detail. Each element $f \in W_r$ has the form

$$f(x) = p(x) + \int_I K(x,t)u(t)dt,$$

where $p \in \pi_{r-1}$ and u lies in the unit ball of $L^\infty(I)$ and $K(x,t)$ is given by

$$K(x,t) = (x-t)_+^{r-1}/(r-1)!$$

(The subscript "plus" means take the maximum of the quantity and zero.) It is reasonably obvious that the n-width of W_r is infinite if $n < r$. (There are never enough dimensions to take care of all of π_{r-1}.) Hence, we assume $r \leqq n$. In this case there is a function p_0 of the special form

$$p_0(x) = \sum_0^{r-1} a_j x^j + (x+1)^r/r! + 2\sum_{k=1}^{n-r} (-1)^k (x-\xi_k)_+^r/r! \tag{41}$$

where $a_0,\dots,a_{r-1}$ are real numbers and $-1 < \xi_1 < \cdots < \xi_{n-r} < 1$ which equioscillates at $n+1$ points of I:

$$p_0(t_i) = (-1)^{i+r+1}\|p_0\|_{L^\infty(I)}, \qquad i = 1,\dots,n+1,$$

where

$$-1 = t_1 < \cdots < t_{n+1} = 1.$$

This function p_0 can be characterized (except for sign) also by the fact that among all functions of the form of the right-hand side of (41), as $a_0,\dots,a_{r-1}$ and $\xi_1,\dots,\xi_{n-r}$ are allowed to vary as prescribed, p_0 has the smallest sup norm on I. Thus, p_0 is a perfect spline of degree n and it has exactly $n-r$ knots in $(-1,1)$; notice that if $n = r$, then p_0 reduces to (a multiple of) the Chebyshev polynomial. The theorem of Tikhomirov on the n-width of W_r is this:

THEOREM 15(bis). *The n-width of W_r is equal to $\|p_0\|_{L^\infty(I)}$ and an optimal subspace is the one spanned by the n functions* $1, x, \ldots, x^{r-1}$, $(x-\xi_1)_+^{r-1}, \ldots, (x-\xi_{n-r})_+^{r-1}$.

Hence, spline functions play a critical role in the determination of the n-width of the function class W_r as they did in the $2n+1$-width of W_r^*. For extensions of Tikhomirov's result, see **[25]**.

7. Optimal recovery. We turn in this section to a topic of recent interest which is closely related to several things we've already covered. The general idea is this: suppose we are given n values of a function f, say $f(t_1), \ldots, f(t_n)$, and suppose we also know that $f \in W_r$. What is the most accurate way of estimating f on $[-1,1]$? That is, we wish to find a scheme which will associate with each function $f \in W_r$ another function, call it Sf, in such a fashion that Sf depends only on $f(t_1), \ldots, f(t_n)$ and $Sf - f$ is small on all of I. It is *not* demanded that the rule $f \to Sf$ be linear but it is required that if $f(t_j) = g(t_j)$ for $j = 1, \ldots, n$, then $Sf = Sg$. Once we have determined an optimal or almost optimal scheme for the points $t_1, \ldots, t_n \in I$ we are then interested in how to select points $t_1^0, \ldots, t_n^0$ so that the scheme associated with these points is best among all possible choices of the points $t_1, \ldots, t_n$.

Let $W^{r,\infty}(I)$ consist of those functions $f \in C^{r-1}(I)$ for which $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L^\infty(I)$; the set W_r just consists of those elements of $W^{r,\infty}(I)$ which have rth derivative bounded by one. If $t_1, \ldots, t_n$ are points of I, set $t = (t_1, \ldots, t_n)$ and

$$U(t) = \{\{f(t_j)\}_{j=1}^n : f \in W_r\}.$$

We shall always assume that $n \geqq r$ to avoid trivialities and we do allow coincidences among the t_j so long as $t_{i+r} > t_i$ for all i; if $t_i = \cdots = t_{i+m}$ we interpret $f(t_{i+j})$ to be $f^{(j)}(t_i)$, $j = 0, \ldots, m$, $m \leqq r-1$. Hence, a recovery scheme (for t) is a mapping from $U(t)$ into $C(I)$. We measure the accuracy of a recovery scheme S by the rule:

$$E(t;S) = \sup_{f \in W_r} \|f - Sf\|_{L^\infty(I)}. \tag{42}$$

We are then interested in the two numbers:

$$E(t) = \inf_S E(t;S) \tag{43}$$

and

$$E = \inf_t E(t) \tag{44}$$

and in determining, if possible, schemes S and S^0 for which equality holds in (43) and (44). The solution of this recovery problem is intimately connected with spline functions.

An argument involving total positivity established that there is a spline function Q_t of degree r on I with exactly $n-r$ knots which satisfies

$$|Q_t^{(r)}| = 1 \quad \text{a.e.} \tag{45}$$

and

$$Q_t(t_j) = 0, \qquad j = 1, \ldots, n. \tag{46}$$

That is, Q_t is given by

$$r!\,Q_t(x) = \sum_{j=0}^{r-1} a_j x^j + (x+1)^r + 2\sum_{i=1}^{n-r} (-1)^i (x-\xi_i)_+^r,$$

where $-1 < \xi_1 < \cdots < \xi_{n-r} < 1$. (The last sum does not appear if $n = r$.) Q_t is unique up to sign. The points $\xi_1, \ldots, \xi_{n-r}$ are *not* free to vary here but are instead determined by (46). The answer to our problem is this.

THEOREM 18. *$E(t)=\|Q_t\|_{L^\infty(I)}$ and hence $E=d_n(W_r)$.*

Once we know the first conclusion the second follows from the properties of the spline function p_0 we described in Section 7: p_0 has exactly $n-r$ knots, n zeros, $|p_0^{(r)}|=1$ a.e., and p_0 has the smallest possible sup norm among all such functions. Hence, an optimal choice $t_1^0,\ldots,t_n^0$ for the points would be the choice of the zeros of p_0 and $E=\inf_t E(t)=\|p_0\|=d_n(W_r)$. Let us now show that

$$E(t)\geqq\|Q_t\|. \tag{47}$$

Since $Q=Q_t$ lies in W_r we have for any scheme S that both $Q-SQ$ and $-Q-S(-Q)$ are functions in $W^{r,\infty}(I)$ and $SQ=S(-Q)$ since $Q(t_j)=0$ for $j=1,\ldots,n$. Hence, either

$$\|Q-SQ\|\geqq\|Q\|$$

or

$$\|-Q-S(-Q)\|\geqq\|Q\|.$$

Thus, (47) holds. The reverse inequality to (47) is more difficult and strongly depends on various special properties of Q. We indicate one of the main ideas. It turns out that the knots $\xi_1,\ldots,\xi_{n-r}$ of Q are interspersed among the zeros $t_1,\ldots,t_n$ in such a fashion that for each n-tuple $y_1,\ldots,y_n$ in $\mathbb{R}^n$ there is a unique spline of degree $r-1$ of the form

$$s(x)=\sum_{j=0}^{r-1}a_jx^j+\sum_{j=1}^{n-r}b_j(x-\xi_j)_+^{r-1} \tag{48}$$

with

$$s(t_j)=y_j,\qquad j=1,\ldots,n. \tag{49}$$

The mapping from $y\in U(t)$ to $s=s_y$ is a scheme and it is possible to show that

$$\|f-s_f\|\leqq E(t) \tag{50}$$

for each $f\in W_r$.

This establishes the desired equality. The reader is referred to **[26]** for the details. Optimal recovery for periodic functions is discussed in **[5]**.

8. Some further comments. Obviously I have touched on only a few of many possible topics in this paper. As I indicated in the introduction, I omitted the entire area of approximation of functions of several variables and also the subject of approximation in norms other than the sup norm. In both these cases the theory in these areas is less complete and there is room for new theorems and new ideas. Some other areas in which there remain unsolved problems are mentioned in the final section of the survey paper by Korneichuk **[16]**. As for new directions in this field optimal recovery and n-widths are now active areas of research; there is also much to be done in investigations concerning approximation by non-linear classes of functions. The books **[21]** and **[22]** contain a number of papers of recent vintage that touch on this last topic as well as a number of others covered here. Finally, the books **[7]**, **[11]**, **[19]**, **[27]**, **[30]**, and **[33]** are good sources for much of the material covered in this paper as well as a great deal more in approximation theory.

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GRADIENT CHARACTERIZATIONS OF ANALYTICITY

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1. Introduction. It is an immediate consequence of the Cauchy–Riemann equations that if $f = u + iv$ or $\bar{f} = u - iv$ is analytic in a domain G, then $\nabla u \cdot \nabla v = 0$ and $|\nabla u| = |\nabla v|$ are identities in G. That is, ∇u and ∇v everywhere have the same length, and the vectors are perpendicular where they are not zero. We shall show that this geometric property of the gradients of u and v actually characterizes analytic functions, at least with the assumption that u and v have continuous partial derivatives up to the second order. We then generalize this result in the following way. We present a family of inner products "$*$", each with its norm "$\| \ \|$". For each such inner product $*$, the equations $\nabla u * \nabla v = 0$ and $\|\nabla u\| = \|\nabla v\|$ characterize those functions $u + iv$ which are analytic after a change of variables determined by $*$. The equations imply that ∇u and ∇v are non-parallel wherever they are non-zero. The converse is essentially true. In particular, if ∇u and ∇v never vanish and are never parallel on a domain, then $u + iv$ is analytic after an appropriate change of variable.

The reader is invited to look first at Section 4, where we give an outline of the development. The principal results are stated in Section 4 without the encumbrance of hypotheses.

These results have applications to the study of algebras of functions satisfying a second order linear partial differential equation **[1]**.

Analyticity of $u + iv$. In this section we show that if $f = u + iv \in C^2(G)$, then f or $\bar{f}$ is analytic in G if $\nabla u \cdot \nabla v = 0$ and $|\nabla u| = |\nabla v|$ hold throughout G. Dzyadyk stated this result in **[4]** for the more general case in which $f \in C^1(G)$, but he did not give a complete proof. The proof we give here is entirely elementary.

THEOREM 1. *Let $u, v \in C^2(G)$. Then $u + iv$ or $u - iv$ is analytic on G if and only if $\nabla u \cdot \nabla v = 0$ and $|\nabla u| = |\nabla v|$ on G.*

Proof. We assume that the two identities hold on G,

$$u_x v_x + u_y v_y = 0, \tag{1}$$
$$u_x^2 + u_y^2 = v_x^2 + v_y^2.$$

The identities (1) are immediate from the Cauchy–Riemann equations, so we prove only that they are sufficient for analyticity of $u + iv$ or $u - iv$. We let $f = u + iv$, so $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$, and

$$f_x^2 + f_y^2 = (u_x^2 + u_y^2) - (v_x^2 + v_y^2) + 2i(u_x v_x + u_y v_y).$$

Hence (1) is equivalent to $f_x^2 + f_y^2 = 0$. In particular, $f_x = 0$ if and only if $f_y = 0$, and $f_x = \pm i f_y$. The condition $f_x = \pm i f_y$, and the continuity of u_x, u_y, v_x, v_y, say that the Cauchy–Riemann equations ($u_x = v_y, u_y = -v_x$; or $u_x = -v_y, u_y = v_x$) hold on each component of $G - Z$, where $Z = \{(x, y) \in G \colon u_x = u_y = v_x = v_y = 0\}$. Let Z^0 be the interior of Z, and $Z^C = G - Z$. To show that

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$Z^0 \cup Z^C$ is dense in G, let U be any open subset of G. If $U \cap Z^C = \emptyset$, then $U \subset Z$, so $U \subset Z^0$. Hence any open U intersects Z^C or Z^0. Since the first partials of u and v are zero on Z, u and v are certainly harmonic on Z^0. We have already seen that $u + iv$ or $u - iv$ is analytic on each component of Z^C, because $f_y = \pm if_x$ and one sign must hold throughout each component by continuity of the partials. Therefore u and v are harmonic on Z^C as well as Z^0. Since $Z^C \cup Z^0$ is dense in G, and $u_{xx} + u_{yy}$, $v_{xx} + v_{yy}$ are continuous on G, u and v are harmonic on G. The functions $g = u_x - iu_y$ and $h = v_x - iv_y$ are analytic on G, since

$$\frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial y}(-u_y), \quad \frac{\partial}{\partial y}(u_x) = -\frac{\partial}{\partial x}(-u_y),$$

and similarly for v. Either u is constant ($g \equiv 0$), or the zeros of g are isolated, and similarly for v and h. We can not have u constant and v not, since $|\nabla u| = |\nabla v|$, so we assume the zeros of both g and h are isolated. Therefore Z consists of isolated points, $Z^0 = \emptyset$, and Z^C is a connected dense subset of G. It follows that f or $\bar{f}$ is analytic on Z^C, and of course continuous on G. An isolated singularity of an analytic function is removable if the function is bounded, so f or $\bar{f}$ is in fact analytic on all of G.

3. Interior mappings. In this section we show that a large class of interior mappings can be characterized in a way which is directly analogous to our earlier characterization of analytic functions. A function f is an *interior mapping* if $f = \phi \circ \zeta$, where ζ is a homeomorphism on G, and ϕ is analytic on $\zeta(G)$. We replace the condition $f_x^2 + f_y^2 = 0$ of Section 1 by a general quadratic equation $af_x^2 + 2bf_xf_y + cf_y^2 = 0$, with $ac - b^2 = 1$, and the condition $\nabla u \cdot \nabla v = 0$, $|\nabla u| = |\nabla v|$ by $\nabla u * \nabla v = 0$, $\|\nabla u\| = \|\nabla v\|$, where $*$ is a new inner product, and $\|\cdot\|$ is the corresponding norm. When ∇u and ∇v do not vanish, these conditions are equivalent to ∇u and ∇v being non-parallel. Hence if ∇u and ∇v are never parallel, then $u + iv$ is an interior mapping.

DEFINITION. Let a, b, c be continuous real functions on G with $a > 0$, and $ac - b^2 = 1$. For two-vectors $\langle p, q \rangle$ and $\langle r, s \rangle$, define

$$\langle p, q \rangle * \langle r, s \rangle = apr + bps + bqr + cqs,$$

and let

$$\|\langle p, q \rangle\|^2 = \langle p, q \rangle * \langle p, q \rangle = ap^2 + 2bpq + cq^2.$$

Since $ac - b^2 = 1$, $\|\langle p, q \rangle\| = 0$ if and only if $p = q = 0$.

LEMMA 2. *If $u, v \in C^1(G)$, and $f = u + iv$, the following two conditions are equivalent*:

$$\nabla u * \nabla v = 0, \quad \|\nabla u\| = \|\nabla v\|, \tag{2}$$

$$af_x^2 + 2bf_xf_y + cf_y^2 = 0. \tag{3}$$

Proof. The real part of the left side of (3) is $\|\nabla u\|^2 - \|\nabla v\|^2$, and the imaginary part is $2\nabla u * \nabla v$.

Observe that if (2) or (3) holds, then $f_x = 0$ if and only if $f_y = 0$, and $\nabla u = \langle 0, 0 \rangle$ if and only if $\nabla v = \langle 0, 0 \rangle$. Hence all of $\nabla u, \nabla v, f_x, f_y$ are zero if any one is. We let Z be the closed subset of G where u_x, u_y, v_x, v_y are all zero. Then all of $\nabla u, \nabla v, f_x, f_y$ are non-zero on $G - Z$.

It is suggestive to apply the terms "perpendicular," "parallel," etc., to the complex quantities f_x, f_y (when they are both non-zero) as well as to the two-vectors ∇u and ∇v. Thus we will say $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$ are parallel if $f_x = \alpha f_y$ for real α, and f_x, f_y are perpendicular if $f_x = i\beta f_y$ for real β; f_x and f_y are not parallel if $f_x = (\alpha + i\beta)f_y$ with $\beta \neq 0$. The hypothesis of Theorem 1 is either of the equivalent conditions:

$$\nabla u \cdot \nabla v = 0, \quad |\nabla u| = |\nabla v|, \tag{i}$$

$$f_x^2 + f_y^2 = 0. \tag{ii}$$

Condition (ii), written as $f_x = \pm i f_y$, can now be expressed by saying f_x and f_y have the same modulus, and are perpendicular where they are non-zero.

LEMMA 3. *Let* $u, v \in C^1(G)$, $f = u + iv$, *and* $J = J(f) = u_x v_y - u_y v_x$. *The following are equivalent (at each point of* G*):*

(i) $J \neq 0$,

(ii) ∇u *and* ∇v *are not* $\langle 0, 0\rangle$, *and are not parallel,*

(iii) f_x *and* f_y *are not zero and are not parallel.*

If $\nabla u, \nabla v, f_x, f_y$ *are all non-zero, then* $\nabla u, \nabla v$ *are non-parallel if and only if* f_x, f_y *are non-parallel; i.e., if and only if* $f_x = (\alpha + i\beta) f_y$ *with* $\beta \neq 0$.

Proof. The Jacobian is the determinant of a two by two matrix whose row entries are the components of ∇u and ∇v, and whose column entries are the real and imaginary parts of f_x and f_y. Since parallelness is the same as rows or columns being proportional, the result is immediate.

LEMMA 4. *Let* $f \in C^1(G)$ *and assume that*

$$f_x = (\alpha + i\beta) f_y \tag{4}$$

where α, β *are continuous real functions and* $\beta \neq 0$. *Then* f *satisfies* (2) *for appropriate* a, b, c, *with* $ac - b^2 = 1$. *In particular, if* $f = \phi \circ \zeta$ *is an interior mapping on* G *with* $\zeta = \xi + i\eta \in C^1(G)$, *and* $J(\zeta) = \xi_x \eta_y - \xi_y \eta_x \neq 0$, *then* f *satisfies* (2), (3), (4) *with* a, b, c, α, β *continuous.*

Proof. If (4) holds, we let $a = 1/|\beta|$, $b = -\alpha/|\beta|$, $c = (\alpha^2 + \beta^2)/|\beta|$, and verify directly that $ac - b^2 = 1$, $a > 0$, and $af_x^2 + 2bf_x f_y + cf_y^2 = 0$.

Now assume $f = \phi \circ \zeta$ with ϕ analytic and $J(\zeta) \neq 0$. By Lemma 3, $\zeta_y \neq 0$ and $\zeta_x = (\alpha + i\beta)\zeta_y$ with $\beta \neq 0$. Clearly α, β are continuous. Since ϕ is analytic, $f_x = (\phi' \circ \zeta)\zeta_x$, $f_y = (\phi' \circ \zeta)\zeta_y$, so (4) holds.

Our aim is to show that (2) or (3) (and *a fortiori* (4)) actually characterize interior mappings. To this end we introduce the following two systems of equations, which play the same role for interior mappings that the Cauchy–Riemann equations ($u_x = v_y$, $u_y = -v_x$, or $u_x = -v_y$, $u_y = v_x$) do for analytic functions:

$$\begin{aligned} v_x &= -bu_x - cu_y & \qquad u_x &= bv_x + cv_y \\ v_y &= au_x + bu_y\,; & u_y &= -av_x - bv_y. \end{aligned} \tag{5}$$

$$\begin{aligned} v_x &= bu_x + cu_y & \qquad u_x &= -bv_x - cv_y \\ v_y &= -au_x - bu_y\,; & u_y &= av_x + bv_y. \end{aligned} \tag{6}$$

The two systems in (5) are equivalent (since $ac - b^2 = 1$), and the two systems in (6) are equivalent. These systems are called Beltrami systems, and have been widely studied [**6**, Ch. 2].

LEMMA 5. *If* $u, v \in C^1(G)$, *then* u, v *satisfy* (5) *or* (6) *if and only if* $\nabla u * \nabla v = 0$ *and* $\|\nabla u\| = \|\nabla v\|$. (Of course (5) and (6) both hold only at points where $u_x = u_y = v_x = v_y = 0$. On each component of the complement of this set, exactly one of (5), (6) holds. We shall see in the proof of Theorem 6 that in fact (5) holds throughout G, or (6) does.)

Proof. If, for example, u and v satisfy (5), then $\nabla u * \nabla v = 0$ and $\|\nabla u\| = \|\nabla v\|$ can be verified directly by using (5) and the definition of $*$.

Now assume $\nabla u * \nabla v = 0$ and $\|\nabla u\| = \|\nabla v\|$, and use the equivalent complex form from Lemma 2

$$af_x^2 + 2bf_x f_y + cf_y^2 = 0. \tag{3}$$

The systems (5) and (6) are clearly satisfied on the set Z where $f_x = f_y = 0$, so we restrict our attention to one component of $G - Z$. Then (3) can be written

$$a\left(f_x - \left(-\frac{b}{a}+\frac{1}{a}i\right)f_y\right)\left(f_x - \left(-\frac{b}{a}-\frac{1}{a}i\right)f_y\right) = 0, \tag{7}$$

and exactly one factor is zero on the component. If the first factor is zero we get

$$u_x = -\frac{b}{a}u_y - \frac{1}{a}v_y,$$

$$v_x = \frac{1}{a}u_y - \frac{b}{a}v_y.$$

This system is equivalent to (6). Similarly, setting the second factor of (7) equal to zero leads to (5).

DEFINITION. A function F is Hölder continuous in G if for any compact set $K \subset G$, there are numbers c and $\alpha, 0 < \alpha \leqq 1$, such that $|F(z_1) - F(z_2)| \leqq c|z_1 - z_2|^\alpha$ for all $z_1, z_2 \in K$.

We will use the result that if a, b, c have Hölder continuous first derivatives in an arbitrary domain G, and $ac - b^2 = 1, a > 0$, then the Beltrami system (5) has a solution $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ such that $\zeta = \xi + i\eta$ is a homeomorphism on G, and $J(\zeta)$ does not vanish. (See [**3**, p. 160] or [**2**], [**5**, p. 132] for more general results.) The solutions ξ, η of (5) necessarily have Hölder continuous derivatives of one order higher than the coefficients [**6**, Theorem 2.4, p. 87], and hence ξ and η are certainly C^2 functions. The mapping $(x, y) \to (\xi, \eta)$ constitutes a change of variables, and since the Jacobian does not vanish, the inverse mapping is also of class C^2.

THEOREM 6. *Let a, b, c have Hölder continuous first derivatives in G, with $ac - b^2 = 1, a > 0$, and let $f = u + iv \in C^2(G)$. If $\nabla u * \nabla v = 0$ and $\|\nabla u\| = \|\nabla v\|$ in G, then there is a homeomorphism $\zeta = \xi + i\eta$ on G such that $\zeta \in C^2(G)$, $J(\zeta) \neq 0$, and $f = \phi \circ \zeta$ where ϕ is analytic on $\zeta(G)$.*

Proof. Let ζ be the homeomorphism described in the preceding paragraph. We show that under the change of variables $(x, y) \to (\xi, \eta)$ the equation $af_x^2 + 2bf_xf_y + cf_y^2 = 0$ transforms into $f_\xi^2 + f_\eta^2 = 0$. Let φ be defined by $\varphi(\xi(x, y), \eta(x, y)) = f(x, y)$, so that $\phi \circ \zeta = f$. The partials of f are

$$f_x = \phi_\xi \xi_x + \phi_\eta \eta_x$$

$$f_y = \phi_\xi \xi_y + \phi_\eta \eta_y.$$

Substituting these in (3) gives

$$\phi_\xi^2\|\nabla\xi\|^2 + \phi_\eta^2\|\nabla\eta\|^2 + 2\phi_\xi\phi_\eta \nabla\xi * \nabla\eta = 0. \tag{8}$$

By Lemma 5, $\|\nabla\xi\| = \|\nabla\eta\|$ and $\nabla\xi * \nabla\eta = 0$. Since $J(\zeta) \neq 0$, $\|\nabla\xi\| \neq 0$ by Lemma 3. Hence we get $\phi_\xi^2 + \phi_\eta^2 = 0$. By Theorem 1, ϕ or $\bar\phi$ is an analytic function of ζ on $\zeta(G)$. Replace ζ by $\bar\zeta$ if $\bar\phi$ is analytic (in which case (5) becomes (6)), and we have the result.

COROLLARY. *If f has Hölder continuous derivatives up to second order, and $J(f) \neq 0$ in G, then $f = \phi \circ \zeta$ with ϕ and ζ as above, and $\phi'(\zeta) \neq 0$ on $\zeta(G)$.*

Proof. By Lemma 3, f_x and f_y are not zero, and $f_x/f_y = \alpha + i\beta$ with $\beta \neq 0$. Clearly α and β have Hölder continuous first derivatives, and hence $a = 1/|\beta|$, $b = -\alpha/|\beta|$, and $c = (\alpha^2 + \beta^2)/|\beta|$ do. As in the proof of Lemma 4, $ac - b^2 = 1, a > 0$, and $af_x^2 + 2bf_xf_y + cf_y^2 = 0$. Hence $f = \phi \circ \zeta$ by the theorem. We have $f_x = (\phi' \circ \zeta)\zeta_x$, $f_y = (\phi' \circ \zeta)\zeta_y$, so $\phi' \neq 0$ because $J(f) \neq 0$.

4. Outline. We list our results below in outline form. All these results hold with appropriate smoothness hypotheses, and it is sufficient to assume that all functions are in $C^3(G)$.

The conditions below refer to a fixed domain G, and functions $f = u + iv$ defined on G.

$$\nabla u \cdot \nabla v = 0, \quad |\nabla u| = |\nabla v|. \tag{A}$$

(B) $$f_x^2 + f_y^2 = 0.$$

(C) $$u_x = v_y,\ u_y = -v_x;\quad \text{or}\quad u_x = -v_y,\ u_y = v_x.$$

THEOREM I. *The conditions* (A), (B), (C) *are equivalent, and equivalent to the fact that f or $\bar{f}$ is analytic on G.*

(D) $$J(f) = u_x v_y - u_y v_x \neq 0.$$

(E) $\nabla u, \nabla v$ *do not vanish, and are not parallel.*

(F) $f_x = (\alpha + i\beta) f_y$ *for functions* α, β *with* $\beta \neq 0$.

LEMMA. (D) *and* (E) *are equivalent, and imply* (F).

Agreement.
$$ac - b^2 = 1, a > 0$$
$$(p, q) * (r, s) = apr + bps + bqr + cqs$$
$$\|(p, q)\|^2 = (p, q) * (p, q)$$
$$a = 1/|\beta|,\ b = -\alpha/|\beta|,\ c = (\alpha^2 + \beta^2)/|\beta|.$$

(G) $$\nabla u * \nabla v = 0,\quad \|\nabla u\| = \|\nabla v\|.$$

(H) $$af_x^2 + 2bf_x f_y + cf_y^2 = 0.$$

(I) $$\begin{cases} v_x = -bu_x - cu_y \\ v_y = au_x + bu_y \end{cases} \text{or} \quad \begin{cases} v_x = bu_x + cu_y, \\ v_y = -au_x - bu_y. \end{cases}$$

THEOREM II. *The conditions* (G), (H), (I) *are equivalent, and are consequences of* (F).

THEOREM III. *There is a change of variables $(x, y) \to (\xi, \eta)$ on G onto G' such that ξ and η satisfy* (I). *After this change of variables,* (H) *becomes* (B), *and* (I) *becomes* (C). *If $f = u + iv$ satisfies* (G), (H) *or* (I), *then f or $\bar{f}$ is an analytic function of ζ; i.e., $f = \phi \circ \zeta$ or $\bar{f} = \phi \circ \zeta$ for ϕ analytic on $\zeta(G)$.*

If $f = \phi \circ \zeta$, where ϕ is analytic and $J(\zeta) \neq 0$ then (F) holds, so finally we have:

THEOREM IV. *Each of the conditions* (F), (G), (H), (I) *is equivalent to the fact that f is a smooth interior mapping*; (D) *or* (E) *implies* (F), (G), (H), (I).

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A CHARACTERIZATION OF THE SPLITTING OF INSEPARABLE ALGEBRAIC EXTENSIONS

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This paper is concerned with the problem as to when an algebraic extension of a field F of characteristic p can be obtained by first extracting pth roots and then making a separable extension, together with related matters. Results are obtained which would seem to supplement nicely some of the standard material in a first-year graduate course which covered algebraic extensions of fields. For example, the problem is connected with simplicity of extensions, and one corollary is that any finite extension of $k(x)$, k perfect and x transcendental over k, is simple. While the material is primarily intended for course usage, possibly as exercises, the proofs are sufficiently elementary that the paper is within the grasp of a graduate student who has already had such a course and wished to read independently.

Throughout this paper the notation and terminology of [1, chapters 5 and 7] will be used unless otherwise noted. In the following, F shall denote a field of characteristic p, K an algebraic extension of F, K_0 and K_i the maximal separable and maximal purely-inseparable extensions of F contained in K.

In general, $K \neq K_0K_i$ (the composition of K_0 and K_i). If $K = K_0K_i$, we say that K splits over F. Theorem 1 gives necessary and sufficient conditions for K to split over F, with Theorem 2 providing a version applicable when K is a simple extension of F (not terminology of [1], an extension, $K \supset F$, is called simple if $K = F(\alpha)$). Theorem 3 discusses certain extensional properties of F, and in particular, if $F = k(x)$, where x is transcendental over k, then we have that these properties are all equivalent to k being a perfect field.

The properties to be considered are intrinsic to F and K, but at times it is convenient in the proofs to have appropriate algebraic extensions of K available. Accordingly some field, for example, a specific algebraic closure of K, sufficient to any needed selection of zeros of polynomials is taken beforehand, and any such zeros actually referenced are to be in that field. In this connection the notation below will be used.

Let $\alpha \in K$, $n = [F(\alpha): F]_i$ where n is a power of p, $f(x) = \mathrm{Irr}(\alpha, F)$, and $\deg f(x) = mn$. Then $f(x) = \phi(x^n)$ with $\phi(x)$ a separable irreducible monic polynomial over F of degree m. If $\alpha_1, \ldots, \alpha_m$ are the m distinct zeros of $f(x)$, then $\alpha_1^n, \ldots, \alpha_m^n$ are the m distinct zeros of $\phi(x)$ and $\phi(x) = \prod_{j=1}^m (x - \alpha_j^n)$. Set $\psi(x) = \prod_{j=1}^m (x - \alpha_j)$. For $\phi(x) = \sum_{j=0}^m c_j x^j$ with $c_m = 1$ and $\psi(x) = \sum_{j=0}^m \gamma_j x^j$ with $\gamma_m = 1$, one has

$$\sum_{j=0}^{m} \gamma_j^n x^{nj} = (\psi(x))^n = \prod_{j=1}^{m} (x^n - \alpha_j^n) = \varphi(x^n).$$

Thus $\gamma_j^n = c_j$. The γ_j's are of course ± 1 times the elementary symmetric functions of the distinct zeros, $\{\alpha_1, \ldots, \alpha_m\}$, of $f(x)$.

The field $F(\gamma_0, \ldots, \gamma_{m-1})$ is purely-inseparable over F. Since $[F(\alpha): F]_s = m$, then

$$[F(\alpha, \gamma_0, \ldots, \gamma_{m-1}): F(\gamma_0, \ldots, \gamma_{m-1})]_s = m.$$

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Since $\psi(\alpha)=0$, $\deg\psi(x)=m$, and $\psi(x)$ has coefficients in $F(\gamma_0,\dots,\gamma_{m-1})$;

$$[F(\alpha,\gamma_0,\dots,\gamma_{m-1}): F(\gamma_0,\dots,\gamma_{m-1})] \leqq m$$

and so must be exactly m. Hence $\psi(x)=\operatorname{Irr}(\alpha, F(\gamma_0,\dots,\gamma_{m-1}))$.

The following lemma follows easily from these remarks.

LEMMA 1. *Let* $\alpha \in K$, $n=[F(\alpha):F]_i$. *Then* $\operatorname{Irr}(\alpha,F)$ *is an* n*-th power in* $K[x]$ *iff* $F(\gamma_0,\dots,\gamma_{m-1})\subset K$.

LEMMA 2. *For* $h \geqq 1$ *and* $\alpha \in F$, $x^{p^h}-\alpha$ *is irreducible over* F *iff* $\alpha^{1/p}\notin F$.

Proof. See Theorem 7, pg. 66 of [**2**]. ■

THEOREM 1. *The following statements are equivalent*:

(a) K *splits over* F,

(b) K *is separable over* K_i,

(c) *If* $\alpha \in K$, $n=[F(\alpha):F]_i$, *then* $\operatorname{Irr}(\alpha,F)$ *is an* n*-th power in* $K[x]$.

Proof. If $K = K_0K_i = K_i(K_0)$, then every element of K_0 is separable over F, so separable over K_i. Thus K is separable over K_i.

If K is separable over K_i, then $K_i(\alpha)$ is separable over K_i since $K\supset K_i(\alpha)\supset K_i$. However, K_i is purely-inseparable over F and $[F(\alpha):F]_s=m$, so $[K_i(\alpha):K_i]=m$. Now, any zero of $\operatorname{Irr}(\alpha,K_i)$ must be a zero of $\operatorname{Irr}(\alpha,F)$, so $\operatorname{Irr}(\alpha,K_i)=\Pi_{i=1}^{m}(x-\alpha_i)$. Thus $\{\gamma_0,\dots,\gamma_{m-1}\}\subset K_i\subset K$, so $\operatorname{Irr}(\alpha,F)$ is an nth-power in $K[x]$, by Lemma 1.

If $\operatorname{Irr}(\alpha,F)$ is an nth-power in $K[x]$, then $\{\gamma_0,\dots,\gamma_{m-1}\}\subset K_i$, $\psi(x)\in K_i[x]$, and $[K_i(\alpha):K_i]\leqq m$. Since $[F(\alpha^n):F]_s=m$ and K_i is purely-inseparable over F, $[K_i(\alpha^n):K_i]\geqq m$. Thus $K_i(\alpha^n)=K_i(\alpha)$ and $\alpha\in K_i(\alpha^n)$. Since $\alpha^n\in K_0$, $\alpha \in K_0K_i$. Thus $K=K_0K_i$. ■

COROLLARY 1. *If* K *is normal over* F, *then* K *splits over* F.

Proof. If K is normal, then $F(\gamma_0,\dots,\gamma_{m-1})\subset F(\alpha)\subset K$. ■

THEOREM 2. *Let* α *be algebraic over* F *and* $[F(\alpha):F]_i=n$. *Then* $F(\alpha)$ *splits over* F *iff there exists a* k *such that* $F(\gamma_0,\dots,\gamma_{m-1})=F(\gamma_k)$.

Proof. If $n=1$ the theorem is routine. If $n>1$, then there exists a j such that c_j is not a pth power in F, x^n-c_j is irreducible over F by Lemma 2, and $[F(\gamma_j):F]=n$.

If $F(\alpha)$ splits over F, then $F(\alpha)\supset F(\gamma_0,\dots,\gamma_{m-1})\supset F(\gamma_j)$, by Theorem 1 and Lemma 1. However, $[F(\gamma_j):F]=n$ and $[F(\gamma_0,\dots,\gamma_{m-1}):F]\leqq[F(\alpha):F]_i=n$, thus $F(\gamma_0,\dots,\gamma_{m-1})=F(\gamma_k)$.

Suppose $F(\gamma_0,\dots,\gamma_{m-1})=F(\gamma_k)$, for some k, so $[F(\gamma_k):F]=n$. By the remarks preceding Lemma 1, we have that $m=[F(\gamma_k,\alpha):F(\gamma_k)]$. Hence $[F(\gamma_k,\alpha):F]=mn$, so $F(\alpha)=F(\gamma_k,\alpha)$, $F(\alpha)$ is separable over $F(\gamma_k)$, and $F(\gamma_k)$ is the maximal purely-inseparable extension of F contained in $F(\alpha)$. Thus $F(\alpha)$ splits over F, by Theorem 1. ■

THEOREM 3. *Consider the following properties of a field* F:

(a) *Every algebraic extension of* F *splits over* F.

(b) *Every finite extension of* F *is a simple extension of* F.

(c) *No extension of* F *contains two distinct purely-inseparable extensions of* F *of the same degree.*

Then (b) *and* (c) *are equivalent and* (c) *implies* (a). *Furthermore, if* $F=k(x)$, *where* x *is transcendental over* k, *then* (a), (b), (c) *are equivalent to* k *being a perfect field.*

Proof. We first show that (c) implies (a).

Assume that F has property (c) and let $\alpha\in K$. If $F(\alpha)$ splits over F for all $\alpha\in K$, then certainly K splits over F. If $[F(\alpha):F]_i=n=1$, then $F(\alpha)$ splits over F, so let $n>1$. Choose k so that c_k is not a pth power in F (recall the definitions of $\phi(x)$ and $\psi(x)$). Then $[F(\gamma_k):F]=n$ and $F(\gamma_k)$ contains

purely-inseparable extensions of degree n' provided $n'|n$. For any j, $F(\gamma_j)$ is a purely-inseparable extension and $[F(\gamma_j):F]|n$. Since F has property c, $F(\gamma_j)\subset F(\gamma_k)$, so $F(\gamma_0,\ldots,\gamma_{m-1})\subset F(\gamma_k)$. Thus $F(\alpha)$ splits over F, by Theorem 2, so (c) implies (a).

We now show that (b) and (c) are equivalent.

Suppose every finite extension of F is a simple extension of F. Let K_1 and K_2 be purely-inseparable extensions of degree n over F contained in some field and let $K = K_1K_2$. The subfield of K consisting of elements with nth powers in F contains K_1 and K_2 and must then be K. Hence $K = F(\beta)$ with $\beta^n \in F$, $[K:F]\leqq n$, and $K_1 = K = K_2$.

Suppose no extension of F contains two distinct purely-inseparable extensions of the same degree. Let K be a finite extension of F and let K' be a field intermediate to F and K. Then $(K')_0 = K'\cap K_0$ and $(K')_i = K'\cap K_i$. Since (c) implies (a), $K' = (K')_0(K')_i$ or $K' = (K'\cap K_0)(K'\cap K_i)$. Since K_0 is separable and finite, thus simple, over F, there are only a finite number of possibilities for $K'\cap K_0$; and since $K'\cap K_i$ is purely-inseparable over F, by hypothesis there are only a finite number of choices for $K'\cap K_i$. There are then only finitely many fields with $F\subset K'\subset K$, and K is a simple extension of F.

Thus (b) and (c) are equivalent.

Now, suppose $F = k(x)$. To show that (a), (b), and (c) are equivalent to k being a perfect field, it suffices to show that (a) implies k is perfect and k perfect implies (c).

Suppose that every algebraic extension of $k(x)$ splits over $k(x)$. Let $a\in k$. Since x is transcendental over k, x is transcendental over $k(a^{1/p})$. Hence $x^{1/p}\notin k(a^{1/p},x)$. Consider $z^{2p} + axz^p + x$ in $F[z]$. This polynomial is Eisenstein over $k[x]$ and so it is irreducible over F. With α as a zero of $z^{2p} + axz^p + x$, we have that $F(\alpha)$ splits over F, by assumption. Hence, by Theorem 2, we have that $F((ax)^{1/p}, x^{1/p}) = F(x^{1/p})$. Thus $F(a^{1/p})\subset F(x^{1/p})$. If $a^{1/p}\notin k$, then $a^{1/p}\notin F$ and $[F(a^{1/p}):F] = p$; so $F(a^{1/p}) = F(x^{1/p})$, that is, $x^{1/p}\in k(a^{1/p},x)$, which is a contradiction. Hence $a^{1/p}\in k$, and k is a perfect field.

Suppose k is perfect and $e>0$. Then $z^{p^e}-x$, in $F[z]$, is irreducible over F. Let $F(\eta)$ be an extension of F with $\eta^{p^e} = x$ so that $F(\eta)$ is purely-inseparable of degree p^e over F. If $f(x)\in F$, then $f(x) = f_1(x)/f_2(x)$, where $f_i(x)\in k[x]$ for $i = 1,2$. Set

$$f_i(x) = \sum_{j=0}^{l_i} a_{ij}x^j,$$

where a_{ij}'s are in k. Since k is a perfect field, $a_{ij}^{1/p^e}\in k$ for all i,j. Since η is also transcendental over k,

$$\left(\sum_{j=0}^{l_1} a_{1j}^{1/p^e}\eta^j \Big/ \sum_{j=0}^{l_2} a_{2j}^{1/p^e}\eta^j\right)\in F(\eta)$$

and its p^e-th power is

$$\sum_{j=0}^{l_1} a_{1j}x^j \Big/ \sum_{j=0}^{l_2} a_{2j}x^j \quad\text{or}\quad f(x).$$

Now let K be an extension of F, $e>0$, and $K(\eta)$ be such that $\eta^{p^e} = x$. Suppose K contains a purely-inseparable extension L of F of degree p^e. If $\alpha\in L$, then $\alpha^{p^e}\in F$. Then there must be an element β in $F(\eta)$ with $\beta^{p^e} = \alpha^{p^e}$; thus $\beta = \alpha$. Hence $L\subset F(\eta)$. Since L and $F(\eta)$ have degree p^e over F, $L = F(\eta)$. Thus no extension of F contains two distinct purely-inseparable extensions of the same degree over F. ■

In general, (a) and (c) are not equivalent, as the following example will show.

Let k be any field of characteristic p and let x, y be algebraically independent transcendentals over k. Take K as the algebraic closure of $k(x,y)$ and F as the subset of elements of K separable over $k(x,y)$. Then every element of K is purely-inseparable over F, so every algebraic extension of F clearly splits over F. Now neither x nor y is a pth power in $k(x,y)$, so $k(x^{1/p},y)$ and $k(x,y^{1/p})$ both have inseparability degree p over $k(x,y)$. Thus $F(x^{1/p})$ and $F(y^{1/p})$ have inseparability degree p over F. However, $k(x^{1/p},y^{1/p})$ has degree of inseparability p over $k(x^{1/p},y)$ and must have degree of

inseparability p^2 over $k(x, y)$. Finally $F(x^{1/p}, y^{1/p})$ must have degree of inseparability p^2 over F and $F(x^{1/p}) \neq F(y^{1/p})$.

Theorem 3 also sheds some light on the following question. It is well known that if K is finite over F, K need not be a simple extension of F. A customary example is $k(x^{1/p}, y^{1/p})$ over $k(x, y)$, where k has characteristic p and x, y are algebraically independent transcendentals over k. Why the need for two transcendentals? The next corollary answers this question.

COROLLARY. *Let k be a perfect field of characteristic p, x a transcendental over k and F a finite extension of $k(x)$, then F has properties* (a), (b) *and* (c).

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HILBERT'S FOURTEENTH PROBLEM

J. E. HUMPHREYS

1. The problem and its significance. Hilbert's 14th Problem arises out of classical invariant theory and leads to some subtle interactions among geometry, group theory, and ring theory.

A special case of the problem can be formulated loosely as follows: Start with a field K (which for Hilbert would probably be $\mathbf{C}$, the complex numbers). Denote by R the polynomial ring $K[T_1, \ldots, T_n]$ in n variables. Then introduce a group G acting on R, and consider the subring of **invariants**, denoted R^G: these are the elements of R left fixed by the action of G. *Is R^G finitely generated, as a ring over K?* This could be thought of as an instance of the tendency to ask which properties of a ring are inherited by its subrings; but of course R^G is a very special sort of subring.

Let us state the problem more carefully. R is an example of a K-**algebra**, a ring with 1 having a compatible structure of vector space over K (and containing a copy of K in the guise of scalar multiples of 1). Here "compatible" means that for all $a, b \in K$, $r, s \in R$: $a(r+s) = ar + as$, $a(rs) = a(rs)$, $a(br) = (ab)r$, $(a+b)r = ar + br$. To say that a K-algebra S is **finitely generated** over K (abbreviated f.g.) is to say that $S = K[t_1, \ldots, t_m]$ for some $t_i \in S$ (which need not be algebraically independent). In other words, S is a homomorphic image of a polynomial algebra in m variables.

As to the group action on R, we begin with the **general linear group** $\mathrm{GL}(n, K)$, consisting of all invertible $n \times n$ matrices over K, with matrix multiplication as the group operation. $\mathrm{GL}(n, K)$ acts on $T_1, \ldots, T_n$ as if they were the standard basis $e_1, e_2, \ldots, e_n$ of n-space: $g \cdot e_j = \Sigma_i a_{ij} e_i$, if $g = (a_{ij})$. This extends naturally to an action of $\mathrm{GL}(n, K)$ on R as a group of K-algebra automorphisms, so that $g \cdot (rs) = (g \cdot r)(g \cdot s)$ for all $r, s \in R$, $g \in G$. If G is an arbitrary subgroup of $\mathrm{GL}(n, K)$, $R^G =$

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$\{r \in R \mid g \cdot r = r \text{ for all } g \in G\}$. Then R^G includes the constant polynomials and inherits from R the structure of K-algebra.

(1.1) *Is R^G f.g. as a K-algebra?*

The reader is probably aware of an important case in which the answer to (1.1) is "yes". The **symmetric group** S_n can be identified with the group G of all $n \times n$ permutation matrices (over any field K): associate with a permutation π the matrix gotten from the identity matrix by permuting its rows according to π. This subgroup of $\mathrm{GL}(n, K)$ acts on $R = K[T_1, \ldots, T_n]$ by permuting the T_i in all possible ways; so R^G consists of the symmetric polynomials. It is well known that R^G is generated (as a K-algebra) by the **elementary symmetric polynomials** $p_k = \Sigma\, T_{i_1} \cdots T_{i_k}$ (sum over sequences $i_1 < \cdots < i_k$). In fact, R^G is isomorphic to a polynomial ring in n variables since $p_1, \ldots, p_n$ are algebraically independent.

Hilbert's problem dates back to his famous address **[14]** at the 1900 Paris congress, where he posed 21 problems whose solution would be a challenge to twentieth-century mathematicians. The original 14th Problem was in fact more general than (1.1). Consider the rational function field $K(T_1, \ldots, T_n)$ in n variables and any subfield L including K. (It is known that L/K is always a f.g. field extension.) Hilbert asked:

(1.2) *Is the K-algebra $L \cap K[T_1, \ldots, T_n]$ f.g.?*

This yields (1.1) as follows: In case G acts as before on $R = K[T_1, \ldots, T_n]$, we take L to be the field of fractions of R^G, so that $L \cap R = R^G$. (We could also let G act directly on $K(T_1, \ldots, T_n)$ and take L to be the subfield of G-invariants. This might be bigger than the fraction field of R^G, but would still intersect R precisely in R^G.) Hilbert apparently believed at the time of his address that a recent publication of L. Maurer had proved (1.1) for arbitrary G, thus leaving open only the generalized version (1.2); but Maurer's argument was later seen to be defective. Earlier Hilbert himself had solved (1.1) affirmatively in some special cases **[12, 13]**.

Lest the reader be kept too long in suspense, we hasten to point out that the answer to Hilbert's question (in either form) is sometimes "no". This was demonstrated around 1958 by M. Nagata. His counterexamples (cf. **[7, 25, 26, 29]** for details) involve polynomial rings in as few as 4 variables, over fields of arbitrary characteristic; the group may be commutative, may coincide with its own commutator subgroup, etc. Nevertheless, (1.1) does admit a positive response for many examples of geometric interest. We shall emphasize the positive aspect below, first surveying some examples and then proceeding more systematically.

The significance of (1.1) may not be readily apparent. Invariants under group actions do arise quite naturally in ring theory and field theory, as the example of symmetric polynomials shows. Nineteenth-century mathematicians were able to reduce a number of geometric questions to the study of invariants; for them finite generation was a matter of actually exhibiting a finite set of generators. But Hilbert shifted some of the emphasis to qualitative finiteness proofs, which are still of interest even in the absence of constructive methods. Today algebraic geometers invoke finite generation of rings of invariants in the study of various delicate "quotient" structures related to the parametrization of families of geometric objects such as curves and surfaces (cf. **[7, 22, 23, 39, 40, 45]**).

Our formulation of (1.1) is too restrictive for some purposes (cf. §6 below). It seems more natural to begin with an arbitrary f.g. K-algebra $\bar{R}$, on which G acts as a group of K-algebra automorphisms. G is no longer given concretely as a group of $n \times n$ matrices (there being no natural choice of n), so conceivably the action of G might be rather wild. We require further that G act **tamely** on $\bar{R}$, meaning by this that the G-translates of an arbitrary element of $\bar{R}$ span only a finite dimensional subspace of $\bar{R}$. (This is true for the earlier action of $\mathrm{GL}(n, K)$ on $K[T_1, \ldots, T_n]$, since the group action respects the degrees of homogeneous polynomials. It is also true for any action of a finite group.) In this situation, the finite generation of $\bar{R}$ implies the existence of a finite dimensional subspace V of $\bar{R}$ which is G-stable and generates $\bar{R}$ as a K-algebra: just take the span of all G-translates of some finite set of

generators. Relative to a basis $v_1, \ldots, v_n$ of V (which already generates $\bar{R}$), G may then be realized as a subgroup of $\mathrm{GL}(n, K)$, acting in the usual way on V. If $R = K[T_1, \ldots, T_n]$, there is a natural map $\phi: R \to \bar{R}$ which respects the actions of G on R and $\bar{R}$. So the picture resembles the original one, with the sole exception that the chosen generators of $\bar{R}$ need not be algebraically independent.

2. Example: Finite groups generated by reflections. We mentioned already the familiar example of S_n acting on polynomials in n variables. When $K = \mathbf{R}$, the real numbers, this example takes on an interesting geometrical aspect. Think of S_n as the group of linear operators on $\mathbf{R}^n$ which permute the standard basis vectors $e_1, \ldots, e_n$ in all possible ways. (To get the action on $K[T_1, \ldots, T_n]$ just identify T_i with e_i.) These operators are clearly *orthogonal* transformations, preserving the usual euclidean inner product.

One special type of orthogonal transformation is a **reflection**, which by definition takes some nonzero vector to its negative and fixes pointwise the orthogonal complement (of dimension $n-1$). We claim that S_n is generated by reflections. The transpositions $(12), (23), \ldots, (n-1, n)$ are known to generate S_n. The operator corresponding to $(i, i+1)$ interchanges e_i and e_{i+1} while fixing the other e_j. Therefore $(i, i+1)$ sends $e_i - e_{i+1}$ to its negative and acts as the identity operator on the subspace spanned by $e_1, \ldots, e_{i-1}$, $e_i + e_{i+1}, e_{i+2}, \ldots, e_n$, which is just the orthogonal complement of $e_i - e_{i+1}$. Similar reasoning shows that an arbitrary transposition (ij) acts as a reflection; but there are no other reflections in S_n, as the reader can verify.

Finite groups generated by reflections have many pleasant properties, and have been intensively studied because of their close connections with semisimple Lie groups and Lie algebras. Chevalley [**4**] was able to generalize the theorem on elementary symmetric polynomials as follows:

(2.1) *Let $G \subset GL(n, \mathbf{R})$ be a finite group generated by reflections, acting on $R = \mathbf{R}[T_1, \ldots, T_n]$ as in* §1. *Then R^G is isomorphic to a polynomial ring over* $\mathbf{R}$ *in n variables, generated by n homogeneous polynomials in the T_i whose degrees d_j are uniquely determined by G and satisfy $d_1 \cdots d_n = |G|$.*

For S_n the degrees in question are of course those of the elementary symmetric polynomials: $1, 2, \ldots, n$, and their product is $n! = |S_n|$. Another familiar example is the dihedral group of order $2n$, viewed as the symmetry group of a regular n-gon centered at the origin in $\mathbf{R}^2$ (which reflections generate this group?). Here the degrees are $2, n$. It is an interesting exercise to construct explicit generators for the ring of invariant polynomials.

For finite groups generated by reflections, (2.1) gives a strongly affirmative answer to Hilbert's problem. Actually, as we shall see in §3, rings of invariants are f.g. for arbitrary finite groups and arbitrary fields K; but only rarely do they have the structure of polynomial rings. In a certain sense, made precise in [**41**], only the groups generated by reflections have this property when $K = \mathbf{R}$ (or $\mathbf{C}$, if "reflection" is understood in a more general sense).

3. Example: Arbitrary finite groups. Hilbert's 14th Problem has an affirmative solution for any finite group G acting on a f.g. K-algebra R. But this is far from obvious. The essential steps were taken by E. Noether [**33, 34**], whose proof of the finite generation of R^G when $\operatorname{char} K = 0$ appeared in 1916, ten years before she disposed of the general case. She was concerned also with providing algorithms for the explicit generation of finite sets of generators of R^G; but we shall discuss just the qualitative aspect (following [**8**, p. 186]).

Two technical notions are needed, both having the flavor of finiteness conditions and both being commonplace to today's graduate students—though hardly commonplace before Noether's work.

Recall first that a commutative ring A (with 1) is called **noetherian** if each ideal of A is finitely generated, or equivalently, if A satisfies the ascending chain condition on ideals. For example, a polynomial ring in n variables over K is noetherian (Hilbert Basis Theorem), and so is any homomorphic image of a noetherian ring, e.g., any f.g. K-algebra. A standard theorem asserts:

(3.1) *If A is noetherian, then any submodule of a finitely generated A-module is also finitely generated.*

Recall next that if A is a commutative ring and B a subring (containing 1), an element $a \in A$ is said to be **integral over** B if for some $b_i \in B$, $a^n + b_{n-1}a^{n-1} + \cdots + b_1 a + b_0 = 0$. In case B is noetherian, this is equivalent to requiring that the subring $B[a]$ of A generated by B and a should be f.g. as a B-module. In general, the elements of A which are integral over B form a subring of A; if this subring is all of A, we say A is integral over B. For example, if A has a set of generators each of which is integral over B, then so is A. Another standard theorem asserts:

(3.2) *If A is integral over a subring B, and f.g. as a ring over B, then A is f.g. as a B-module.*

Now consider an arbitrary finite group G, acting on a f.g. K-algebra $R = K[r_1, \ldots, r_n]$. If T is an indeterminate, $p_i(T) = \Pi_{g \in G}(T - g \cdot r_i)$ is a polynomial in $R[T]$ with lead coefficient 1, having r_i as a root. Moreover, applying an element of G to this polynomial just permutes the factors, so the various coefficients c_{ij} of $p_i(T)$ must lie in R^G. Let S be the K-algebra generated by all the c_{ij} $(1 \leqq i \leqq n)$. Then S is a f.g. K-algebra (hence noetherian), $S \subset R^G \subset R$, and R is integral over S (since each r_i is integral over S). According to (3.2), R is a f.g. S-module. In turn, S being noetherian, (3.1) implies that R^G is also a f.g. S-module. But then the c_{ij}, together with a finite set of S-module generators for R^G, obviously suffice to generate R^G as a K-algebra.

The proof is slick (cf. [**1**] for an even more polished version). But of course the slickness conceals the difficult mixture of finiteness arguments which go into (3.1) and (3.2).

4. Example: Completely reducible actions. Finite generation of R^G is sometimes a consequence of the fact that the action of G on R is especially well behaved. To make this precise we have to introduce a couple of notions.

For our purposes, a G-**module** is a vector space V over K on which G acts by linear transformations, denoted $g \cdot v$ $(g \in G, v \in V)$. For any subgroup G of $\mathrm{GL}(n, K)$, $K[T_1, \ldots, T_n]$ is, as described earlier, a G-module; similarly, any G-stable subspace is a G-module. A G-module $V \neq 0$ is called **irreducible** if it has no proper nonzero G-submodule. (Example: The standard action of $\mathrm{GL}(n, K)$ on n-space is transitive on the set of nonzero vectors, hence surely irreducible.) V is called **completely reducible** if for each G-submodule W there is at least one complementary G-submodule W': $V = W \oplus W'$. Clearly, irreducible implies completely reducible. It is easy (at least when V is finite dimensional over K) to see that $V \neq 0$ is completely reducible if and only if it is a direct sum of irreducible G-submodules.

In §9 we shall exhibit some infinite groups whose finite dimensional "rational" modules are all completely reducible. But for the moment we have to be content with another foray into finite groups, where Maschke's Theorem states:

(4.1) *Let G be a finite group whose order $|G|$ is not divisible by* char K. *Then every finite dimensional G-module V is completely reducible.*

The proof is not difficult: Given a G-submodule W of V, choose any vector subspace W' of V for which $V = W \oplus W'$ (direct sum of subspaces). That W' exists follows from the well-known fact that a basis of W can be extended to a basis of V. Let $\pi : V \to W$ be the projection having W' as kernel. The idea is to replace π by another projection π^* which is a G-module homomorphism; then Ker π^* will be a G-submodule complementary to W, as required. We obtain π^* by "averaging" over G:

$$\pi^*(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1}\pi \qquad g \cdot v.$$

Here $1/|G|$ is being viewed as an element of K, which makes sense because of our restriction on char K. It is routine to verify that (1) π^* maps V to W, (2) π^* is the identity map on W, and (3)

$\pi^*(g' \cdot v) = g' \cdot \pi^*(v)$ for $g' \in G$, $v \in V$. Indeed, (2) follows from the fact that W is a G-submodule, since we have built the factor $1/|G|$ into π^*, while (3) just depends on the observation that summing over $g \in G$ is the same as summing over $g'g$ for any fixed g'.

When $K = \mathbf{R}$ or $\mathbf{C}$, there is a more geometric way to prove (4.1), which generalizes to compact topological groups if summing over the finite group is replaced by integrating over the topological group with respect to Haar measure. Start with any inner product (v, w) on V, and define

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} (g \cdot v, g \cdot w).$$

Then $\langle v, w \rangle$ is also an inner product, with the additional invariance property $\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle$. This insures that the orthogonal complement $W^\perp$ of a G-submodule W is again a G-submodule. Since in any case $V = W \oplus W^\perp$, this shows that V is completely reducible.

The restriction on char K in (4.1) is really essential, as the following example shows. Let K be the field of p elements (for a prime p), and let G be the group of 2×2 matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{for } a \in K;$$

so $|G| = \operatorname{char} K = p$. G acts on 2-space, fixing pointwise the line spanned by the first basis vector; but this line has no G-complement, since the matrices in G have no further eigenvectors. (Similar examples occur whenever char K divides G.)

What is the relevance of complete reducibility to Hilbert's 14th Problem?

Say G acts tamely on the f.g. K-algebra $\bar{R}$, that is, for each $\bar{r} \in \bar{R}$, $\{g \cdot \bar{r} \mid g \in G\}$ spans a finite dimensional subspace of $\bar{R}$. As in the last paragraph of § 1, we can find an n-dimensional G-submodule of $\bar{R}$ which generates $\bar{R}$. This yields a natural homomorphism ϕ of $R = K[T_1, \ldots, T_n]$ onto $\bar{R}$ which respects the action of G on each (G acts on R as a subgroup of $\mathrm{GL}(n, K)$). If we further assume that G acts completely reducibly on every finite dimensional G-submodule of R, it follows that R is completely reducible.

(4.2) *In the situation just described, assume that R is completely reducible. Then both R^G and $\bar{R}^G$ are f.g.*

The proof is worth discussing in some detail. First we show that R^G is f.g. For any irreducible G-module $M \subset R$, either G fixes all vectors (forcing $\dim M = 1$ and $M \subset R^G$) or else G fixes no nonzero vector of M. Denote by S the sum of all irreducible submodules of R of the latter type. Clearly $R^G \cap S = 0$. Complete reducibility implies readily that $R = R^G \oplus S$ (direct sum of G-modules). In a similar spirit, since $R^G \cap R^G S = 0$ and since $R^G S$ is G-stable and hence a direct sum of irreducibles, we have $R^G S \subset S$.

As a first approximation to showing that R^G is f.g., let us show that R^G is at least a *noetherian* ring. If I is any ideal in R^G, a typical element of the extended ideal IR can be written as $f = \Sigma f_i r_i + \Sigma f_j s_j$ (where $f_i \in I$, $r_i \in R^G$, $s_i \in S$), thanks to the decomposition $R = R^G \oplus S$. Moreover, the respective sums lie in R^G and S, since $R^G S \subset S$. In case $f \in R^G$, it follows that f had to belong to I in the first place. So extending and then restricting an ideal of R^G gets us back to that ideal. Now the ascending chain condition (A.C.C.) on ideals of the noetherian ring R implies A.C.C. on ideals of R^G.

Recall that A.C.C. on ideals is equivalent to the finite generation of all ideals. Apply this to the ideal I in R^G consisting of polynomials without constant term. (Clearly R^G is the direct sum of I and the constants.) If a polynomial is G-invariant, so are its homogeneous components, so a finite generating set for I can be chosen to consist of homogeneous polynomials, say $f_1, \ldots, f_t$. It is not too hard to see that any polynomial $f \in I$ is itself a polynomial in the f_i (using induction on the degree of f), so in fact the f_i suffice to generate R^G as a K-algebra.

This argument for finite generation of R^G depends on special features of polynomial rings. To

show in turn that $\bar{R}^G$ is f.g., it will be enough to show that ϕ maps R^G *onto* $\bar{R}^G$. For this we make another appeal to the complete reducibility of R. Start with $\bar{r} \in \bar{R}^G$ and choose any pre-image $r \in R$; it need not be G-invariant. The G-translates of r, all of which map to $\phi(r) = \bar{r}$, span a finite dimensional G-submodule M of R. Since M maps onto $K\bar{r}$, $M \cap \operatorname{Ker} \phi = N$ is a G-submodule of dimension one less than $\dim M$. Choose a G-complement: $M = N \oplus N'$. Then write $r = n + n'$ $(n \in N, n' \in N')$, so $N' = Kn'$. It follows that $\bar{r} = \phi(r) = \phi(n) + \phi(n') = \phi(n')$. If $g \in G$, then $g \cdot n' = an'$ for some $a \in K$, and $\bar{r} = g \cdot \bar{r} = g\phi(n') = \phi(g \cdot n') = \phi(an') = a\phi(n') = a\bar{r}$. This forces $a = 1$, so $n' \in R^G$. Therefore, ϕ maps R^G onto $\bar{R}^G$.

5. Example: The additive group. The additive group of K may be thought of as the group of 2×2 matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \in K$, which shows already the lack of complete reducibility of its natural action on 2-space. Nevertheless:

(5.1) *If $G \subset \mathrm{GL}(n, K)$ is isomorphic (in the sense of algebraic groups, Section 7 below) to the additive group of K, an algebraically closed field of characteristic 0, then $K[T_1, \ldots, T_n]^G$ is f.g.*

Formulated somewhat differently, this result goes back to R. Weitzenböck [**42**]. A modern proof is due to C. S. Seshadri [**37**]; cf. [**8**, pp. 200–205]. His strategy is to embed G into the **special linear group** $G' = \mathrm{SL}(2, K)$ (consisting of matrices of determinant 1) in the way indicated above, and then to make G' act on $R = K[T_1, \ldots, T_n]$ as well as on a related f.g. K-algebra R' (having two more generators than R), in such a way that $R'^{G'}$ maps naturally *onto* R^G. For G', Hilbert's 14th Problem is already known to have an affirmative solution in the framework of §4 (cf. §9 below), so $R'^{G'}$ is f.g., forcing R^G to be f.g.

A similar idea, still requiring char $K = 0$, has been used by G. Hochschild and G. D. Mostow [**15**] to get a finite generation theorem for certain analogous groups G, such as the full group of upper triangular matrices with diagonal entries all 1, sitting inside $G' = \mathrm{SL}(n, K)$. In this situation they have to assume that the action of G on R extends to an action of G'—an assumption known to be automatically satisfied when G is the additive group. (Similar results were found independently by F. Grosshans [**10**].)

6. Some conditional results. Suppose N is a normal subgroup of a group G. Then any action of G on a ring R induces an action of the quotient group G/N on R^N: all elements of the coset gN act on an N-invariant just as g does. It is immediate that:

(6.1) $(R^N)^{G/N} = R^G$.

Even when R is a polynomial ring in n variables and G is a subgroup of $\mathrm{GL}(n, K)$, R^N might be f.g. without itself being a polynomial ring; nor is G/N given to us in any definite way as a matrix group. This points up the necessity for our earlier reformulation (at the end of §1) in terms of tame actions of arbitrary groups on f.g. K-algebras: the action of G/N on R^N is tame if the action of G on R is tame. In particular, (6.1) implies:

(6.2) *Hilbert's 14th Problem has an affirmative solution for G if it does for both N and G/N.*

Another conditional statement can be made, this time concerning the field K. Say L is a field extension of K, and R is a K-algebra. The tensor product $L \otimes_K R$ becomes an L-algebra in a natural way, and is f.g. over L if R is f.g. over K. (For example, if $R = K[T_1, \ldots, T_n]$, the extended algebra is naturally isomorphic to $L[T_1, \ldots, T_n]$.) Moreover, any K-algebra automorphism of R extends in a natural way to $L \otimes_K R$. Say G acts on R. It can then be shown that:

(6.3) $(L \otimes_K R)^G = L \otimes_K R^G$; *moreover, this algebra is f.g. over L if and only if R^G is f.g. over K.*

This allows us, for example, to pass to an algebraically closed field when that is convenient. The reader might try to supply a proof of (6.3) in the special case $K = \mathbf{R}$, $L = \mathbf{C}$. (Cf. [**29**, Chapter I].)

7. Zariski closure of a linear group. So far we have examined just some isolated cases in which Hilbert's 14th Problem has an affirmative solution. To study the problem systematically, we have to dig deeper into the structure of linear groups, using the powerful tools of algebraic geometry and the theory of linear algebraic groups [**3**, **8**, **18**]. We assume henceforth (on the strength of (6.3)) that K is algebraically closed. This is a technical requirement for the theory about to be sketched, insuring the existence of "enough" zeros for polynomials in many variables.

Affine space K^m, the cartesian product of m copies of K, has a topology in which a closed set is defined to be the set of common zeros of a finite collection of polynomials. This is called the **Zariski topology**. Here a closed set might be a point, a curve, a surface, etc. Notice that the zeros of polynomials $f_1, \ldots, f_t$ are the same as the zeros of the ideal in $K[T_1, \ldots, T_m]$ generated by the f_i. On the other hand, any ideal in $K[T_1, \ldots, T_m]$ is f.g. (Hilbert Basis Theorem) and therefore arises in this way. The axioms for a topology are not hard to check. For example, the intersection of any collection of closed sets is the closed set defined by the ideal generated by all polynomials defining the various given sets. The Zariski topology is T_1 but not T_2; in contrast to the euclidean topology on $\mathbf{C}^m$, it does not have "many" open or closed sets, and the nonempty open sets are all very "big" (think of the complement of a curve).

$\mathrm{GL}(n, K)$ is an *open* set in K^{n^2}, being the complement of the set of zeros of the polynomial $\det(T_{ij})$. It is convenient, however, to embed $\mathrm{GL}(n, K)$ in K^{n^2+1} by identifying a matrix (a_{ij}) with the point $(a_{ij}, 1/\det(a_{ij}))$. In this way $\mathrm{GL}(n, K)$ becomes a *closed* set, the zeros of the polynomial $T \cdot \det(T_{ij}) - 1$, where T is a new indeterminate corresponding to the extra coordinate. A subgroup of $\mathrm{GL}(n, K)$ is called a **linear algebraic group** if it is closed in the topology thus induced from K^{n^2+1}. For example, the matrices of determinant 1 form a closed subgroup: the special linear group $\mathrm{SL}(n, K)$. Similarly, the diagonal subgroup, the upper (or lower) triangular subgroup, and all finite subgroups of $\mathrm{GL}(n, K)$ are algebraic groups. While an algebraic group is not a topological group in the usual sense (the topology being T_1 but not T_2), there are some obvious parallels.

The notion of algebraic group just sketched is not at all intrinsic—it depends on a specific embedding in some $\mathrm{GL}(n, K)$. This is eventually too restrictive, e.g., when one tries to give a quotient group the structure of algebraic group. If I is the ideal consisting of all polynomials in $K[T_{ij}, T]$ which vanish on the algebraic group G, the f.g. K-algebra $K[G] = K[T_{ij}, T]/I$ can be viewed as the ring of **polynomial functions** on G; it can be shown to characterize G completely. (For a f.g. K-algebra to play the role of $K[G]$ for some G, it turns out to be necessary and sufficient that it have no nonzero nilpotent elements and that it admit appropriate "coproduct," "counit," and "coinverses." The resulting category of K-algebras and algebra homomorphisms is isomorphic to the category of linear algebraic groups, where a "morphism" is defined to be a group homomorphism given by polynomial functions in the matrix entries.)

What bearing does all of this have on Hilbert's 14th Problem? Say G is an arbitrary subgroup of $\mathrm{GL}(n, K)$, acting in the usual way on the polynomial ring $R = K[T_1, \ldots, T_n]$. The Zariski closure $\bar{G}$ of G is again a group (the proof resembles the familiar one for topological groups), and it can be shown that:

(7.1) $R^G = R^{\bar{G}}$.

As a result, no generality is lost if we take G to be closed in the first place. This is advantageous because there is a detailed structure theory for algebraic groups, to be outlined in the following section.

The idea behind (7.1) can be explained simply. Only the inclusion $R^G \subset R^{\bar{G}}$ has to be discussed. Take $r \in R^G$. Then the $\bar{G}$-translates of r span a finite dimensional subspace of R, for which we can

choose a basis $r_1, \ldots, r_t$ with $r_1 = r$. The action of $\bar{G}$ on this basis yields a homomorphism (of algebraic groups) $\phi : \bar{G} \to \mathrm{GL}(t, K)$. The subgroup H of $\mathrm{GL}(t, K)$ fixing r is closed, being the zero set of the polynomials $T_{21}, T_{31}, \ldots, T_{t1}$, and includes $\phi(G)$. So $\phi^{-1}(H)$ is closed in $\bar{G}$ and includes G, hence coincides with $\bar{G}$. It follows that $\bar{G}$ fixes r.

Similar reasoning can be used to adapt (7.1) to the general case, in which G acts tamely on a f.g. K-algebra R.

8. Structure of algebraic groups. Remember that K denotes an algebraically closed field, of arbitrary characteristic. The structure of a closed subgroup of $\mathrm{GL}(n, K)$ has been investigated thoroughly, beginning with the fundamental work of A. Borel and C. Chevalley in the mid-1950's (for up-to-date accounts see [**3**, **18**]). Unfortunately, we cannot hope to do justice here to the rich array of arguments marshaled by Borel and Chevalley. The most we can do is to outline carefully their final results.

One basic fact is that the quotient G/N of an algebraic group G by a closed normal subgroup N has a natural structure of algebraic group. "Natural" means that any homomorphism $G \to G'$ of algebraic groups having N in its kernel admits a unique factorization $G \to G/N \to G'$. In view of the discussion in §7, the proof requires the construction of a suitable f.g. K-algebra (without nilpotent elements) to play the role of $K[G/N]$. The right candidate is $K[G]^N$, whose finite generation is far from obvious.

A second basic fact is that each element g of an algebraic group G has a unique decomposition as the product of commuting **semisimple** and **unipotent** elements in G, written $g = g_s g_u$. To explain these notions, suppose for a moment that G is a closed subgroup of $\mathrm{GL}(n, K)$. A matrix s is called semisimple if its minimal polynomial has distinct roots, or equivalently (since K is algebraically closed), if s is diagonalizable. A matrix u is called unipotent if all its eigenvalues are 1, or equivalently, if it is the identity plus a nilpotent matrix. It follows from the Jordan normal form for matrices that each $g \in \mathrm{GL}(n, K)$ has a unique decomposition $g = g_s g_u$ of the type indicated above. It is not obvious that these parts lie in G when g does, nor is it obvious that the choice of these parts is independent of the choice of an embedding of G in some $\mathrm{GL}(n, K)$. But this can be shown, along with the fact that any homomorphism $\phi : G \to G'$ of algebraic groups preserves semisimple and unipotent parts, i.e., $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$.

We remark that an algebraic group consisting entirely of unipotent elements is called a **unipotent group**, e.g., the group of all upper triangular matrices having 1's along the diagonal. (But, perversely, a group consisting entirely of semisimple elements is not called a "semisimple group"; this name is reserved for something else.)

Now we can begin to describe the structure of an arbitrary algebraic group.

(8.1) *An algebraic group G has only finitely many connected components in the Zariski topology. The component G^0 which contains 1 is called the* ***identity component****. It is a closed normal subgroup of G, and the other connected components are its cosets.*

This is similar to what happens in a topological group. Since G/G^0 is finite, it will not cause us any trouble, in view of §3 and (6.1).

Suppose next that G is connected. G has a unique largest closed connected normal unipotent subgroup $R_u(G)$, called the **unipotent radical**. If $G \neq 1$ but $R_u(G) = 1$, then G is called **reductive**. $\mathrm{GL}(n, K)$ is easily seen to be reductive, as is the group $D(n, K)$ of diagonal matrices in $\mathrm{GL}(n, K)$. The latter example deserves a little more attention. $D(n, K)$ is just the direct product of n copies of the multiplicative group of K, and is called an n-dimensional **torus**. (There is a connection with the usual notion of topological torus when one drops down from **C** to **R**; but we won't try to explain this here.)

For any connected algebraic group G, the derived group (G, G) is closed and connected. In case G is reductive and $G = (G, G)$, we call G **semisimple**; a good example is $\mathrm{SL}(n, K)$. If G is semisimple

and has no proper closed normal subgroups except for finite ones, we call G **simple**; again $\mathrm{SL}(n, K)$ is a good example. This is not the usual group-theoretic notion of "simple," but it is close: the quotient of a simple algebraic group by its (finite) center turns out to be simple in the usual sense.

(8.2) *Let G be reductive. Then $G = (G, G)\cdot Z$, where (G, G) is semisimple (or trivial), Z is the center of G, Z^0 is a torus, and $Z \cap (G, G)$ is finite. If $G = (G, G)$ is semisimple, then G is isomorphic to $(G_1 \times \cdots \times G_t)/F$, where the G_i are simple algebraic groups and F is a finite central subgroup of the direct product.*

By way of illustration, the reductive group $\mathrm{GL}(n, K)$ is the product of its derived group $\mathrm{SL}(n, K)$ and the group of scalar matrices (a 1-dimensional torus), which intersect in the center of $\mathrm{SL}(n, K)$: the subgroup of scalar matrices having as sole eigenvalue an nth root of 1.

To crown this edifice we have only to cite Chevalley's classification of simple algebraic groups [**18**, §32–33]. These correspond precisely to the list of *simple Lie algebras* over **C** found more than half a century earlier by W. Killing and E. Cartan. There are four infinite families A_n, B_n, C_n, D_n, and five exceptional types E_6, E_7, E_8, F_4, G_2, cf. [**17**]. In this correspondence, more than one group may belong to a given Lie type, e.g., to A_{n-1} belong $\mathrm{SL}(n, K)$ and its various quotients by subgroups of its center. Types B, C, D correspond to other "classical" linear groups (orthogonal, symplectic). [The author takes this opportunity to point out that Theorem 32.1 in [**18**] is overstated; it overlooks the "half-spin" groups of type D_n, n even. But Theorem′ is correctly stated.]

9. Linearly reductive groups. The detailed structure theory of algebraic groups sketched in §8 enables us to see where the criterion of (4.2) can be used to get finite generation of rings of invariants. Call an algebraic group G **linearly reductive** if all its (finite dimensional) "rational" modules are completely reducible. To say that an n-dimensional G-module V is **rational** is just to say that the $n \times n$ matrices describing the action of G on any basis of V have entries given by polynomial functions on G. Concretely, if $G \subset \mathrm{GL}(m, K)$ and $\dim V = n$, the action is described by a homomorphism $\phi : G \to \mathrm{GL}(n, K)$ such that $\phi(g_{ij}) = (\pi_{st}(g_{ij}))$, where $\pi_{st}(T_{ij})$ are polynomials in m^2 variables T_{ij}. (This is accurate at least when $G \subset \mathrm{SL}(m, K)$; otherwise 1/det has to be brought into the picture.) We get rational G-modules automatically when G is finite or when G is a subgroup of $\mathrm{GL}(n, K)$ acting in the standard way on n-space.

Which algebraic groups are linearly reductive? It was observed already in (4.1) that a finite group G is linearly reductive provided $\mathrm{char}\, K$ does not divide $|G|$. The general criterion is as follows:

(9.1) *The algebraic group G is linearly reductive if and only if either (a) $\mathrm{char}\, K = 0$ and G^0 is reductive (or trivial), or (b) $\mathrm{char}\, K = p > 0$, p does not divide $[G; G^0]$, and G^0 is a torus.*

Let us consider briefly what goes into the proof, limiting ourselves to the case $G = G^0$. If G is a torus, its image under any homomorphism of algebraic groups is a commutative group consisting of semisimple elements (cf. §8). Therefore any rational action of G on n-space is completely reducible, the matrices by which G acts being simultaneously diagonalizable over K; so G is linearly reductive. In case $\mathrm{char}\, K = 0$, and G is semisimple, an argument for linear reductivity can be based on the close relationship between G and its Lie algebra [**18**, 14.3]. The latter is "semisimple" and its finite dimensional modules are all completely reducible, thanks to a celebrated theorem of H. Weyl. Weyl's original method was to transfer the problem to an associated compact Lie group, where there exists an invariant integral analogous to the averaging sum used for a finite group in §4; cf. [**44**] for the application to invariant theory. (Weyl's theorem also admits a purely algebraic proof; cf. [**17**, 6.3].)

The "only if" part of (9.1) was proved by Nagata [**27**; **29**, Chapter VI], using fairly elementary facts about algebraic groups; but his argument is rather complicated. Once one has more detailed information about the groups, it is not too hard to locate specific modules which fail to be completely reducible if G fails to meet the stated conditions.

10. Geometric reductivity. When char $K = 0$, many of the interesting groups one encounters are in fact linearly reductive (9.1), so (4.2) is relevant. At the other extreme, a unipotent group may or may not yield f.g. rings of invariants, as exemplified by §5 along with Nagata's counterexamples (some of which involve commutative unipotent groups). It is not clear how much more one can expect to say in this direction within the framework of algebraic groups.

When char $K = p > 0$, (9.1) suggests that linear reductivity may be an unrealistically strong condition to impose. Indeed, finite groups always behave well *vis-à-vis* Hilbert's problem (§3) but are not always linearly reductive. In the early 1960's the needs of his geometric invariant theory led D. Mumford [22] to conjecture that reductive groups—GL(n, K) in particular—would satisfy a condition which is markedly weaker than linear reductivity but still adequate to insure finite generation of rings of invariants. One way to state Mumford's condition (roughly dual to his formulation) is as follows:

(10.1) *Let V be a (finite dimensional) rational G-module, containing a nonzero G-invariant vector v. Then there exists a G-invariant homogeneous polynomial function f on V of positive degree for which $f(v) \neq 0$.*

Recall that the polynomial functions on a vector space V are the elements of the symmetric algebra of its dual space V^*; these may be thought of concretely as polynomials in $f_1, \ldots, f_n$ (any fixed basis of V^*). The action of G on V^* is dual to the given action on V: $(g \cdot f)(v) = f(g^{-1} \cdot v)$ for $g \in G$, $f \in V^*$, $v \in V$.

If (10.1) is always true for G, we call G **geometrically reductive** (the term "semi-reductive" has also been used in the literature). For example, any finite group is geometrically reductive: Given V and v as above, take a linear function l on V for which $l(v) \neq 0$, and set $f = \Pi_{g \in G}\, g \cdot l$. As another example, any linearly reductive group is geometrically reductive: The annihilator of v in V^* is a G-submodule, hence admits a 1-dimensional G-complement, which is spanned by a G-invariant f not vanishing at v (here f is a homogeneous polynomial function of degree 1).

Condition (10.1) may still strike the reader as somewhat artificial. But it arises naturally in the following context. Let S be an n-dimensional irreducible rational G-module, and denote by End S the n^2-dimensional vector space consisting of all linear operators on S. G acts naturally on End S: $(gt)(s) = gt(g^{-1}s)$ for $g \in G$, $t \in \text{End}\, S$, $s \in S$. (Relative to a fixed basis of S, End S may be viewed as the space of $n \times n$ matrices, where this action amounts to *conjugation* by the element of GL(n, K) corresponding to g.) In this way End S itself becomes a rational G-module, playing the role of V in (10.1). It has a unique 1-dimensional subspace consisting of G-invariants: the scalar multiplications. Indeed, $g \cdot t = t$ just means that t is a G-module map (the matrix corresponding to t commutes with all the matrices corresponding to G); but then t is multiplication by a scalar, thanks to Schur's Lemma.* There is a very well-known homogeneous G-invariant polynomial function on End S which is nonzero on the nonzero scalars: the *determinant*. (G-invariance results from the fact that similar matrices have the same determinant.)

Mumford conjectured that all reductive groups are geometrically reductive. In turn, Nagata was able to prove:

(10.2) *Let G be a closed subgroup of $GL(n, K)$ which is geometrically reductive. Then R^G is f.g. (where $R = K[T_1, \ldots, T_n]$), as is $\bar{R}^G$ in the situation of* (4.2).

The proof (cf. [**28**; **8**, pp. 190–193]) resembles that of (4.2) up to a point, but involves some intricate inductive steps. It would be interesting to have a more conceptual proof of this key result.

* Recall how the proof goes: Say $t \neq 0$. Since Ker t and Im t are G-submodules, while S is irreducible, it is clear that Ker $t = 0$, Im $t = S$. Thus t is invertible. The G-module maps $S \to S$ therefore form a division subring of End S, finite dimensional over the algebraically closed field K; this division ring must then coincide with K, since otherwise we could generate finite field extensions of K.

For some time after Mumford formulated his conjecture there was only slight progress toward proving it. Meanwhile, Nagata and others [**30**, **31**, **32**] explored the notion of geometric reductivity further and showed, for example, that the geometrically reductive groups in characteristic 0 are precisely the linearly reductive ones (finite-by-reductive).

11. Mumford's conjecture. The conjecture reads:

(11.1) *Reductive algebraic groups are geometrically reductive.*

It may as well be assumed that $\operatorname{char} K = p > 0$. T. Oda [**35**] verified (11.1) for $\mathrm{GL}(2, K)$ when $p = 2$; then C. S. Seshadri [**38**] treated $\mathrm{GL}(2, K)$ without any restriction on p (cf. [**8**, pp. 194–196]). The proofs adapt easily to $\mathrm{SL}(2, K)$; but for quite a while there was no further progress.

In 1974 W. J. Haboush [**11**] succeeded in proving (11.1) in full generality. His key ideas, to be sketched below, took shape around the time of the A.M.S. Summer Institute on Algebraic Geometry (Arcata, Calif.). (See [**6**, **40**] and Mathematical Reviews 52 #3179 for other accounts of Haboush's proof.)

At roughly the same time, E. Formanek and C. Procesi [**9**] were developing a proof of (11.1) just for $\mathrm{GL}(n, K)$; their method is closely related to the classical invariant theory of the general linear and symmetric groups. Afterward other proofs of (11.1) in the general case were found, first by the present author [**19**] and then by the authors of [**5**]. In each case some of Haboush's argument is retained, but [**19**] emphasizes finite dimensional representation theory while [**5**] emphasizes the cohomology of algebraic groups.

Thanks to (6.1), it is enough to prove (11.1) for semisimple groups: tori are linearly reductive, and a reductive group modulo a central torus is either trivial or semisimple. Haboush's approach is as follows. Begin with a rational G-module V containing a nonzero G-invariant vector v. *Suppose* we are able to construct a G-module map $\phi: V \to \operatorname{End} S$ for.some irreducible rational G-module S, in such a way that $\phi(v) \neq 0$. (Then $\phi(v)$ has to be a nonzero scalar.) The pullback $\det \circ \phi$ will be the sought-for G-invariant homogeneous polynomial function of positive degree on V not vanishing at v. This is where the special case $\operatorname{End} S$, discussed in §10, comes into play.

The difficulty is of course to find a suitable module S and to construct the map ϕ. Haboush selects S from an especially nice family of irreducible G-modules, the **Steinberg modules** $St(q)$, which occur for each power q of p and have dimension q^m (m being determined by G—for example, $m = 1$ if $G = \mathrm{SL}(2, K)$). Given V, his construction works for any sufficiently large q. The idea is (1) to map V into the ring $K[G]$ of polynomial functions on G described in §7; (2) to follow this with the projection $K[G] \to K[G]^T$, where T is a torus in G of maximum dimension, acting on $K[G]$ on the right; (3) to write $K[G]^T$ as an ascending union of (left) G-submodules isomorphic to $\operatorname{End} St(q)$ for increasing powers q. By making sure that v does not get lost along the way (i.e., get mapped to 0), he finally gets the desired map ϕ.

Step (1) is fairly standard and goes as follows. Choose $l \in V^*$ with $l(v) = 1$ and then map $V \to K[G]$ by the recipe $v' \mapsto f$, where f is the polynomial function on G defined by $f(g) = l(g \cdot v')$. The reader can check that this is a G-module map, if we make G act on $K[G]$ (on the left) by $(g' \cdot f)(g) = f(gg')$. Moreover, v is sent to the scalar 1.

Step (2) is easy, once we realize that the torus T acts completely reducibly on $K[G]$ (tori are linearly reductive in arbitrary characteristic). Then the projection of $K[G]$ onto the space of (right) T-invariants is defined and is a (left) G-module map. Moreover, under the composite map $V \to K[G] \to K[G]^T$, v is still sent to 1.

Step (3) is much more sophisticated, and cannot easily be explained without going deeply into the structure of semisimple groups (e.g., the relationship between G and its maximal tori). We shall mention only that $\operatorname{End} St(q)$ appears in $K[G]^T$ in the guise of $St(q) \otimes St(q)$ (based on the fact that $St(q)$ is isomorphic to its dual space as a G-module). Here $St(q)$ has first been constructed explicitly as a certain space of polynomial functions on G—the sections of a holomorphic line bundle.

Apart from the deeper geometrical applications discussed in [**40**], (11.1) helps to settle the characteristic p case of the following theorem about homogeneous spaces of algebraic groups (cf. [**2**, **21**, **36**]):

(11.2) *Let G be a reductive group, H a closed subgroup. Then G/H is an affine variety* (*i.e., is isomorphic to a closed set in some affine space*) *if and only if H^0 is reductive* (*or trivial*).

It has to be explained that a homogeneous space G/H always has a natural structure of open subset in some affine variety, but may not itself be affine (it may, for example, be realizable as a closed set in some projective space and hence appear "compact"). Suppose H^0 is reductive. Then it is geometrically reductive (11.1) and thus $K[G]^{H^0}$ is f.g. The affineness of G/H is intimately related to the finite generation of $K[G]^H$, which follows since H/H^0 is finite (cf. (6.1) and §3). For this direction of (11.2), G can in fact be arbitrary. The "only if" part is more subtle and involves (11.1) less directly.

12. Conclusion. With Haboush's proof of the Mumford conjecture, we arrive at a satisfying general statement: *For an algebraic group G such that G^0 is reductive* (*or trivial*), *Hilbert's 14th Problem always has an affirmative answer.* In view of Nagata's counterexamples involving unipotent and other non-reductive groups, it is unlikely that any larger class of groups can be treated so definitively in arbitrary characteristic. However, for a given group G, it is still reasonable to ask about the finite generation of R^G for special choices of R; cf. the work of Hochschild and Mostow [**15**] mentioned in §5, which might also have an analogue in prime characteristic.

As the question of finite generation of rings of invariants recedes into the background, other questions of a more refined nature may be expected to occupy the foreground; cf. [**16**]. To say that R^G is f.g. is, after all, only a first step. It remains to see what other properties of R are bequeathed to R^G.

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THE SIXTH U.S.A. MATHEMATICAL OLYMPIAD

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The sixth U.S.A. Mathematical Olympiad took place on May 3, 1977. The preliminary examination, the Annual High School Mathematics Examination, was taken on March 8 by approximately 365,000 students. From among these students, all those who scored 105 or better out of the maximum possible score of 150 were invited to participate in the Olympiad. There were 111 of these. Because

four of the acceptances were not complete, 107 students finally took the Olympiad. Table 1 shows the results.

H.S. Exam \ Olympiad	0–9	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89	90–100
150				1	1					1
145–149										
140–144					1	1			1	
135–139						1		1		
130–134	1	1	1	1		1				
125–129	1	4	1	2		1				
120–124	5	3	2		5	2	1			
115–119	2	1		3		2	1			
110–114	6	5	3	10	1	1	2			
105–109	11	7	7	2	1	1	1			

TABLE 1—Comparative scores on High School Mathematics Exam and Olympiad

As has been the case with previous results, these scores show no significant correlation.

The Olympiad papers were graded during the weekend of May 13 by Professor Murray Klamkin, Professor John Bender and the writer. The top eight scorers were:

Mark Kleiman	Stuyvesant High School	New York, N.Y.
Randall Dougherty	W. T. Woodson High School	Fairfax, Va.
Peter Shor	Tamalpais High School	Hill Valley, Cal.
Michael Larsen	Lexington High School	Lexington, Mass.
James Propp	North Senior High School	Great Neck, N.Y.
Ronald Kaminsky	Albany High School	Albany, N.Y.
Victor Milenkovic	New Trier H.S. East	Winnetka, Ill.
Paul Weiss	Stuyvesant High School	New York, N.Y.

These students were honored at ceremonies held in Washington, D.C. on June 6 and 7, thanks to a grant from IBM. They received a variety of awards from MAA (including the Arenstorf Memorial Medal), IBM, Hewlett-Packard, NCTM, Mu Alpha Theta, John Wiley & Sons, and Springer Verlag.

After training for three weeks at the US Military Academy, in West Point, the students traveled as a team to Belgrade, Yugoslavia, where they scored first in the International Mathematical Olympiad over twenty other nations.

So that readers may compare the nature of each problem with its relative difficulty as indicated in the scores, we supply Table 2.

Problem no.	blank	0	1–5	6–10	11–15	16–20	21–25
1	6	26	45	5	8	17	
2	17	52	14	7	3	13	1
3	16	44	19	10	7	10	1
4	26	32	10	7	10	21	1
5	12	24	35	18	10	7	1

TABLE 2—Grades on each of the problems.

The scores in the "blank" column indicate that students still find geometry problems most

difficult. However, the rest of the table would seem to indicate that they found the algebra problems hard as well.

For the amusement of the reader, the problems proposed on the Olympiads are given below:

Sixth U.S.A. Mathematical Olympiad — May 3, 1977

1. Determine all pairs of positive integers (m,n) such that $(1+x^n+x^{2n}+\cdots+x^{mn})$ is divisible by $(1+x+x^2+\cdots+x^m)$.

2. ABC and $A'B'C'$ are two triangles in the same plane such that the lines AA', BB', CC' are mutually parallel. If ΔABC denotes the area of triangle ABC with an appropriate $\pm$ sign, etc., prove that

$$3(\Delta ABC+\Delta A'B'C')=\Delta AB'C'+\Delta BC'A'+\Delta CA'B'+\Delta A'BC+\Delta B'CA+\Delta C'AB.$$

3. If a and b are two of the roots of $x^4+x^3-1=0$, prove that ab is a root of $x^6+x^4+x^3-x^2-1=0$.

4. Prove that if the opposite sides of a skew (non-planar) quadrilateral are congruent, then the line joining the midpoints of the two diagonals is perpendicular to these diagonals, and, conversely, if the line joining the midpoints of the two diagonals of a skew quadrilateral is perpendicular to these diagonals, then the opposite sides of the quadrilateral are congruent.

5. If a, b, c, d, e are positive numbers bounded by p and q, i.e., $0<p\leqslant a,b,c,d,e\leqslant q$, prove that

$$(a+b+c+d+e)(1/a+1/b+1/c+1/d+1/e)\leqslant 25+6\left(\sqrt{p/q}-\sqrt{q/p}\right)^2$$

and determine when there is equality.

The XIX International Mathematical Olympiad

July 4, 5, 1977

1. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon.

2. In a finite sequence of real numbers the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

3. Let n be a given integer >2, and let V_n be the set of integers $1+kn$, where $k=1,2,\ldots$. A number $m\in V_n$ is called *indecomposable* in V_n if there do not exist numbers $p,q\in V_n$ such that $pq=m$. Prove that there exists a number $r\in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions which differ only in the order of the elements of V_n will be considered the same.)

4. a, b, A, B are given constant real numbers and

$$f(\theta)=1-a\cos\theta-b\sin\theta-A\cos 2\theta-B\sin 2\theta.$$

Prove that if $f(\theta)\geqslant 0$ for all real θ, then $a^2+b^2\leqslant 2$ and $A^2+B^2\leqslant 1$.

5. Let a and b be positive integers. When a^2+b^2 is divided by $a+b$ the quotient is q and the remainder is r. Find all pairs (a,b) given that $q^2+r=1977$.

6. Let $f(n)$ be a function defined on the set of all positive integers and taking on all its values in the same set. Prove that if $f(n+1)>f(f(n))$ for each positive integer n, then $f(n)=n$ for each n.

Solutions to these problems have appeared in a pamphlet, obtainable from Dr. Walter E. Mientka,

Executive Director, Annual High School Mathematics Examination, 917 Oldfather Hall, University of Nebraska, Lincoln, NE 68588, at a cost of 50¢ per copy (postpaid); payment must accompany the order; minimum order $1.50.

I would like to take the liberty of expanding somewhat on the Olympiads. I have been asked what effect these contests have had. In the five years since we started this event, we have had some nineteen team members. A good number of them have already begun publishing in mathematics, while others have solved problems posed in this MONTHLY and other publications. I note that the May 1977 issue of the MONTHLY has a solution for one problem by Mark Kleiman, and another by Russell Lyons. I believe the Olympiad Program is developing mathematical talent.

I have also been asked about the training of our team members. In preparation for the International Olympiad, we have held a three-week training session at which we invited the eight team members and sixteen above-average students not yet in their senior year. This year's team consists of eight students, six of whom were in last year's training session. Yet we teach only algebra, geometry, trigonometry, and perhaps a bit of number theory, combinatorics, and the like. Without this training, our team would be slaughtered. I have been hoping that the secondary schools would take over some of this training, and am happy to note that the MAA agrees. I have just received the pamphlet, "Math in High School," which contains recommendations for secondary school mathematics. It can be obtained from the Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C., 20036. I urge mathematics coordinators to read it and act upon its recommendations.

In summary, the USA Olympiad selects, trains and encourages future mathematical talent. It appears to be helping in the search for a more reasonable curriculum in the secondary schools. It provides the opportunity for our students to meet with their peers from other nations.

It might be mentioned that our committee has, in the past few years, had requests for information about the U.S.A. Olympiad from other nations wishing to start their own contests. These have come from, for example, Sri Lanka, Greece, Australia, and within the past few days, Brazil. We would like to think that our Olympiad is part of an international striving for better communication among nations.

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PROGRESS REPORTS

EDITED BY P. R. HALMOS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal:

usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

THE SERRE CONJECTURE

W. H. Gustafson, P. R. Halmos, J. M. Zelmanowitz

What does the first row of an invertible matrix look like? That depends: where are the entries? If the entries are real numbers (or, for that matter, elements of an arbitrary field), the first row of an invertible matrix can be anything except $(0,0,\ldots,0)$. (If the given row of length n has a non-zero entry, assume, with no loss of generality, that that entry is in the first position, and construct the square matrix of size n with that first row, the rest of the first column all 0's, and the lower right corner equal to the identity matrix of size $n-1$.)

If the matrices about which the question is being asked are restricted to have integer entries, the answer is less obvious. Thus, for instance, $(2,4,6)$ cannot be the first row of an invertible integer matrix, because the determinant of any such matrix must be ± 1, whereas the determinant of an integer matrix whose first row is $(2,4,6)$ is necessarily even. There is a clue here: if the entries of a prescribed row have a non-trivial common divisor, then that row cannot occur in an invertible matrix. It turns out that the necessary condition thus discovered (that the greatest common divisor of the prescribed row be 1) is sufficient as well: any row satisfying it *can* occur in an invertible matrix.

The last assertion is true for the ring of integers, or, for that matter, for any principal ideal domain: the condition that a row $(a_1,a_2,\ldots,a_n)$ must satisfy is the existence of a related column $(t_1,t_2,\ldots,t_n)$ such that $a_1t_1+a_2t_2+\cdots+a_nt_n=1$. Such rows are called unimodular rows. (The t's exist if and only if the greatest common divisor of the a's is 1. Equivalently: the t's exist if and only if the ideal generated by the a's is the entire domain. Note that in case the domain is a field the conditions are equivalent to $(a_1,a_2,\ldots,a_n)\neq(0,0,\ldots,0)$.)

There are various approaches to the proof of sufficiency; one that works smoothly for Euclidean domains goes as follows. Regard $(a_1,a_2,\ldots,a_n)$ as a matrix with one row and n columns, and use the existence of the t's to infer the possibility of performing a sequence of elementary column operations on it so as to convert it to $(1,0,\ldots,0)$. The performance of an elementary column operation has the same effect as multiplying the given matrix on the right by certain elementary matrices. If the product of all the multipliers is U, so that

$$(a_1,a_2,\ldots,a_n)\cdot U=(1,0,\ldots,0),$$

then

$$(a_1,a_2,\ldots,a_n)=(1,0,\ldots,0)\cdot U^{-1},$$

so that U^{-1} is an invertible matrix with first row $(a_1,a_2,\ldots,a_n)$.

It is now tempting to jump to a general algebraic conclusion: if R is a commutative ring with unit, and if $(a_1,a_2,\ldots,a_n)$ is a unimodular row over R, then $(a_1,a_2,\ldots,a_n)$ is fit to be the first row of an invertible matrix over R. For $n=1$ the conclusion is trivial, and for $n=2$ it is almost equally trivial: given (a_1,a_2) and (t_1,t_2) with $a_1t_1+a_2t_2=1$, the matrix

$$\begin{pmatrix} a_1 & a_2 \\ -t_2 & t_1 \end{pmatrix}$$

does the trick.

For $n \geq 3$, however, the conclusion is false; the standard counter-example makes contact with a well-known part of elementary topology. Let R be the ring of all real-valued continuous functions defined on the 2-sphere, i.e., on the locus of the equation $x^2+y^2+z^2=1$ in $\mathbf{R}^3$. If $a_1(x,y,z)=x$, $a_2(x,y,z)=y$, and $a_3(x,y,z)=z$, then (a_1,a_2,a_3) is a unimodular row (because $x^2+y^2+z^2=1$). The row (a_1,a_2,a_3) cannot, however, occur in an invertible matrix over R. Reason: the second row (b_1,b_2,b_3) of such a matrix would be, at each point (x,y,z), linearly independent of the first, and, therefore, would have a non-zero projection in the plane tangent to the sphere at (x,y,z). This is impossible: there is no non-singular continuous tangent vector field on the sphere (or you can't comb a porcupine).

Serre (1955) made the conjecture that for certain important special rings the general conclusion is true; the rings are the polynomial rings $R=k[x_1,\ldots,x_m]$ in m variables, with coefficients in a field k. The original formulation of Serre's conjecture had to do with modules over these R's. Recall that a module over R is "a vector space with respect to scalars from R". A module over R is of special importance if it has a finite basis (equivalently, if it is isomorphic to R^m for some m). If a module over R has a basis, and is the direct sum of two submodules, do they too necessarily have bases? A module with a basis is called *free*, and a direct summand of a free module is called *projective*; the original formulation of Serre's conjecture was that every projective module over $k[x_1,\ldots,x_m]$ is free.

The subject makes surprising contact with topology. A *vector bundle* over a base space X is a generalization of the concept of the Cartesian product of X with a vector space. Intuitively, a bundle is a "twisted" Cartesian product, in the sense in which a Möbius strip is a twisted cylinder. The Cartesian product itself is the easiest vector bundle (called the *trivial* bundle); the generalized ones retain the projection map onto X, just as if they were Cartesian products, but are like Cartesian products only locally.

A *section* of a vector bundle is a continuous function s from X to the bundle, such that s followed by the projection onto X is the identity mapping on X. The vector structure of the bundle makes possible a natural definition of addition of sections. If, moreover, R is the ring of scalar-valued continuous functions on X, then the vector structure of the bundle makes possible a natural definition of multiplication of a section by an element of R. In other words, the set of all sections is a module over R. If the base space is at all decent, it turns out that this module is always projective; the module is free if and only if the bundle is trivial. There is, thus, a correspondence between projective modules and vector bundles, and, in that correspondence, the free modules correspond to trivial bundles.

What is known in the topological context is that a bundle over m-dimensional Euclidean space $\mathbf{R}^m$ is in a natural sense always (isomorphic to) the trivial bundle. How much of that conclusion continues to make sense and remains true in an algebraic context? If, in other words, k is an arbitrary field, does it make sense to speak of vector bundles over k^m? The answer is yes; a topology that gives the phrase its sense (the Zariski topology) is definable in purely algebraic terms. (A closed set in k^m is the locus of common zeros of a set of polynomials in m variables.) A vector bundle over k^m, with "fiber" given by the vector space k^n, is a "twisted" version of $k^m \times k^n$. The role of the ring of scalar-valued continuous functions on the base space k^m is played by the ring $R=k[x_1,\ldots,x_m]$ of polynomials. The sections form a projective module over R; Serre's conjecture is that such a module is necessarily free, i.e., that such a bundle is necessarily trivial.

The conjecture remained open for more than 20 years. The first non-trivial step was Seshadri's (1958); he proved the conjecture for $m=2$. A later and important step was taken by Horrocks (1964) who proved an analogous result for local rings (rings with only one maximal ideal). The final step was taken simultaneously and independently by Quillen and Suslin (1976).

Quillen in effect reduced the general case to the one treated by Horrocks. The essence of Quillen's method is induction on the number m of variables. The step from $m-1$ to m is similar to the procedure of complexifying a real vector space. It involves showing that every projective module over

$k[x_1,\ldots,x_m]$ is a tensor product of a projective module over $k[x_1,\ldots,x_{m-1}]$ with $k[x_1,\ldots,x_m]$. The reason the induction step is successful is that the property of having a basis survives the construction.

References

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2. A. A. Suslin, Projective modules over a polynomial ring are free, Soviet Math. Dokl., 17 (1976) 1160–1164.
3. T. -Y. Lam, Serre's conjecture, Springer-Verlag, Berlin, 1978.

MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

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ILLUMINATION OF BOUNDED DOMAINS

JEFFREY RAUCH

This note shows how simple geometric properties of the ellipse and an elementary compactness argument can be used to study how many light sources are needed to illuminate the interior of a bounded domain.

Suppose $\Omega \subset \mathbf{R}^2$ is a bounded, open, connected set such that $\partial\Omega$ has continuously turning tangent and Ω lies on one side of its boundary. A *ray* in Ω is a piecewise linear path with corners at $\partial\Omega$ where the usual law of reflection is satisfied, namely, the segments make equal angles with the tangent line. We say that Ω *is illuminated from a point* P if for every $Q \in \Omega$, there is a ray containing both P and Q. We say that Ω *is directly illuminated from* P if for every $Q \in \Omega$ the line segment $\overline{PQ}$ lies in Ω. In the first case, Q is hit by a ray beginning at P. In the second case, it is hit by a ray before the ray reaches $\partial\Omega$. The set Ω is convex if and only if it is directly illuminated from each point $P \in \Omega$. It is starlike if and only if it is directly illuminated from some $P \in \Omega$. We say that Ω *is directly illuminated from* $\{P_1,\ldots,P_k\}$ if for every $Q \in \Omega$ there is a $j \in \{1,\ldots,k\}$ such that the segment $\overline{P_jQ}$ lies in Ω. The definition of Ω *is illuminated from* $\{P_1,\ldots,P_k\}$ should be obvious. The basic question we discuss is how many points are required to illuminate a "reasonable" set Ω?

The first example (due to R. Penrose) shows that, in general, one must place the sources wisely. We give a set Ω and a $P \in \Omega$ such that Ω is *not* illuminated from P. Incidentally, for wave optics (as opposed to geometrical optics) this is not possible, see **[1]**. Consider the upper half of an ellipse completed by a curve which connects the foci by a straight segment and forms two lobes as in Figure 1.

CLAIM: Any ray beginning in the left hand lobe never hits the line segment connecting the foci.

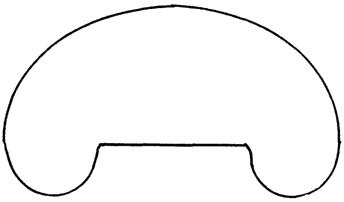

FIG. 1.

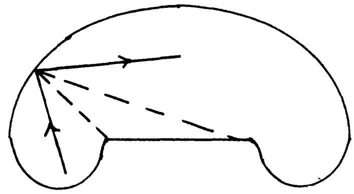

FIG. 2.

To see this, consider the first reflection of a ray which leaves the lobe. The ray from the left hand focus which reflects at the same point on $\partial\Omega$ is used as a comparison (see Figure 2).

The solid ray beginning in the lobe lies outside the dotted ray connecting the foci. For the next reflection consult Figure 3.

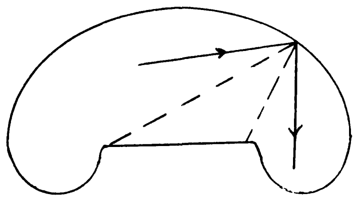

FIG. 3.

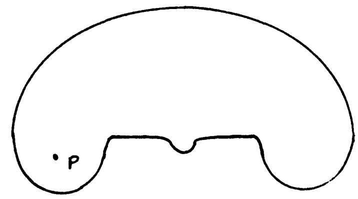

FIG. 4.

By repeating this process, one sees that the ray moves steadily to the right, always "shielded" from the segment between the foci by a dotted ray. The process continues until the ray passes into the right hand lobe where it bounces around. If it escapes the argument is repeated but in the opposite direction. The proof of the claim is complete.

The example of Ω and P is given in Figure 4. The little dimple below the line connecting the foci is not illuminated from P.

The region of Figure 4 can be directly illuminated from three points since it is the union of three starlike sets. One may ask if every region is illuminated from a finite number of points. We prove the following stronger assertion. *Any bounded region Ω lying on one side of its C^1 boundary is the finite union of starlike sets.*

Proof. For any $P \in \Omega$, choose an open disc $D_P \subset \Omega$ with center P. For $P \in \partial\Omega$ choose an open disc D_P with center P so that $D_P \cap \bar{\Omega}$ is starlike. This is possible because $\partial\Omega$ is C^1 and Ω lies on one side of its boundary, so if the disc is chosen with radius sufficiently small, then $D_P \cap \bar{\Omega}$ is well approximated by the intersection of D_P and a closed half plane. For any $P \in \bar{\Omega}$ the set $D_P \cap \bar{\Omega}$ is a relatively open subset of $\bar{\Omega}$ so $\{D_P \cap \bar{\Omega} \mid P \in \bar{\Omega}\}$ is an open covering of $\bar{\Omega}$ by starlike subsets. By compactness of $\bar{\Omega}$, there is a finite subcover and the proof is complete.

One might ask to what extent the smoothness of $\partial\Omega$ is needed to insure that Ω be illuminated from finitely many points. We give an example where $\partial\Omega$ is smooth with the exception of a single point P and such that

(i). Ω is not illuminated from any finite set of points.

(ii). Any ray beginning in Ω does not reach P.

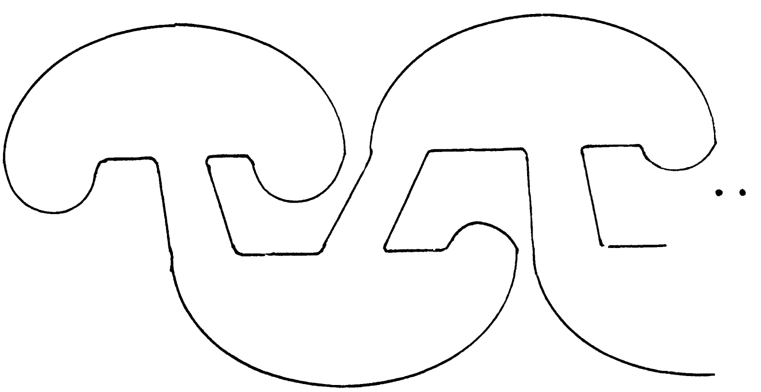

FIG. 5.

The reason (ii). is desirable is that if a ray were to reach P it would not be clear how to reflect it. Consider first the unbounded set in Figure 5 where $\cdots$ means that the figure repeats indefinitely. Any ray beginning in a lobe of the first Penrose type mushroom will never leave the first mushroom. Therefore to illuminate this lobe requires at least one source inside the first mushroom. Similarly, a ray beginning in the nth mushroom spends its entire life in the $(n-1)$st, nth and $(n+1)$st mushrooms. Thus, to illuminate the nth mushroom, there must be a source in at least one of these three mushrooms. Thus, to illuminate every mushroom requires an infinite number of sources. To create a bounded example with this property, consider the building block in Figure 6.

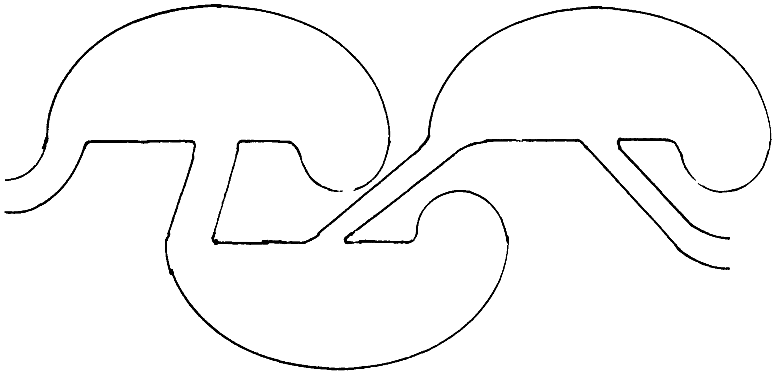

FIG. 6.

Connect such a unit to one on the right, which is half its size. Connect that one to one on the right half again as large, and so on. After the region is stoppered on the left in any simple way, one gets a bounded region Ω, which squeezes down to a single singular point on the right. As before, to illuminate Ω requires a source in each unit, hence an infinity of sources. Furthermore, a ray beginning in the nth unit never passes beyond the mushrooms immediately adjacent to this unit, so no ray ever reaches the singular point. The construction is complete.

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A PROBABILISTIC ESTIMATE OF INVARIANCE FOR GROUPS

GARY SHERMAN

If the finite group G acts on the finite non-empty set X (i.e., G is represented as a group of permutations of X) and we denote the probability that an element chosen at random from G fixes an element chosen at random from X by $P_G(X)$, then $P_G(X) = k/|X|$, where k is the number of orbits of G on X (see **[4]**). If G acts on itself by conjugation, then $P_G(G)$ is the probability that two elements chosen at random from G commute (see **[1]**). Using the class equation, the reader can readily show $P_G(G)$ to be at most $5/8$ for nonabelian groups. Gustafson **[1]** showed the bound $5/8$ to be valid for infinite nonabelian compact groups and, more recently, MacHale **[2]** established the same bound for non-commutative rings. This author **[4]** considered $P_A(G)$ where A is the group of automorphisms of the finite abelian group G. It was shown that $P_A(G) \to 0$ as $|G| \to \infty$ and if $G \neq Z_2$, then $P_A(G) \leqq 3/4$.

Since the automorphisms and the inner automorphisms of a group induce a natural action on the power set of the group, both $P_G(G)$ and $P_A(G)$ can be viewed as conditional probabilities, i.e., the probability that a randomly chosen subset is fixed by a randomly chosen inner automorphism or automorphism, respectively, given that the subset is a singleton. In this note we estimate the probability that a group automorphism fixes a subset of the group.

It is convenient to proceed in the following more general setting. Let G and H be finite groups of orders m and n with smallest prime divisors p and q, respectively. Assume that G acts as a group of automorphisms on H. The action of G on H induces an action of G on 2^H, the power set of H, defined by $Sg = \{sg \mid s \in S\}$ for $g \in G$ and $S \in 2^H$. The common kernel of both actions is denoted by K.

THEOREM. *If* $P_G(2^H) \neq 1$, *then* $P_G(2^H) \leqq 1/p + (1-1/p)(\frac{1}{2})^{[n(1+pq-p-q)]/pq}$.

Proof. From the previous remarks, we have $P_G(2^H) = k/2^n$ where k is the number of orbits of G on 2^H. A theorem of Burnside [3, page 49] gives $k = 1/m \sum_{g\in G} f(g)$ where $f(g)$ is the number of subsets of H that are fixed by g. To determine $f(g)$, we need to investigate the orbit structure of the permutation of 2^H determined by g. Thus, we define the orbit type of $g \in G$ to be $(l_1, l_2, \dots, l_n)$ where $l_i = l_i(g)$ is the number of orbits of g on H of length i. Since $Sg = S$ if, and only if, S is the union of orbits of g on H, we have $f(g) = 2^{l_1+l_2+\cdots+l_n}$. Therefore,

$$(*) \qquad P_G(2^H) = \frac{1}{m\cdot 2^n}\sum_{g\in G} 2^{l_1(g)+l_2(g)+\cdots+l_n(g)}.$$

Each element of K contributes 2^n to the sum of (*) since it has orbit type $(n, 0, \dots, 0)$. If $g \in G - K$, then $1 \leqq l_1(g) \leqq n/q$ since $\{x \in H \mid xg = x\}$ is a proper subgroup of H. Viewing each orbit O_g of g on H as an orbit of $\langle g\rangle$ on H, we find that $|O_g| \,\big|\, |\langle g\rangle|$. If $|O_g| \neq 1$, this implies $p \leqq |O_g|$. From the cycle type of g, we obtain $\sum_{j=p}^{n} j\cdot l_j = n - l_1$ which implies

$$\sum_{j=p}^{n} l_j \leqq \sum_{j=p}^{n} \frac{j}{p} l_j = \frac{n-l_1}{p}.$$

Thus

$$2^{l_1(g)+l_2(g)+\cdots+l_n(g)} \leqq 2^{l_1+(n-l_1)/p} \leqq 2^{(n(p+q-1))/pq}.$$

It follows that

$$P_G(2^H) \leqq \frac{1}{m2^n}(|K|2^n + (m-|K|)2^{(n(p+q-1))/pq})$$

$$\leqq \frac{1}{p} + \left(1-\frac{1}{p}\right)\left(\frac{1}{2}\right)^{(n(1+pq-p-q))/pq}.$$

The second inequality follows since $\beta = (\frac{1}{2})^{(n(1+pq-p-q))/pq}$ is less than 1, $[G\colon K] \geqq p$ by hypothesis and $h(\alpha) = \alpha + (1-\alpha)\beta$ is increasing on $(0,1)$.

It is easy to see that $P_G(2^H)$ is at most $3/4$. Simply take $p = q = 2$ and $n = 4$. This absolute upper bound is sharp: let Z_2 act as the group of automorphisms of Z_4.

Some observations in the case of a non-abelian group, G, acting on itself by conjugation are in order. Since K is the center of G, it follows that $[G\colon K] \geqq p^2$. From the proof of the theorem, we obtain

$$P_G(2^G) \leqq \frac{1}{p^2} + \left(1-\frac{1}{p^2}\right)\left(\frac{1}{2}\right)^{n(1-(1/p))^2} \leqq \frac{1}{4} + \frac{3}{4}\left(\frac{1}{2}\right)^{n/4}$$

and direct computation yields $P_{S_3}(2^{S_3}) = 3/8$. Therefore $P_G(2^G) \leqq 7/16$. The reader can verify that the bound $7/16$ is sharp by examining the quaternion group or the dihedral group of order eight. Indeed, the general bound

$$\frac{1}{p^2}+\left(1-\frac{1}{p^2}\right)\left(\frac{1}{2}\right)^{n(1-(1/p))^2}$$

is assumed if and only if $[G\colon K]=p^2$, which implies G is nilpotent.

For the class of nonabelian simple groups, we have

$$P_G(2^G)\leqq\frac{1}{n}+\left(1-\frac{1}{n}\right)\left(\frac{1}{2}\right)^{2n/5}.$$

This is a corollary to the proof of the theorem because $|K|=1$ and G does not contain a proper subgroup of index less than five. Since A_5 is the nonabelian simple group of smallest order

$$P_G(2^G)\leqq P_{A_5}(2^{A_5})\leqq\frac{1}{60}+\left(\frac{59}{60}\right)\left(\frac{1}{2}\right)^{24}<.0167.$$

Moreover, if $|G|\to\infty$, then $P_G(2^G)\to 0$, as one might expect.

The author wishes to thank the referees for their helpful suggestions.

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THE BOREL–CANTELLI LEMMA AND PRODUCT-SUM FORMULAS

FREDERICK STERN

The Borel–Cantelli lemma and stochastic independence can be used to produce product-sum formulas. Three examples are given below. This lemma [1, p. 201] states that for a sequence of events $\{A_n\}$ with respective probabilities $\{a_n\}$, if $\sum_{n=1}^{\infty} a_n$ converges, then with probability one only finitely many events $\{A_n\}$ occur. Let p_k with $k=0,1,2,\ldots$, be the probability that exactly k of the events $\{A_n\}$ occur. According to the lemma, if $\sum_{n=1}^{\infty} a_n<\infty$ then $\sum_{k=0}^{\infty} p_k=1$. Let us further suppose that the events $\{A_n\}$ are stochastically independent events and each probability a_n is positive. In that case, the probability that none of the events occurs is given by the infinite product $p_0=\prod_{n=1}^{\infty}(1-a_n)$. The probability that exactly one of the events occurs is given by

$$p_1=p_0\sum_{n=1}^{\infty}\frac{a_n}{(1-a_n)}.$$

For any positive integer k, the probability that exactly m of tne events occur is

$$p_k=p_0\sum_{S_k}\frac{a_{n_1}}{(1-a_{n_1})}\,\frac{a_{n_2}}{(1-a_{n_2})}\cdots\frac{a_{n_k}}{(1-a_{n_k})},$$

where $S_k=(n_1,n_2,\ldots,n_k)$ runs through the sets of positive integers such that $n_1<n_2<\cdots<n_k$. By combining the above remarks, we have the non-probabilistic result that, for any sequence of positive numbers $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n<\infty$ and $a_n<1$,

$$\left(\prod_{n=1}^{\infty}(1-a_n)\right)^{-1}=1+\sum_{k=1}^{\infty}\sum_{S_k}\frac{a_{n_1}}{(1-a_{n_1})}\cdot\frac{a_{n_2}}{(1-a_{n_2})}\cdot\cdots\cdot\frac{a_{n_k}}{(1-a_{n_k})},$$

where $S_k = (n_1, n_2, \ldots, n_k)$ runs through the sets of positive integers such that $n_1 < n_2 < \cdots < n_k$.

EXAMPLE 1 [2]. Suppose $a_n = x^n$ where $0 \leqq x < 1$. We obtain for Euler's product,

$$\left(\prod_{n=1}^{\infty} (1-x^n)\right)^{-1} = 1 + \sum_{k=1}^{\infty} \sum_{S_k} \frac{x^{n_1+n_2+\cdots+n_k}}{(1-x^{n_1})(1-x^{n_2})\cdots(1-x^{n_k})}.$$

EXAMPLE 2. Let $\Pi_1, \Pi_2, \Pi_3, \ldots$ be an enumeration of primes greater than or equal to 2 and s be any number greater than 1. Let $a_n = 1/\Pi_n^s$. We have for the Riemann zeta-function

$$\prod_{n=1}^{\infty} \frac{1}{(1-\Pi_n^{-s})} = 1 + \sum_{k=1}^{\infty} \sum_{S_k} \frac{1}{(\Pi_{n_1}^s - 1)(\Pi_{n_2}^s - 1)\cdots(\Pi_{n_k}^s - 1)}.$$

EXAMPLE 3 [4, p. 245]. Let $\{\rho_n\}$ be a sequence of increasing positive numbers and let r be greater than zero. Let $a_n = r/(r+\rho_n)$. Define the entire function $f(z) = \Pi_{n=1}^{\infty}(1 + z/\rho_n)$ with power series $\Sigma_{k=0}^{\infty} c_k z^k$. Then we get

$$f(r) = \prod_{n=1}^{\infty}\left(1 + \frac{r}{\rho_n}\right) = 1 + \sum_{k=1}^{\infty} r^k \sum_{S_k} (\rho_{n_1}\rho_{n_2}\cdots\rho_{n_k})^{-1}$$

and $c_k = \Sigma_{S_k} (\rho_{n_1}\rho_{n_2}\cdots\rho_{n_k})^{-1}$.

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RESEARCH PROBLEMS

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In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

WHEN ARE PERMUTATIONS ADDITIVE?

A. KOTZIG AND P. J. LAUFER

Suppose $n = 2k+1$ is odd and call an n-vector a **permutation** if its n coordinates comprise the set $\{0, \pm 1, \pm 2, \ldots, \pm k\}$. The vector $f = (-k, -k+1, -k+2, \ldots, k)$ is called the **fundamental permutation.** A permutation p is said to be a **σ-permutation** if $p + f$ is also a permutation. Denote by S_n the set of σ-permutations.

1. *How big is S_n?*

S_n is never empty since $(0, 1, 2, \ldots, k, -k, -k+1, \ldots, -1)$ is a σ-permutation. For $n > 1$, $|S_n|$ appears to be even, but we cannot prove this by observing that the negative reversal of a σ-permuta-

tion is also a σ-permutation, since some σ-permutations, such as

$$(1,3,-2,0,2,-3,-1) \quad \text{and} \quad (2,-1,3,0,-3,1,-2)$$

are their own negative reversals.

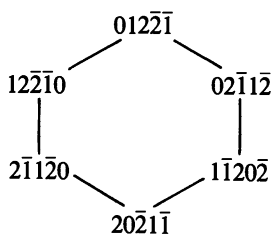
Two σ-permutations $\boldsymbol{u}, v$, are called an **additive pair** (α-pair) if $\boldsymbol{u}+v+\boldsymbol{f}$ is a permutation.

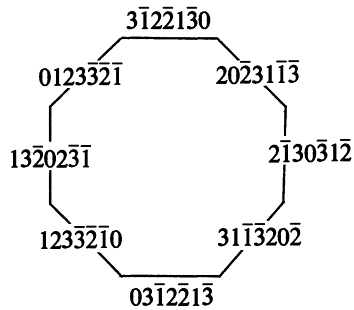
2. *For what values of n are there α-pairs, and how many α-pairs are there?*

Here are the answers to the questions for the first few values of n:

$n=$	1	3	5	7	9	11	13
$\lvert S_n\rvert=$	1	2	6	28	244	2544	35600
# of α-pairs	0	0	6	8	0	0	?

For each n we can draw a graph with the σ-permutations as vertices, joining two by an edge just if they form an α-pair. It is easy to see that no σ-permutation forms an α-pair with itself (except in the trivial case $n=1$) so for $n>1$ the graph has no loops. Here are the graphs for $n=5$ and 7,





except that 20 isolated vertices are not shown in the latter.

We can make an α-pair as the product of two smaller ones. For example the product of

$$\begin{matrix} 0\,1\,2\,\bar{2}\,\bar{1} \\ 1\,2\,\bar{2}\,\bar{1}\,0 \end{matrix} \quad \text{and} \quad \begin{matrix} 0\,1\,2\,3\,\bar{3}\,\bar{2}\,\bar{1} \\ 1\,3\,\bar{2}\,0\,2\,\bar{3}\,\bar{1} \end{matrix} \quad \text{is}$$

$$0\;1\;2\;\bar{2}\;\bar{1}\;5\;6\;7\;3\;4\;10\;11\;12\;8\;9\;15\;16\;17\;13\;14\;\overline{15}\;\overline{14}\;\overline{13}\;\overline{17}\;\overline{16}\;\overline{10}\;\bar{9}\;\bar{8}\;\overline{12}\;\overline{11}\;\bar{5}\;\bar{4}\;\bar{3}\;\bar{7}\;\bar{6}$$

$$6\;7\;3\;4\;5\;16\;17\;13\;14\;15\;\bar{9}\;\bar{8}\;\overline{12}\;\overline{11}\;\overline{10}\;1\;2\;\bar{2}\;\bar{1}\;0\;11\;12\;8\;9\;10\;\overline{14}\;\overline{13}\;\overline{17}\;\overline{16}\;\overline{15}\;\bar{4}\;\bar{3}\;\bar{7}\;\bar{6}\;\bar{5}$$

so that a partial answer to Question 2 is:

all n of the form $5^a 7^b$.

Are there any others?

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CLASSROOM NOTES

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AN ELEMENTARY EVALUATION OF THE CATALAN NUMBERS

DAVID SINGMASTER

1. Introduction. The Catalan number C_n is the number of ways of associating a product of $n+1$ terms. Thus $C_0 = C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14, \ldots$. These numbers arise in many other situations and are well known to have the values

$$C_n = \binom{2n}{n} \Big/ (n+1).$$

The most common derivation uses the recurrence $C_n = \sum_{i=0}^{n} C_i C_{n-i}$ applied to the generating function $f(x) = \sum_{i=0}^{\infty} C_i x^i$ to show $1 + x(f(x))^2 = f(x)$, so $f(x) = (1 - \sqrt{(1-4x)})/2x$ and the Taylor series of $f(x)$ can be computed to give the result. This seems like rather heavy machinery for the level at which students first meet the problem.

In this note, I present a proof that

$$C_n = \binom{2n}{n} \Big/ (n+1) \text{ in the form } C_n = \binom{2n+1}{n} \Big/ (2n+1)$$

by a direct combinatorial argument. First I show a bijection between the products of $n+1$ terms and the well-formed reverse Polish strings of $n+1$ operands and n operators. There are $\binom{2n+1}{n}$ strings of $n+1$ operands and n operators. Then I show that exactly one of the $2n+1$ cyclic shifts of such a string is well-formed, giving the result. I believe that this proof is the most suitable for a student's first exposure to the problem. It can be presented either in lecture or as an exercise with hints.

2. The derivation. Consider any product of $n+1$ terms. We insist on putting in the final set of parentheses for the final operation, so we have n pairs of parentheses. For example, when $n = 2$, we have the products $((ab)c)$ and $(a(bc))$. It is remarkable, though well known in Logic and Computer Science, that only the left or only the right parentheses are necessary. Indeed, if we omit the right (left) parentheses and replace all the left (right) parentheses by some operator symbol, say X, then we have the Polish (reverse Polish) notation of Łukasiewicz. For example, when $n=2$, the products above have the Polish forms $XXabc$ and $XaXbc$ and the reverse Polish forms $abXcX$ and $abcXX$. Henceforth we shall only deal with the reverse Polish forms, which are more convenient.

Any product of $n+1$ terms or operands gives a unique reverse Polish string of $n+1$ operands and n operators. A simple induction verifies that we have more operands than right parentheses occurring in any initial segment of a product. Thus the same relation must hold for operands and operators in the corresponding reverse Polish string. We say a string of $n+1$ operands and n operators is *well-formed* if this is the case. Given a well-formed string, we can recover the product of $n+1$ terms as follows. Read the string from left to right. The first symbol must be an operand and we write this down. Continuing, if the next symbol is an operand, we write it at the right of the last symbol written. If the next symbol is an operator, we write a parenthesis about the last two operands and consider the parenthesized expression as a new operand, replacing the enclosed two. Since the string is well-formed, we will get a correctly parenthesized product as the result of continuing this process. So we have a bijection between the products of $n+1$ operands and the well-formed strings of $n+1$ operands and n operators.

Now the actual symbols or names of the operands are irrelevant to our problem—only the positions occupied by operands are important. Let us denote a position occupied by an operand by an O and continue to use X for an operator. Then our reverse Polish strings become strings of $n+1$ O's and n X's. For example, when $n=2$, we have the strings $OOXOX$ and $OOOXX$. Our definition of well-formed then asserts that every initial string must have more O's than X's.

There are $\binom{2n+1}{n}$ possible strings of $n+1$ O's and n X's. How many of these are well-formed? Consider any string A of $n+1$ O's and n X's. We now show that A has exactly one cyclic shift which is well-formed. We must consider A as a cycle, i.e., its first entry is considered to be the successor of its last entry.

For $n=1$, the result is true. When $n>1$, there must be a segment of length 2 in A of the form OX. Deleting this pair leaves a string of n O's and $n-1$ X's. By induction, we can assume this string has a unique well-formed shift, and its starting point is a starting point for a well-formed shift of A. For example, given $OOXXO$, deletion of the first OX pair leaves O—XO or OXO, whose unique well-formed shift is OOX or OO—X, giving the well-formed shift $OOOXX$. Given $OXOOX$, deletion gives —OOX or OOX whose unique well-formed shift is OOX or OOX—, giving the well-formed shift $OOXOX$. Note that when the starting point of the reduced sequence occurs where the OX was deleted, the OX is reinserted at the end. This shows that A has a well-formed shift. Now a well-formed shift of A cannot begin just before the X or the O of an adjacent OX pair, and the deletion of such a pair from a well-formed shift of A gives a well-formed shift of the reduced sequence. Hence if A has two distinct well-formed shifts, then deletion of the same pair from each will give two distinct well-formed shifts of the reduced sequence, contrary to the inductive hypothesis. So A has a unique well-formed shift.

Now we have $\binom{2n+1}{n}$ sequences of $n+1$ O's and n X's and hence

$$\binom{2n+1}{n}\Big/(2n+1)$$

cycles of $n+1$ O's and n X's consisting of $2n+1$ equivalent sequences. Exactly one sequence of each cycle is well-formed so there are

$$\binom{2n+1}{n}\Big/(2n+1)$$

well-formed sequences.

3. Notes. We can translate the situation into random walk terms either by letting O and X correspond to horizontal and vertical unit steps or by letting O and X correspond to diagonal steps parallel to $(1,1)$ and $(1,-1)$. For example, $OOXOX$ gives either Figure 1 or Figure 2. The criterion

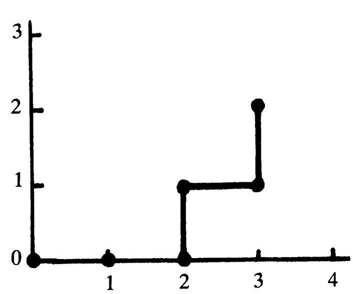

FIG. 1

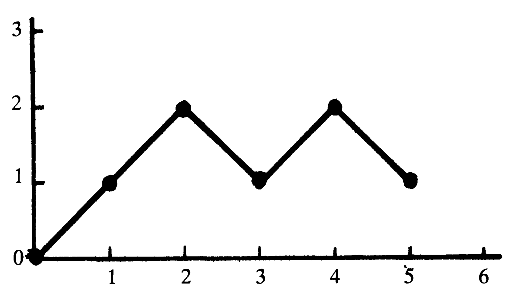

FIG. 2

for being well-formed is, in Figure 1, that the path remains below the diagonal, and, in Figure 2, that the path remains above the x-axis, excepting the initial point at the origin in both cases. Figure 2 is thus the classical ballot problem of Bertrand **[2]**, with numbers $n+1$ and n.

After finding a version of the proof above, I examined many of the relevant references in the first edition of Gould's bibliography and in other articles, all of which are given in the revised edition of Gould's bibliography **[4]**. Most of the ideas used above have been used before, but I did not find any place where they were used to provide as simple a proof as I have presented. Silberger's paper **[7]** gives all the basic ideas but not as simply. I found the elegant deletion argument in Silberger, who gives it in the more general form suitable for Bertrand's ballot problem: if candidates receive m and n votes in an election, with $m > n$ and ballots counted one by one, the probability of the winner being ahead at every point is $(m-n)/(m+n)$. I suspect the argument is much older, since I recall once hearing it from a statistician. My original proof shifted ill-formed initial segments to the end of the sequence. I have recently learned that Sands **[6]** has found my original proof and applied it to k-ary operations.

Many other derivations for C_n have been given, and many different sequences of sets have been shown to be enumerated by C_n; see [**3**, **4**]. There is another very simple, though slightly indirect, derivation due to Rodrigues [**5**, **1**], which could also be used in an elementary class. Let $R_n = (n+1)!C_n$, so R_n is the number of products of $n+1$ operands, taken in any order. Then $R_n = (4n-2)R_{n-1}$ since the $n+1$st operand can be combined on the right or the left of any of the n operands or any of the $n-1$ partial products corresponding to the operations in any of the ways of forming a product of n operands. Then $R_0 = 1$ determines R_n and C_n.

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REAL DIVISION ALGEBRAS AND DICKSON'S CONSTRUCTION

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Real division algebras are often encountered in algebra courses. They are mentioned in many undergraduate algebra texts because they are rich in algebraic structure and because they have been completely classified. Unfortunately, many authors restrict their discussion to the associative case. In fact, nonassociative real division algebras are included in the complete classification.

There is one part of the nonassociative theory which could cause problems for an undergraduate student. A student who is familiar only with associative objects might assume that division algebras are related to algebras in the same way division rings are related to rings; that to solve linear equations it is sufficient to require that each nonzero element has a multiplicative inverse.

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MATHEMATICAL EDUCATION

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CALCULUS AND LINEAR ALGEBRA IN APL

JAMES W. ENGLAND

This paper is a report on experimental courses in calculus and linear algebra which have been given for the past three years at Swarthmore College. These courses make up the first year of a two-year introductory mathematics program given by the department. The second year consists of a semester of several variable calculus followed by a "Moore Method" course in introductory analysis. The experimental calculus and linear algebra courses are offered as alternatives to conventional courses in the same subjects. In the conventional courses and the experimental courses we stress the constructive and computational features of calculus and linear algebra.

In the experimental courses, which are the subjects of this report, we use computing in a central way: to represent the ideas and concepts of calculus and linear algebra, and facilitate their analysis and exploration. To do this we use APL (A Programming Language) as the notation. In doing so we use the advantages of the APL notation to alter the presentation of a number of topics contained in calculus and linear algebra. The advantages we see in the APL notation are its relative conciseness, its lack of ambiguity, its ability to handle arrays in a straightforward manner, and its demand for explicitly defined functions.

A result of using the APL notation in these courses is a high degree of integration of the subjects of linear algebra and calculus. This notation makes it possible for the students to easily use the computer as an experimental device. This is possible because the notation which is used to express the concepts and techniques which arise in the courses is capable of being used to direct a machine to implement those ideas.

In general, the notation one uses for a subject tends to grow from the ideas in the subject. That is, topics considered important are easy to express while those considered less important become difficult to state. However, in using APL as the notation for calculus and linear algebra we are faced with the reverse situation. That is, we are using a new notation for well-established subjects. It is, therefore, important to remember that the notation (or language) one uses to describe a subject will shape one's view of the subject. There is a danger that important ideas may not be easy to express. Of course, there is also the potential that previously difficult and important ideas are easier to express and hence more accessible. Hence we have tried to identify the effect of using APL as the notation in these

courses. APL as the notation for calculus and linear algebra causes one to stress the constructive and computational results of the subjects. Many of the analytic tools, algorithms, and results of theorems are described as explicit functions. Once so defined, they can easily be experimented with in order to gain a clearer understanding of their meanings and effect. They also remain available for use throughout the remainder of the course. Every function considered in these courses is introduced by at least one explicit construction. In addition to an understanding of the topics covered in the course, we think the students leave the course with a rather different view of mathematics. This view includes seeing experimentation as a part of mathematics. The demand for explicitly defined functions stresses a constructive view of these subjects and probably results in the students taking a similar view of mathematics. The nature of this program is one in which the notion of approximation is fundamental. Generally, we proceed from a class of well-understood functions to a new class of functions via approximation. In this way students see the idea of approximation used to define new objects as well as approximating known objects for computational purposes.

Polynomials are studied extensively in the calculus course. A polynomial is identified with its coefficient vector. Functions defined on polynomials are therefore defined on vectors. A large number of these functions are linear. This makes it useful to introduce the notion of linear functions and their matrix representation together with some elementary matrix algebra. For example, the function *POLY* is a function of two arguments, the left argument being the coefficient vector of a polynomial, and the right argument being a vector consisting of the numbers at which the polynomial is to be evaluated. For example, $(^{-}4, 0, 1)$ *POLY* $(^{-}2, 0, 2, 4)$ results in $(0, ^{-}4, 0, 12)$. The polynomial associated with $(^{-}4, 0, 1)$ in standard notation is $^{-}4 + X^2$. When the right argument is held fixed, say V, the function *POLY V* is linear in its left argument. Therefore, it has a matrix representation, the size of which depends upon the degree of polynomials in the left domain. For polynomials of degree 2 and $V \leftarrow (^{-}2, 0, 2, 4)$ the matrix representation of *POLY V* is

$$B \leftarrow \begin{array}{rr} 1 & ^{-}2 \\ 0 & 0 \\ 1 & 2 \\ 1 & 4 \end{array} \quad \text{and for the polynomials of degree 4, it is} \quad A \leftarrow \begin{array}{rrrr} 1 & ^{-}2 & 4 & ^{-}8 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \end{array}$$

We determine these matrices in the standard way. That is, we apply the function *POLY V* to columns of the N by N identity matrix, *ID N*. The name of the function which does this is *REP* so that '*POLY* $(^{-}2, 0, 2, 4)$' *REP ID* 4 produces the matrix A as given above. Notice that for a vector V, if we let ρV denote the size of V (in this case, the number of elements in V), then '*POLY V*' *REP ID* ρV is a Vandermonde matrix. Provided the entries in V are distinct, this matrix is easily seen to be invertible. From this we can conclude that given N points in the plane, the first coordinates of which are distinct, there exists a unique polynomial of degree at most $N-1$ which passes through these points. In fact, we can now easily define a function, which we will call *CFP*, with the property that if P and Q are vectors of the same length and the entries in P are distinct, then P *CFP* Q is the coefficient vector of the unique polynomial of degree at most $^{-}1 + \rho V$ which passes through the points whose coordinates are determined by P and Q. In order to complete the description of the function *CFP*, it is necessary to give a brief introduction to the APL notation.

An expression written in APL is computed by proceeding from right to left. Most parentheses I will use are unnecessary, but should help the reader unfamiliar with APL notation. Most functions I define have a much more general definition than what I give! That is, the following at best is an introduction to the parts of the notation necessary to understand the concepts discussed in this paper. The operator $\circ.$ is monadic (has one argument written on the right) with the argument of $\circ$ being a dyadic function, e.g., $+$, $-$, $\times$, $\div$, etc. If F denotes a dyadic function $\circ.F$ is a matrix whose $[I;J]$

element is the result of applying F to the Ith element of the left argument and the Jth element of the right argument. For example $(0,1,2)\circ.+(5,6)$ results in

$$\begin{array}{cc} 5 & 6 \\ 6 & 7 \\ 7 & 8. \end{array}$$

The symbol $*$ denotes the power function and ι is a monadic function defined on the positive integers with ιN resulting in the first N non-negative integers. For example, $8=2*3$, $9=3*2$ and $(0,1,2,3)=\iota 4$. Recall from the last paragraph ρ is a monadic function which results in the size of its argument. Thus, $3=\rho(7,2,6)$ and $(3,2)=\rho((0,1,2)\circ.+(5,6))$.

Returning to the above example let $V\leftarrow(^{-}2,0,2,4)$ and notice that $V\circ.*(\iota(\rho V))$

$$\begin{array}{cccc} 1 & ^{-}2 & 4 & 8 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64. \end{array}$$

The function denoted by $+.\times$ (called the inner product) is dyadic. When both the left and right arguments are matrices of the proper size, the result is the matrix product. When the left argument is a matrix and the right argument a vector, the result is the standard inner product of a matrix and vector.

Let C be a vector satisfying $(\rho C)=(\rho V)$. That is, the degree of the polynomial associated with C is at most $^{-}1+(\rho V)$. Then $(V\circ.*(\iota(\rho C)))+.\times C$ is the value of the polynomial C at the elements of V. The last statement can serve as a definition of the function $POLY$. We return (at last) to the definition of the function CFP. Given P and Q as above, we want a polynomial C, with $(\rho C)=(\rho P)$ satisfying $Q=(C\ POLY\ P)$. Written in matrix form, we want to solve the system of linear equations $Q=(P\circ.*(\iota(\rho C)))+.\times C$. Notice that since I am only interested in polynomials satisfying $(\rho C)=(\rho P)$, I can replace (ρC) by ρP in the definition of $POLY$. Hence, we can rewrite the above equations as $Q=(P\circ.*(\iota(\rho P)))+.\times C$. This system of equations has a unique solution if and only if its matrix of coefficients, $P\circ.*(\iota(\rho P))$, is nonsingular. Parenthetically, note that an easy way to see that the Vandermonde matrix is nonsingular, using the definition of $POLY$, is that the unique polynomial of degree less than N, which is zero at N distinct numbers, is the zero polynomial. Finally, if A is a nonsingular matrix, $\boxed{\div}A$ denotes the inverse of A. When $\boxed{\div}$ is used dyadically with a nonsingular matrix, A, as a right argument and an appropriate vector or matrix, V, as a left argument, the result is

$$(\boxed{\div}A)+.\times V. \qquad \{(V\boxed{\div}A)=(\boxed{\div}A+.\times V)\}.$$

Thus, we can now define the function

$$P\ CFP\ Q \quad \text{by} \quad Q\boxed{\div}(P\circ.*(\iota\rho P)).$$

The function CFP is now available for use throughout the remainder of the course.

The following example may help explain some of the points I have raised. The example is an outline of how numerical integration can be presented. To describe this, I need to introduce a few additional APL functions.

Scalar functions extend to arrays element by element. For example, $(6,4,3,5)=((6,8,9,20)\div(1+\iota 4))$. The symbol ϕ denotes the function which reverses the order of the elements of a vector, e.g., $(3,2,1,0)=\phi(\iota 4)$. The reduction operator, denoted by /, takes a dyadic function as its left argument and a vector as its right argument and returns the value of the expression obtained by inserting the left argument between succeeding elements of the right argument. For example, $6=+/(0,1,2,3)$ and $(.5)=\div/(1,2)$. If V is a vector and I is an array whose elements are a subset of the indices of V, then $V[I]$ is the array formed by replacing each element of I by the element in V whose index it is. The indices of a vector V are $\iota(\rho V)$. Recall that ιN includes zero. Let $V\leftarrow(2,3,6,8)$, $V[0]=2$; $V[2]=6$; $V[3]=8$; $V[2\ 0\ 3]=(6,2,8)$ and if

$$I \leftarrow \begin{matrix} 1 & 3 \\ 2 & 0 \\ 3 & 2 \end{matrix} \quad \text{then } V[I] \text{ is } \begin{matrix} 3 & 8 \\ 6 & 2 \\ 8 & 6. \end{matrix}$$

We can now give a description of numerical integration. At the end of this paper we give numerical examples of the functions we will define. The reader may wish to consult these examples while reading this section. The function PII (polynomial indefinite integral) is defined on the coefficient vector of a polynomial C, by the expression $(O, C \div (1 + \iota(\rho C)))$ so that $PII\ C$ is an indefinite integral of the polynomial C. If C is a polynomial, and if V is a vector whose entries are the endpoints of an interval, then the expression $-/((PII\ C)\ POLY\ (\phi\ V))$ will define a dyadic function written $V\,PDI\,C$ (polynomial definite integral), which produces the definite integral of C over the interval V. Assume the function N is integrable on the interval $V \leftarrow (A, B)$. For N a positive integer $P \leftarrow A + (((B - A) \div N) + \iota(N + 1))$ is a partition of V into N subintervals of equal length. With these functions, we can define a (bad) approximation to the integral of F over the interval V by the expression

$$(*) \qquad (P[0, {}^{-}1 + (\rho P)])\ PDI\ (P\ CFP\ (FP)).$$

The difficulty with this approximation is that in order for it to be reasonably accurate over a large interval, we must take N to be quite large. In turn, this means the matrix which must be inverted in CFP is quite large. Hence, we need a more sophisticated technique.

Recall that a Kth order approximate integral of F over V ($K = 2$ for the Trapezoid Rule, $K = 3$ for Simpson's Rule) is obtained by adding the integrals of the $(N - 1) \div (K - 1)$ polynomials obtained by fitting polynomials to K successive points of P and $F\,P$. Now, if in expression (*) we replace $F\,P$ by a vector Q with $(\rho P) = (\rho Q)$, the resulting expression defines a function of P and Q. With P held fixed, this is linear in Q. We denote this function by AI. That is, $P\ AI\ Q$ is defined by $(P[0, {}^{-}1 + (\rho P)])\ PDI\ (P\ CFP\ Q)$. The function $COEFF$ is defined so that it sends the partition P to the matrix representation of the linear function $P\,AI$. The expression '$P\,AI$' $REP\ ID\ (\rho P)$ will serve as a suitable definition for $COEFF\ P$. We have $((COEFF\ P) + . \times Q) = P\ AI\ Q$. We need the following identity, which is straightforward to prove. For A and S scalars and P a partition $(COEFF\ S \times (A + P)) = (S \times COEFF\ P)$. Thus, for an equally spaced partition P we have $(COEFF\ P) = (S \times COEFF(\iota\rho P))$ where S is the distance between adjacent points of P. Finally, to easily lump together successive elements of a partition into groups of K, we need only produce the proper matrix of indices. As long as $(N - 1) \div (K - 1)$ is a positive integer, this is given by $(\iota K) \circ . + (K - 1) \times \iota((N - 1) \div (K - 1))$. The dyadic function $LUMP$ is defined by letting $K\ LUMP\ P$ equal $P[(\iota K) \circ . + (((K - 1) \times \iota(\rho P) - 1) \div (K - 1))]$.

To finish this problem, recall F is defined on the interval V, P is a partition of this interval, and K is chosen so that $((\rho P) - 1) \div (K - 1)$ is an integer. If $M \leftarrow K\ LUMP\ FP$, then the 0th column of $M, M[;0]$ is the value of F at the first K points of P so that $(COEFF\ P[\iota K]) + . \times M[;0]$ is $P[\iota K]\ AI\ M[;0]$ or the value of the integral of the polynomial fit to the first K points with X-coordinates P and Y-coordinates $F\,P$. We can simplify this by the identity we gave above to $S \times (COEFF\ \iota K) + . \times M[;0]$ where S is the distance between adjacent points of P. This same argument holds for each group of K points, which corresponds to the various columns of M. Thus $S \times (+/(COEFF\ \iota K) + . \times (K\ LUMP\ (F\,P))$ is the value of a Kth order approximate integral of the function F over the interval with respect to the partition P. We conclude by having the students define a dyadic function INT so that $(A, B)\,INT$ 'F' gives an approximate value for the integral of the function F over the interval $(A \leqq X) \wedge (X \leqq B)$. The function INT (integral) is defined early in the calculus course. It is used to compute definite integrals until techniques of integration are covered.

The section dealing with series is primarily devoted to power series. Series of numbers are considered only to check the convergence of power series at the endpoints of the interval of

convergence. Topics in differentiation are covered in a rather standard way. However, since APL allows one to easily manipulate arrays, the students define a symbolic differentiator; that is, a function whose argument is a symbol string which represents an elementary function, and whose result is a symbol string representing the derivative of the original function.

The text we use is *Calculus in a New Key* by D. L. Orth, A.P.L. Press, Swarthmore, Pa. As far as we know, this is currently the only text available. The discussion we gave of numerical integration is, in a different form, contained in Orth's book.

Courses such as these, involving a rather radical departure from the standard courses in calculus and linear algebra, can create certain difficulties. The first and ultimately the most serious difficulty is its acceptance within the local scientific community. Calculus and linear algebra are subjects which are fundamental to the physical and social sciences. Hence, any departure from the beaten path is suspect. At the same time, much elementary mathematics is learned by students as they attempt to use it in other science courses. Thus, there is a danger that students will not get reinforcement of their calculus and linear algebra in other courses. This problem seems to be one which can be decreased in its severity by a careful job of explaining the program to other departments. The precision of APL sometimes makes it difficult for people to use it informally, particularly in lectures. Our attitude in this regard has been to use the notation informally once it is understood by the students. Related to this is the fact that people often have difficulty looking at an APL expression and finding it as meaningful as a similar expression in conventional notation. This problem lessens as the course proceeds and people become more familiar with the APL notation.

Our experience with this program is such that we are attempting, whenever possible, to use the APL notation in other mathematics courses. In one sense this is an attempt to see to what extent APL can be used as a notation for mathematics. We hope that others will join us in this work.

Examples of the functions used in numerical integration

(1) $PII\ C$ (Polynomial indefinite integral)

$$(0,(C \div 1+\iota(\rho C)))$$

EXAMPLE: $PII\ (3,8,12)$

(2) $(0,3,4,4)\cdot(A,B)\ PDI\ C$ (Polynomial definite integral)

$$-/(PII\ C)\ POLY\ (B,A)$$

$$((PII\ C)\ POLY\ B)-((PII\ C)\ POLY\ A)$$

EXAMPLE: $(0,3)\ PDI\ (3,8,12)$
$((0,3,4,4)\ POLY\ 3)-((0,3,4,4)\ POLY\ 0)$
$153-0$
153

(3) $P \leftarrow (^{-}3,^{-}2,6,^{-}2.2,^{-}1.8,-1.4,^{-}1,^{-}.6,^{-}.2,.2,.6,1,1.4,1.8,2.2,2.6,3)$

P is a partition of the interval $(^{-}3 \leqq X) \wedge X \leqq 3$.
$P[0,^{-}1+\rho P]=(^{-}3,3)$
F is the function defined by

$$(*-(X*2)\div 2)\div(\bigcirc 2)*.5 \leftrightarrow (1/\sqrt{2\pi})\exp(-X^2/2)$$

$$F=(.004,.014,.035,.079,.150,.242,.333,.391,$$

$$.391,.333,.242,.150,.079,.035,.014,.004)$$

$P\ AI\ Q$
$(P[0,{}^{-}1+(\rho P)])\ PDI\ (P\ CFP\ Q)$
$P\ AI\ FP$
.99744

(4) A table of values of $COEFF\ \iota N$ for various values of N is given below:

N	$COEFF \iota N$
2	$(1,1)\div 2$
3	$(1,4,1)\div 3$
4	$(3,9,9,3)\div 8$
6	$(475,1875,1250,1250,1875,475)\div 1440$

(5) Letting $K=6$ we get

6 *LUMP P*

¯3.00	¯1.00	1.00
¯2.60	¯.60	1.40
¯2.20	¯.20	1.80
¯1.80	.20	2.20
¯1.40	.60	2.60
¯1.00	1.00	3.00

6 *LUMP F P*

.00	.24	.24
.01	.33	.15
.04	.39	.08
.08	.39	.04
.15	.33	.01
.24	.24	.00

$S \leftarrow P[1]-P[0]$

The approximate value of the integral of F over $({}^{-}3 \leqq X) \wedge X \leqq 3$ using 5th degree polynomials is

$$S \times \ +/(COEFF\ \iota 6)+.\times 6\ LUMP\ F\ P$$

0.9971193436

Letting $K=4$ we get

4 *LUMP P*

¯3.00	¯1.80	¯.60	.60	1.80
¯2.80	¯1.40	¯.20	1.00	2.20
¯2.20	¯1.00	.20	1.40	2.60
¯1.80	¯.60	.60	1.80	3.00

4 *LUMP F P*

.00	.08	.33	.33	.08
.01	.15	.39	.24	.04
.04	.24	.39	.15	.01
.08	.33	.33	.03	.00

The approximate value of the integral of F over the same interval using 3rd degree polynomials is

$$S \times \ +/(COEFF\ \iota 4)+.\times 4\ LUMP\ F\ P$$

0.9972530128

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MATRIX EXAMPLES IN MODERN ALGEBRA

GREGORY DOBBINS AND GORDON STRATE

During a recent modern algebra course taught at Wheaton College, I tried something a bit different. There were about twenty-five students in the class, mostly sophomores who were very able and well prepared, and they had just finished linear algebra the previous quarter. Although I was not their teacher then, I had also been involved considerably with linear algebra and matrices in some topological algebra research work. Specifically, in Section 4 of [1], I had used some examples of matrix groups that I thought would actually be of interest to my students and helpful in teaching. So this course was planned to concentrate slightly more than usual on groups, yet with adequate time at the end for an introduction to rings and fields, which are the focus of the sequel, Modern Algebra II. We made these matrix examples an integral part of the course and used them profitably to illustrate various group concepts along the way.

The purpose of this note is to report on something novel and extra that worked out nicely in the class. It is not to introduce a new way for teaching algebra or to convince students that all examples of groups are like these. The text used was *A First Undergraduate Course in Abstract Algebra* by Hillman and Alexanderson, and the usual finite group examples, as well as others given there, were covered.

Introduction of these matrix examples added a fresh excitement to the course. Moreover, the students were amazed to find out that linear algebra and modern algebra are actually related to one another. They appreciated the emphasis on the unity of ideas between these two areas and the teacher was happy to see time spent in research pay off in the classroom. In this note, we hope to share some of the excitement.

Here, by way of summary, are some of the things noticed by the class in evaluating their experience. In these examples, the various concepts and their interrelationships do not just correspond to different subsets, but rather they take on interesting geometric significance. By utilizing the euclidean representations of the matrix groups, we not only have a good illustration of the use of an isomorphism to gain a new viewpoint, but, in addition, subgroups become rays and half planes with commutator subgroups and group centers perpendicular to one another. The important fact that right and left cosets of a subgroup by the same group element are not always the same is dramatized when one can visualize them as sets with a geometric structure, such as intersecting rays, rather than sets that are simply unequal. If one takes a suitable matrix M, he can adjoin the identity matrix I to form $M|I$ and successively row-reduce to obtain $I|M^{-1}$. In this way, he can see the inverse appear in a "get your hands dirty" fashion and then later check that $MM^{-1}=I=M^{-1}M$. The abstract concept of an "inverse element" is forced to materialize in a concrete example. Often, in studying a group concept in this matrix context, one can decide by solution of specific equations which elements in the group have the given property. For instance, this was used to find the general forms of matrices in the commutator subgroup and group center. In what follows, we illustrate these ideas and others with a few comprehensive examples. Many of the observations summarized herein were parts of individual exercises that involved substantial computation for the students.

Example 1. Let G be the group of all 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

such that $a>0$. This group provides two illustrations of the isomorphism concept in that it may be viewed as the group of affine orientation preserving transformations on the reals R by $x\to ax+b$ or as the noncommutative group of the half plane, consisting of $\{(a,b)\,|\,a>0\}$ with the multiplication $(a,b)(c,d)=(ac,ad+b)$. In the next example, we shall denote this group as $\mathrm{Aff}^{+}(R)$. We sketch the half-plane representation in Figure 1 below.

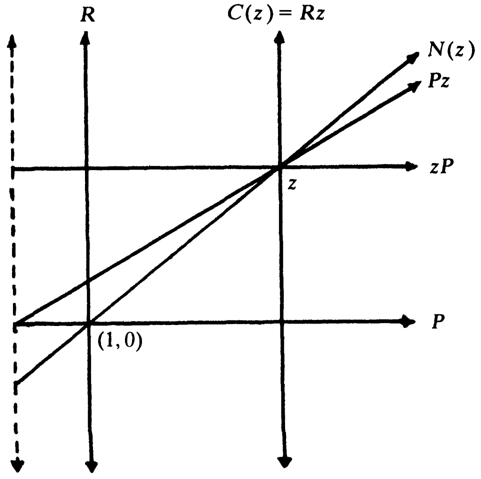

FIG. 1

It is surprising how many ideas can be illustrated by this one example. The vertical line $x = 1$ is the commutator subgroup and is isomorphic to the additive reals R while the horizontal ray P, consisting of all points of the form $(x, 0)$, is isomorphic to the multiplicative group of positive reals and is a nonnormal subgroup. The group G is the semidirect product of R with P and admits the double coset decomposition $G = PwP \cup P \cup Pw^{-1}P$ for any $w \in G\backslash P$. If we choose an arbitrary point $z = (a, b)$ in $G\backslash P$, then the left coset zP is the horizontal ray $y = b$, whereas Pz is a ray with nonzero slope b/a. Since R is a normal subgroup $Rz = zR$ and Rz is the vertical line through z. The coset Rz also happens to be the conjugacy class $C(z)$ of z in G for $z \notin R$ and as a matrix we see what diagonal matrix z is conjugate to by the intersection with P. The normalizer $N(z)$ for $z \in R$ can be calculated as follows: if $(x, y) \in N(z)$ then $(x, y)(a, b) = (a, b)(x, y)$ or $(xa, xb + y) = (xa, ay + b)$ or

$$y = \frac{b}{a-1} x - b.$$

Thus, $N(z)$ is the ray through the identity $(1, 0)$ and (a, b); it is also the conjugate subgroup wPw^{-1} where $w = (1, b/(1-a))$. All of these observations are presented in Figure 1.

Example 2. Let G be the group of 2×2 upper triangular matrices of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ such that $x, z > 0$. This group has the 3-space representation as the set of all triples (x, y, z) such that $x, z > 0$ with the multiplication $(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1x_2, x_1y_2 + y_1z_2, z_1z_2)$. The group center Z is the ray $x = z > 0$ and the commutator subgroup G' is the perpendicular $x = z = 1$ and is separated by the identity $(1, 0, 1)$ into three conjugacy classes. The remaining conjugacy classes are lines parallel to G' excluding those passing through points on the ray $x = z > 0$, $y = 0$. Each of those points being in the center Z divides the corresponding line into three conjugacy classes. The subgroup $H = \{(x, 0, z)\ x, z > 0\}$ is isomorphic to the diagonal matrices $\begin{bmatrix} x & 0 \\ 0 & z \end{bmatrix}$ and divides the group in half. In Figure 2 below we show the right half of G determined by the half plane H. The diagonalizable matrices correspond to points (x, y, z) such that $x \neq z$; that is, points not on horizontal lines piercing Z.

Using some linear algebra, we see that the coset $Z(a, b, c)$ of the center corresponds to the set of all positive scalar multiples of the vector (a, b, c). Thus, the cosets of the center are the rays emanating from the origin into G. Furthermore, if we pick an arbitrary (a, b, c) in G but not in Z, then with the help of a little more linear algebra, one can show that the normalizer of (a, b, c) is the half plane resulting from the intersection of the vector span of $(1, 0, 1)$, (a, b, c) and $(a, b, c)^{-1}$ with G.

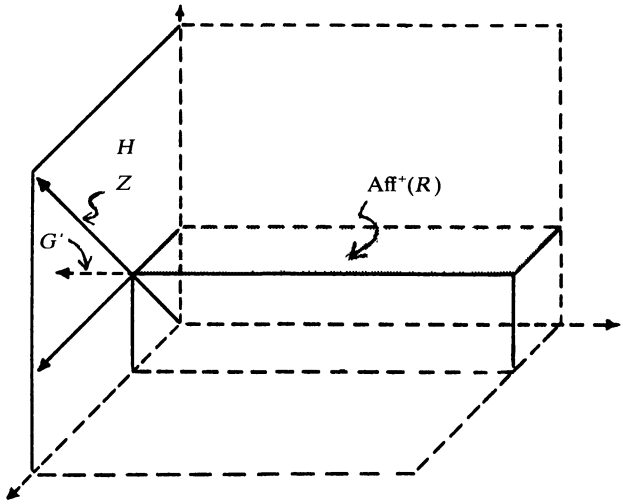

FIG. 2

Example 2 also provided good illustration for further topics such as subgroup products, semidirect and direct products, homomorphisms and quotient groups. It is interesting to discover that the subgroup $G'Z$ is simply the half plane in G that is determined by the rays G' and Z. Is this group isomorphic to the group $\mathrm{Aff}^+(R)$ from Example 2? Denoting the formation of a semidirect product by x_s we can show $G \cong (G'Z)x_s\{(1,0,z)\,|\,z>0\}$ and since the last written set is a group isomorphic to P, we have $G \cong (G'Z)x_sP$. Quotient groups can be identified by thinking of elements of G as factored into certain matrix products. For example, $G/(G'Z) \cong P$ since any $g \in G$ can be factored as $\begin{bmatrix} r & s \\ 0 & r \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$ for suitable r, s, t; $G/G' \cong P \times P$ since any $g \in G$ can be factored as $\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}$ for suitable r, s, t and $G/Z \cong \mathrm{Aff}^+(R)$ since any $g \in G$ can be factored as $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix}$ for suitable r, s, t. One very distinctive feature of these quotient examples is that, both algebraically and geometrically, the student can clearly see the subgroup being "divided or moded out" as well as what remains.

A copy of the group $\mathrm{Aff}^+(R)$ is indicated in Fig. 2 as the half plane $z = 1$. Is this group conjugate to the other half plane $G'Z$ mentioned above? Geometrically, what is the action of conjugation on these half planes?

The next example presented in the form of a problem was not used in the class and is more directly from [1]. It is given here to illustrate how these examples could be extended for use in a course where there is more time to devote to group theory.

Problem. Let G denote the group of all 3×3 upper triangular matrices of the form

$$\begin{bmatrix} x & y & z \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix} \quad \text{such that} \quad x, a, c > 0.$$

Let H be the subset of all matrices of the form

$$\begin{bmatrix} x & 0 & z \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix} \quad \text{and } H_1 \text{ all matrices of the form} \quad \begin{bmatrix} x & y & z \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

in G. Show that H and H_1 are nonnormal subgroups, that an element in the Core(H) has the form

$$\begin{bmatrix} x & 0 & z \\ 0 & x & b \\ 0 & 0 & c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

for an element in Core(H_1). Observe that $G = \text{Core}(H)\text{Core}(H_1)$ and let $K = \text{Core}(H) \cap \text{Core}(H_1)$ and note that an element in K looks like

$$\begin{bmatrix} x & 0 & z \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}.$$

This subgroup K is equal to G^2Z where G^2 is the second commutator subgroup of G and Z is the center. Show that G/K is isomorphic to the direct product group $\text{Aff}^+(R) \times \text{Aff}^+(R)$. Is there a general result suggested here?

To sum up, then, these matrix examples served to bring more computation and hence involvement and interest into the course as well as to point out the unity of ideas in linear and modern algebra. They were also the starting place of various questions and conjectures by several students who wondered things like whether this algebraic and that geometric property were always linked together in general, etc. A search of my bookshelf has turned up numerous texts on modern algebra at this level but all of them seem to follow pretty much the same pattern regarding examples. Except for the brief references, such as the problem # 11 on page 150 in [2], these texts dwell primarily on the finite examples. Our experience is that matrix examples are valuable in the course and add an extra dimension that we have tried to relate here.

The second author was a student in the Modern Algebra class reported here.

References

1. J. G. Dobbins, Well bounded semigroups in locally compact groups, Math. Z., 148 (1976) 155–167.
2. N. H. McCoy, Introduction to Modern Algebra, rev. ed., Allyn and Bacon, Boston, 1968.

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MATHEMATICS IN THE INTEGRATED SCIENCE PROGRAM AT NORTHWESTERN UNIVERSITY

M. PINSKY AND R. C. SPEED

Introduction. Faculties of the departments of mathematics and natural sciences at Northwestern University have recently implemented a new degree-producing interdepartmental curriculum, the Integrated Science Program (ISP). The breadth, integration, and rigor of ISP differ from those of any other multidisciplinary educational program, past or present, of which we are aware. Because of the quantitative theme of the ISP courses, mathematics is the foundation of the ISP structure. We shall briefly describe ISP and then discuss its mathematics component—in part to solicit comments from readers.

Integrated Science Program. ISP is a three-year curriculum of 24 specially developed one-quarter courses that present a broad range of natural sciences and mathematics. The program is designed for a small group, 30 per year, of academically superior students who are interested in careers in science

and/or mathematics. Taken with 12 one-quarter courses in the liberal arts, ISP leads to the B.A. in Science in three years. Instructors are senior faculty of science and mathematics departments at Northwestern. ISP courses are more accelerated and more rigorous than most regular departmental offerings. Through coordinated instruction, use of applications, and sequential arrangement of topics, ISP courses attempt to integrate the natural sciences and mathematics and to show that the basic relationships and techniques of one discipline are commonly applicable to others. Besides formal course work and laboratories, ISP includes a regular seminar and frequent visits of small groups of ISP students with Northwestern researchers to introduce the students to current problems and to show how research is done.

Faculty study committees spent several years (1973–1976) designing ISP. The program, aided by a three-year grant from the National Science Foundation, began in the fall of 1976, with a class of 30 specially recruited students.

We think of the beginning years of ISP as an experiment in high-level undergraduate science-mathematics education. One philosophical concept that we are testing is that there is an extensive common base to the scientific disciplines and that the exercise of intellect, analysis, and innovation, and the organization of knowledge are more effective if we emphasize the common base rather than the conventions and techniques unique to each discipline. A second idea is that the undergraduate years are the time for students, especially those gifted intellectually, to learn in breadth and depth the full realm of science. The appropriate time for such students to specialize is in graduate school. As a corollary, we consider graduate work at the Ph.D. level to be essential for ISP graduates who plan to become practicing scientists. Thus, ISP is designed to provide a comprehensive and rigorous background for graduate studies in all fields of science and in mathematics. An evaluation of the successes and pitfalls of the principles and practices of our program, together with a curriculum syllabus, will be available in the summer of 1979.

Curriculum. The first year of the ISP curriculum includes a central sequence of mathematics, classical physics (mechanics, electricity and magnetism, and waves), and inorganic and organic chemistry. The topics of mathematics and physics are closely timed, so that the students rapidly reach a level of sophistication in physics considerably beyond that normally given in lower-division courses. Chemical kinetics is treated at some length in the freshman year in concert with the development of first-order differential equations in mathematics. The second and third years of ISP comprise a number of advanced sequences: mathematics (3 units), modern physics (3 units), thermodynamics (1 unit), biochemistry and biological sciences (5 units), geophysics and astrophysics (3 units). The advanced sequences incorporate integrative material. For example, a course in geophysics will serve to relate topics of classical physics to the earth while exploiting the mathematics of partial differential equations, given concurrently with the geophysics course. A course in astrophysics, following quantum theory and nuclear physics, will employ this physics in the context of stellar evolution. A course emphasizing physical aspects of biochemistry will dwell on quantitative treatments of ion migration, membrane diffusion, and X-ray scattering, using the tools of physics and mathematics developed in previous courses.

Specific Mathematics Content of ISP. The mathematics program in ISP consists of a two-year sequence of six one-quarter courses, taken in the first two years of the student's career. The first-year course assumes a working knowledge of one-variable calculus, usually attained during a high-school calculus course in the senior year.

The first one and one-half quarters of mathematics cover calculus in two and three dimensions, integrated as much as possible with mechanics and electromagnetism. The topics covered include vectors, equations of lines and planes, curves in 3-space, line integrals, double and triple integrals, partial differentiation (including the usual related topics), surface integrals, the divergence theorem, and Stokes' theorem. This material is taught mainly from chapters 5 through 8 of the book *Calculus,*

An Introduction to Applied Mathematics, by H. P. Greenspan and D. J. Benney (McGraw-Hill, New York, 1973).

The final one and one-half quarters of the first-year mathematics sequence contain a treatment of ordinary differential equations and the necessary background material on infinite series and linear algebra. This part of the sequence uses the book *Differential Equations and Their Applications*, by M. Braun (Springer-Verlag, 1975). Besides providing applications to physics (coupled oscillators, normal modes), this part of the sequence illustrates the importance of developing mathematics (infinite series, linear algebra) for the purpose of solving certain classes of differential equations—many of which arise in applications.

Throughout the year the instructor made use of computer programming. This was applied to Taylor's formula with remainder and other problems in the first quarter. Fortran IV was presented in a special lecture series to ISP students in the winter quarter and was used in the numerical solution of ordinary differential equations and other problems.

The second-year mathematics sequence begins with a one-quarter course in boundary value problems. This course uses *Fourier Series and Boundary Value Problems*, by R. V. Churchill. It treats the classical problems of wave motion and heat conduction using eigenfunction expansions, with applications to phenomena in the earth presented in a concurrent geophysics course. With the proper emphasis, the student emerges from the course in boundary value problems well prepared to assimilate the rudiments of Schrödinger's equation in the winter-quarter physics course. Meanwhile, the second-quarter mathematics course turns to complex analysis, using *Complex Variables and Applications*, by Churchill, Brown, and Verhey. In addition to the standard topics, including residue calculus and conformal mapping, the course includes a section on asymptotic methods (Laplace's method, method of stationary phase). Finally, in the third quarter, the student takes a one-quarter course in stochastic methods. This includes a unit on probability on finite sample spaces and introduces the statistical concepts of estimation and hypothesis testing. The central limit theorem is taught, together with several applications. With sufficient time, some elementary notions of stochastic processes are introduced.

The general flavor of the second-year course is that of applied analysis. Although the lectures contain the statement (and even an occasional proof) of many theorems, the students are not expected to master the material at this level. In this sense, the course resembles a course in "Methods of Mathematical Physics," commonly taught to first-year graduate students in physics.

Role of Mathematics in a Science Program. The central importance of mathematics is an article of faith in ISP. We are reminded of the physicist Eugene Wigner [**1**] who referred to the "unreasonable effectiveness of mathematics in the natural sciences." Wigner pointed out, among other things, that mathematical concepts turn up in entirely unexpected connections. In ISP we are able to demonstrate the usefulness of various kinds of mathematics in areas of biochemistry, geophysics, physical chemistry, population biology, and theoretical physics.

We believe that the six-quarter ISP mathematics sequence is a minimum for the scientist of the future. Among working scientists of today we find highly disparate mathematics backgrounds. We believe that if science as a whole is to rise to a higher common level, it can do so only through mathematics as the medium of communication.

Students entering the program have studied calculus in high school and are eager to continue their mathematical training in college. All courses are taught by senior faculty members who have volunteered because of an interest in the program. As a rough guideline, we teach problem-solving and have resisted the temptation to teach theorem-proving. Consequently the course can proceed at a rate that is accelerated in comparison with conventional service courses. We estimate that the first-year course covers from 25 to 40 percent more material than a conventional sequence.

We feel that ISP has strong attraction for students who may be interested in a career in mathematics. It may be objected that in mathematics ISP avoids an early confrontation with difficult

foundational matters. Although we acknowledge this aspect of the program, we respond first that much of mathematics itself has origins in the physical and biological sciences. The study of real systems invariably leads to well-posed mathematical problems. By studying several areas of science in depth, the future mathematician has a rich supply of scientific problem areas on which to base advanced work.

We go one step further and argue that ideas from physics, for example, have actually helped in the solution of many mathematical problems. One could cite numerous examples of this phenomenon. A recent example of wide mathematical interest is the use of conserved quantities in the study of ordinary and partial differential equations.

Although we cannot demonstrate all of these subtle interactions in the classroom, it is our firm belief that ISP will enable the future mathematician to pursue a career in the finest traditions of mathematical physics.

ISP Students. The first ISP class consisted of thirty students selected from a group of about 120 applicants whose credentials were considered satisfactory (year of calculus, excellent achievement record, evidence of strong science-mathematics motivation). The only easily transmitted measures of ability of the first class are average SAT scores: 670v, 750m. Most were in the upper percent of their graduating class, had grade averages of 3.8 or better, and had strong recommendations from science and mathematics teachers. The performance of the majority of these students in their freshman courses in ISP has met our expectations for an intellectually stimulated, achieving group of undergraduate science majors.

From a larger applicant pool for the second class for ISP for the fall of 1977, the 30 students selected had the following average SAT scores: 683v, 755m. We are confident that in years to come ISP will continue to attract students with outstanding academic credentials.

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Reference

1. E. P. Wigner, The unreasonable effectiveness of mathematics in the natural sciences, Comm. Pure Appl. Math., 13 (1960) 1–14.

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PROBLEMS AND SOLUTIONS

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All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before August 31, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2713*. *Proposed by Saul Singer, Brooklyn, N.Y.*

A stack of x rings is given, decreasing in size from the bottom up. In addition, y empty stacks are provided ($y \geqslant 2$). Let $N(x,y)$ be the minimum number of moves necessary to transfer the rings to one of the empty stacks subject to the following two rules:

(1) Move just one ring at a time,

(2) at no time can a larger ring be placed atop a smaller one.

It is conjectured that

$$N(x,y)=\sum_{k=1}^{m} 2^{k-1}\binom{k+y-3}{y-2}+2^{m}\left[x-\binom{m+y-2}{y-1}\right],$$

where m is the largest integer such that the expression in the brackets is $\geqslant 0$.(The case $y=2$ is the famous "Tower of Hanoi" problem.)

E 2714. *Proposed by M. J. Pelling, Balliol College, Oxford, England*

Let G_1, G_2 be two bounded convex regions in $\mathbf{R}^2$ and suppose G_1 is translated to $G_1(t)$ by the transformation $x \to x+ta$ where a is a fixed unit vector. Consider the area $A(t)$ of $G_1(t) \cap G_2$ as a function of t. Is it always true that there is a constant c such that $A(t)$ is monotonic increasing for $t \leqslant c$ and monotonic decreasing for $t \geqslant c$?

What happens in $\mathbf{R}^n$?

E 2715*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let G be the centroid of the triangle $A_1A_2A_3$ and let

$$\theta_i = \measuredangle(\overrightarrow{A_iA_{i+1}}, \overrightarrow{A_iG}), \qquad (i=1,2,3).$$

Prove or disprove that $\Sigma \sin \theta_i \leqslant 3/2$.

E 2716*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let ABC be a triangle with P an interior point. Let A', B', C' be the points where the perpendiculars drawn from P meet the sides of ABC. Let A'', B'', C'' be the points where the lines joining P to A, B, C meet the corresponding sides of ABC.

Prove or disprove that $A'B'+B'C'+C'A' \leqslant A''B''+B''C''+C''A''$.

E 2717. *Proposed by E. Ehrhart, Strasbourg, France*

Find the number of symmetric 4×4 matrices whose entries are all the integers from 1 to 10 and whose row-sums are all equal.

E 2718. *Proposed by Gordon D. Prichett, Hamilton College, Clinton, N.Y.*

Find all prime numbers p which have the following two properties:

(i) all numbers obtained from p by permuting its digits are also prime,

(ii) the sum and the product of the digits of p are also prime.

SOLUTIONS OF ELEMENTARY PROBLEMS

Characteristic Polynomial of a Matrix

E 2635 [1977, 134]. *Proposed by Kirby C. Smith, Texas A & M University*

Let F be a field of characteristic $p \neq 0$. Let $A = CD$ where C is a cyclic $p \times p$ matrix over F and D is the diagonal matrix with diagonal entries $0, 1, 2, \ldots, p-1$. Compute the characteristic polynomial of A. Generalize.

Solution by D. Ž. Djoković (Associate Editor). All indices will belong to $P = \{0, 1, \ldots, p-1\}$ which is considered as a subset of F. Let

$$f(\lambda) = \lambda^p - c_1\lambda^{p-1} + c_2\lambda^{p-2} - \cdots + (-1)^p c_p$$

be the characteristic polynomial of A.

For each subset $I \subset P$ let $A(I)$ be the principal minor of A corresponding to the rows and columns with indices in I. Similar notation will be used for C.

Clearly we have

$$A(I) = \Big(\prod_{i \in I} i\Big) \cdot C(I), \qquad I \subset P. \tag{1}$$

It is well known that

$$c_k = \sum_{|I| = k} A(I), \qquad I \subset P. \tag{2}$$

Let us say that two k-subsets I_1 and I_2 of P are equivalent if $I_2 = I_1 + j$ for some $j \in P$. Let R_k be a set of representatives of these equivalence classes. If $1 \leqq k \leqq p-1$ then each equivalence class has p elements because $I \subset P$, $|I| = k$ and $I + j = I$ imply that $j = 0$.

Since C is cyclic, i.e., $C = (\gamma_{ij})$, $\gamma_{ij} = \sigma_{i-j}$ $(i, j \in P)$, $\sigma_i \in F$, we have that $C(I + j) = C(I)$ for $I \subset P$ and $j \in P$. Using this and (1) we can rewrite (2) as follows

$$c_k = \sum_{I \in R_k} \Big[C(I) \cdot \sum_{j \in P} \Big(\prod_{i \in I + j} i\Big)\Big], \qquad 1 \leqq k \leqq p-1. \tag{3}$$

If $I = \{i_1, i_2, \ldots, i_k\}$ then

$$\sum_{j \in P} \Big(\prod_{i \in I + j} i\Big) = \sum_{j=0}^{p-1} (i_1 + j)(i_2 + j) \cdots (i_k + j).$$

This sum is zero if $k \leqq p-2$ because we have

$$\sum_{j=0}^{p-1} j^m \equiv 0 \pmod{p}, \quad 0 \leqq m \leqq p-2.$$

Hence $c_k = 0$ for $1 \leqq k \leqq p-2$. Clearly $c_p = 0$ since the first column of A is zero.

Finally, it follows from (3) that $c_{p-1} = -\Delta$ where $\Delta = C(I)$ and $I = \{1, 2, \ldots, p-1\}$. One just has to use Wilson's Theorem $\Pi_{i=1}^{p-1} i \equiv -1 \pmod{p}$. Thus we have $f(\lambda) = \lambda^p - \Delta\lambda$.

Microbe Culture

E 2636 [1977, 134]. *Proposed by D. E. Knuth, Stanford University*

A pair of microbes was recently discovered which reproduce in a very peculiar way. The male microbe (a diphage) has two receptors on its surface, and the female (a triphage) has three receptors.

When a culture of diphages and triphages is irradiated with a psi-particle, exactly one of the receptors absorbs the particle (each receptor being equally likely). If it was a diphage, it changes to a triphage; but if it was a triphage, it splits into two diphages.

Give a simple formula for the average number of diphages present if we begin with a single diphage and irradiate the culture n times with psi-particles.

Solution by K. F. Andersen, University of Alberta, Canada. Let d_n and t_n denote respectively the average number of diphages and triphages present following the nth irradiation with a psi particle. The number of receptors is then given by $r_n = 2d_n + 3t_n$ and we must have

$$\Delta d_n \equiv d_{n+1} - d_n = -(2d_n/r_n) + 2(3t_n/r_n) = (6t_n - 2d_n)/r_n \tag{1}$$

$$\Delta t_n \equiv t_{n+1} - t_n = (2d_n/r_n) - (3t_n/r_n) = (2d_n - 3t_n)/r_n. \tag{2}$$

Thus it follows that

$$\Delta r_n = \Delta(2d_n + 3t_n) = 1$$

and hence $r_n = n + r_0 = n + 2$; and similarly

$$\Delta(d_n - 2t_n) = -6(d_n - 2t_n)/r_n$$

so that

$$d_{n+1} - 2t_{n+1} = \frac{n-4}{n+2}(d_n - 2t_n).$$

In particular, $d_n = 2t_n$ for $n \geqq 5$, so that $d_n = 2(n+2)/7$ in that case, while direct successive evaluation in (1) and (2) shows that $(d_0, d_1, d_2, d_3, d_4) = (1, 0, 2, 1, 9/5)$.

Also solved by 54 other readers and the proposer.

Comment. As noted by a number of solvers, the relation $d_n = 2t_n$ is true for all but a finite number of n if $r_0 \leqq 6$.

An Old Result

E 2637 [1977, 134]. *Proposed by Armond E. Spencer, State University College, Potsdam, N.Y.*

If $a_0, a_1, \ldots, a_{n-1}$ are integers show that

$$\prod_{0 \leqslant i < j \leqslant n-1} \frac{a_i - a_j}{i - j}$$

is also an integer.

Comments. M. S. Klamkin informs us that this is the same as problem 132 in G. Pólya and G. Szegö, *Problems and Theorems in Analysis* II, Springer 1976, p. 134. The solution appears as a special case of Problem 96 on pp. 96, 229:

By row manipulations we have

$$\begin{vmatrix} 1 & \cdots & 1 \\ \binom{x_1}{1} & & \binom{x_n}{1} \\ \vdots & & \vdots \\ \binom{x_1}{n-1} & & \binom{x_n}{n-1} \end{vmatrix} = \left(\prod_{i=1}^{n-1} i^{-(n-i)} \right) \cdot \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & & x_n^{n-1} \end{vmatrix} = \prod_{1 \leqslant i < j \leqslant n} \frac{x_i - x_j}{i - j}.$$

Klamkin, Henry Ricardo and the proposer refer to H. W. Segar, *Messenger of Math.*, 22 (1892), 57–67 where this result is stated. The proofs appear in the same journal by Cayley 22 (1892), 186–190, and by Segar 23 (1893), 31–37. Ricardo has also located this problem in D. K. Faddeev and I. S.

Sominskii, *Problems in Higher Algebra*, Freeman 1965, Problem 269 and in Shklarsky, Chentzov and Yaglom, *The USSR Olympiad Problem Book*, Freeman 1962, Problem 62.

Ronald Evans refers to his joint paper with I. M. Isaacs, Proc. Amer. Math. Soc. 58 (1976), 51–54 where the result is attributed to O. H. Mitchell, Amer. J. Math. 4 (1881), 341–344.

Weyl's formula for the dimensions of simple $SU(n)$-modules shows that the number in the problem is equal to the dimension of such a module and hence is an integer.

Also solved by M. T. Bird, Aage Bondesen (Denmark), Robert Breusch, Peter de Buda, Charles Delzell, Guy Fourt (France), Emden Gansner, Irving Gerst, Douglass Grant (Canada), Alan Hartman (Australia), Sidney Heller, Douglas Hensley, L. Kuipers (Switzerland), J. Lagarias, Jordan Levy, O. P. Lossers (Netherlands), L. E. Mattics, Albert Nijenhuis, David Richman, Louis Thurston, Kåre Villanger (Norway), L. J. Warren, Carroll Webber, Jr., Edward Wong, University of Wyoming Problem Group, and Paul Yiu (Hong Kong).

Leaders of a Maximal Clique

E 2638 [1977, 135]. *Proposed by Robert McNaughton, Rensselaer Polytechnic Institute*

Call a set of positive integers a *clique* if no two of its elements are relatively prime. Call a member of a clique a *leader* if it is not a proper multiple of another member of the clique. Construct a maximal clique with infinitely many leaders. (The set of all cliques is partially ordered by inclusion.)

Solution by Eli L. Isaacson, New York University. Let $p_1=2, p_2=3, p_3=5,\ldots$ be the sequence of primes. Let C be the set of all integers of the form

$$p_1p_3\cdots p_{2n-1}p_{2n} \qquad (n\geqslant 1)$$

or of the form

$$p_2p_4\cdots p_{2n}p_{2n+1} \qquad (n\geqslant 1).$$

It is easy to see that C is a clique. Let M be the set of all positive integral multiples of the elements of C. Then clearly M is also a clique and we claim that it is maximal.

Let $O_n=\{1,3,\ldots,2n-1,2n\}$, $E_n=\{2,4,\ldots,2n,2n+1\}$ $(n\geqslant 1)$. To prove the maximality of M it suffices to show that if I is a *finite* set of positive integers which meets each of the sets O_n and E_n $(n\geqslant 1)$ then I contains one of these sets.

I cannot consist of even integers only because $I\cap O_n\neq\varnothing$ implies $2n\in I$ for each $n\geqslant 1$. Similarly, I cannot consist of odd integers only. Let $2s$ (resp. $2t+1$) be the smallest even (resp. odd) number in I. If $s>1$ then $I\cap O_1\neq\varnothing$ implies $1\in I$ and $I\cap E_n\neq\varnothing$ implies $2n+1\in I$ for $2n+1<2s$. Hence $I\supset O_s$. Now let $s=1$ and $t\geqslant 1$. Then $I\cap O_n\neq\varnothing$ implies $2n\in I$ for $2n<2t+1$. Thus, in this case $I\supset E_t$. In the remaining case $s=1, t=0$ we have $I\supset O_1=\{1,2\}$.

It remains to note that each element of C is a leader in M.

Also solved by S. Baron, David Bienenfeld (Israel), D. M. Bloom, Stephen Bronn, James Davis, Charles Delzell, R. B. Eggleton & A. Hartman (Australia), Richard Enison, Lorraine Foster, William Gorman III, Gustaf Gripenberg (Finland), Jerrold Grossman, David Hammer, A. Hartman (Australia), Ellen Hertz, I. M. Isaacs, Joel Levy, Jordan Levy, L. E. Mattics, J. G. Mauldon, Roy Olson, August Sardinas, Gilbert Traub, Matt Wyneken (Scotland), The University of Wyoming Problem Group, and the proposer.

Two Perpendicular Lines

E 2639 [1977, 135]. *Proposed by G. Tsintsifas, Thessalonika, Greece*

Let ABC be a triangle with $\measuredangle A=40°$, $\measuredangle B=60°$. Let D and E be points lying on the sides AC and AB, respectively, such that $\measuredangle CBD=40°$ and $\measuredangle BCE=70°$. Let F be the point where the lines BD and CE intersect. Show that the line AF is perpendicular to the line BC.

Solution by the University of Maine at Machias Ms 106B class. The claim follows from

$$BA\cos 60^\circ = \frac{1}{2}BA = \frac{1}{2}BC\frac{\sin 80^\circ}{\sin 40^\circ} \qquad \text{(by the law of sines)}$$
$$= BC\cos 40^\circ = BF\cos 40^\circ.$$

Also solved by 72 other readers.

Powers of Two and Binomial Coefficients

E 2640 [1977, 135]. *Proposed by James E. Desmond and William R. Hastings, Pensacola Junior College*

Prove or disprove: The largest power of 2 which divides

$$\binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}}, \qquad (n>1)$$

is 2^{3n}.

Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands. We have

$$(2^n)! = 2^{2^n-1}P_1P_2\cdots P_n \text{ where } P_k = 1\cdot 3\cdot 5\cdots(2^k-1),$$

and so

$$\binom{2^n}{2^{n-1}} = \frac{2P_n}{P_1P_2\cdots P_{n-1}}.$$

It follows that

$$\binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}} = \frac{2(P_{n+1}-P_n^2)}{P_1P_2\cdots P_n}. \tag{1}$$

With $a=2^n$ we have

$$P_{n+1}-P_n^2 = (a^2-1^2)(a^2-3^2)\cdots\left(a^2-(2^n-1)^2\right) - 1^2\cdot 3^2\cdots(2^n-1)^2.$$

Thus $b=(P_n^2-P_{n+1})a^{-2}$ is an integer and we have

$$b \equiv P_n^2\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots+\frac{1}{(2^n-1)^2}\right) \qquad (\operatorname{mod} a^2). \tag{2}$$

Since the numbers $P_n/(2k-1)$ for $k=1,2,\ldots,2^{n-1}$ form a complete system of representatives of odd integers $\operatorname{mod} a$, it follows from (2) that

$$b \equiv 1^2+3^2+\cdots+(2^n-1)^2 = 2^{n-1}\cdot\frac{4^n-1}{3} \qquad (\operatorname{mod} a). \tag{3}$$

It now follows from (1) and (3) (and the definition of b) that the assertion of the problem is true.

Also solved by W. J. Blundon (Canada), Aage Bondesen (Denmark), Robert Breusch, L. Carlitz, Irving Dodes, F. M. Hoppe & I. E. Leonard & A. Meir & A. Rhemtula (Canada), Irving Gerst, S. H. Gould (Taiwan), Sidney Heller, Allan Johnson, Jr., L. E. Mattics, K. Robert (Canada), Harold Shapiro, Edith Sloan, L. van Hamme (Belgium), Roger Weitzenkamp, and the University of Wyoming Problem Group.

Comments. Heiko Harborth remarks that more general results are reviewed in *Reviews in Number Theory* (W. J. LeVeque, ed., AMS, Providence, 1974) pp. 340–341. Bondesen notes that L. Comtet, in his book *Advanced Combinatorics* (Dordrecht 1974), p. 79, attributes to Fjeldstat the fact that 2^{3n} divides the left member of (1).

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before August 31, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6183* [1977, 829] **(correction)**. *Proposed by Albert A. Mullin, Redstone Arsenal, Alabama.*

Let R be a ring with a finite number n of multiplicative idempotents.
(a) If R is commutative, show that n is a power of 2.
(b) If R has a unit, show that n is even but need not be a power of 2.
(c) Is there an R for which n is an odd prime?

6210. *Proposed by Olga Taussky, California Institute of Technology*

A theorem of Minkowski (*Gesammelte Werke* I, p. 212) states: Let A be an integral square matrix which is congruent to the unit matrix I modulo an odd prime number. Then A is either equal to I or it is of infinite order. Give a proof based on the eigenvalues of A.

6211*. *Proposed by Alvin J. Paullay and Sidney Penner, Bronx Community College of CUNY*

Suppose that each square of an $n \times n$ chessboard is colored either black or white. A square, formed by the horizontal and vertical lines of the board, will be called *chromatic* if its four distinct corner squares are all of the same color.

(a) Exhibit a black-white coloring of a 9×9 board in which every such square, as described above (there are 204), is not chromatic.

(b) Find the smallest n, say s, such that, with any such coloring, every $s \times s$ board must contain a chromatic square.

6212*. *Proposed by A. A. Mullin, USA Advanced Technology Center, Huntsville, Alabama*

Prove that $[\pi^n]$ is prime for only finitely many positive integers n.

6213. *Proposed by C. G. Mendez, Metropolitan State College, Denver*

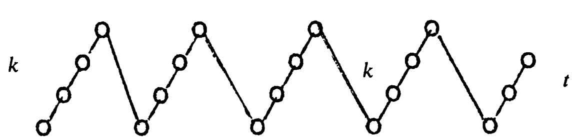
Let G be an open dense subset of the Cantor set C. Is the boundary $\mathrm{Fr}(G)$ of G countable?

6214*. *Proposed by Leonard Carlitz, Duke University*

Let k and t be fixed integers, $k \geqq 2$, $t \geqq 0$ and let $A_k(kn+t)$ denote the number of permutations of $Z_{kn+t} = \{1, 2, 3, \ldots, kn+t\}$ such that

$$a_{kj+1} < a_{kj+2} < \cdots < a_{kj+k},\ a_{kj+k} > a_{kj+k+1} \quad (j = 0, 1, \ldots, n-1)$$
$$a_{kn+1} < a_{kn+2} < \cdots < a_{kn+t}.$$

It has recently been proved as a corollary of a general result (*Combinatorial properties of a special polynomial sequence.* Canadian Math. Bull., 1976) that $A_4(2n+1) = 2^{-n}A_2(2n+1)$. Prove this identity by a direct combinatorial argument.



6215. *Proposed by Ki Hang Kim and Fred Roush, Alabama State University*

Heawood's system for the four-color theorem for a map with n faces amounts to a linear system of rank $n-2$ in a $2n-4$ dimensional vector space over GF(3). Prove that for a random rank n system in a $2n$ dimensional vector space over GF(3), the probability that there is at least one solution vector with no zero component tends to 1 as $n \to \infty$.

SOLUTIONS OF ADVANCED PROBLEMS

Combinatorics in Finite Sets

6060 [1975, 1016]. *Proposed by Daniel Sokolowsky, Antioch College*

Let $N(S)$ denote the number of elements in a finite set S, and S_n denote the set of integers $\{1,2,\ldots,n\}$. Suppose for $k \geqq 3$ we have $2k$ sets $a_i, b_i, i=1,\ldots,k$ and n elements $w_1,\ldots,w_n$ contained in these $2k$ sets so that the following conditions are satisfied:

(i) $a_i \cap b_i = \varnothing$, $N(a_i \cup b_i) = n-1$, $i=1,\ldots,k$.

(ii) For each j, $j=1,\ldots,n$, $(w_j) = a_{p_1} \cap \cdots \cap a_{p_r} \cap b_{q_1} \cap \cdots \cap b_{q_s}$ for appropriate subsets $\{p_1,\ldots,p_r\}$, $\{q_1,\ldots,q_s\}$ of S_k.

Show that the maximum possible value of n is $2^k - 1$.

Partial solution by R. A. Christiansen, Grinnell College. For $k = 3$ I propose to show that $n = 9$ is possible. Let

$$a_1 = \{a,b,c,d,j,k\}, \quad b_1 = \{e,f\}$$

$$a_2 = \{b,g,l,m\}, \quad b_2 = \{d,e,h,i\}$$

$$a_3 = \{a,g,h,n,o\}, \quad b_3 = \{c,i,f\}.$$

Then $a = a_1 \cap a_3$, $b = a_1 \cap a_2$, $c = a_1 \cap b_3$, $d = a_1 \cap b_2$, $e = b_1 \cap b_2$, $f = b_1 \cap b_3$, $g = a_2 \cap a_3$, $h = b_2 \cap a_3$, $i = b_2 \cap b_3$.

Note. The editor is still seeking solutions where the subsets $\{p_i\}$ and $\{q_j\}$ are proper subsets of S_k.

A Fourier and Probability Integral

6111 [1976, 661]. *Proposed by Barthel W. Huff, Queen's University, Kingston, Ontario*

The following problem was encountered when studying certain stochastic processes. Evaluate

$$M(\lambda) = \lim_{n\to\infty}\left(-2^n\int_{-\infty}^{\infty}\left[(2\pi^3\lambda)^{-1/2}\int_0^{|x|}\exp\{-y^2/2\lambda\}\,dy\right]\cdot \left[\int_{-\infty}^{\infty}\exp\{-|u|^\alpha/2^n\}e^{-iux}du\right]dx\right),$$

where $0<\alpha<1$ and $\lambda>0$.

Solution by R. W. K. Odoni, Exeter, England. $M(\lambda) = -4(2\pi^3\lambda)^{-1/2}$ times the limit as $\varepsilon \to 0^+$ of $J(\varepsilon)$, where

$$J(\varepsilon) = \varepsilon^{-1}\int_0^\infty\left\{\int_0^x e^{-y^2/2\lambda}dy\right\}\left\{\int_0^\infty e^{-\varepsilon u^\alpha}\cos ux\,du\right\}dx.$$

We write

$$f(x) = \int_0^x e^{-y^2/2\lambda}dy, \; g(x) = \int_0^\infty e^{-\varepsilon u^\alpha}\frac{\sin xu\,du}{u}.$$

Then

$$J(\varepsilon) = \varepsilon^{-1}\int_0^\infty f(x)(dg(x)/dx)dx.$$

Integrating by parts we find

$$\varepsilon J(\varepsilon)=\sqrt{2\pi\lambda}\,/2\lim_{x\to\infty} g(x)-\int_0^\infty g(x)e^{-x^2/2\lambda}dx.$$

However, as $x\to\infty$, $g(x)\to\pi/2$, since, for each fixed $\varepsilon>0$, $e^{-\varepsilon u^\alpha}$ is a monotone function of u. (See H. S. Carslaw, *Introduction to the Theory of Fourier's Series and Integrals*, p. 219.) We now observe that the last equation may be rewritten as

$$J(\varepsilon)=\int_0^\infty e^{-x^2/2\lambda}\left\{\int_0^\infty \frac{1-e^{-\varepsilon u^\alpha}}{\varepsilon}\,\frac{\sin ux\,du}{u}\right\}dx.$$

Letting $\varepsilon\to0^+$ and interchanging limits (which may be justified by a uniform convergence argument) we find

$$\begin{aligned}J(\varepsilon)&\to\int_0^\infty e^{-x^2/2\lambda}\left\{\int_0^\infty u^{\alpha-1}\sin ux\,du\right\}dx\\&=\left\{\int_0^\infty e^{-x^2/2\lambda}x^{-\alpha}dx\right\}\left\{\int_0^\infty t^{\alpha-1}\sin t\,dt\right\}\\&=\left\{\frac{1}{2}(2\lambda)^{(1-\alpha)/2}\Gamma\left(\frac{1-\alpha}{2}\right)\right\}\left\{\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\right\}\end{aligned}$$

(see Carslaw, p. 214), and so

$$M(\lambda)=-2\pi^{-3/2}(2\lambda)^{-\alpha/2}\Gamma(\alpha)\Gamma\left(\frac{1-\alpha}{2}\right)\sin\frac{\pi\alpha}{2}.$$

Also solved by Paul Chauveheid (Belgium), M. L. Glasser (Canada), M. A. Pinsky, and the proposer.

A Series of Iterates

6112 [1976, 661]. *Proposed by Jan Mycielski, University of Colorado*

Let $f(x)$ be a differentiable function such that $f(0)=0$, $0<f(x)<x$ for $x>0$, and $f'(0)=1$. We put $f^0(x)=x$ and $f^{n+1}(x)=f(f^n(x))$ for $n=0, 1,\cdots$. Find conditions under which the series $\Sigma_{n=0}^\infty f^n(1)$ converges (diverges).

Solution by Bruce Reznick, Duke University. Write $a_n=f^n(1)$, so that $a_{n+1}=f(a_n)$ and we consider $\Sigma_{n=0}^\infty a_n$. As $0<a_{n+1}<a_n$, $\lim a_n$ exists, and $\lim a_n=\lim f(a_n)=f(\lim a_n)$, so $\lim a_n=0$. We will write $b_n=a_n^{-1}$. $\{b_n\}$ is a sequence increasing strictly to infinity. We define $g(x)=x^{-2}(x-f(x))$ for $x>0$.

Case 1. Suppose $|g(x)|<M$ for all $x>0$; then Σa_n diverges. For then, $0<x-f(x)<Mx^2$, so $a_n-a_{n+1}<Ma_n^2$, $a_{n+1}>a_n-Ma_n^2=a_n(1-Ma_n)$. For n sufficiently large, $a_n<1/2M$ and $1/(1-\varepsilon)<1+2\varepsilon$ for $\varepsilon<1/2$. Hence, by taking the reciprocal, $b_{n+1}<b_n(1+2Ma_n)=b_n+2M$. Therefore $b_n<2Mn+C$ and $a_n>2Mn+C)^{-1}$, so Σa_n diverges.

Case 2. Suppose $g(x)>Ax^{-\varepsilon}$ as $x\to0$; we can take $\varepsilon\leqslant\frac{1}{2}$. Then Σa_n converges. For then, $Ax^{2-\varepsilon}<x-f(x)$, so $Aa_n^{2-\varepsilon}<a_n-a_{n+1}$, $a_{n+1}<a_n(1-Aa_n^{1-\varepsilon})$, $b_{n+1}>b_n(1-Aa_n^{1-\varepsilon})^{-1}$. For n sufficiently large we have $Aa_n^{1-\varepsilon}<1$, so $b_{n+1}>b_n(1+Aa_n^{1-\varepsilon})=b_n+Ab_n^\varepsilon$. Now write $b_n=r_nn^{1/(1-\varepsilon)}$; for each n, $r_n>0$. For n sufficiently large, $(1+1/n)^{1/(1-\varepsilon)}<1+2/n(1-\varepsilon)$. Hence, letting $k=1/(1-\varepsilon)$, we have $r_{n+1}(n+1)^k>r_nn^k+Ar_n^\varepsilon n^{k\varepsilon}=n^k(r^n+Ar_n^\varepsilon n^{-1})$.

Thus

$$r_{n+1}\left(1+\frac{2}{1-\varepsilon}\cdot\frac{1}{n}\right)>r_n\left(1+\frac{1}{n}\right)^k>r_n+Ar_n^\varepsilon n^{-1}.$$

So

$$r_{n+1}>r_n\left(1+Ar_n^{\varepsilon-1}n^{-1}\right)\cdot\left(1+\frac{2}{1-\varepsilon}n^{-1}\right)^{-1}.$$

Thus, if $Ar_n^{\varepsilon-1}>2/(1-\varepsilon)n$, i.e., $r_n<(\frac{1}{2}(1-\varepsilon)A_n)^{1/(1-\varepsilon)}$, then $r_{n+1}>r_n$. On the other hand, for n sufficiently large,

$$r_{n+1}>r_n\left(1+(Ar_n^{\varepsilon-1}n^{-1})\right)(1-2/(1-\varepsilon)n)>r_n\left(1+(Ar_n^{\varepsilon-1}-2/(1-\varepsilon))\frac{1}{n}\right)$$

$$>r_n+\frac{1}{n}A-\frac{2}{1-\varepsilon}\cdot\frac{r_n}{n}.$$

So, for n sufficiently large, if $r_n>(\frac{1}{2}(1-\varepsilon)A_n)^{1/(1-\varepsilon)}$, then $r_{n+1}>\frac{1}{2}r_n$. We see now that the $r_n>c>0$, hence $a_n<c\cdot n^{-1/(1-\varepsilon)}$, so Σa_n converges.

Also solved by G. Laugwitz, R. W. K. Odoni (England), Robert Quackenbush, and the proposer.

Note. Odoni gives the following convergence condition: $f''(x)$ exists for $0<x<\delta$ and for some $k>1$, $|\log x|^k=o(f''(x))$ as $n\to 0^+$.

A Class of Stieltjes–Riemann Integrable Functions

6113 [1976, 661]. *Proposed by Claudia Simionescu, University of Brasov, Romania*

Let X be a compact metric space, F a real finitely additive set function not of bounded variation, and let T_F be the set of Riemann-Stieltjes integrable functions. Then T_F is of first category in $C(X)$. (Simionescu, *Remarque sur l'intégrabilité...*, Communicările Acad. R. P. R. XIII (1973, #12). Can this result be improved to: T_F is nowhere dense?

Solution by the proposer. The suggested improvement is not possible. Consider, for instance

$$F(x)=\begin{cases} x\sin 1/x, & x\neq 0,\\ 0, & x=0.\end{cases}$$

This is a continuous function but is not of bounded variation.

It can be proved that every polynomial in x is Stieltjes-Riemann integrable with respect to $F(x)$. By the Weierstrass theorem T_F is dense on $C([0,1])$.

The Covariance Distribution

6114 [1976, 748]. *Proposed by R. M. Norton, College of Charleston*

Let $Z_1=XY$, where X and Y are independent standard normal random variables, and Z_1 and Z_2 be independent and identically distributed. Derive the density function $f(x)$ of Z_1+Z_2.

Solution by A. J. Bosch, Technological University, Eindhoven, the Netherlands. $Z_1+Z_2=X_1Y_1+X_2Y_2$ is the sample covariance (with 2 degrees of freedom) of a bivariate standard normal population. Hence the characteristic function is known:

$$\phi_{Z_1+Z_2}(t)=(1+t^2)^{-1}$$

(cf. *On the Sample Covariance from a Bivariate Normal Distribution*, Annals of the Institute of Statistical Mathematics, vol. 19 (1967 b) page 357 with $\rho=0$, $|\Sigma|=1$, $n=2$.) So $f(x)=\frac{1}{2}e^{-|x|}$, $-\infty<x<\infty$.

REMARK. Z_1+Z_2 is the off-diagonal element of a two-dimensional Wishart matrix with parameters $\nu=2$ and $\Sigma=\mathrm{I}$.

We may prove a generalization of the above. Let $Z=\Sigma_{i=1}^{2m}X_iY_i$ all independent. Then $f(x)=\Sigma_{j=0}^{m-1}c_j|x|^je^{-|x|}$, where

$$c_j\doteq\binom{m-1}{j}\frac{\Gamma(2m-j-1)}{\Gamma^2(m)2^{2m-j-1}},\qquad \text{(cf. above reference.)}$$

Also solved by Barry Arnold, Theodore Bolis, L. E. Clarke (England), Michael Driscoll, Bennett Eisenberg, Peter Enis & M. M. Desu, Ann Goodsell, G. R. Grimmett (England), Allan Gutjahr, Norman Johnson, Franklin Kemp, O. P. Lossers (Netherlands), Edward Rockover, G. S. Rogers, R. J. Serfling, Wolfe Snow, Lajos Takács, Anthony Zappetella, and the proposer.

Note. Johnson refers us to Exercise 11.15 of M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, vol. 1, for the methods of obtaining the generalization in the above solution.

n-Dimensional Distributions with Given Marginals

6115 [1976, 748]. *Proposed by L. Franklin Kemp, Tulsa, Oklahoma*

Let $F_1(x_1)$, $F_2(x_2),\dots,F_n(x_n)$ be n probability distribution functions (d.f's). If $H(x_1,x_2,\dots,x_n)$ is any n-dimensional d.f. with marginals $F_i(x_i)$, then the solution to Problem 5894 [1974, 413] showed that $\min_i F_i(x_i)$ is an n-dimensional d.f. such that $H(x_1,x_2,\dots,x_n)\leqslant \min_i F_i(x_i)$. What is the n-dimensional d.f. that bounds any $H(x_1,x_2,\dots,x_n)$ from below?

Solution by Stamatis Cambanis, University of North Carolina, Chapel Hill. The answer to this problem (as well as that of Problem 5894) is contained in G. Dall'Aglio, *Fréchet classes and compatibility of distribution functions*, Symposia Mathematica, Vol. 9(1972), Academic Press. Specifically, Dall'Aglio shows that an n-dimensional distribution function $H(x_1,\dots,x_n)$ has marginals $F_1(x_1),\dots,F_n(x_n)$ if and only if

$$F'(x_1,\dots,x_n)\leqslant H(x_1,\dots,x_n)\leqslant F''(x_1,\dots,x_n)$$

for all $x_1,\dots,x_n$, where

$$F''(x_1,\dots,x_n)=\min\{F_1(x_1),\dots,F_n(x_n)\}$$

is a distribution function with marginals $F_1,\dots,F_n$, while

$$F'(x_1,\dots,x_n)=\max\{F_1(x_1)+\cdots+F_n(x_n)-n+1,0\}$$

for $n>2$ *is not in general a distribution function.*

Moreover, for any fixed $x_1,\dots,x_n$,

$$H(x_1,\dots,x_n)=F_1(x_1)+\cdots+F_n(x_n)-n+1$$

if and only if for every $1\leqslant i<j\leqslant n$

$$H_{ij}(x_i,x_j)=F_i(x_i)+F_j(x_j)-1,$$

where H_{ij} is the (i,j) marginal of H. The same result was also proved at about the same time by A. Sklar in *Random variables, joint distribution functions, and copulas*, Kybernetika (Prague), Vol. 9 (1973), pp. 449–460. Sklar states the result in terms of "n-dimensional copulas" rather than n-dimensional distributions.

Also solved by Theodore Bolis, O. P. Lossers (Netherlands), Joshua Seeger, Lajos Takács, Edward Wolff, and the proposer.

MISCELLANEA

9. The author follows the widely spread custom of thinking that replacing $y=f(x)$ by $f\colon N\to P$ one obtains a new theorem.

V. I, Arnol'd

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REVIEWS

Edited by J. Arthur Seebach, Jr. and Lynn A. Steen
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
Collaborating Editor for Films: Seymour Schuster, Carleton College

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Computer Power and Human Reason: *From Judgment to Calculation.* By Joseph Weizenbaum. W. H. Freeman, San Francisco, California, 1976. xii+300 pp. \$9.95 cloth, \$5.95 paper. (Telegraphic Review, June-July 1976.)

This is an important book, must reading for those who, like this reviewer, are seriously concerned about the misapplication of technology to the problems of contemporary society. It is a pugnacious book, written by a professor of computer science at MIT but remarkably urging us to recognize the limitations of computational technology. And this is a timely book, coming as it does at a point in history when Banquo's warning

> ...oftentimes, to win us to our harm,
> The instruments of darkness tell us truths,
> Win us with honest trifles, to betray's
> In deepest consequence...

is most appropriate.

Here the instruments of darkness are the mechanizers of the non-mechanizable: some psychologists, in particular many members of the artificial intelligence (AI) community, some linguists, and compulsive programmers. Not too prepossessing a group of adversaries perhaps, but when you replace Macbeth and Banquo as antecedents of the quotation's "us" with the general public, the issue is worth joining.

Weizenbaum's central metaphor lies in one of this own creative contributions to "natural language" interactive computers. He developed a widely used program called DOCTOR. (In the Buffalo area alone several dozen high schools have terminal access to it.) The program parodies—the author's word—a non-directive psychologist, as the following excerpt from the middle of a longer "conversation" suggests (computer in caps, terminal input by a person):

I AM SORRY TO HEAR YOU ARE DEPRESSED
It's true. I am unhappy.
DO YOU THINK COMING HERE WILL HELP YOU NOT TO BE UNHAPPY
I need some help. That much seems certain.
WHAT WOULD IT MEAN TO YOU IF YOU GOT SOME HELP
Perhaps I could learn to get along with my mother.
TELL ME MORE ABOUT YOUR FAMILY
My mother takes care of me.
WHO ELSE.... (pp. 3–4)

To his surprise and chagrin Weizenbaum found that: (1) psychiatrists proposed the program as an automatic form of psycho-therapy, (2) users of the program anthropomorphized and became emotionally involved with the computer, and (3) many believed that the computer, so programmed, actually understood natural language.

To these and other widespread misunderstandings about computers, Weizenbaum answers directly, but the central thrust of the book is more profound. "The contestants on the one side are those who, briefly stated," says the author, "believe that computers can, should, and will do everything, and on the other side those who, like myself, believe there are limits to what computers ought to be put to do.... We can count, but we are rapidly forgetting how to say what is worth counting and why." (pp. 11, 16)

Of particular interest to this reader are the reasoned responses to the AI crowd. (A recent AI proposal is to construct a computer MOLIM—a model of a learner in mathematics—against which mathematics curricular materials would be tested.) Weizenbaum argues persuasively that there is no computer brain at the end of the rainbow; to the Heisenberg uncertainty principle and the Gödel incompleteness theorem, we may add this Weizenbaum computer limitation proposition.

Some flavor of the book (and some initial ammunition for those who agree with its purpose) may be gained from some quotations:

> [T]he computer, used as a "number-cruncher" (that is, merely as a fast numerical calculator), and it is so used especially in the behavioral sciences, has often... put muscles on analytic techniques that are more powerful than the ideas those techniques enable one to explore. (p. 159)

> [AI problem solving] heuristics express no interesting general principles. Furthermore, such principles cannot be discovered merely by expanding the range of a system in a way that enables it to get more knowledge of the world. Even the most clever clock builder of the seventeenth century would never have discovered Newton's laws by building ever fancier and more intricate clocks. (p. 197)

> The relationship between understanding and writing... remains as problematical for computer programming as it has always been for writing in any other form. (p. 110)

> "Can anything we may wish to do be described in terms of an effective procedure?" The answer to that question is "No." (p. 65)

> When Skinner contrasts science with common sense and claims the first to be much superior, he means his "behavioral science" and he means the "common" in "common sense" pejoratively. He does not mean a common sense informed by a shared cultural perspective, or a common sense that, for no "reason" at all, balks at the idea that freedom and dignity are absurd and outmoded concepts. (p. 255)

> [T]he computers in the Pentagon were "fixed" to transform the genuine strike reports coming in from the field into the false reports to which government leaders were given access. George Orwell's Ministry of Information had become mechanized. (pp. 238–239)

> [T]heory based programs enjoy the enormously important advantage that, when they misbehave, their human monitors can detect that their performance does not correspond to the dictates of their theory and can diagnose the reason for the failure from the theory. But most existing programs, and especially the largest and most important ones, are not theory based in this way.... Construction is based on rules of thumb, strategems that appear to "work" under most foreseen circumstances, and on other ad-hoc mechanisms.... [I]f the program has outrun the understanding of the agents who created it, what can it mean for it to "grow in effectiveness," or, for that matter, to "get worse"? (pp. 232, 235)

The electronic computer is providing us with power scarcely dreamed of three decades ago. In these few years it has already changed our daily lives in many ways and steered our heroes to the moon. It will do much much more in the years ahead. But there are also things that the computer will not do now, and indeed will never do, things on which we should not waste our valuable and limited intellectual resources. Professor Weizenbaum provides a valuable service in this book, fleshing out arguments of earlier critics like Hubert Dreyfus and Noam Chomsky and adding much that only he, an insider and an expert in the field, could provide. His book should be read, his message heeded.

GERALD R. RISING, State University of New York at Buffalo

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, L, *Handbook of Applied Mathematics, Fourth Edition.* Ed: Edward E. Grazda. Krieger, 1977, v + 1119 pp, $20.50. [ISBN: 0-88275-615-X] Formulas and tables from shop mathematics: brickwork, plumbing, electronics, etc. Although it contains little that is more mathematically advanced than algebra, this book should be useful for any home owner, entrepreneur, energy saver, or renaissance man who wants to know something of the basic tables and formulae used by architects, bankers, and engineers. Reprint of the 1966 original Van Nostrand edition. JAS

GENERAL, S, P, *Introduction à la Théorie des Sous-Ensembles Flous a l'Usage des Ingénieurs, Tome IV: Compléments et Nouvelles Applications.* A. Kaufmann. Masson (US Distr: SMPF, 14 E. 60th St, NY 10022), 1977, x + 334 pp. [ISBN: 2-225-44033-6] "Probabilization" of fuzzy subsets, fuzzy topology, fuzzy relations, and fuzzy hypergraphs. Applications to problems of human behaviour. FLW

GENERAL, P, *Transactions of the Moscow Mathematical Society for the Year 1975, V. 32.* AMS, 1977, iii + 258 pp, $37.20. [ISBN: 0-8218-1632-2]

HISTORY, P, L**, *A History of Numerical Analysis from the 16th Through the 19th Century.* Herman H. Goldstine. Springer-Verlag, 1977, xiv + 348 pp, $24.80. [ISBN: 0-387-90277-5; 3-540-90277-5] A faithful modern exposition of major documents by the masters (especially Napier and Briggs, Newton, Euler, Lagrange, Laplace, Gauss, Jacobi, Cauchy) on logarithms, interpolation, summation, least squares, quadrature and related themes. Not a comprehensive "search-for-origins" history, but a monument to the author's feeling that the great masters were those responsible for the most significant accomplishments. An invaluable sourcebook in the classics of numerical analysis. LAS

FOUNDATIONS, T(15-17), S, L, *A Course in Mathematical Logic.* J.L. Bell, M. Machover. North-Holland, 1977, xviii + 599 pp, $20.50. [ISBN: 0-7204-28440] Thorough introduction to a variety of topics in mathematical logic, based on a beginning graduate course at the University of London. Strives to present a "balanced diet" and to reveal the interplay of "structural" (i.e., set theoretical) ideas and "constructive" methods. Includes significant, fundamental results in boolean algebras, model theory, recursion.theory, undecidability and incompleteness, intuitionistic first order logic, axiomatic set theory, nonstandard analysis. Could serve as a reference book. Includes exercises. GM

FOUNDATIONS, S(13), L, *Precision, Language and Logic.* F.H. George. Pergamon Pr, 1977, xi + 216 pp, $17.50. [ISBN: 0-08-019650-0] In the author's words: "This book is intended to provide a text to guide people in the main ingredients of clear thinking and logical discussion." Includes sections on propositional and predicate calculus, on logic, probability and philosophy, and on automata theory and neural nets. Seems to ramble considerably and reads like a first draft (e.g., "Ostensive rules are at the roots of communicable signs"). GM

COMBINATORICS, S(17-18), P, *Combinatorial Set Theory.* Neil H. Williams. Stud. in Logic and Found. of Math., V. 91. North-Holland, 1977, xi + 208 pp, $26.75. [ISBN: 0-7204-0722-2] Combinatorics on infinite sets, including some material meaningful only in this domain. Many references are made to the generalized continuum hypothesis. Virtually all results are proved in detail. MU

NUMBER THEORY, T*(18: 1), S, P, L?, *Algebraic Number Theory.* Robert L. Long. Pure and Appl. Math., V. 41. Dekker, 1977, ix + 192 pp, $18.50. [ISBN: 0-8247-6540-0] This second course in algebraic number theory treats Dedekind domains, localization, completions, p-adic fields, ramification; cyclotomic and abelian extensions, zeta functions and L-series, class numbers, and the action of a group ring in a Galois extension. This text is well thought out and nicely written. Includes good problems and a bibliography. CEC

ALGEBRA, P, *The Collected Papers of Alfred Young 1873-1940.* U of Toronto Pr, 1977, xxvii + 684 pp, $10. [ISBN: 0-8020-2267-7] Reprints, with biographical essay, of all but Young's book *Algebra of Invariants.* Young's pioneering work inspired the 1975 Oberwolfach symposium "Combinatorics: Young Tableaux and Combinatorics in Symmetric Group Representation." JAS

ALGEBRA, S(18), P, *Lecture Notes in Mathematics-592: Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen.* Detlef Voigt. Springer-Verlag, 1977, 413 pp, $16.30 (P). [ISBN: 0-387-08251-4; 3-540-08251-4]

ALGEBRA, P, *Moufang Loops of Small Order.* Orin Chein. Memoirs No. 197. AMS, 1978, iv + 131 pp, $8.40 (P). [ISBN: 0-8218-2197-0] All nonassociative Moufang loops of order ≤ 63 are determined, and their properties are investigated. SS

ALGEBRA, T(18: 1), S, P, *Algèbre commutative. Languages géométrique et algébrique.* Jean Pierre Lafon. Hermann (US Distr: SMPF, 14 E. 60th St, NY 10022), 1977, 454 pp, 68F (P). [ISBN: 2-7056-5849-1] Rings of fractions, localization and globalization, noetherian and artinian rings, prime decomposition, algebraic integers, field theory, projective and affine geometric algebra, a-adic topologies and completions, and derivations and differentials are treated. Includes good lists of problems and bibliography. A very formal style of writing. CEC

ALGEBRA, P. *Lecture Notes in Mathematics-586: Séminaire d'Algèbre Paul Dubreil, Paris 1975-1976 (29ème Année).* Ed: M.P. Malliavin. Springer-Verlag, 1977, vi + 188 pp, $8 (P). [ISBN: 0-387-08243-3; 3-540-08243-3]

ALGEBRA, P. *Contributions to Universal Algebra.* Ed: B. Csákány, J. Schmidt. North-Holland, 1977, 607 pp, $81.75. [ISBN: 0-7204-0725-7; 963-8021-01-2] Proceedings (expanded and edited) from the colloquium held at József Attila University in Szeged, Hungary, August 26-29, 1975. JAS

ALGEBRA, P. *Lecture Notes in Mathematics-616: Abelian Group Theory.* Ed: D. Arnold, R. Hunter, E. Walker. Springer-Verlag, 1977, ix + 423 pp, $17.70 (P). [ISBN: 0-387-08447-9; 3-540-08447-9] Proceedings of the second New Mexico State University conference held at Las Cruces, New Mexico, December 9-12, 1976. JAS

ALGEBRA, S(18), P. *Numbers of Generators of Ideals in Local Rings.* Judith D. Sally. Lect. Notes in Pure and Appl. Math., V. 35. Dekker, 1978, viii + 93 pp, $12.50 (P). [ISBN: 0-8247-6645-8] An algebraic (rather than geometric) treatment of recent developments in the subject. Topics include local rings of small dimension, local complete intersections, powers of ideals, Hilbert functions, and connections between ideals I in a local ring R and R/I. The author makes special efforts to provide necessary background material, hence these notes are accessible to second-year graduate students. SG

CALCULUS, T(13: 1). *Calculus for Business and Life.* Howard B. Beckwith. Wadsworth, 1978, x + 341 pp, $13.95. [ISBN: 0-534-00551-9] Applications are used to motivate an intuitive and uncomplicated presentation of elementary concepts needed for business and social sciences. Includes sections on partial derivatives (early in the text) and separable differential equations. JNC

DIFFERENTIAL EQUATIONS, T*(14-15: 1), L. *Ordinary Differential Equations with Modern Applications.* N. Finizio, G. Ladas. Wadsworth, 1978, xv + 380 pp, $15.95. [ISBN: 0-534-00552-7] Standard topics of a first course in differential equations, embellished with a huge supply of applications: every equation type and solution method considered is illustrated by models of modern problems in the physical, biological, and social sciences. Nice treatments of second order equations with variable coefficients, boundary value problems, nonlinear equations, and numerical solutions. Assumes only calculus. TRS

DIFFERENTIAL EQUATIONS, P. *Geometrie der Berührungstransformationen, Second Corrected Edition.* Sophus Lie, Georg Scheffers. Chelsea, 1977, xi + 693 pp, $29.50. [ISBN: 0-8284-0291-4] A corrected reprint of a volume which Lie published in 1896. JAS

DIFFERENTIAL EQUATIONS, S(17-18), P. *Differentialgleichungen. Lösungsmethoden und Lösungen, I: Gewöhnliche Differentialgleichungen.* E. Kamke. Teubner, Stuttgart, 1977, xxvi + 668 pp. [ISBN: 3-519-02017-3] Reprint of the eighth (1965) edition. JD-B

DIFFERENTIAL EQUATIONS, P. *Applications of Bifurcation Theory.* Ed: Paul H. Rabinowitz. Acad Pr, 1977, ix + 389 pp, $15.50. [ISBN: 0-12-574250-9] The papers presented at an advanced seminar held in Madison, Wisconsin, October 27-29, 1976. JAS

DIFFERENTIAL EQUATIONS, P. *Proceedings of the Conference on Differential Equations and Their Applications.* Adolf Haimovici. Editura Academiei (Romania), 1977, 160 pp, Lei 9 (P). Proceedings of the conference held at Iaşi, Romania, October 24-27, 1976. JAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-594: Singular Perturbations and Boundary Layer Theory.* Ed: C.M. Brauner, B. Gay, J. Mathieu. Springer-Verlag, 1977, vii + 539 pp, $18.90 (P). [ISBN: 0-387-08258-1; 3-540-08258-1] Proceedings of the conference held at the École Centrale de Lyon, December 8-10, 1976. JAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-615: Turbulence Seminar.* Ed: P. Bernard, T. Ratiu. Springer-Verlag, 1977, v + 155 pp, $8.30 (P). [ISBN: 0-387-08445-2; 3-540-08445-2] The main lectures presented at the seminar in Berkeley 1976-77 (except for one by W. Kline of Stanford) together with some relevant shorter papers appended to appropriate main talks. JAS

NUMERICAL ANALYSIS, P. *Numerical Analysis.* Ed: J. Descloux, J. Marti. Int. Ser. Num. Math., V. 37. Birkhäuser, 1977, 248 pp, sFr. 44 (P). [ISBN: 3-7643-0939-3] Proceedings of the colloquium on numerical analysis held at Lausanne, October 11-13, 1976. JAS

NUMERICAL ANALYSIS, P. *Numerische Prozeduren.* Heinz Rutishauser. Int. Ser. Num. Math., V. 33. Birkhäuser, 1977, 127 pp, sFr. 48 (P). [ISBN: 3-7643-0874-5] A presentation of five algorithms with theory, "subset Algol 60" programs, and further technical details based on the work of Heinz Rutishauser, and prepared by three of his colleagues. JAS

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-571: Constructive Theory of Functions of Several Variables.* Ed: W. Schempp, K. Zeller. Springer-Verlag, 1977, vi + 290 pp, $11 (P). [ISBN: 0-387-08069-4; 3-540-08069-4] Proceedings of the conference held at Oberwolfach, April 25 to May 1, 1976. JAS

FUNCTIONAL ANALYSIS, S(18), P. *Integral Operators in Spaces of Summable Functions.* M.A. Krasnoselskii, *et al.* Trans: T. Ando. Noordhoff, 1976, xv + 520 pp, $82.75. [ISBN: 90-286-0294-1]

FUNCTIONAL ANALYSIS, P. *C*-Algebras.* Jacques Dixmier. Math. Lib., V. 15. North-Holland, 1977, xiii + 492 pp, $48.95. [ISBN: 0-7204-0762-1] Translation of the author's *Les C*-Algèbres et leurs Représentations,* 1968. Bibliography updated through 1975. TRS

FUNCTIONAL ANALYSIS, P. *Factorization and Model Theory for Contraction Operators with Unitary Part.* Joseph A. Ball. Memoirs No. 198. AMS, 1978, iii + 68 pp, $7.60 (P). [ISBN: 0-8218-2198-9]

OPTIMIZATION, T(16-17: 1, 2), S, P, L, *The Art and Theory of Dynamic Programming*. Stuart E. Dreyfus, Averill M. Law. Math. in Sci. and Eng., V. 130. Acad Pr, 1977, xv + 284 pp, $18.50. [ISBN: 0-12-221860-4] Considers first the deterministic and then the stochastic aspects of many problems. The student is introduced to the art of problem solving by many carefully constructed exercises whose solutions are given. FLW

OPTIMIZATION, T(14-15: 1), S, *Computer Methods in Operations Research*. Arne Thesen. Acad Pr, 1978, xiii + 268 pp, $19.50. [ISBN: 0-12-686150-1] Presents algorithms, some coded in Fortran, for solving problems in networks, critical paths, linear programming, 0-1 programming, and resource constrained scheduling. Includes some elementary preliminaries and descriptions of simulations. Examples and problems. RWN

ANALYSIS, P, *The Markov Moment Problem and Extremal Problems*. M.G. Kreĭn, A.A. Nudel'man. Trans. Math. Mono., V. 50. AMS, 1977, v + 417 pp, $47.20. [ISBN: 0-8218-4500-4] An up-to-date study of the generalized moment problem in the framework of the geometry of convex bodies and generalizations of the St. Petersburg school's problems on limiting magnitudes of integrals and on least deviating functions. JAS

ANALYSIS, S(16-17), *Métodos Topológicos en Análisis*. William R. Derrick. Sociedad Colombiana de Matemáticas (Bogota), 1977, v + 58 pp, (P). A brief presentation of some fixed point theorems with the usual applications in analysis. JAS

ANALYSIS, T(17-18: 1, 2), L, *Topological Vector Spaces*. Romulus Cristescu. Trans: Mihaela Suliciu. Noordhoff Inter, 1977, ix + 232 pp, Lei 27. [ISBN: 90-286-0116-3] A revised translation of the Romanian *Spatii liniare topologice* published in 1974. A substantial and reasonably standard presentation with an index and bibliography but no problems. JAS

ANALYSIS, T(16-17: 1), S, P, *Integral Equations*. B.L. Moiseiwitsch. Longman, 1977, ix + 161 pp, $9.50 (P). [ISBN: 0-582-44288-5] This text treats equations of the convolution type, equations with singular kernals, Fredholm theory, and Hilbert-Schmidt theory. It assumes some previous exposure to introductory analysis, complex variable and linear algebra. It also includes an introduction to Hilbert space, problems, and a brief bibliography. CEC

ANALYSIS, P, *Theory of Functions and Its Applications*. Ed: L.S. Pontrjagin. Proc. of Steklov Inst. of Math., No. 134. AMS, 1977, v + 458 pp, $50 (P). Proceedings for the year 1975 with an essay in honor of Sergeĭ Mihaĭlovič Nikol'skiĭ whose colleagues, students and disciples produced the research published here. JAS

ALGEBRAIC GEOMETRY, P, *Lecture Notes in Mathematics-589: Séminaire de Géométrie Algébrique du Bois-Marie 1965-66, SGA 5*. A. Grothendieck, *et al*. Springer-Verlag, 1977, xii + 484 pp, $16.30 (P). [ISBN: 0-387-08248-4; 3-540-08248-4] The proceedings of Grothendieck's 1965-66 seminar and the last of the 8 SGA tomes. SG

DIFFERENTIAL GEOMETRY, P, *On the Number of Simply Connected Minimal Surfaces Spanning a Curve*. A.J. Tromba. Memoirs No. 194. AMS, 1977, v + 121 pp, $7.60 (P). [ISBN: 0-8218-2194-6] Global nonlinear analysis techniques are used to study the Plateau problem. Certain incomplete finiteness results for the spanning surfaces are obtained. JAS

GEOMETRY, P, *Lecture Notes in Mathematics-610: Higher Order Contact of Submanifolds of Homogeneous Spaces*. Gary R. Jensen. Springer-Verlag, 1977, xii + 154 pp, $8.30 (P). [ISBN: 0-387-08433-9; 3-540-08433-9] "...An exposition of E. Cartan's theory of higher order invariants of submanifolds of homogeneous spaces treated by the method of moving frames." Five applications of the theory are given: surfaces in R^3, curves in a Grassmann manifold, holomorphic curves (in CP^2 and $CG_{4,2}$) and affine surface theory. SG

GEOMETRY, P, *Lecture Notes in Mathematics-588: Théorie des G-Structures: Le Problème d'Equivalence*. Pierre Molino. Springer-Verlag, 1977, 163 pp, $8 (P). [ISBN: 0-387-08246-8; 3-540-08246-8] Revised notes from a 1974-75 course at the Université de Montpellier which presents a thorough study of the problem of equivalence for G-structures. JAS

TOPOLOGY, P, *Surgery on Codimension 2 Submanifolds*. Michael H. Freedman. Memoirs No. 191. AMS, 1977, iv + 93 pp, $7.20 (P). [ISBN: 0-8218-2191-1] The author gives a partial classification of smooth submanifolds whose inclusions have the same connectivity properties as the inclusion of a complex hypersurface. JAS

PROBABILITY, T(18), P, *Stochastic Processes, A Survey of the Mathematical Theory*. John Lamperti. Appl. Math. Sci., V. 23. Springer-Verlag, 1977, xvii + 266 pp, $9.80 (P). [ISBN: 0-387-90275-9; 3-540-90275-9]

PROBABILITY, P, *Probabilities and Metrics*. R.M. Dudley. Lect. Notes Ser., No. 45. Aarhus U, ii + 123 pp, (P). A series of expository lectures on probabilities on metric spaces wherein one may discuss convergence of probabilities and the like. Contains what is probably the first English proof of the Kantorovich-Rubinštein theorem. JAS

Reviewers Whose Initials Appear Above

Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; George Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Milton Ulmer, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D.C. 20036. Items must be submitted at least five months before publication can take place.

PERSONAL ITEMS

Professor Albert Wilansky, Lehigh University, has been appointed University Distinguished Professor.

Associate Professor Lawrence Conlon, Washington University, St. Louis, has been promoted to Professor.

Professor Robert C. Campbell, Rockhurst College, Kansas City, Missouri, resigned to do actuarial work with Milliman and Robertson, Denver.

Assistant Professor R. Michael Range, SUNY at Albany, has been promoted to Associate Professor.

Associate Professor Robert Main, Oregon College of Education, Monmouth, has been promoted to Professor.

Assistant Professor Andre H. Tchen, New York University, has been appointed Assistant Professor at Carnegie-Mellon University.

Assistant Professor V. K. Srinivasan, University of Texas at El Paso, has been promoted to Associate Professor.

Professor Edward J. Dudewicz, Ohio State University, has been elected a Fellow of the New York Academy of Sciences by its Board of Governors.

Associate Professor Helene Reschovsky, University of Connecticut, has retired.

Dr. Elaine Pavelka, Morton College, Cicero, Illinois, has been elected Midwest Vice President of the American Mathematical Association of Two Year Colleges.

Assistant Professor Joseph Wiest, West Virginia Wesleyan College, has been promoted to Associate Professor.

Associate Professor Subhash Saxena, University of South Carolina, has been promoted to Professor.

Assistant Professor Kenneth V. Turner, Jr., Anderson College, Anderson, Indiana, has been promoted to Associate Professor.

Professor Seymour Ginsburg, University of Southern California, has been appointed the first Fletcher Jones Professor of Computer Science at USC.

Associate Professor Hugh D'Alarcao, Bridgewater State College, Bridgewater, Massachusetts, has been promoted to Professor.

Dr. Harold N. Hauser has been appointed Chairman of the Mathematics-Physics Division of Mt.Hood Community College, Gresham, Oregon.

Assistant Professor Watson L. Chin, Southwest Union College, Keene, Texas, has been promoted to Associate Professor.

Associate Professor Simon A. Stricklen, Jr., Southern Technical Institute, Marietta, Georgia, has been appointed Head of the Mathematics Department.

Assistant Professor Mary J. Koelz, Mississippi State University, has been promoted to Associate Professor.

Professor Paul R. Meyer, Herbert H. Lehman College, New York City, has been appointed Chairman of the Department of Mathematics.

Associate Professor Richard K. Williams, Southern Methodist University, has been promoted to Professor.

Dr. Ioannis A. Koutrouvelis, Postdoctorate Fellow at the University of Georgia, has left to become Assistant Professor at Virginia Commonwealth University.

Associate Professor Laurence D. Hoffmann, Claremont Men's College, has been appointed Chairman of the Department of Mathematics.

Dr. John C. Bailar has been appointed Visiting Professor at the Harvard School of Public Health.

Dr. Ronald Shepler has been appointed Assistant Professor at Ferris State College, Big Rapids, Michigan.

Assistant Professor Glenn C. Fenneman, Wartburg College, has been promoted to Associate Professor.

Assistant Professor James Joiner, University of Missouri-Rolla, has been promoted to Associate Professor.

Assistant Professor Kenneth H. Price, Stephen F. Austin State University, has been promoted to Associate Professor.

Professor Robert C. Walls, University of Arkansas for Medical Sciences, has been appointed as Acting Chairman of the Division of Biometry.

University of Arkansas at Little Rock: Assistant Professors Mary Jane Gates and David Hsu have been promoted to Associate Professors.

Stanford University: Dr. Larry Washington has left to become Assistant Professor at the University of Maryland. Dr. Peter Winker has left to become Assistant Professor at Emory University.

Michigan Technological University, Houghton: Assistant Professor John L. Lowther has been promoted to Associate Professor. Associate Professor Arthur B. Boggs has retired.

Winthrop College, Rock Hills, South Carolina: Associate Professor Edward P. Guettler has been appointed Chairman of the Department of Mathematics. He succeeds Professor Billy G. Hodges, who has returned to full-time teaching in the Department.

San Diego State University: Dr. Nicholas A. Branca, Penn State University, has been appointed Associate Professor. Professor Vincent C. Harris has retired with the title of Professor Emeritus.

Temple Junior College, Temple, Texas: Instructor Kent MacDougall has been appointed Coordinator of Mathematics. Director Ethel Haag has retired with the title of Coordinator of Mathematics Emeritus.

Iowa State University: Assistant Professor Deane E. Arganbright has left to become Associate Professor at Whitworth College. Assistant Professor Stephen J. Willson has been promoted to Associate Professor. Professor Clarence H. Lindahl has retired with the title of Professor Emeritus. Dr. Robert B. Feinberg, Clarkson College, has been appointed Assistant Professor. Dr. Elgin H. Johnston, University of Illinois, has been appointed Instructor. Dr. Harry F. Smith, James Madison University, has been appointed Visiting Assistant Professor.

Kansas State University: Dr. David B. Surowski, Loyola-Marymount University, has been appointed Assistant Professor. Associate Professors Richard J. Greechie and George E. Strecker have been promoted to Professors.

Arizona State University: Assistant Professor Dennis Weis has left to accept a position at U.S. International University. Professor Abraham Sinkov has retired with the title of Professor Emeritus.

University of Illinois-Urbana: Assistant Professor George A. Converse has left to become Associate Professor at Monmouth College, Monmouth, Illinois. Assistant Professor C. Ward Henson III has been promoted to Associate Professor.

University of Hawaii: Assistant Professors Thomas C. Craven and Gerald N. Hile have been promoted to Associate Professors. Associate Professor John T. Baldwin, University of Illinois, has been appointed Visiting Associate Professor.

North Carolina State University: Dr. James M. Ortega, Director of the Institute for Computer Applications in Science and Engineering at NASA's Langley Research Center, has been appointed Professor and Head of the Mathematics Department. Assistant Professor Robert Silder has been promoted to Associate Professor.

University of Vermont: Dr. Theodore R. Hatcher has left to accept a position with the E.G.G. Corporation, Idaho Falls, Idaho. Associate Professor Roger L. Cooke has been promoted to Professor. Donald E. Moser has been appointed Chairman of the Department of Mathematics.

McNeese State University: Dr. Allen Scholnick, CUNY, has been appointed Faculty Research Assistant. Assistant Professor Harlin Brewer has been promoted to Associate Professor.

Naval Postgraduate School, Monterey, California: Professor Carroll O. Wilde was appointed Chairman of the Analysis Panel for the Undergraduate Mathematics Applications Project. Professor Craig Comstock was also appointed to the panel.

University of Maine at Machias: Dr. Jerald Bope, University of Wyoming, has been appointed Assistant Professor. Dr. Howard Eves, Adjunct Professor of Mathematics, has retired.

Eastern New Mexico University: Dr. Howard Campaigne has left to accept a position at the University of Alabama in Birmingham. Associate Professor R. F. Fawcett has been promoted to Professor.

University of Houston: Associate Professor Paul J. Knopp has left to accept another position. Associate Professor Jutta Hansen has been promoted to Professor.

University of Colorado: Assistant Professor Roger K. Alexander has left to become Assistant Professor at Rensselaer Polytech Institute. Professor Stanislaw M. Ulam has retired with the title of Professor Emeritus.

Colorado College: Dr. Frederick C. Tinsley, University of Wisconsin, has been appointed Instructor. Assistant professors John M. Karon and James T. Wood have left to accept other positions. Assistant Professor David W. Roeder has been promoted to Associate Professor.

California State University, Chico: Associate Professors Judith Clark, Frank Burk, and Stephen Bemiller have been promoted to Professors.

Central Missouri State University: Dr. Terry A. Goodman, University of Texas, has been appointed Instructor. Assistant Professor Gerald C. Schrag has been promoted to Associate Professor.

Southern Connecticut State College: Assistant Professors William Berlinghoff and Michael Mack have been promoted to Associate Professors.

Professor Edward G. Begle, Stanford University, died on March 2, 1978, at the age of 63. He was a member of the Association for forty-two years.

Professor Clarence A. Swanson, University of Southern Colorado, died on June 29, 1977, at the age of 59. He was a member of the Association for fifteen years.

Retired Professor Charles Fox, McGill University, died on May 1, 1977, at the age of 80. He was a member of the Association for twenty years.

Professor Emeritus Edgar L. Swanson, South Dakota School of Mines and Technology, died on January 9, 1978, at the age of 68. He was a member of the Association for thirty-eight years.

Dr. Theodore Bennett, Marietta College, Marietta, Ohio, died on March 10, 1976 at the age of 76. He was a member of the Association for fifty years.

Retired Professor James W. Blincoe, Randolph-Macon College, Ashland, Virginia, died on June 11, 1977 at the age of 77. He was a member of the Association for fifty-two years.

Mrs. Edna C. Schnefel, former Chairperson of the Mathematics Department, Ft. Lee High School, Cliffside Park, New Jersey, died on February 14, 1975 at the age of 66. She was a member of the Association for thirty-nine years.

Professor Walter R. Talbot, Morgan State University, died on December 26, 1977.

Professor Emeritus Robert B. Lyon, Arizona State University, died on October 13, 1977.

OPPORTUNITIES ABROAD FOR TEACHERS

Opportunities to attend a summer seminar or to teach abroad will be available under the Fulbright-Hays Act for the 1979-80 school year. Elementary and secondary teachers, college instructors, and assistant professors are eligible to participate in the teacher exchange program. Basic requirements are: U.S. citizenship, a bachelor's degree, three years of teaching experience for one-year positions and two years of experience for seminars. As most of the positions are on an interchange basis, applicants must be employed currently. Application should be made before November 1, 1978. A brochure and application form should be obtained in September by writing to: Teacher Exchange Section, Division of International Education, U.S. Office of Education, Washington, D.C. 20202.

POETRY IN MATHEMATICS?

We need poems for a contemporary anthology. We are seeking poems which are mathematically oriented and relate logical concepts and mathematical symbiology to language and inner experience. Please send contributions to: Primary Press, Box 105, Parker Ford, Pennsylvania 19457, c/o Ernest Robson. Deadline: June 21, 1978.

COMPUTING CONFERENCE

The ninth Conference on Computing in the Undergraduate Curriculum (CCUC/9), June 12th, 13th, and 14th at the University of Denver, Denver, Colorado. All disciplines including Agriculture, Biological Sciences, Business, Chemistry, Computer Science, Earth Sciences, Fine Arts, History, Home Economics, Humanities, Environmental Sciences, Geography, Music, Physics, Social Sciences, Psychology and Statistics will be represented. Fur further information, contact the conference chairman, Professor William S. Dorn, Department of Mathematics, University of Denver, Denver, Colorado 80208.

SLOAN FOUNDATION FELLOWSHIPS

Alfred P. Sloan Foundation Fellowships for Basic Research totaling $1,564,200 have been awarded to 79 outstanding young scientists in 46 colleges and universities in the United States and Canada. The scientists, whose average age is 31, were selected from among hundreds of highly qualified nominees on the basis of their exceptional potential to make creative contributions to scientific knowledge in the early stages of their careers. Their research is expected to advance the frontiers of physics, chemistry, mathematics, and neuroscience. The fellowships run for two years and are in the amount of $9,900 a year.

The Sloan Fellowships for Basic Research were established by the Alfred P. Sloan Foundation in 1955 as a means of stimulating advances in fundamental research by young faculty scientists at a time in their careers when government support is difficult to obtain. The grants, which are administered by the Fellows' institutions, are designed to permit added freedom and flexibility for the most creative young researchers.

Candidates for fellowships are nominated by senior scientists familiar with their talents. No formal research proposal is required, and the Fellow is free to shift the direction of his research at any time. The Foundation requires a brief annual progress report.

During the 1978-79 academic year 174 scientists at 76 institutions will be receiving support through the two-year fellowships. Since the program began, 1,485 scientists at 155 institutions have received fellowships aggregating $29,634,800.

The fellowship funds may be used for support of technical assistance, professional travel, summer support, computer time, support of predoctoral and postdoctoral fellows, release from teaching where this is compatible with the needs of the Fellow's department, equipment and supplies, and other purposes approved by the Fellow's institution.

ANNOUNCEMENT FOR USES OF CLEP EXAMINATION IN MATHEMATICS

The College Board and the Mathematics Association of America have issued a joint statement clarifying the purpose, as well as the uses and possible misuses, of the CLEP General Examination in Mathematics. The statement was developed in response to concerns expressed by some professors of mathematics about what they felt were improper uses of the CLEP General Examinations in Mathematics.

Appropriate uses cited in the statement include granting credit for or exemption from courses in general mathematics for non-science majors where courses with objectives comparable to those of the General Mathematics Examination are given by the institution, and meeting suitable general mathematics area requirements as required by some institutions.

The statement also calls attention to possible misuses of the examinations: Granting credit for courses in algebra, trigonometry or precalculus mathematics (CLEP Subject Examinations in algebra and trigonometry are intended for this purpose); granting mathematics course credit when the institution does not offer a course that matches the examination; and requiring performance levels that are either lower or higher than those required of students in comparable courses.

In addition, the statement underscores the importance of consulting with the mathematics faculty when an institution establishes policies and standards for granting credit by examination.

The Statement was proposed by the College Board/MAA Committee on Mutual Concerns, with representatives of both organizations. It was endorsed by the MAA Board of Governors in August, and accepted by College Board Trustees at their September meeting.

FUNCTION THEORY ON THE UNIT CIRCLE

Principal Lecturer, Professor Donald Sarason, University of California, Berkeley. Sponsored by the Mathematics Department and Research Division of Virginia Polytechnic Institute and State University at Blacksburg, Virginia 24061, June 19-June 23, 1978.

The lectures will present a survey of function theory on the unit circle, with emphasis on recent developments. A large number of topics will be covered so as to give an overall picture of the area. This will preclude going into any single topic in great depth. In particular, it will not be feasible to present many detailed proofs in the lectures. The aim will be to stimulate those who attend the conference to pursue the material in greater depth on their own.

Following is a tentative syllabus for the lectures. It will be assumed that those attending the lectures have a basic knowledge of one-dimensional harmonic analysis, including some familiarity with Hardy spaces. Some background material will be presented in the first three lectures.

1. Conjugate functions (existence of the conjugate function, classical results on boundedness of the conjugation operator)
2. The Hardy space H^1_∞ (including the maximal function characterization)
3. The Hardy space H^∞ (including the interpolation and the corona theorem)
4. Functions of bounded mean oscillation (the John-Nirenberg characterization, Fefferman's characterizations)
5. Functions of vanishing mean oscillation (characterizations, various subspaces, quasicontinuous functions)
6. Douglas algebras (results of Chang and Marshall in the closed algebras between L^∞ and H^∞)
7. Best approximation in H^∞ and $H^\infty + C$ (Hankel operators, interpolation by inner functions, best H^∞ approximation of a continuous function, existence of best approximations in $H^\infty + C$)
8. Weighted L^p boundedness of the conjugation operator (theorems of Helson-Szegö and Hunt-Huckenhoupt-Wheeden)
9. Analytic functions of bounded mean oscillation (recent results of Pommerenke)
10. Toeplitz operators (invertibility, multiplication).

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The Annual Meeting of the Northern California Section was held on February 18, 1978, jointly with the Northern California Section of SIAM, at the College of Notre Dame, Belmont. There were 208 registered participants.

At the business meeting, with Jane Day, Section Chair, presiding, William H. Landis was presented with the first Certificate of Merit of the Northern California Section for his many years of devoted service to the section as Chairman of the High School Mathematics Contest. Stephen T. Tschantz, Andrew Z. Fire, and Silver Lee were honored for their performance on the 1977 Putnam Competition. Gerald Alexanderson, Section Governor, made a slide presentation of the new National office building and Jane Day discussed various section projects. The membership voted to raise section dues to $2.00 and elected the following section officers: Section Chair: William Chinn, City College of San Francisco; Vice Chair: Carroll Wilde, Naval Postgraduate School; Program Chair: Jane Day, College of Notre Dame; and Secretary-Treasurer: Leonard Kosinski, University of Santa Clara.

The following invited papers were presented:

A Short History of Topology; by Hans Samelson, Stanford University.

The Steiner Tree Problem, by R. L. Graham, Bell Laboratories.

The New (?) Math, by Edward Teller, Lawrence Livermore Laboratory.

On Hilbert's Sixteenth Problem: Limit Cycles of Polynomial Differential Equations, by Stephen Smale, University of California, Berkeley.

A special showing of mathematics films followed the presentation of the above papers. A luncheon was held at the college dining hall and featured a talk by Constance Reid on *The Answer to the Question that Everyone Asks.*

Newman Fisher, *Secretary-Treasurer*

MATHEMATICAL ASSOCIATION OF AMERICA

THE SIXTY-FIRST ANNUAL MEETING OF THE ASSOCIATION

The Sixty-First Annual Meeting of the Association was held at the Hyatt Regency Hotel, Atlanta, Georgia, from Friday to Sunday, January 6-8, 1978, in conjunction with meetings of the American Mathematical Society, the Association for Women in Mathematics, the Conference Board of the Mathematical Sciences, and the Mathematicians Action Group. There were registered approximately 3100 persons.

Sessions were held on Friday morning, Saturday morning, and Sunday morning and afternoon. On Sunday morning there were parallel sessions, one in the Regency Ballroom and one in Hanover Hall. All other sessions were held in the Regency Ballroom. Presiding officer at the Retiring Presidential Address by Dr. Henry O. Pollak was MAA President Henry L. Alder; at the lecture by Professor Charles L. Fefferman, Professor Trevor Evans; at the lecture by Professor Ivan Niven, Professor John D. Neff; at the lecture by Professor Paul Erdős, Professor Richard A. Duke.

The Program Committee consisted of Richard A. Duke, *Chairman;* James C. Abbott, Joseph E. Cicero, Roy A. Dobyns, Trevor Evans, H. Eugene Hall, Emilie V. Haynsworth, Jeremy Kilpatrick, Benjamin J. Martin, John D. Neff, Arthur Schlissel.

FIRST SESSION OF THE ASSOCIATION

Chauvenet Symposium.

A pair of lectures by authors of winners of the Chauvenet Prize on the occasion of publication of a two-volume series, *The Chauvenet Papers.* Professor James C. Abbott, U. S. Naval Academy, served as moderator.

First Lecture: Offbeat Integral Geometry, by Professor Lawrence Zalcman, University of Maryland.

Naive questions about determining the properties of a function from an examination of its integrals over a given family of lines, circles, spheres, discs, etc. lead to surprising results involving a number of diverse subject areas, including special functions, number theory, complex analysis, differential geometry, harmonic analysis, partial differential equations, and numerical analysis. Applications of related mathematics range from nuclear engineering to brain surgery. The emphasis here will be on results having maximum "shock value" and on the (numerous) tantalizing open questions.

Second Lecture: Boolean-Valued Models in Set Theory, Analysis, and Quantum Mechanics, by Professor Martin D. Davis, Courant Institute of Mathematical Sciences, New York University.

Panel Discussion: Numerical Analysis in the Classroom

A panel discussion with Professor William S. Dorn, University of Denver, Professor Thomas A. Porsching, University of Pittsburgh, and James S. Vandergraft, University of Maryland. Moderated by Professor Gunter H. Meyer, Georgia Institute of Technology.

Professor Dorn said a brief survey of numerical methods courses in U.S. colleges and universities reveals that, although numerical analysis courses still are predominantly FORTRAN based and require students to write their own computer programs, there is an increasing tendency towards the use of "canned" programs. These courses usually are at a junior-senior level and almost all contain the shopworn list of topics: solution of equations, interpolation, numerical integration, ordinary differential equations, eigenvalues, and least squares. Such things as partial differential equations, linear programming and Fourier analysis appear very infrequently. At best all numerical methods courses appear to be little more than a series of disconnected topics and there are those who claim that the time for a more unified approach has arrived.

Professor Vandergraft noted that undergraduate courses in numerical analysis at a large university must serve a wide variety of students, including mathematics majors, computer science majors, and engineers. This diversity in background and interests complicates the prerequisite requirements, and necessitates certain compromises between "theory" and "practice". Furthermore, there is the quite general problem of how to give students some practical computer experience without overburdening them with long programming assignments. Various aspects of these problems were discussed, with special emphasis on programming considerations.

SECOND SESSION OF THE ASSOCIATION

Panel Discussion: *A Course in Applied Mathematics Based on Problems from Regional Industries.*

A panel discussion with a faculty presentation by Professor Marvin S. Keener, Oklahoma State University, and student presentations by Ms. Jerri Ezell, Cities Service, Mr. Bob Hayes, Oklahoma State University, and Ms. Kathy Stewart, Southwestern Bell Telephone. Moderated by Professor Jeanne L. Agnew, Oklahoma State University.

Mathematics students considering nonacademic employment need experience in modeling a problem, using mathematics learned in a variety of courses, extracting appropriate new mathematics from the literature, communicating with nonacademic persons, and writing about a problem, its solution, and the interpretation of this solution. A course which provides these experiences with a unique flavor of authenticity was designed and taught at Oklahoma State University by Professors Marvin Keener and Jeanne Agnew. The basic idea involves the cooperation of individuals in the nonacademic community. These industrial consultants come to campus and state for the class a mathematical problem with which they have been involved. The students jointly work out a solution to the problem and individually write reports describing their work. Dr. Keener described in detail the preparation of this course, the method of presentation, and the problems and pleasures involved in this effort.

As an example of the type of activity involved in the course, a statement was given of a problem presented to the class by Dr. Darrell Gimlin of the Research Division of Amoco Production Company. Three different methods of solution were devised by the students for this problem and were presented by Kathy Stewart, Jeri Ezell, and Bob Hayes.

Business Meeting of the Association: Announcement of the recipients of the Award for Distinguished Service to Mathematics and the Chauvenet Prize. Presentation of Certificates of Merit.

Retiring Presidential Address: *An Inside Outsider Looks at Mathematics Education,* by Dr. Henry O. Pollak, Bell Telephone Laboratories.

THIRD SESSION OF THE ASSOCIATION

Panel Discussion: *Using the History of Mathematics to Teach Mathematics.* Moderated by Professor Arthur Schlissel, John Jay College of Criminal Justice, City University of New York.

Pedagogical Values of the History of Mathematics, by Professor Morris Kline, New York University.

Historical accounts properly coordinated with conventional topics aid teaching immensely. False proofs given by great mathematicians can precede the modern correct proof. Such examples call for critical thinking as to what is wrong and give students reassurance that they are not stupid when they make mistakes. History provides motivation for many topics. Gauss's resort to huge amounts of calculation to arrive at and verify a conjecture should be the prototype for the introduction of many topics. The historical approach to many topics, e.g., Galois theory, is far more illuminating than the modern one. The older one should certainly be given first. Other values and specific examples were given.

Two Case Histories, by Professor Fredrick V. Pohle, Adelphi University.

Two case histories were reviewed to illustrate their value in teaching the topics in courses. The first referred to the method of variation of parameters (differential equations) and the second referred to "Student's" t-distribution (statistics). In the former, the origins in celestial mechanics were reviewed to show the motivation and application of the method. In the latter, the original paper of "Student" was reviewed to show the importance and the significance of the distribution.

Using Contemporary Sources, by Professor Edward J. Barbeau, University of Toronto.

Many standard results were originally presented in a different context or given another proof to that currently used. Even when the context is obsolete or the proof is erroneous or incomplete, a deeper insight into their significance can be obtained. For this reason, it is worth spending the time to resurrect and present them to a class in their initial form. Examples can be found in the work of Kepler, Rolle, Newton, Euler, Fourier, Cauchy and Stieltjes. Some excerpts, paraphrases and expository articles are already available for classroom use, but more work on producing these is needed.

Panel Discussion: *Remediation.*

A panel discussion by Professors Marvin L. Bittinger, Indiana University-Purdue University at Indianapolis, Ronald M. Davis, Northern Virginia Community College, Leonard Feldman, San Jóse State University, Edward B. Wright, Linn-Benton Community College. Moderated by Professor Joseph E. Cicero, Clayton Junior College.

Professor Bittinger said that mathematics learning centers and other approaches to individualized instruction are having wide spread effect on remedial mathematics teaching, at the college level. Why? Why do some approaches succeed where others fail? Some answers to these questions were discussed.

Professor Feldman said that, in order to help college students learn or review elementary mathematics, it is necessary to recognize that remedial mathematics involves a significant amount of anxiety and avoidance. Then it is possible to offer instruction in a way that leads to a minimum of resistance and a maximum of motivation - if one knows something about the insecurities and needs of the students. This part of the panel presentation discussed: (1) questionnaires* about students' needs and concerns; (2) methods* developed as a result of students' responses. (*Available from Leonard Feldman.)

Professor Davis stressed the faculty skills necessary to conduct a program of remediation. He said that the faculty should be competent at higher levels of mathematics and aware of basic theories of learning such as those described by Piaget and Gagne. One of the most essential skills is to be able to dissect a mathematical concept to its most basic components, he said.

Professor Wright spoke about the conditions that must be considered from the student viewpoint. For example, a remedial class may contain students who have been out of school for a long period, others who have recently graduated from secondary school, and still others who never completed secondary school. This and other differences--such as age, interests, and experience--dictate that one may not use a group-based mode of instruction in most remedial classes.

The Serre Problem Concerning Polynomials over a Field, by Professor Barbara L. Osofsky, Rutgers University.

In 1956, J. P. Serre raised the following question: "Are all projective modules over the ring $K[x_1,\ldots,x_n]$ of polynomials in n variables over a field free?" That is the homological algebra statement of the question. Serre was concerned with its geometric translation: "Are all vector bundles over affine n-space trivial?" There is a linear algebra equivalent of the question: "Given k polynomials $\{f_i(x_1,\ldots,x_n) \mid 1 \le i \le k\}$ such that for some polynomials $\{r_i \mid 1 \le i \le k\}$, $f_1r_1 + f_2r_2 + f_kr_k=1$, does there exist a k x k matrix over $K[x_1,\ldots,x_n]$ with first row the f_i's and determinant 1?" The question remained open for almost 20 years before relatively elementary methods yielded two different proofs and the answer is "yes".

Some Old and Some New Ideas in Partial Differential Equations, by Professor Charles L. Fefferman, Princeton University.

The talk gave an introduction to some methods used to study differential equations, which are motivated by Fourier analysis. The methods include pseudodifferential and Fourier integral operators and the wave-front set, as well as the wave-packet transform.

Combinatorics Really Does Count, by Professor Ivan Niven, University of Oregon.

Many results in elementary combinatorics are important for the first two years of college mathematics. Counting arguments arise in almost all courses, as shown for example by the repeated occurrences of binomial coefficients and factorials. The purpose of the talk was to identify, with several illustrations, the rich variety of ideas and methods that are available.

FOURTH SESSION OF THE ASSOCIATION

Panel Discussion: *Credit and Placement by Examination.*

A panel discussion with Professor Beverly L. Brechner, University of Florida and University of Michigan, Professor Betty J. Hinman, University of Houston, Downtown Campus, and Professor Alfred L. Putnam, University of Chicago. Moderated by Professor Donald L. Kreider, Dartmouth College.

The four panelists discussed various aspects of the practice of granting college credits through the taking of standardized examinations. The idea of credit-by-examination grew out of the need to recognize knowledge and achievement acquired through means other than the traditional college course.

The first panelist discussed the kind of problems that arise when credit-by-examination policies are mandated at the state level without regard to the purpose and proper use of the examination instruments. The specific example concerns a State of Florida policy regarding the CLEP General Examination in Mathematics.

The second panelist traced the general background and philosophy of the credit-by-examination movement, followed by a discussion by the remaining two panelists of the placement examination program recently introduced by the Mathematical Association of America and of the policies and programs of the College Board in the area of placement and credit-by-examination. The MAA's placement examinations in algebra and in trigonometry/elementary functions have been introduced on a national scale through a subscription program. And the Mathematical Association and the College Board have worked together through a joint committee to formulate a statement of principles for the appropriate and proper use of the CLEP General Examination in Mathematics, or comparable standardized tests, in granting credit of course exemption in mathematics.

Panel Discussion: *Interim Report of the National Research Council Committee on Applied Mathematics Training.*

A panel discussion with Professor Peter J. Hilton, Battelle Memorial Institute and Case Western Reserve University, Professor George D. Mostow, Yale University, Professor Stephen Smale, University of California, Berkeley, and Professor Jean E. Taylor, Rutgers University, Douglass College. Professor Hilton also served as moderator.

Professor Mostow remarked on the formation and charge to the Committee, Professor Hilton reviewed the Committee's activities, Professor Smale commented on a draft proposal for a mathematics major, and Professor Taylor addressed non-curricular programs.

Combinatorial Problems in Elementary Geometry, by Professor Paul Erdös, Hungarian Academy of Sciences.

The author discussed various geometrical problems all of which are of a combinatorial nature. Denote by $f(n)$ the largest integer so that if $x_1,\ldots,x_n$ are n distinct points in the plane, then there are at most $f(n)$ pairs x_i, x_j whose distance is one. It is surprisingly hard to get good bounds for $f(n)$. Thus, $f(n) = 0(n^{1.5})$ is fairly hard to prove. Probably $f(n) = 0(n^{1+\varepsilon})$ for every $\varepsilon > 0$. Many other related problems were discussed.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in the Phoenix Room of the Hyatt Regency on Friday and Saturday at 7:00 P.M. The following films were shown on Friday:

7:00-7:14 P.M.	Regular Homotopies in the Plane: Part I-a film from the Topology Films Project
7:20-7:25 P.M.	Butterfly Catastrophe (Nelson Max, Mark Thiel)
7:30-7:40 P.M.	Geometric Introduction to Partial Differential Equations (Roy E. Myers)
7:45-8:15 P.M.	Weather by Numbers
8:20-8:33 P.M.	Orthogonal Projection (Daniel Pedoe)
8:35-9:35 P.M.	What is an Integral? (Edwin Hewitt) b&w-a film from the MAA Collegiate Films Series

The following films were shown on Saturday:

7:00-7:22 P.M.	Shapes of the Future: Some Unsolved Problems in Geometry-Two Dimensions (Victor L. Klee, Jr.)-a film from the MAA Individual Lectures Film Project
7:25-7:33 P.M.	Powers of Ten (Charles Eames)
7:35-8:00 P.M.	Numbers Now and Then (B. L. van der Waerden)-a BBC production for the Open University's History of Mathematics course
8:05-8:12 P.M.	Similar Triangles (Bruce and Katharine Cornwell)
8:15-8:41 P.M.	Isometries (W.O.J. Moser, Seymour Schuster)
8:45-8:53 P.M.	Accidental Nuclear War (David S. Gillman)
8:55-9:17 P.M.	Adventures in Perception (Maurits Escher)

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Thursday, January 5, 1978 at 9:00 A.M. in the Strasbourg-Vienna Room of the Atlanta Hilton Hotel with 42 members present. Among the items of business transacted were the following:

The Board of Governors elected Donald L. Kreider, Dartmouth College, as a member of the Finance Committee for the four-year term 1978-81.

The Board of Governors elected Katherine P. Layton, Beverly Hills High School, and Timothy J. Robertson, University of Iowa, as Governors-at-Large for the three-year term 1978-1980. These individuals were elected to represent the constituencies of secondary school teachers and statisticians on the Board.

The following were elected Associate Editors of the AMERICAN MATHEMATICAL MONTHLY for the period extending through December 31, 1981:

Abraham P. Hillman, University of New Mexico
William E. Mastrocola, Colgate University.

Professor Hillman will edit the Problems sections and Professor Mastrocola will edit the Mathematical Education section.

It was voted to approve the recommendation of the Committee on Earle Raymond Hedrick Lectures that Professor Richard K. Guy, University of Calgary, be invited to deliver the twenty-sixth set of Earle Raymond Hedrick Lectures at the Association's meeting at Brown University, August 8-10, 1978.

It was voted to approve revised By-Laws of the North Central and Wisconsin Sections.

Certain editorial changes in the By-Laws were approved and the Secretary was instructed to announce these changes at the business meeting of the Association in August, 1978.

The Board heard the report of the Committee on the Teaching of Undergraduate Mathematics in which it was reported that the Committee has drafted a revision of the pamphlet, *Suggestions on the Teaching of College Mathematics*, as well as a second pamphlet, *Training Programs for Teaching Assistants in Mathematics*.

The operating budget for 1978 was approved and the Board discussed budgets for 1979 and 1980. The Finance Committee reported that the total accumulated surpluses and losses in the Association's operating budget for the period 1969-77 is approximately zero. This calculation does not include several large gifts received during that period.

The Board heard the report of President Alder as Chairman of the Committee to Oversee Dissolution of the Special Projects Office. President Alder paid highest tribute to Professor George B. Pedrick and Mrs. Kathy Magann for the superb job they have done in closing the Office. President Alder said he hoped very much that there may be a time soon when the MAA will be able to make use again of the special talents and highest devotion of both Professor Pedrick and Mrs. Magann. Resolutions of appreciation were approved for Mrs. Magann and Professor Pedrick.

The Board ratified an agreement with the Humanities Research Center of the University of Texas to establish the ARCHIVES OF AMERICAN MATHEMATICS. The Association's purpose in entering into this agreement is to establish a location where mathematical archival material belonging to the MAA can be deposited and serve as a nucleus for additional gifts of mathematical archival material.

The Board of Governors approved purchase of the property located at 1527 and 1529 Eighteenth Street, N.W., Washington, D.C. for the purchase price of $735,000. These two townhouses and carriage house are intended to serve as the Association's headquarters and to provide additional space for rental or resale purposes. The Board also authorized a building fund drive.

The Executive Director reported that, as of November 16, 1977, there were 17,884 members in good standing compared to 17,874 one year earlier. On the same date, there were 413 Academic Members compared to 408 one year earlier.

President Alder reported on the highly favorable reactions which have been received on the brochures *The Math in High School....you'll need for college*, *Recommendations for the Preparation Needed by Students for Collegiate Mathematics Courses*, and the *CEEB/MAA Statement on the CLEP Examination*. President Alder reported that the Sloan Foundation has funded a proposal for a

"Washington Conference II." This Conference will be modeled after the conference held in Washington in 1958 to discuss the priorities in mathematics education in the era then beginning. The present conference will be a small, working conference attended by about forty individuals and will be held in late April in the Washington area.

The Board voted to receive the following grants:

a. A grant of $1000 for support of "Women and Mathematics" during the 1977-78 academic year from the Johnson and Johnson Foundation.

b. A grant of $5986 for completion of the "Sourcebook in Applied Mathematics at the High School Level" from the National Science Foundation.

c. A grant of $14,300 from the Alfred P. Sloan Foundation for support of the Washington Conference.

d. A grant of $8,500 from the International Business Machines Corporation for support of the 1978 Olympiad Awards Ceremony.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The Annual Business Meeting of the Association was held on Saturday, January 7, 1978, in the Regency Ballroom of the Hyatt Regency Atlanta, with President Alder presiding.

The Association's Seventeenth Award for Distinguished Service was made to Professor Richard Davis Anderson of Louisiana State University. The citation (which appears on pages 73-74 in the February issue of the MONTHLY) was prepared and read by Professor R. H. Bing. Professor Anderson, in accepting the Award, said that he was grateful for the opportunity to serve and is looking forward to the challenges of the future.

A bound copy of the citation was presented to Jeanette Anderson in appreciation of her support of Professor Anderson's activities.

The Chauvenet Prize for 1978 was presented to Professor S. S. Abhyankar of Purdue University for his paper, *Historical Ramblings in Algebraic Geometry and Related Algebra,* MONTHLY (83), 409-48. Further details concerning this Prize and its recipient appears in the February, 1978, issue of the MONTHLY, pages 74-75.

President Alder next announced that Professors Samuel L. Greitzer, Rutgers University, and Murray S. Klamkin, University of Alberta, were to be presented the Association's Certificate of Merit. This presentation was made in honor of their hard and effective work as coaches of the U.S.A. team in the International Mathematical Olympiad. In addition to the Certificate of Merit, Professors Greitzer and Klamkin were presented a U.S.A. Mathematical Olympiad Medal.

The Secretary presented his report announcing first that Professor Dorothy L. Bernstein of Goucher College had been elected President-Elect for 1978. Professor Bernstein will serve as MAA President for the 1979-80 term.

It was next announced that Professor Peter J. Hilton, Battelle Memorial Institute and Case Western Reserve University, had been elected First Vice-President for the term 1978-79. The Secretary thanked, on behalf of all members of the Association, all of the candidates in the recent election. He also thanked Professors Donald J. Albers, Lida K. Barrett, and Ralph P. Boas for their work as members of the Nominating Committee.

On behalf of all members of the MAA, the Secretary thanked Professor R. Creighton Buck, whose term as First Vice-President expired at the conclusion of this meeting, for his service to the Association during the previous two years. He particularly thanked Professor Buck for organizing the upcoming Washington Conference.

Next announced was the establishment of THE ARCHIVES OF AMERICAN MATHEMATICS at the University of Texas. The Secretary said that the MAA is greatly indebted to Lucille E. Whyburn for her considerable aid in establishing these Archives.

The Secretary called attention to the fact that, prior to 1977, the MAA journals had routinely been received on schedule. Unfortunately, the combination of a strike by the printer and poor performance by the compositor combined to interrupt that good record in 1977. Since the strike has been settled and the compositor replaced, the journals should soon be on schedule again, he said. The Secretary stressed that restoring timely delivery of the MAA journals was of the highest priority.

The Secretary called attention to the excellent program for this meeting. He thanked all members of the Program Committee, especially Professor Richard A. Duke, Chairman.

The Secretary also thanked the Committee on Arrangements for having attended to the many details necessary for the conduct of this meeting. He particularly thanked Professor John D. Neff for having chaired the Committee and Professor Joseph E. Cicero for having served as Publicity Director.

POSTER SESSION

Posters were on display in Ivy Hall of the Hyatt Regency after 9:00 A. M. on Friday, and authors were available between noon and 1:30 P. M. on Saturday to discuss their displays. The following abstracts were submitted:

GERALD L. ALEXANDERSON, The University of Santa Clara, Santa Clara, CA 95053. High School Guidance Materials Published Recently by the MAA and NCTM.

In March, 1976, the MAA and NCTM appointed a joint committee, the Committee on the Reported Decline in the Preparation of Students for Collegiate Mathematics. From the work of that Committee came two new joint releases from the MAA and NCTM: *Recommendations for the Preparation of High School Students for College Mathematics Courses* and *The Math in High School ... you'll need for college*. The latter brochure was produced by the MAA Committee on Advisement and Personnel, with the help of several NCTM members. The first is a set of recommendations for teachers, school administrators, and parents on the background needed by students for collegiate level courses. The second is a brochure for guidance counselors, teachers, and students, advising students on the amount of mathematics needed for various careers.

These and other materials were available and discussion and comments were invited.

RONALD M. DAVIS, Northern Virginia Community College, Alexandria Campus, Alexandria, Virginia, 22311. The Development of a Diagnostic and Placement Program in Mathematics at NVCC.

As an open-door community college, NVCC accepts students with varied backgrounds. It is essential that the mathematical level of a student be determined at or before registration. During the 1975-76 academic year the math faculty on the five campuses of NVCC, in conjunction with a complete reexamination of all math courses, developed extensive item-specific diagnostic tests for all math courses. At the Alexandria Campus an extensive placement program was developed which utilized counselors, math faculty, a sophisticated placement guide, and selective testing with retests and sample tests available. Samples of the tests, the placement guide, course entry level criteria and the record-keeping procedures were displayed.

LEONARD FELDMAN, San Jose State University, San Jose, CA 95192. Remediation: Math Anxiety and Avoidance.

In order to help college students learn or review elementary mathematics, it is necessary to recognize that remedial mathematics involves a significant amount of anxiety and avoidance. Then it is possible to offer instruction in a way that leads to a minimum of resistance and a maximum of motivation - if one knows something about the insecurities and needs of the students. The display was related to the panel presentation which discussed: (1) questionnaires* about students' needs and concerns; (2) methods* developed as a result of students' responses. (*Available from author.)

LESTER H. LANGE, San Jose State University, San Jose, CA 95192. How Many Numbers Do We Really Need?

The very useful rational numbers, while plentiful, constitute only a small part of the set of real numbers. We will look at a certain very sparsely distributed subset of the rationals (connected with Pythagoras) to see how much of our work could be done if these were the only numbers we had. The investigation provides various opportunities for student projects (some involving computers in several ways) and also leads very naturally to the consideration of density questions.

JOHN J. MORRELL, Ball State University, Muncie, Indiana 47306. Interactive Computer Assistance for Calculus.

The paper describes an attempt to further students' understanding of several key concepts in Calculus through the use of interactive programs written for use on a time-sharing computer.

The student's interaction with the program, the novelty factor in using a computer, and the resulting speed of calculation help to maintain (create) both interest and motivation in more fully understanding such concepts as limits, ε-δ proofs, the derivative as a limiting case of quotients, and the integral as a limiting case of Riemann sums.

The student's control over the internal programs distinguishes this from both canned programs that simply provide "answers" and from courses that require knowledge of a computer language.

Several programs and student runs were exhibited.

JANE I. ROBERTSON, Schoolcraft College, Livonia, MI 48151. Silhouettes: A Visual Aspect

Arithmetic is one of the few areas of our lives in which there is such a thing as THE RIGHT ANSWER. Since a right answer exists, we can each teach our students to get it, then check it. The algorithms that evolved for people to use with place value decimal notation show the operands in horizontal rows, positioned in columns to weight them according to place value. Hence the calculations have characteristic shapes--or silhouettes--that reveal information about the operation, operands, and result. Silhouettes are a visual aspect of the much wider topic of the pattern and structure of algorithms.

MEETINGS OF OTHER ORGANIZATIONS

The AMS held sessions from Tuesday, January 3, through Saturday, January 7. Invited AMS addresses were delivered in the Regency Ballroom according to the following schedule:

WEDNESDAY

9:00 A.M.	*Ergodic Theorems in Demography* Professor Joel E. Cohen, Rockefeller University
10:30 A.M.	*Pseudoconvexity and the Problem of Levi* Professor Yum Tong Siu, Yale University
1:00 P.M.	COLLOQUIUM LECTURE I: *Algebraic K-theory* Professor Hyman Bass, Columbia University
3:30 P.M.	*Some Mathematical Aspects of Quantum Field Theory* Professor Thomas Spencer, Rockefeller University
8:30 P.M.	JOSIAH WILLARD GIBBS LECTURE: *Mathematical Typography* Professor Donald E. Knuth, Stanford University

THURSDAY

9:00 A.M.	*The Topology of the 3-Sphere* Joan S. Birman, Columbia University
10:30 A.M.	*Recursively Enumerable Sets and Degrees* Professor Robert I. Soare, University of Chicago
1:00 P.M.	COLLOQUIUM LECTURE II
2:15 P.M.	*Aspects of Geometry and Topology of the Spectrum* Professor Jeff Cheeger, SUNY at Stony Brook
8:30 P.M.	*An Employer's Viewpoint of Nonacademic Employment* Panelists: Drs. James A. Dewar, Turpin Systems Company; Brockway McMillan, Bell Telephone Laboratories; Daniel H. Wagner, Daniel H. Wagner Associates; Shmuel Winograd, IBM

FRIDAY

1:00 P.M. COLLOQUIUM LECTURE III

2:30 P.M. *Representations of Finite Groups of Lie Type*
Professor Charles W. Curtis, University of Oregon

4:00 P.M. *Propogation, Reflection, and Diffraction of Singularities of Solutions to Wave Equations*
Professor Michael E. Taylor, Rice University

SATURDAY

1:00 P.M. COLLOQUIUM LECTURE IV

3:30 P.M. *Images of Manifolds Under Cell-like Maps*
Professor Robert D. Edwards, UCLA

The AMS Business Meeting was held at 4:30 P.M. on Thursday. The George David Birkhoff Prize in Applied Mathematics was awarded at a session at 3:15 P.M. on Thursday. Both meetings were in the Regency Ballroom.

The Association for Women in Mathematics sponsored a panel discussion, *Black Women Mathematicians,* at 7:30 P.M. on Saturday in the Regency Ballroom. AWM held a meeting of its Council at 5:30 P.M. on Saturday in the Grecian Room.

The Conference Board of the Mathematical Sciences sponsored a panel discussion, *The Growing Role of Applications in Mathematical Higher Education.* Professor Clayton V. Aucoin, Clemson University, served as moderator. This meeting was held at 2:00 P.M. on Friday in the Phoenix Room of the Hyatt.

The Mathematicians Action Group held an open meeting of its Steering Committee at 9:00 A.M. on Tuesday. MAG held its business meeting at 4:30 P.M. on Wednesday in the Phoenix Room and a panel discussion at 5:00 P.M. on Friday.

NSF staff were available in the Embassy Room of the Atlanta Hilton Hotel to provide counsel and information on NSF programs on January 5, 6, and 7 from 9:00 A.M. to 5:00 P.M. At noon on Friday, Dr. William H. Pell, Head of the NSF Mathematical Sciences Section, spoke on recent changes in Foundation policy. This talk was given in the Regency Ballroom.

Professor Mark P. Hale, Jr., University of Florida, scheduled a session for persons interested in computer-assisted test construction and other forms of technical support for instruction. This meeting was held in the Grecian Room of the Hyatt on Friday, January 6, from 7:30-9:30 P.M. The program consisted of several short presentations followed by an open discussion.

David P. Roselle
Secretary

CALENDAR OF FUTURE MEETINGS

Fifty-eighth Summer Meeting, Brown University, Providence, August 8–10, 1978.

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 weeks before meeting.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, St. Peter's College, Englewood Cliffs, November 4, 1978.

NORTH CENTRAL, University of Saskatchewan, Saskatoon, October 20–21, 1978.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO

OKLAHOMA–ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 weeks before meeting.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 16–17, 1978.

PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.

ROCKY MOUNTAIN, University of Denver, Denver, spring 1979.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN, University of Tennessee, Chattanooga, spring 1979.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

Texas, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Chicago, January 3–8, 1979.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Stouffer's Greenway Plaza Hotel, Houston, Texas, October 10–14, 1978.

AMERICAN MATHEMATICAL SOCIETY, Brown University, Providence, Rhode Island, August 9–12, 1978.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of British Columbia, Vancouver, June 19–22, 1978.

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton Park Hotel, Washington, D.C., December 4–6, 1978.

ASSOCIATION FOR SYMBOLIC LOGIC, Madison, Wisconsin, June 18–24, 1978.

ASSOCIATION FOR WOMEN IN MATHEMATICS, Brown University, Providence, Rhode Island, August 8–12, 1978.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHEMATIQUES, University of Western Ontario, London, Ontario, June 1–2, 1978.

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INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, Stevens Point, Wisconsin, August 6–9, 1978.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Bonadventure Hotel, Los Angeles, California, November 12–16, 1978.

PI MU EPSILON

SCHOOL, SCIENCE AND MATHEMATICS ASSOCIATION, Hotel Fort Des Moines, Des Moines, Iowa, November 2–4, 1978.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Hyatt Regency Hotel, Knoxville, Tennessee, October 30–November 1, 1978.

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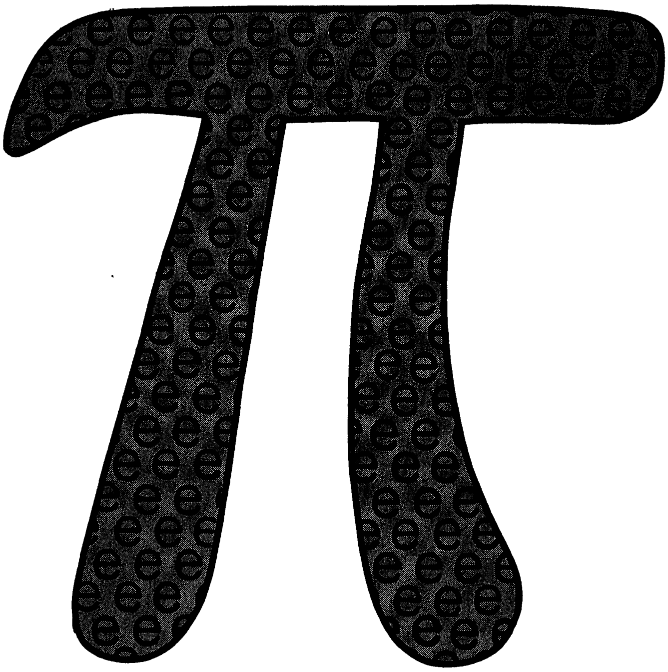

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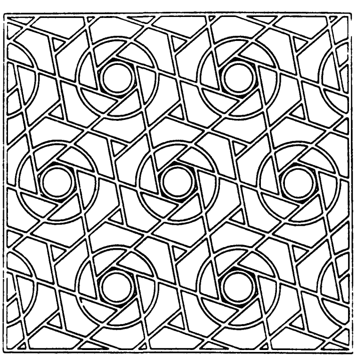
JUNE - JULY

THE AMERICAN MATHEMATICAL MONTHLY

Volume 85, Number 6

Headquarters Building for the MAA

Computerized Tomography



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Indeterminate Forms; Films; and "Proofs" to Grade

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1978

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 85 NUMBER 6

CODEN: AMMYAE

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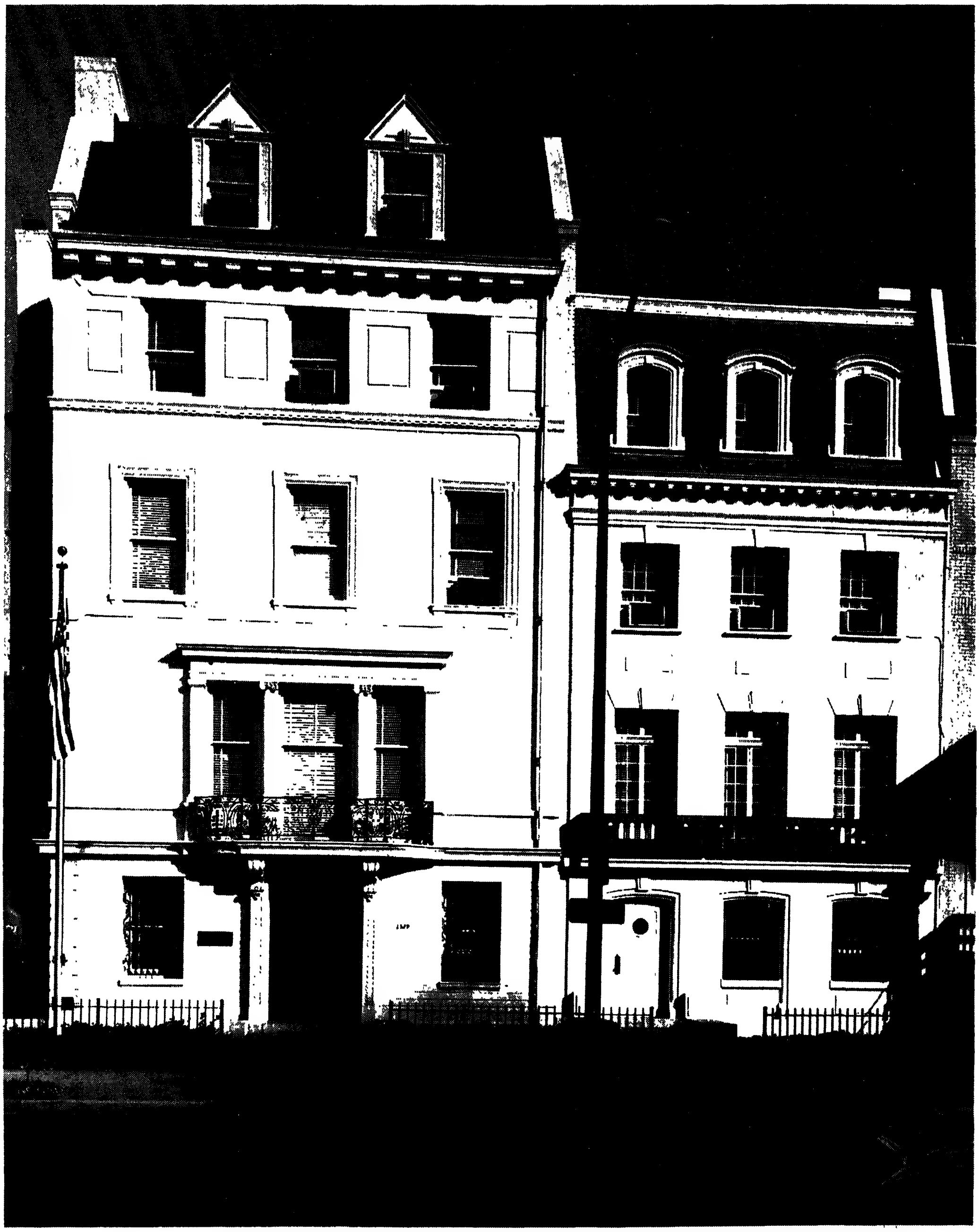


1529 EIGHTEENTH STREET, N. W., WASHINGTON, D. C. 20036

A HEADQUARTERS BUILDING FOR THE ASSOCIATION

HENRY L. ALDER
President, Mathematical Association of America

Action by the Board of Governors

For the past ten years, the Association has been renting office space in Washington, D.C. About three years ago, it became clear that we would need more space and would have to move, and a search for suitable quarters was begun, along with a study of whether to rent or buy. I am happy to announce a successful conclusion. At its meeting on January 5, 1978, the Board of Governors unanimously approved the purchase of a complex of two connecting townhouses and an annex located at 1527–1529 Eighteenth Street, N.W., a short block from Dupont Circle and about three blocks from our present offices. These buildings are perfect for the needs of the MAA and have space to spare that can be used for rental income or future expansion. A contract has been signed, and the MAA will move in during the summer of 1978.

As the MAA approaches this significant event in its history, it is appropriate to review briefly what the MAA is, its spectacular growth in recent years, and how it does the many things for which it has assumed responsibility.

What Is the MAA?

The Mathematical Association of America is an organization devoted to the interests of mathematics teaching in the colleges and universities, having 18,500 individual and 425 institutional members and 6,500 additional subscribers. The MAA provides a wide range of services to its members, the mathematical community, and the general public:

MEETINGS	Two national meetings each year and about forty sectional meetings
JOURNALS	Three journals addressed to a wide audience ranging from undergraduate students to college and university faculty
BOOKS	Eighty-five books spanning the same wide range, available to members at minimal cost
PAMPHLETS	Free pamphlets on careers in the mathematical sciences the mathematics needed in careers outside the mathematical sciences the teaching of mathematics the undergraduate mathematics curriculum
FILMS	Several dozen educational films
CURRICULUM STUDIES	Continuing intensive study of the curriculum by the *Committee on the Undergraduate Program in Mathematics (CUPM)*
PRIZES	The *Award for Distinguished Service to Mathematics* and several prizes for excellence in expository writing
LECTURERS	Three visiting lecturer programs for schools and colleges
FACULTY SERVICES	Employment services for college faculty

ARCHIVES	A cooperative program to preserve historical documents
TESTS	A placement test program for colleges
CONTESTS	Several mathematical contests for high school students, involving 350,000 students each year, and the Putnam competition for college students
SECTION ACTIVITIES	Twenty-nine Sections, each holding at least one meeting a year; in addition, Sections conduct lecture programs, publish newsletters, and sponsor seminars, workshops, contests, and other activities

How Has the MAA Been Growing?

Since 1970 there has been a dramatic increase in the scope of MAA activities and services:

JOURNALS	1970	Two journals, THE AMERICAN MATHEMATICAL MONTHLY and MATHEMATICS MAGAZINE
	Today	These two plus THE TWO-YEAR COLLEGE MATHEMATICS JOURNAL
BOOKS	1970	25 books, 6,000 copies a year, for colleges, including *The Carus Mathematical Monographs* and *MAA Studies in Mathematics*
	Today	85 books (plus 20 in production), 40,000 copies a year, including several new *Carus* and *Studies* volumes, the *Dolciani Mathematical Expositions* for colleges, the *Raymond W. Brink Selected Mathematical Papers* on topics in the undergraduate curriculum, and the *New Mathematical Library* series for students and teachers
PAMPHLETS	1970	Pamphlets issued in editions of a few thousand
	Today	Almost 100,000 copies distributed (by MAA and co-sponsor) of *Recommendations for the Preparation of High School Students for College Mathematics Courses* and over 250,000 copies distributed of *The Math in High School You'll Need for College*
PRIZES	1970	The *Award for Distinguished Service to Mathematics*, and the *Chauvenet Prize* and *Lester R. Ford Awards* for expository writing
	Today	These plus the *Carl B. Allendoerfer Awards* for articles in MATHEMATICS MAGAZINE and the *George Pólya Awards* for articles in THE TWO-YEAR COLLEGE MATHEMATICS JOURNAL
LECTURERS	1970	The *Visiting Lecturers and Consultants* program for colleges, supported by the National Science Foundation, with all but two of the several dozen lecturers from colleges and universities
	Today	This program, now supported entirely from MAA funds and now with 16 lecturers from industry and government, plus "*Women and Mathematics*" and "*Blacks and Mathematics*," lecture programs designed to guide more school students from these groups into mathematics courses

FACULTY SERVICES	1970	An employment register and information service for college faculty (co-sponsored)
	Today	This service plus the *Sabbatical Exchange Information Service* for college faculty
ARCHIVES	1970	No organized program
	Today	*The Archives for American Mathematics* at the Humanities Research Center of the University of Texas
TESTS	1970	No program of tests
	Today	A growing *Placement Test Program* for colleges
CONTESTS	1970	The *Putnam* competition for college students and the *High School Contests*
	Today	These two plus the *USA Mathematical Olympiad* and a team in the *International Mathematical Olympiad*; in our four competitions against teams from twenty countries, the U.S. team came in second, third, third, and first.

How Do These Things Get Done?

The Association is governed by a *Board of Governors*, composed of representatives from the twenty-nine *Sections*, Governors-at-Large, editors, and past and present officers—about fifty people in all. The Board meets twice a year. The *Executive and Finance Committees* handle MAA affairs between meetings of the Board.

The programs of the Association are carried out through national committees, some fifty in number, and by the Sections and their committees.

All the work is coordinated by the *Executive Director*, three assistants, in charge of finance, membership, and publications, and their staff of a dozen at the Washington office. Here are some administrative comparisons between 1970 and today.

FINANCES	1970	All financial accounting done by hand on a simple cash basis
	Today	Financial accounting done by advanced accrual methods with the aid of a computer, allowing us to assess indirect costs accurately and enabling us to respond to vastly increased governmental requirements (tax records, postal regulations, etc., etc.) and to provide the month-by-month information needed for careful budget control
MEMBERSHIP	1970	Membership records kept by hand; no accurate demographic information
	Today	Membership records handled by computer; up-to-date information available about our members on education, class of employer, number of years of membership, history of MAA activities, choice of journal subscriptions, and so on
PUBLICATIONS	1970	All our books (25) published and distributed by commercial publishers
	Today	All our books (85) published and distributed by MAA

What Is the New Headquarters Like?

The headquarters complex consists of two connecting townhouses and an annex, located at 1527–1529 Eighteenth Street, N.W., in the Dupont Circle section of Washington, D.C. This is prime territory, where one will find other association headquarters (e.g., the Brookings Institution and AAAS, a block away), a section of the famous "Embassy Row" along Massachusetts Avenue, some medical offices, and several attractive apartment buildings and private homes—and a Metro station. Our buildings are 1900 vintage, well cared for, and in excellent condition. No. 1529 was the home of Charles Evans Hughes during the twenties, when he was Secretary of State; the annex was his carriage house.

No. 1529 has five stories and No. 1527 has four, plus basements. There are three stairways and an elevator. Because the buildings are now housing another association, the interiors have already been divided into convenient offices. There is even a computer room with special flooring and its own cooling system. The carriage house is ideal for a mail room, print shop, and storage. The surrounding area has parking space for nineteen cars and is conveniently accessible from two directions.

What Will the New Headquarters Do For the MAA?

Provide needed space for our headquarters. Our present facilities are overcrowded.

Protect us from rent increases. Our lease expires soon. Rents are rising rapidly.

Provide a sound, long-term investment. Real estate in the Dupont Circle area has been appreciating at a rate of 10% a year.

Offer us an annual income. The total space is more than we need for ourselves, so we can rent some of it out (including the parking spaces!).

Allow for greater efficiency, since certain equipment that is too expensive for us alone (a versatile copying machine, for instance) could be shared with a tenant.

Provide room for centralization. Eventually, we may wish to consolidate our administrative work, some of which is now performed elsewhere.

Provide room for expansion. We will have space for more sophisticated mailing equipment and for storage space for inventories of books, as well as space for new equipment such as a printer or a computer.

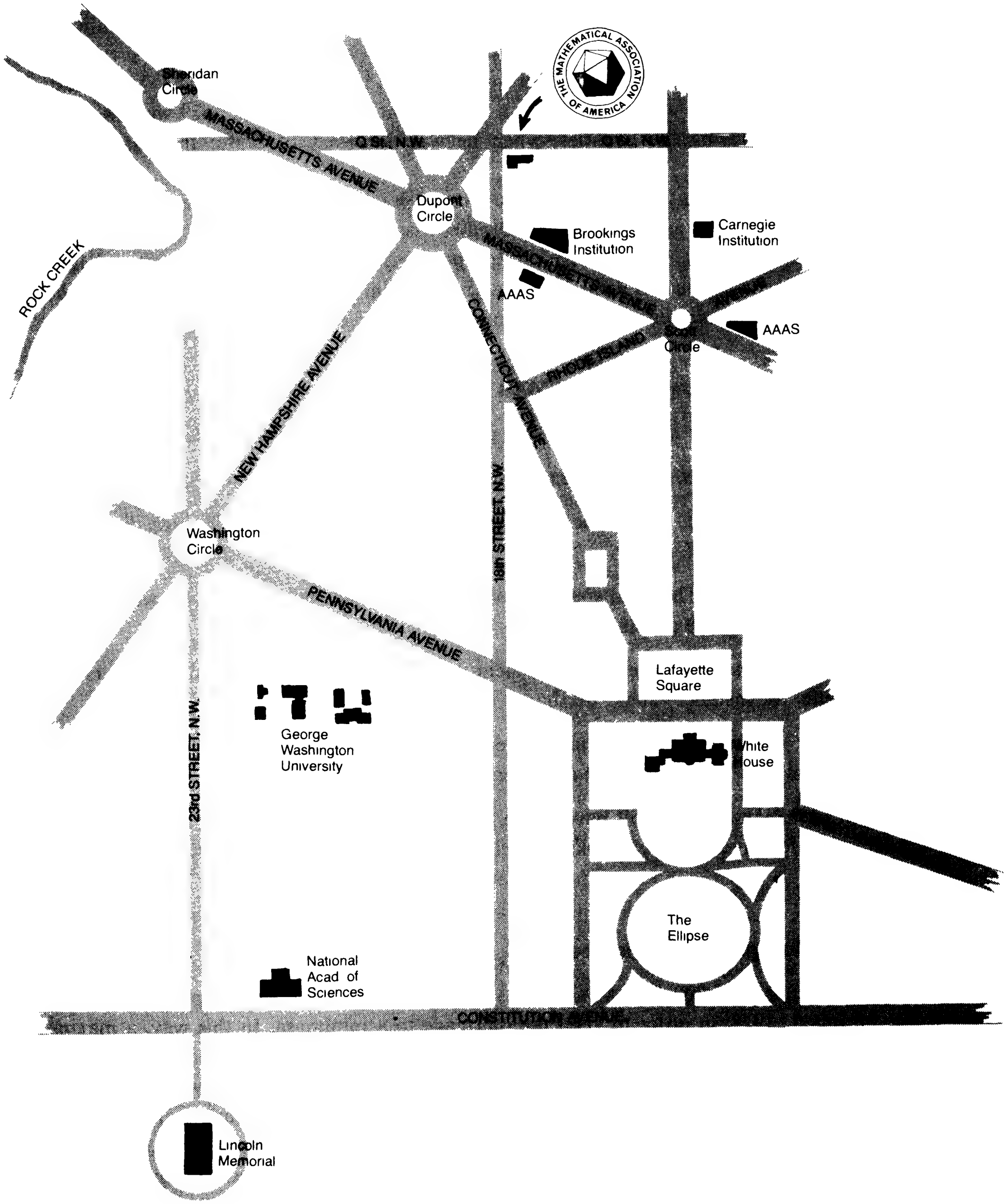
How Will the New Headquarters Be Financed?

The purchase price is $735,000. The MAA wants to pay off as much of this as possible before the closing date of July 1, 1978, in order to keep the mortgage to a minimum. In this way, a maximum amount is freed for projects needed to advance the interests of the mathematical sciences.

Hence, an appeal for funds was sent to every MAA member in early March. Two handsome gifts for the building had already been received at that time: $80,000 from the Vaughn Foundation of Tyler, Texas (pledged), and $70,000 from Professor Mary P. Dolciani. This leaves $585,000 to go.

This appeal for funds is unlike any ever made by the MAA. A contribution toward the new headquarters building is an investment that will assist the MAA in its work for many years. Contributions of all sizes are welcome, and all contributions will be gratefully acknowledged. (All contributions are tax deductible.) The most helpful form of contribution is cash now, before September 1 (an extension of the time specified in the letter sent to the membership). Each $10 gift received now will lower the interest on the mortgage by $1 a year.

For gifts as high as $50, there are alternative forms of payment. The following are equivalent for most members: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White House
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George Washington University
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National Acad of Sciences
CONSTITUTION AVENUE
Lincoln Memorial

- A *cash gift* of $50 before September 1.
- A *pledge* of $60, one-fourth to be paid before September 1 and the rest in equal installments over the next three years (smaller pledges are also welcome).
- A *dues advance* before September 1 of $225, to be credited to your account. This amounts to an interest-free loan to the Association, to be paid back to you in annual installments equal to your then applicable dues. This form of gift may have immediate tax advantages for you. (To be helpful, a dues advance should be at least $225 so that it will have time to accrue interest before being paid back.)

Contributions of $50 or more (or equivalent as suggested above) will be given special designations:

Donor	Cash gift of $ 50 or equivalent pledge or dues advance
Sponsor	Cash gift of $ 100 or equivalent pledge
Patron	Cash gift of $ 500 or equivalent pledge
Benefactor	Cash gift of $1,000 or equivalent pledge

and their names will be displayed in the reception area of the building (except for those who ask to remain anonymous). Those who contribute still more generous amounts will be given further recognition. If you own securities which have appreciated substantially in value, you may wish to consider offering them as a gift, as this would have tax advantages for you. I will be delighted to discuss this or other possibilities with you.

One idea which you might wish to consider is a gift as a memorial. For example, students of a favorite professor could make a gift in his or her honor. A department of mathematics or a Section might make a gift to honor one of its distinguished members. For an appropriate gift one of the rooms in the building could be named for such an individual. Other forms of memorials are also possible.

If you have not yet had a chance to do so, I hope that you are now able to respond to this appeal. Please indicate how much you wish to pledge (the amount not enclosed with your pledge to be paid in three equal installments with your dues for 1979, 1980, 1981) and/or whether you wish to send a dues advance of $225 or more. Also, please advise if you prefer your contribution to remain anonymous. Please respond to the Mathematical Association of America at the address below.

We are proud of the new headquarters and enthusiastic about the prospects of increased service to the mathematical profession it will enable the MAA to offer. We look forward to your generous support.

Send contributions or requests for further information to:

Dr. Alfred B. Willcox, Executive Director
Mathematical Association of America
1225 Connecticut Avenue, N.W.
Washington, D.C. 20036

COMPUTERIZED TOMOGRAPHY: THE NEW MEDICAL X-RAY TECHNOLOGY

L. A. SHEPP AND J. B. KRUSKAL

Abstract. Computerized X-ray tomography is a completely new way of using X-rays for medical diagnosis. It gives physicians a more accurate way of seeing inside the human body and permits safe, convenient, and quantitative location of tumors, blood clots and other conditions which would be painful, dangerous, or even impossible to locate by other methods. Although each tomography machine costs hundreds of thousands of dollars, hundreds of tomography machines are already in use.

A mathematical algorithm to convert X-ray attenuation measurements into a cross-sectional image plays a central role in tomography. Sophisticated mathematical analysis using Fourier transforms has led to algorithms which are much more accurate and efficient than the algorithm used in the first commercial tomography machines. We show how some of the algorithms in actual use have been developed. We also discuss some related mathematical theorems and open questions.

1. Introduction. In computerized tomography, X-ray transmission measurements are recorded on a computer memory device rather than on film, and a sophisticated mathematical algorithm is applied. This produces a numerical description of tissue density as a function of position within a thin slice through the body. The physician examines this function by use of visual displays.

In the ordinary medical use of X-rays, the picture is something like a shadow; any feature in line with denser bone tissue tends to be blocked out. In other words, if we could make a great many pictures, each of a thin slice perpendicular to the beam of X-rays, the actual X-ray picture is formed by superposition of all these hypothetical pictures, i.e., it is a kind of "multiple exposure." Computerized X-ray tomography provides a picture of a single thin slice through the body, without superposition. The word tomography is related to the Greek word "tomos" meaning cut or slice.

Imagine a thin slice, say through the head, perpendicular to the main body axis. Several hundred parallel X-ray pencil beams are projected through the head in the plane of this slice, and the attenuation of X-rays in each beam is measured separately and recorded. (In earlier machines a single beam has been used by translating it parallel to itself within the plane; some of the later machines discussed in Section 3 use fan rather than parallel arrays of beams.) Another set of parallel beams is used within the same plane but at an angle of perhaps 1° or so with the first set, and measurements are taken again. The process is repeated until measurements have been taken for a grid covering all directions in the plane. An elaborate calculation then permits approximate reconstruction of the X-ray attenuation density as a function of position within the slice.

In appropriate units, tissue density in the head varies roughly between 1.0 and 1.05 with the exception of bone which has a density of about 2. Some features of medical interest are indicated by variations of density as small as .005. Reconstructing tissue density with adequate accuracy at a sufficiently fine grid of points is thus a challenging project.

Mathematically we may describe the problem as follows. Consider a fixed plane through the body. Let $f(x,y)$ denote the density at the point (x,y), and let L be any line in the plane. Suppose we direct a thin beam of X-rays into the body along L, and measure how much the intensity is attenuated by going through the body. It is easy to see that the logarithm of the attenuation factor is given

---

Lawrence Shepp received his Ph.D. in Mathematics in 1961 at Princeton University under W. Feller and has been at Bell Laboratories, Murray Hill, N.J. since 1962, where he is a member of the Mathematics Center. He has made many contributions to probability theory and received a Paul Lévy prize. He became interested in tomography in 1972. He presented this paper as invited lecturer at the MAA and AMS meetings in Toronto, 1976.

Joseph B. Kruskal received his Ph.D. in mathematics at Princeton in 1954. He has been a member of the Bell Laboratories Mathematics Center since 1958, but has taught at Princeton, Madison, Ann Arbor, Yale, Cambridge (England), and Columbia at various times. He has made contributions to combinatorial mathematics, statistics, the mathematics of psychology, and statistical linguistics, and has been president of the Psychometric Society and The Classification Society (NAB).—*Editors*

approximately by the projection or line integral of f along L,

$$P_f(L) = \int_L f(x,y)\,ds, \tag{1.1}$$

where s indicates length along L. The formula (1.1) is only an approximation because: (1) it assumes the X-ray beam is infinitely thin, and (2) it assumes the beam is monochromatic, or alternatively that the physical attenuation coefficient is independent of the energies of the different X-ray photons, and (3) it ignores the significant statistical fluctuations due to the limited number of photons actually transmitted during each measurement. None of these three assumptions is quite correct, but the error in using (1.1) can in principle be made arbitrarily small.

Although complete mathematical analysis of the above errors is impossible, the effects of the approximations have been studied by various means. Thus consider the first approximation. Suppose we replace the line integrals $P_f(L)$ by integrals $P_f(S)$ over strips S. It is shown in Appendix 1 that the strip integrals of f are the line integrals of the function $k*f$, the convolution of f with a circularly symmetric kernel k, where k is the Abel transform of the strip shape. Although the strip integral is still only an approximation, it suggests that an X-ray beam of nonzero width will merely produce a slight smoothing in the reconstruction. The last approximation can be analyzed with the techniques of signal-in-noise communication theory [**20**]. All three approximations have been studied using simulations [**21**].

The mapping $f \to P_f$ in (1.1) is known as the Radon transform because Radon (1917) studied it extensively. He showed that if f is continuous and has compact support, then f is uniquely determined by the values of $P_f(L)$ for all lines L (i.e., the Radon transform is one-one). Furthermore, Radon gave the fairly simple inversion formula, (2.1) below, for f.

Around 1970, G. N. Hounsfield of EMI, Ltd. invented the first computerized tomography machine to give an image accurate enough to be of value in medical diagnosis [**11**]. Since that time computerized X-ray tomography has assumed great medical importance. It is interesting to note that several years prior to Hounsfield's work, Bracewell and Riddle [**1**] used the tomography principle in radioastronomy, while Cormack [**2a, 2b**] proposed it for medical use.

Mathematics has played a central role in tomography from the very beginning, because without a mathematical algorithm of some sort, reconstruction of the density f from its projections $P_f(L)$ is impossible. Rigorous deduction plays an important but limited role in these algorithms. Without it, tomography would be far less effective. For example, Radon's work, which appears to have been pure mathematics carried out for its own sake, has played a very important role in the development of tomography. Lebesgue-Stieltjes integration theory provided the foundation on which the present formulation of algorithms is based. Classical theorems by Fourier, Poisson and others play an important role as we shall illustrate. On the other hand, when comparing the value of different algorithms, rigorous deduction hardly enters the picture. For example, consider one important aspect of reconstruction, the accuracy to which the density values are reconstructed. In principle, it might be possible to give rigorous error bounds and these could be used to compare the accuracy of different algorithms. However, this has not yet been possible (and might not even provide a useful comparison). Instead the accuracy is compared by several types of experiments which we shall describe in a moment.

It is interesting to consider the method by which algorithms have been "justified" mathematically in this field. While this method consists of mathematical reasoning in a certain sense, the reasoning is far from rigorous. Approximations are introduced at many steps with only intuition as a guide to the error involved. We do not know of a single instance in which a tomographic algorithm has been justified in a truly rigorous sense. Thus, in contrast to some other workers in this field, we do not feel that one derivation is more rigorous than another, whether it is based on Radon's inversion formula, the Fourier inversion formula or any other foundation.

It is interesting to see how non-rigorous mathematical thinking has played the major role in comparing different algorithms. The most widely used technique for comparing algorithms has been to compare the reconstructions when applied to data taken from real human subjects. Another technique is to use what physicians call "phantoms": this means taking data from a physical object of known structure instead of a human subject. This is useful because we know what the true object is. Errors in the reconstruction, however, may be due to errors in the data or to errors in the algorithm. To separate these, Shepp introduced a "mathematical phantom." As described more fully in [**20**], this involves simulating a body section by a mathematically describable function. It has been convenient to use piecewise constant functions where the pieces are made from circles or ellipses (since for these the projection data is easily obtained). In a mathematical phantom there is no measurement error, so any errors in the reconstruction are due to the algorithm. Furthermore, any desired type of measurement error can be simulated to study its effects, which has been very useful [**21**] in the design of the newer high speed scanning machines.

FIG. 1. Simulation of human head using 11 ellipses. The density of the skull is 2.0 and of the ventricles, tumors, etc. is 1.0-1.05 (see [**20**] for more details).

Figure 1 shows a mathematical phantom which is a reasonable imitation of a slice through the human head, subject to the limitations mentioned above. The skull has density 2.0 while the diagonal ellipses represent ventricles of the brain which are filled with fluid of density 1.0. The surrounding gray matter has density 1.02; the various tumors, and the blood clot just inside the skull, have densities ranging between 1.03 and 1.05. Thus all interior head tissue has a very narrow range of variation. See [**20**] for further details.

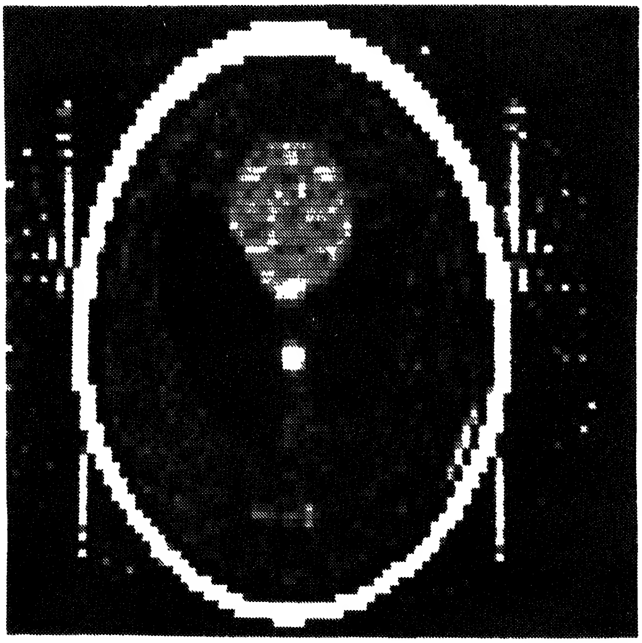

FIG. 2. Reconstruction using the algorithm embodied in the first commercial machine (EMI Ltd.) from 180×160 strip projection data obtained by exact calculation from Fig. 1.

Figures 2,3,4 show reconstructions of this phantom by three different algorithms in chronological order. Figure 2 shows a reconstruction using the algorithm embodied in the first commercial machine, which was constructed and sold by EMI Ltd. We shall call this the Hounsfield iterative algorithm (Hounsfield 1971) since it was based on an iterative relaxation method. Figure 3 shows the reconstruction from the same data by the Fourier-based convolution algorithm due to Shepp (1974). Figure 4 shows the reconstruction by the algorithm now used in EMI machines and due to C. Lemay (1974) of EMI. It is evident that Figs. 3 and 4 are great improvements over Fig. 2 which has artifacts which make it difficult to detect the small tumors. It also has a ring around the inside of the skull, which was also observed in human reconstructions and was believed to be a genuine aspect of human anatomy. Through the use of mathematical phantoms it was first learned that this ring is an artifact due to the algorithm. Figure 4 also shows this artifact, much decreased, and some tomographers argue that such "overshoot" is desirable because it enhances some otherwise less notable features of importance, such as edges of tumors. Other tomographers feel otherwise, pointing out that the overshoot can conceal a feature immediately adjacent to the skull such as a blood clot. We feel that at this stage of tomography the goal should be to make the tomographs as accurate as possible. Enhancement may have value, but it also has dangers which are better deferred until the field has matured further.

The comparison shown in the figures is not quite fair for various reasons, most notably because Fig. 4 is based on a finer grid of simulated X-ray beams than Fig. 2. (However, Fig. 3 is based on the same data as Fig. 2.) Nevertheless, there is little question that the conclusions are correct, because they are supported by a wealth of other experiments, using various algorithms and all three methods of comparison described above.

Some experimental details may be worth mentioning. The reconstructions in Figs. 2 and 4 were made by placing the simulated projection data from the mathematical phantom in the appropriate memory area of an EMI machine (at Columbia-Presbyterian Hospital, NYC) so that the machine could process the data through its algorithm and display the reconstruction in its normal manner. The reconstruction in Fig. 3 was made at Bell Laboratories using a computer program due to Shepp [**20**].

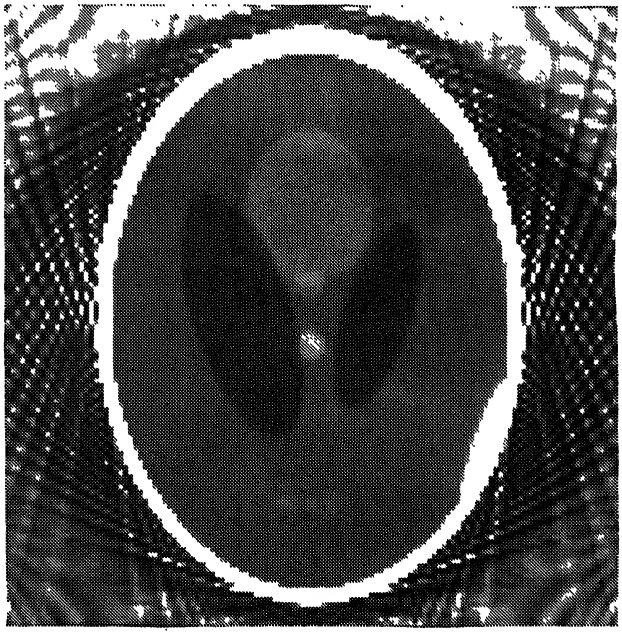

FIG. 3. Reconstruction from the same data using the Fourier based algorithm of Shepp [20] (see [20] for more details).

FIG. 4. Reconstruction using the algorithm now embodied in the EMI machine from 180×239 strip projection data obtained by exact calculation from Fig. 1.

In §2, we survey the mathematical foundations and some rigorous models of tomography and state some theorems of mathematical interest which have grown out of tomography. In §3 we briefly describe some of the major tomographic algorithms now in use.

Another survey [22] of the field of tomography has recently appeared and deserves comparison. [22] treats some interesting mathematical questions related to tomography, but has little to do with the way mathematical algorithms are being used in the field.

2. Mathematical foundations. Radon [18] gave a simple formula to invert the transform (1.1). Assume the projections $P_f(L)$ are given for all lines L where f is continuous with compact support. If Q is any point in the plane, denote by $F_Q(q)$ the average value of $P_f(L)$ over all lines L at distance $q>0$ from Q. Then for any Q, f is reconstructed by

$$f(Q)=-\frac{1}{\pi}\int_0^\infty \frac{dF_Q(q)}{q}, \tag{2.1}$$

where the integral converges as a Stieltjes integral in spite of the apparent singularity at $q=0$.

Some pure mathematicians seem to have the mistaken notion that a formula like (2.1) is a complete answer to the tomographic reconstruction problem, and that little more remains to be done. However, it will come as no surprise to most applied mathematicians that the work has just begun. First, an inversion formula is not enough. The stability properties of the inverse mapping are of vital importance since if the inverse mapping is not sufficiently stable then impractically precise measurements of $P_f(L)$ may be required to determine f to sufficient accuracy. Second, in practice it is necessary to use sums rather than integrals and this discretization involves subtle and difficult questions.

To illustrate the latter point first, the natural discrete approximations to (2.1) do not give good reconstructions. This is partly because of the singularity at $q=0$ but mainly because finding a good natural approximation to $F_Q(q)$ is possible only if there are many lines L available at distance exactly q from Q. This occurs only for a few points Q and for a few distances q, no matter how the lines L are chosen. Thus another inversion formula for the mapping $f\to P_f$ is needed which lends itself better to discrete approximations.

To illustrate the point about stability of the inversion mapping, let us consider a more general inversion problem. Since the Radon transform uses all lines L, while discrete approximations deal with only a finite number of lines, it is natural to consider how the inversion depends on the set $\mathfrak{L}$ of available lines. Furthermore, the density functions $f(x,y)$ must have smoothness properties, so it is natural to specify that f belongs to some linear space $\mathfrak{F}$ of functions, and to consider how the reconstruction depends on $\mathfrak{F}$.

Now consider three different sets of lines: the set of all lines, the set $\mathfrak{L}_D$ of all lines which intersect a disk D and the set $\mathfrak{L}_D'$ of all lines which are exterior to D. We shall refer to the complete Radon transform, the interior Radon transform, or the exterior Radon transform, according to whether L ranges over the first, second, or third of these sets. Then Radon's theorem (2.1) shows that the complete Radon transform is a 1-to-1 mapping. That is, P_f determines f everywhere if f belongs to $\mathfrak{F}_\infty$, the set of functions which are continuous and have compact support. The complete Radon transform is even 1-to-1 on the larger set $\mathfrak{F}_k$, the set of continuous functions f with $f(x,y)=o\big((x^2+y^2)^{-k}\big)$ as $(x,y)\to\infty$, $k>1$, [18].

Inversion of the interior or exterior Radon transform is of medical interest. For example, the fewer measurements required for the interior transform could reduce patient dosage in imaging a specific organ or part of the body within D. Inversion of the exterior transform could solve the problem caused by rapid heart motion in imaging the chest region around the heart. If there were an accurate inverse exterior transform, the measurements through the beating heart could simply be omitted.

Let us first consider inversion of the interior transform. There is an elementary "solution" which is often rediscovered, sometimes called "old tomography" because there are older devices (not very

successful) which used this method. Old tomography merely forms the average $P_f(L)$ over all lines L which go through each point (x,y). This however gives not f itself but a smoothed version of f, namely

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+u,y+v)\frac{du\,dv}{\sqrt{u^2+v^2}}. \tag{2.2}$$

Old tomography is actually a particular algorithm for reconstructing f at a point Q from its parallel projections in each of n directions by merely averaging the line integrals $P_f(L)$ over the lines L nearest to Q in each direction. Figure 5 was obtained in this way and shows that this algorithm does not give good reconstructions. It is easy to see that the interior Radon transform does not determine f inside D, i.e., there are nonzero functions f with zero interior Radon transform. Nevertheless, practical inversion of the interior transform may be possible for certain more limited classes $\mathcal{F}$.

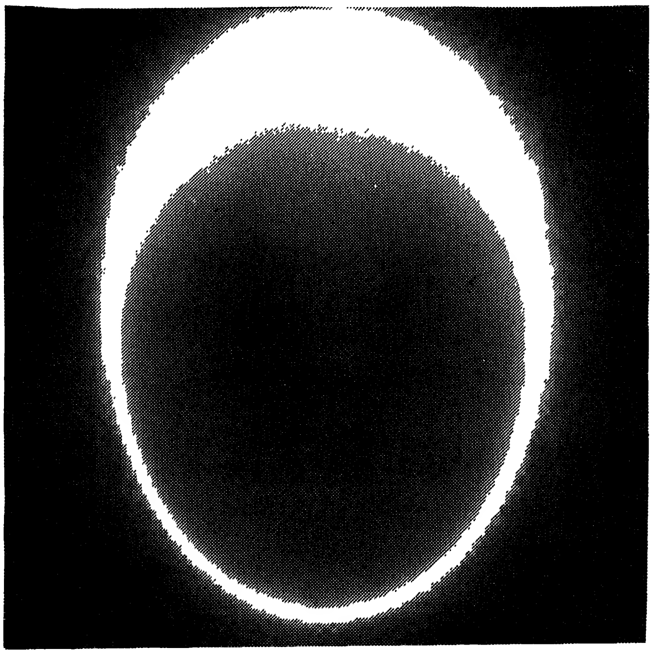

FIG. 5. Reconstruction using the algorithm of "old tomography" which gives for each point Q the average of that line integral in each direction whose line is nearest to Q. There are 128 lines in each of 64 directions and the line integrals were obtained by exact calculation from Fig. 1.

For the exterior Radon transform, theorems of Helgason [8] and Ein-Gal [4] show that the exterior Radon transform is 1-to-1 on $\mathcal{F}_\infty$ and these authors provide explicit inversion formulas for $f\in\mathcal{F}_\infty$. However, the inversion mapping for the exterior Radon transform has a property which suggests to us that a practically useful numerical inversion may never be possible. In particular let us extend the Radon transform to the class of functions $\mathcal{F}_k$ for $k>1$ as above. The complete Radon transform remains 1-to-1 over $\mathcal{F}_k$. However, an example of D. J. Newman below shows that the exterior Radon transform is not 1-to-1 for any k. In fact the same example shows that if D_1 is a disk containing D, there exist $f\in\mathcal{F}_k$ with zero exterior transform and which are arbitrarily small outside of D_1 but large in D_1-D. Of course this does not constitute a proof that exterior inversion is unstable in some sense for $\mathcal{F}_\infty$ itself, but it certainly suggests trouble. What is needed is a study of the stability of the inversion formulas of [8] and [4] in the presence of noise in the measurements.

Newman's examples of $f\in\mathcal{F}_k$ for which $P_f(L)=0$ for $L\in\mathcal{L}_D'$ are constructed as follows: Let

$$f(x,y)=\operatorname{Re}\frac{1}{z^n},\quad z=x+iy,\qquad n\geqslant 2. \tag{2.3}$$

For all lines L not passing through $z=0$,

$$\int_L f(x,y)\,ds = \operatorname{Re} c \int_L \frac{1}{z^n}\,dz = 0, \tag{2.4}$$

because, along L, $ds = c\,dz$ (ds and dz are proportional) and $z^{-n}dz$ is an exact differential. It is easy to modify f inside a disk D containing $x=y=0$ so as to make f continuous. Note $f \in \mathcal{F}_k$ for $n > k$.

Another example of $\mathcal{L}$ and $\mathcal{F}$ where a mathematical inversion formula exists, but which is believed to be impractical due to instabilities, occurs with $\mathcal{L} = \mathcal{L}_1 =$ the set of all lines which make an angle of say $\leqslant 1°$ with the x-axis. Here again for $f \in \mathcal{F}_\infty$ explicit inversion is possible but the inversion involves a process of analytic continuation. Measuring $P_f(L)$ for only those $L \in \mathcal{L}$, would present significant engineering and cost advantages if a practical realization of the analytic continuation inversion were possible but this seems unlikely, again because of noise instabilities. What is needed but not available is a theory which would assign a measure of stability to the various cases above. We shall return to this point at the end of §2.

For a restricted set of lines $\mathcal{L}$ there are usually many $f \in \mathcal{F}$ with the given projections. However, a unique reconstruction may be obtained by placing an additional criterion of optimality on f. Thus Marr [**16**] has considered $\mathcal{L} = \mathcal{L}_n =$ the (finite) set of $n(n-1)/2$ lines joining all pairs of n equally spaced points on the circumference of the unit disk D, and $\mathcal{F}_m =$ the set of polynomials in x and y of degree m. For $m \leqslant n-2$, Marr gives a formula for the polynomial $g \in \mathcal{F}_m$ which minimizes the sum-of-squares error,

$$\sum_{L \in \mathcal{L}_n} \left(P_f(L) - P_g(L)\right)^2. \tag{2.5}$$

Although Marr's model is realistic in the sense that $\mathcal{L}_n$ is finite, it does not seem desirable in practice to restrict g to be a polynomial. Also if $m > n-2$, g is not unique since (2.5) can be made zero in many ways.

Moving closer to the situation of central interest, let $\mathcal{L}$ consist of all lines at one of n distinct angles $\theta_1, \dots, \theta_n$. Let $L_{t,\theta}$ be the line with equation

$$x\cos\theta + y\sin\theta = t \tag{2.6}$$

and use $P_f(t,\theta)$ for convenience to denote $P_f(L_{t,\theta})$. We assume $f \equiv 0$ outside a disk D which we take for convenience to be the unit disk. Consequently, $P_f(t,\theta) = 0$ for $|t| \geqslant 1$. We want to keep $\mathcal{F}$ reasonably wide so as not to force an inappropriate structure on f, and hence use $\mathcal{F} = L^2(D)$. Denoting the given projection data by $P_j(t)$, f must satisfy the basic equations

$$P_f(t,\theta_j) = P_j(t), \quad -1 < t < 1, \quad j = 1, \dots, n. \tag{2.7}$$

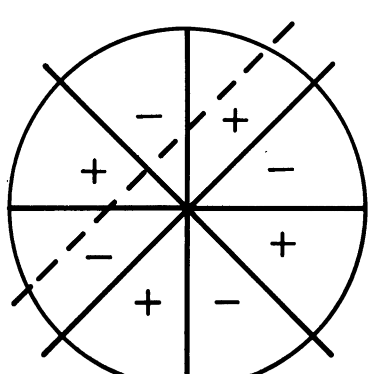

FIG. 6. Example of a function which projects to zero in each of the 4 directions, 0°, 45°, 90°, 135°.

We see from Fig. 6 that if there is one solution f to this equation there will be many solutions. (For a stronger assertion of nonuniqueness of f, see [22].) Therefore it is necessary to impose an optimality condition on f in order to obtain a unique inversion formula. One natural condition which leads to interesting results is to select among the solutions of (2.7) that solution g which has minimum possible L^2-oscillation, i.e.,

$$\int_D\int (g(x,y)-\bar{g})^2 dx\,dy = \text{minimum} \tag{2.8}$$

where $\bar{g}$ is the average of g over D. Note that $\bar{g}$ is determined from the known projections $P_j(t)$ since

$$\bar{g}=\frac{1}{\pi}\int_D\int g\,dx\,dy=\frac{1}{\pi}\int_{-1}^{1}P_j(t)dt, \qquad j=1,\ldots,n. \tag{2.9}$$

From (2.9) it is easy to see that (2.8) is equivalent to the simpler condition

$$\int_D\int g^2 dx\,dy = \text{minimum}. \tag{2.10}$$

This formulation of the problem has the advantage that the optimal g is computable explicitly in terms of P_j, at least for the case of equally spaced angles θ_j. The solution was found by Logan and Shepp [15] and the same problem was posed and solved independently by O. Tretiak [23]. To describe the solution we introduce the useful notion of a ridge function due to B. F. Logan, which is simply a function constant along lines perpendicular to the direction θ,

$$\rho(x,y)=\rho(x\cos\theta+y\sin\theta), \qquad (x,y)\in D. \tag{2.11}$$

Let $\mathfrak{R}$ be the subspace of $L^2(D)$ formed by sums of ridge functions in the directions $\theta_1,\ldots,\theta_n$. It is proved in [15] for the case of equally spaced angles θ_j that the solution to (2.7) and (2.10) has the form

$$g(x,y)=\sum_{j=1}^{n}\rho_j(x,y), \tag{2.12}$$

where ρ_j is a ridge function in direction θ_j. In fact, g is the orthogonal projection of any f satisfying (2.7) onto $\mathfrak{R}$. Further, in the case of equally spaced angles, it is possible to obtain an explicit expression for ρ_j. For this purpose introduce the change of variables,

$$h_j(\tau)=\rho_j(\cos\tau)\sin\tau, \qquad 0<\tau<\pi \tag{2.13}$$

and expand h_j and P_j into sine series

$$h_j(\tau)=\sum_{\omega=1}^{\infty}\hat{h}_j(\omega)\sin\omega\tau, \tag{2.14}$$

$$P_j(\cos\tau)=\sum_{\omega=1}^{\infty}\hat{\pi}_j(\omega)\sin\omega\tau. \tag{2.15}$$

This allows a complete diagonalization and an explicit solution for $\hat{h}_j(\omega)$ in terms of $\hat{\pi}_k(\omega)$, $k=1,\ldots,n$ for each $\omega=1,2,\ldots$, i.e., $\hat{h}_j(\omega)$ depends only on $\hat{\pi}_k(\omega)$, $k=1,\ldots,n$ for the same value of ω. This leads to an expression for each ridge function ρ_j of g in terms of convolutions with two fixed kernels operating on the projections. Unfortunately each ridge function involves all projections, which makes the formula apparently impractical (except perhaps for circularly symmetric functions).

For any $f\in L^2(D)$, there is of course a unique polynomial $R(x,y)$ of degree $n-1$ which best approximates f in the L^2 sense on D. Curiously every function f satisfying (2.7) has the *same* best fitting polynomial R. In fact this polynomial R is merely the first n terms (in ω) of the expansion of g generated by (2.13)-(2.15). This gives a fairly explicit algorithm for the optimal polynomial $R(x,y)$ of degree $n-1$. Unfortunately, for functions f of medical interest, R does not give sufficiently good approximations to be of value.

It was asserted in [15] without proof that the space $\mathfrak{R}$ of sums of ridge functions is a *closed* subspace of $L^2(D)$. This follows from the explicit inversion formula of [15] for the case of equally

spaced angles, but as several readers of **[15]** pointed out, is not proved in general in **[15]**. Fortunately the general result has recently been proved by Hamaker and Solmon [7].

B. F. Logan **[14]** proved a very interesting theorem whose practical implications for tomography are not yet entirely clear, but which deserve further study. Very roughly the theorem states that knowledge of the full projections in each of n directions is sufficient to reconstruct f up to but not beyond bandwidth n. A little more precisely, Logan shows that a function in $L^2(D)$ of essential bandwidth $n(1-\varepsilon)$ can be essentially reconstructed from any n views. On the other hand, he proves that there exist functions of essential bandwidth $n(1+\varepsilon)$ which project to zero in any n given directions. Still more precisely, suppose that n distinct directions $\theta_1,\ldots,\theta_n$ are given and that $\hat{f}$ denotes the Fourier transform of $f \in L^2(D)$. The concentration of the energy of f in the frequency band of radius ρ is defined as the ratio

$$\lambda(f;\rho)=\int_{u^2+v^2\leqslant\rho^2}\int|\hat{f}\,(u,v)|^2\,du\,dv\Big/\int_{u^2+v^2<\infty}\int|\hat{f}\,(u,v)|^2\,du\,dv. \tag{2.16}$$

To measure how n projections limit the concentration, let

$$\lambda_n(\rho)=\lambda_n(\rho;\theta_1,\ldots,\theta_n)=\sup_g\lambda(g;\rho), \tag{2.17}$$

where the sup is taken over all $g \in L^2(D)$ which project to zero in the given directions θ_j, i.e., for which $P_g(t,\theta_j)\equiv 0$. Logan proves that $\lambda_n(\rho)$ is independent of $\theta_1,\ldots,\theta_n$. He also proves that $\lambda_n(\rho)$ rapidly increases around $\rho=n$,

$$\lim_{n\to\infty}\lambda_n(n(1+\varepsilon))=\begin{cases}1 & \varepsilon>0\\ 0 & \varepsilon<0\end{cases}. \tag{2.18}$$

Actually much more precise estimates of $\lambda_n(\rho)$ are obtained in **[14]** but (2.18) will suffice for our purposes. Logan shows further that if $\lambda(f;\rho)$ is near one and $\lambda_n(\rho)$ is near zero, then n projections of f suffice to make a good L^2 estimate of f. On the other hand, by definition, if $\lambda_n(\rho)$ is near one then there exist $f \in L^2(D)$ which are well-concentrated in frequency to a circle of radius ρ, but which project to zero in the given directions and hence cannot be reconstructed.

Thus, roughly speaking, to reconstruct functions f of bandwidth ρ one needs $n=\rho$ projections of f. Unfortunately it has not yet been established what value of ρ is needed for functions f of medical interest.

It seems very surprising that the bandwidth does not depend on the angles θ_j but only on how many different angles there are. This suggests the possibility that one could use n angles within a very narrow range, which would limit the motion of the X-ray tube and allow much faster data acquisition. This would have many benefits, including perhaps the possibility of imaging the beating heart. However, it seems quite likely that any algorithm for this purpose (i.e., which would give the same resolution as in the case of equally-spaced angles) would require unrealistically precise estimates of $P_f(L)$. But, as long as such limitations have not been firmly established the narrow angle approach remains a tantalizing possibility, deserving further mathematical quantification.

3. Algorithms for practical use. In practice, measurements $P_l=P_f(L_l)$ are made for a finite set $\mathfrak{L}$ of lines, L_l, and it is desired to find an approximation $\bar{f}_q$ to $f(Q_q)$ for a grid of points Q_q, generally taken to be a square grid. It is not surprising that the properties of the set $\mathfrak{L}$ play an important rôle. For so-called "parallel-mode" tomography machines, which include all the earlier machines, $\mathfrak{L}$ consists of many equally-spaced *parallel* lines at each of many equally-spaced angles. Most of our discussion will be devoted to algorithms suitable for parallel-mode machines.

Some of the newer machines have adopted a different mode of scanning, in order to reduce the scanning time from minutes to seconds. In such "fan-beam" machines, $\mathfrak{L}$ consists of several "fans" of lines. Each fan consists of many lines through a single focal point. The focal points lie on a circle concentric to the disk on which f is supported, which has a radius R typically about 3 times that of D.

The parallel-mode set of lines can be regarded as a limiting ($R \to \infty$) special case of the fan-beam mode set, but we shall only touch lightly on algorithms for fan-beam machines.

There are two major types of algorithms for reconstruction, iterative and convolutional. The first uses an iterative procedure [11], [6], [20] which updates the current estimate of the density using each projection measurement in turn and which converges to the desired solution after about 5 or 10 complete cycles through every measurement. The first successful tomographic reconstruction algorithm (Hounsfield, 1972) used an algorithm of this type. A convolution algorithm (in the parallel mode) forms the density estimate by applying to the set of measurements a linear mapping which has a certain property, namely, that the coefficient which weights the measurement $P_f(L_l)$ in estimating the density $f(Q_q)$ is a function ϕ only of the distance from Q_q to L_l. Many different weight functions, or filters, ϕ have been suggested [1], [11], [20], [2]. One important advantage of the convolutional algorithm over the iterative algorithm is in speed of computation, since experience shows that an iterative algorithm usually takes about i times longer, where i is the number of complete iteration cycles. At one time the iterative procedures were asserted by many people [11], [5], [6], [17] to have advantages in accuracy. It was also commonly believed that there was little to be gained by improvements in the algorithm, and the iterative procedures were considered attractive because they are inherently "digital" or "discrete," or because they can in principle be used interactively for as many iterations as desired (although they are seldom used this way). It is now generally agreed, however, that the convolutional algorithms are not only faster but give reconstructions with much better accuracy and spatial resolution than the earlier iterative procedures.

Recently still further light has been shed on the comparison between iterative and convolutional algorithms. Experiments by Shepp, which were confirmed by others [9], have pinpointed the chief reason for inaccuracy in the early iterative algorithms [11], [6], and he has given an iterative algorithm [20] whose accuracy is roughly comparable with that of the better convolutional algorithms. To understand the source of the earlier inaccuracy, note that the iterative algorithms attempt to find a function f which is piecewise constant on the squares in a grid, and which has the given projections. The X-ray beam is represented by a strip rather than a line (see Appendix) and each strip gives an equation in the unknown values of f on the grid squares. The equations are solved by Gauss-Seidel relaxation. It turns out that in setting up the equation for each strip it is important to use the actual area of intersection of the strip with each grid square rather than an average value approximation to this area as described in [11] or the even cruder approximation used in [6], where weight one or zero is assigned to each square, according as its center is or is not inside the strip. That the exact coefficients are important did not become clear until the experiments by Shepp with mathematical phantoms [20].

Another limitation of the iterative procedure has to do with spatial resolution. To avoid excessive computation time, iterative procedures have been used only with grids up to 80×80, while the more efficient convolutional algorithms typically use 160×160 or larger. This is a real increase in spatial resolution and has great practical significance. We suspect that even if computation time did not limit the iterative procedures in this way, the necessity of having more equations than unknowns in iterative methods might prevent them from achieving the same spatial resolution as the convolutional algorithms. However, the question is now moot and iterative algorithms are no longer used in commercial machines. Of course it must be acknowledged that for certain oscillatory functions f and noisy measurements of $P_f(L_l)$, the iterative method produces a reconstruction which is closer in some sense to f than the convolution reconstruction.

The convolutional algorithms are fast because advantage can be taken of the special structure of the linear mapping: Direct computation of

$$\bar{f}_q = \sum_{l=1}^{N_L} c_{ql} P_f(L_l), \qquad q = 1, \dots, N_Q, \tag{3.1}$$

for a general set of weighting constants c_{ql} would involve $N_Q N_L$ multiplications and additions, where

N_Q and N_L are the numbers of grid points and measurement lines. Since typical values are $N_Q = 160 \times 160 = 25{,}600$ and $N_L = n \cdot m =$ (number of directions) $\times$ (numbers of lines or strips per direction) $= 180 \times 160 = 28{,}800$, the direct computation via (3.1) would be rather time consuming. The convolutional algorithm, as is clear from the explicit Fortran program **[20]**, involves fewer than $4m^2n + 4N_Qn$ multiplications and additions. In the convolution method c_{ql} is a function ϕ

$$c_{ql} = \phi(d_{ql}) \tag{3.2}$$

of the distance, d_{ql}, from Q_q to L_l. The linear mapping (3.1) is then carried out in 2 stages. The first stage is to perform convolutions. Each convolution is over the set of parallel projections in one of the directions with the function ϕ. The convolution is stored and the second stage is to back-project each convolution to a ridge function as will be described below. The back-projections are summed to obtain the reconstruction. (In the fan-beam case a convolution is carried out for each fan, but the back-projection is not a ridge function but depends also on the distance of Q_q to the focal point of the fan. The calculation and use of this distance makes the fan-beam computation somewhat more difficult than the parallel-beam case.)

There have been two major approaches to choosing the weight function ϕ in the convolutional algorithms. One of them, which is based on the Fourier inversion formula, we refer to as the Fourier approach. The other approach seeks to compensate for the smoothing introduced by old tomography. Because this approach results in seeking to give good approximate reconstructions of a δ-like function, we call this approach the δ-function approach. The Fourier approach has a long history, going back at least to **[3]**. Further contributions were made in **[1]**, **[19]**, **[20]**. In **[20]** the earlier works were unified and generalized by focusing attention on the role of ϕ (**[1]** and **[19]** had proposed specific but different ϕ's) which can be thought of as a filter function, in analogy with problems in communication theory and antenna design. The δ-function approach is due to Cho **[2]** and was later independently rediscovered and extended **[13]**. While both approaches lead to more or less equivalent algorithms, we feel the Fourier approach has provided a framework in which ϕ may be varied conveniently. One wants to vary ϕ in order to: (a) diminish artifacts (see the discussion in Chapter 1 of the artifact inside the skull in Fig. 4); (b) trade-off between density and spatial resolution **[20]**; (c) sharpen the smoothing effect (Appendix A) resulting from the X-ray beam having nonzero width (the beam width cannot be decreased without losing X-ray flux). The Fourier approach allows one to limit the search for a good ϕ to those with an appropriate Fourier transform, i.e., which satisfy (3.13), below.

In estimating the density at Q_q, should the weight given to the measurement $P_f(L_l)$ depend only on the distance from L_l to Q_q? This seems natural enough intuitively; why should two measurements at the same distance from Q be given different weights? It is also suggested by Radon's formula (2.1) which can be interpreted roughly speaking to say that $f(Q)$ is the integral of $P_f(L)$ over every L with a weight function which depends only on the distance q between Q and L. This intuition led Shepp to the idea that it would be no loss to restrict attention to algorithms which have this property in discrete form. This means that the weight c_{ql} in (3.1) is a function only of the distance d_{ql} from Q_q to L_l as in (3.2). As we shall see below, this restriction has provided helpful guidance in choosing an algorithm for tomographic machines operating in the parallel mode. However, the emergence of "fan-beam" machines, and the important advance in fan-beam algorithms by Lakshminarayanan **[12]** has shown that this restriction excludes valuable algorithms.

The general theory of parallel-mode algorithms having "function-of-the-distance" form due to Shepp and Logan **[20]**, has proved very useful. Both this theory and the Lakshminarayanan fan-beam algorithm are obtained by suitable (nonrigorous) approximation from the Fourier reconstruction formula. This rigorous formula for reconstructing a function from all its projections is an alternative to Radon's formula (2.1), and is almost as old (**[24]** seems to be the earliest reference, but see also **[3]**). To obtain this formula define the two-dimensional Fourier transform $\hat{f}$ of f in polar coordinates (ω, θ) by

$$\hat{f}(\omega,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-i\omega(x\cos\theta+y\sin\theta)}dx\,dy \tag{3.3}$$

and write the Fourier inversion formula,

$$f(x,y)=\left(\frac{1}{2}\pi\right)^2\int_0^{\pi}\int_{-\infty}^{\infty}\hat{f}(\omega,\theta)e^{i\omega(x\cos\theta+y\sin\theta)}|\omega|\,d\omega\,d\theta. \tag{3.4}$$

This formula can be obtained from the ordinary Fourier inversion formula by changing from cartesian to polar coordinates (ω,θ). The factor $|\omega|$, which will be very important, comes from the Jacobian of the transformation. Next we express $\hat{f}(\omega,\theta)$ in terms of P_f by changing variables in the definition, and simplifying. Set in (3.3)

$$\begin{aligned} t&=x\cos\theta+y\sin\theta\\ s&=-x\sin\theta+y\cos\theta \end{aligned} \tag{3.5}$$

and note the Jacobian of this rotation is 1, so

$$\begin{aligned}\hat{f}(\omega,\theta)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-i\omega t}ds\,dt\\ &=\int_{-\infty}^{\infty}P_f(t,\theta)e^{-i\omega t}dt=\hat{P}_f(\omega,\theta),\end{aligned} \tag{3.6}$$

where $\hat{P}_f(\omega,\theta)$ is the one-dimensional Fourier transform of $P_f(t,\theta)$ with respect to t. Substituting this in the Fourier inversion formula, we obtain

$$f(x,y)=\left(\frac{1}{2\pi}\right)^2\int_0^{\pi}\int_{-\infty}^{\infty}\hat{P}_f(\omega,\theta)e^{i\omega(x\cos\theta+y\sin\theta)}|\omega|\,d\omega\,d\theta. \tag{3.7}$$

If there were a function ϕ whose Fourier transform $\hat{\phi}$ had the form

$$\hat{\phi}(\omega)=|\omega|, \tag{3.8}$$

we could use Parseval's identity,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{P}(\omega)\hat{\phi}(\omega)e^{i\omega\tau}d\omega=\int_{-\infty}^{\infty}P(t)\phi(\tau-t)dt \tag{3.9}$$

with $\tau=x\cos\theta+y\sin\theta$ in (3.7) to obtain

$$f(x,y)=\frac{1}{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty}P_f(t,\theta)\phi(x\cos\theta+y\sin\theta-t)\,dt\,d\theta. \tag{3.10}$$

Note this has the function-of-the-distance form since $x\cos\theta+y\sin\theta-t$ is the distance from (x,y) to $L(t,\theta)$.

Unfortunately, if ϕ is an "honest" function, its Fourier transform $\hat{\phi}(\omega)$ tends to zero as $\omega\to\infty$, so (3.8) is impossible. However, by using a function ϕ whose Fourier transform $\hat{\phi}$ approximates $|\omega|$ in a suitable sense we may be able to use (3.10) as an approximate formula. We want to insure that the error in (3.10), namely

$$\left(\frac{1}{2\pi}\right)^2\int_0^{\pi}\int_{\infty}^{\infty}\hat{P}_f(\omega,\theta)e^{i\omega(x\cos\theta+y\sin\theta)}(\hat{\phi}(\omega)-|\omega|)\,d\omega\,d\theta \tag{3.11}$$

is small. Now it is plausible to assume that

$$\hat{f}(\omega,\theta)\text{ is small for large }\omega\text{ (say }|\omega|>\Omega) \tag{3.12}$$

since this is a smoothness assumption on f, i.e., it means that f contains little high frequency energy. In the region where $\hat{f}(\omega,\theta)\equiv\hat{P}_f(\omega,\theta)$ is small, we can afford a large discrepancy between $\hat{\phi}(\omega)$ and $|\omega|$. Thus if we weaken (3.8) to

$$\hat{\phi}(\omega)\approx|\omega|\quad\text{for small }\omega\text{ (say }|\omega|<\Omega), \tag{3.13}$$

then (3.10) may be a good approximation. Of course the same Ω must be used in (3.12) and (3.13).

The earlier tomography machines all operate in the "parallel mode." Without going into the details

this means that the measurements $P_f(t,\theta)$ are available for equally-spaced parallel lines at each angle θ_j. If a is the spacing between adjacent parallel lines then t takes on the values ka for $k = 0, \pm 1, \pm 2, \ldots, \pm 1/a$, for each angle. In practice, equally-spaced angles are always used, so $\theta_j = (j-1)\pi/n, j = 1,2,\ldots,n$. Then the natural discrete approximation to (3.10) is

$$f_\phi(x,y) = \frac{a}{2n} \sum_{j=1}^{n} \sum_{k=-\infty}^{\infty} P_f(ka,\theta_j)\phi(x\cos\theta_j + y\sin\theta_j - ka). \tag{3.14}$$

Of course there are only finitely many terms in the inner sum since $P_f(ka,\theta_j) = 0$ for $ka > 1$ because f is supported on the unit disk. (3.13) gives a simple algorithm for reconstructing f which depends upon the choice of ϕ.

How ϕ should be chosen is not completely understood except for guiding principles. Intuitively, it seems likely that if the spacing between the parallel lines is not fine relative to the high frequencies in $f(x,y)$, it will not be possible to reconstruct f very well. Since the spacing between the grid lines is a (a itself is chosen to achieve a desired resolution), we may assume that f has little energy above the Nyquist frequency of π/a. Thus we may plausibly take $\Omega \approx \pi/a$. This still leaves a lot of room for choice of ϕ. One possibility, suggested by Bracewell and Riddle (1956), who were the first to use the Fourier method in an algorithmic context, is

$$\hat{\phi}(\omega) = \begin{cases} |\omega|, & |\omega| < \Omega \\ 0, & |\omega| > \Omega \end{cases}. \tag{3.15}$$

However, this choice does not work well, apparently because the corresponding $\phi(t)$,

$$\phi(t) = \Omega \frac{\sin \Omega t}{\pi t} - \frac{1 - \cos \Omega t}{\pi t^2} \tag{3.16}$$

only decays like $1/t$ for large t so that the weight given by (3.14) to a measurement line L far away from (x,y) is still too large. Intuitively it seems undesirable to permit such long range effects. If $\hat{\phi}(\omega)$ is absolutely continuous, then ϕ will decay at least as fast as $1/t^2$ for large t. On the other hand, if $\hat{\phi}'(\omega)$ has the same discontinuity at $\omega = 0$ as $|\omega|$ (this is expected if (3.13) holds) then it can be expected that $\phi(t)$ will decay no faster than $1/t^2$.

In order that (3.14) should be a good approximation to f, the inner sum in (3.14) should be a good approximation to the inner integral in (3.10), i.e., we should have

$$\int_{-\infty}^{\infty} P_f(t,\theta)\phi(u-t)\,dt \approx \sum_{k=-\infty}^{\infty} P_f(ka,\theta)\phi(u-ka)a. \tag{3.17}$$

However, a function ϕ satisfying (3.13) for $\Omega \approx \pi/a$ has the shape shown in Fig. 7. Note that there are only one or two sampling points in the main lobe of ϕ. Under such sparse sampling one ordinarily would not expect the Riemann sum in (3.17) to be a good approximation to the integral. Fortunately, however, the sampling is regular and the Poisson summation formula (identity)

$$\sum_{k=-\infty}^{\infty} \hat{\psi}\left(\frac{2\pi k}{a}\right) = a \sum_{k=-\infty}^{\infty} \psi(ka) \tag{3.18}$$

may be applied. In (3.18) we use

$$\psi(t) = \psi_u(t) = P_f(t,\theta)\phi(u-t) \tag{3.19}$$

and note that the right side of (3.18) is the right side of (3.17) whereas the left side of (3.17) is the $k = 0$ term of (3.18). Therefore the error in (3.17) consists of the $k \neq 0$ terms on the left of (3.18). Using (3.6),

$$\begin{aligned} \hat{\psi}_u\left(\frac{2\pi k}{a}\right) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}\left(\omega - \frac{2\pi k}{a}\right) \hat{P}_f(\omega,\theta) e^{i\omega u}\, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}\left(\omega - \frac{2\pi k}{a}\right) \hat{f}(\omega,\theta)\, d\omega \end{aligned} \tag{3.20}$$

since the transform of a product is the convolution of the transforms. We want (3.20) to be small for $k \neq 0$.

By (3.12) $\hat{f}(\omega,\theta)$ may be large for $\omega < \Omega$. Indeed, since the X-ray density typically stays fairly constant within many organs we may expect that $\hat{f}(0,\theta)$ is large. To make (3.20) small, we want $\hat{\phi}(\omega - (2\pi k/a))$ to be small whenever $|\omega| < \Omega$ and $k \neq 0$ and particularly small whenever $\omega = 2\pi k/a$.

These arguments may seem crude to those accustomed to pure mathematics, but they are typical of much applied mathematics. The guidance they provide in choosing ϕ has been very helpful. Reconstructions of f with different ϕ's are compared by the method discussed in Chapter 1.

As we shall see later on, in the case of fan-beam projection data, the values of $P_f(t,\theta)$ are only available for unequally-spaced values of t. In this situation the Poisson sum argument used to obtain (3.17) breaks down, and it is not surprising that there seems to be no choice of ϕ which yields good reconstructions. However, by interpolating in t to obtain approximate values of $P_f(t,\theta)$ at regularly spaced values of t, it is possible to obtain good reconstructions. Unfortunately this interpolation leads to significant errors in case the measurements of $P_f(t,\theta)$ are corrupted by drift and gain variation in X-ray detectors. This is a crucial point which will be referred to later in discussing the adaptation of the parallel-mode algorithm to fan-beam machines.

Many convolution algorithms can be thought of in the framework we have described above, e.g., old tomography corresponds to use of the special ϕ where $\phi(t) = 1$ for $|t| \leqslant a$ and 0 otherwise. Bracewell and Riddle **[1]**, who gave the first Fourier based convolution algorithm, used (3.16). Ramachandran and Lakshminarayanan **[19]** used a function ϕ which is the same as (3.16) for $\Omega = \pi/a$ at the points $t = ka$, $k = 0, \pm 1, \ldots$ and is piecewise linear in between. They introduced this function to achieve substantial savings in computation time (since it avoids a great many sine and cosine evaluations required by (3.16)) but assumed that the modification would degrade the image somewhat. Within our framework, however, there is no reason to expect the piecewise linear modification to do worse than the original (see **[20]** for more details) and experimental results bear this out.

Shepp and Logan **[20]** were the first to look at the choice of ϕ in a systematic way. They found the principles above to provide good guidance. They also expressed the trade-off relation between spatial and density resolution in terms of the choice of ϕ. The choice (3.16) and the modification both have the property that $\phi(t)$ continues to oscillate even for large t.

To avoid these oscillations, and to achieve the computational efficiency of a piecewise linear filter, a new filter was designed by the following approach. We wish to emphasize the approach, rather than the particular filter which was chosen, since many other choices might be quite as effective. As a source of piecewise linear filters, we may take even functions $\hat{\phi}(\omega)$ which are approximately equal to (3.15) in a suitable sense, calculate their inverse transforms, and then take piecewise linear approximations. To evaluate the piecewise linear filters, we may examine their transforms, the image reconstructions which they yield, and various other considerations. The filter used in **[20]** was chosen after examining several filters which were generated in this way. It is a piecewise linear approximation to the inverse transform of

$$\hat{\phi}(\omega) = \begin{cases} 2|\sin(\omega a/2)|/a & \text{for } |\omega| \leqslant 2\pi/a, \\ 0 & \text{elsewhere,} \end{cases}$$

a function which was suggested by B. F. Logan. This filter is given by:

$$\begin{cases} \phi(ka) = -\dfrac{4}{\pi^2 a}\dfrac{1}{4k^2-1}, & k = 0, \pm 1, \ldots \\ \phi \text{ piecewise linear in between these values.} \end{cases} \tag{3.21}$$

The piecewise linearity saves substantial computation time (as was first observed in **[19]** with no apparent loss in image quality). Using any such function ϕ, the calculation of (3.14) may be arranged to be very efficient (see **[20]** for more details and a simple 60-line Fortran program).

As we mentioned earlier, there is another approach to convolution algorithms which may be

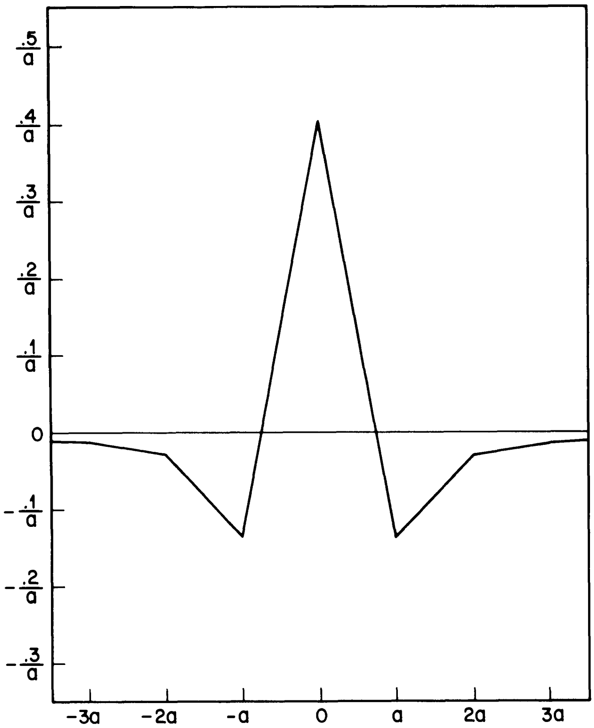

FIG. 7. The reconstruction filter ϕ of (3.21). The algorithm assigns weight $\phi(d)$ to a line L at a distance d from the point of reconstruction. Note that a is the distance between adjacent centerlines of ray measurements. This is typical of reconstruction filters, having positive main lobe and negative side lobes.

discussed in the same framework. This interesting and natural approach, which is due to Z. H. Cho [2] but was also arrived at by LeMay [13], avoids Fourier transforms altogether. Roughly speaking, the idea is to start with "old tomography" and correct its inadequacies. Old tomography forms a ridge function from $P_f(t,\theta_j)$ in the direction orthogonal to θ_j. The ridge function, which is called the "back projection" of $P_f(\cdot,\theta_j)$, has $P_f(\cdot,\theta_j)$ as a cross-section. Old tomography then sums the back projections. The result, as we saw in (2.2), is a smeared version of the desired function f.

To correct this smearing, Cho decided to convolve $P_f(t,\theta)$ with some function $\phi(t)$ before back projecting. By this means he arrived at (3.14) using a much more direct route than the one we described. To achieve the desired effect, one can see that ϕ should have a central peak surrounded by negative "sidelobes."

Roughly speaking, any function $f(x,y)$ can be formed as a linear combination of functions which are sharply peaked at one location and zero elsewhere, in other words, approximate δ-functions. Therefore a linear algorithm which correctly reconstructs an approximate δ-function will correctly reconstruct any density function. Cho used a cylindrical spike S of radius a and height one as an approximate δ-function and chose ϕ so that the average of the back projections

$$\frac{1}{n}\sum_{j=1}^{n}\phi(x\cos\theta_j+y\sin\theta_j)\approx\frac{1}{\pi}\int_0^{\pi}\phi(x\cos\theta+y\sin\theta)d\theta \tag{3.22}$$

would approximate S well.

Cho's algorithm [2] also involves several other features of the back projection which are quite outside the simple framework we have discussed. Experiments by Shepp seem to show, however, that these other features do not play an important role in his algorithm, and that the algorithm (3.14) using Cho's function ϕ (modified to be piecewise linear) gives results quite like Cho's. LeMay's algorithm [13] has much in common with Cho's.

The algorithms of [2] and [13] give good results, as illustrated in Fig. 4. However, as we discussed in the introduction, they give a characteristic artifact just inside the skull, which could conceal medically important conditions. Also the functions ϕ which Cho and LeMay obtained are defined in terms of a sequence of numerical values. This type of ϕ is more complicated to use in noise analyses [21] than is the analytically explicit and especially simple ϕ in (3.21). Furthermore, this approach does not provide as much guidance in modifying ϕ to achieve the desirable trade-off of density vs. spatial resolution.

An important virtue of any theoretical framework in which to obtain ideas or algorithms is flexibility in adapting to new situations. Some of the new tomography machines do not operate in the "parallel-mode" mentioned earlier, but in the "fan-beam mode," in order to reduce the X-ray scanning time from minutes to seconds. In this mode $P_f(L)$ is measured for n "fans" of lines L, where the lines in each fan have a common focal point (at an X-ray detector). The n focal points lie on a circle, concentric with the unit circle on which f is nonzero, which has a radius R typically about 3 times as large. (The limiting case as $R\to\infty$ is the parallel mode.)

It is interesting to see how the Fourier approach to convolutional algorithms can be adapted to this situation. While it is possible to regroup the lines into sets of (almost) parallel lines at each of several angles, the parallel lines are not equally-spaced. One approach would be to maintain the restriction to linear algorithms satisfying (3.2), which states that the reconstruction weight for each line L depends only on the distance from L to the point Q of reconstruction. However, as discussed earlier, the Poisson-sum argument breaks down because the parallel lines are not equally spaced. No function ϕ has been found which gives good reconstructions in this situation.

Another approach is to interpolate P_f between the irregularly spaced and not quite parallel lines to obtain pseudomeasurements at an equally spaced set of exactly parallel lines. Although this interpolation is neither simple nor elegant, this method can be made to work reasonably well, subject to one important proviso which makes it quite useless in practice.

The proviso is that if there are any systematic differences between the fans, due to differences between the X-ray detectors with which they are associated, the image is seriously marred by streaks. Unfortunately, it seems difficult to control the drift and gain problems in detector behavior to a sufficient degree of uniformity to avoid this problem.

Fortunately, an elegant effective algorithm has been found by a young biophysicist, A. V. Lakshminarayanan [12], by an approach which uses the fan-beam geometry rather than fighting it. Starting with the Fourier inversion formula (3.7) he made changes of variables to adapt these formulas to the fan-beam coordinates. He then succeeded in making approximations similar to those described above, and obtained an algorithm [12] which is now widely used in fan-beam scanners. Subsequently another derivation for this algorithm, based on Radon's inversion formula and using different approximations, has been found by Herman and Naparstek [10].

Appendix. We study the relationship between the line integral of f

$$P_f(t,\theta)=\int_{-\infty}^{\infty} f(t\cos\theta+s\sin\theta,t\sin\theta-s\cos\theta)\,ds \tag{A.1}$$

along the line $L(t,\theta)$ in (2.6) and the strip integral of f

$$Q_f(t,\theta)=(1/2\delta)\int_{-\delta}^{\delta} P_f(t-u,\theta)\,du \tag{A.2}$$

obtained by integrating f over a strip of width 2δ about $L(t,\theta)$ (and dividing by the width of the strip). We show that (A.2) is actually the line integral of a smoothed version, $f^k(x,y)$, of f obtained by convolving f with a centrally symmetric two-dimensional kernel k,

$$f^k(x,y)=\int_0^{2\pi}\int_0^{\infty} f(x-\rho\cos\alpha,y-\rho\sin\alpha)k(\rho)\rho\,d\rho\,d\alpha. \tag{A.3}$$

In the more general case, where the strip has weight function W, denote the W-weighted strip integral by

$$Q_f^W(t,\theta)=\int_{-\infty}^{\infty} W(u)P_f(t-u,\theta)du. \tag{A.4}$$

Note (A.2) is the special case

$$W(u)=\begin{cases} 1/2\delta; & |u|\leqslant\delta \\ 0; & |u|>\delta \end{cases}. \tag{A.5}$$

We show that the line integrals of f^k are the strip integrals of f, i.e.,

$$P_{f^k}(t,\theta)=Q_f^W(t,\theta) \tag{A.6}$$

where k is the kernel given by the Abel transform of W, i.e.,

$$W(u)=\int_{-\infty}^{\infty} k(\sqrt{u^2+v^2}\,)dv; \qquad k(\rho)=-\frac{1}{\pi\rho}\frac{d}{d\rho}\int_{\rho}^{\infty}\frac{W(u)u}{\sqrt{u^2-\rho^2}}du. \tag{A.7}$$

This shows that regarding actual measurements as line measurements, when of course the beam has a nonzero width, is not serious, in the sense that one is merely reconstructing f^k rather than f, which for thin beams is a reasonable approximation to f. However, we hasten to point out that in actual X-ray measurements neither $P_f(t,\theta)$ nor $Q_f^w(t,\theta)$ is measured (neglecting statistical errors, the nonzero height of the beam, and polychromatic effects) but rather a nonlinear weighting is obtained. Thus if a parallel X-ray beam with profile $W(u)$ is passed through an object of attenuation f, the log of the ratio input/output intensities is given by

$$\log I_{\text{in}}/I_{\text{out}}=-\log\int_{-\infty}^{\infty} W(u)\exp^{-P_f(t-u,\theta)}du. \tag{A.8}$$

If W is δ-function like; i.e., if the beam is very narrow and if $P(t,\theta)$ is smooth in t, then the linearized approximation to (A.8) is (assuming W integrates to unity)

$$\log I_{\text{in}}/I_{\text{out}}\doteq\int_{\infty}^{\infty} W(u)P(t-u,\theta)du=Q_f^W(t,\theta) \tag{A.9}$$

which is a good approximation. This approximation is the spatial analogue of the linearization in energy with a polychromatic beam.

To prove (A.6) and (A.7), note that from (A.1) and (A.3)

$$P_{f^k}(t,\theta)=\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} f(t\cos\theta+s\sin\theta-\rho\cos\alpha,\ t\sin\theta-s\cos\theta-\rho\sin\alpha)k(\rho)\rho d\rho d\theta ds. \tag{A.10}$$

Make the change of variables

$$\begin{aligned}\rho\cos\alpha&=u\cos\theta+v\sin\theta\\ \rho\sin\alpha&=u\sin\theta-v\cos\theta\\ s'&=s-v\end{aligned} \tag{A.11}$$

and use the definition of W to obtain (A.6) after a short calculation. The second part of (A.7) is the well-known Abel inversion formula which is easily proved.

From (A.7), for a square beam as in (A.5),

$$k(\rho)=\begin{cases}(1/2\pi\rho)(\delta^2-\rho^2)^{-1/2} & 0\leqslant\rho<\delta\\ 0 & \delta\leqslant\rho.\end{cases} \tag{A.12}$$

For a circular beam of radius δ, from (A.7),

$$W(u)=\begin{cases}(1/\pi\delta^2)(\delta^2-u^2)^{1/2} & |u|\leqslant\delta\\ 0 & u>\delta\end{cases} \tag{A.13}$$

$$k(\rho)=\begin{cases}(1/\pi\delta^2), & \rho<\delta\\ 0, & \rho>\delta\end{cases} \tag{A.14}$$

which is a uniform weight.

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THE PLANE SYMMETRY GROUPS: THEIR RECOGNITION AND NOTATION

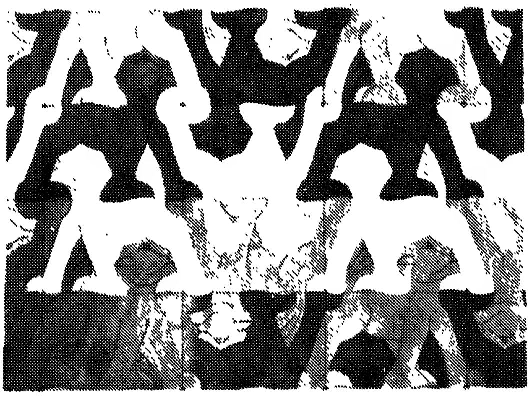
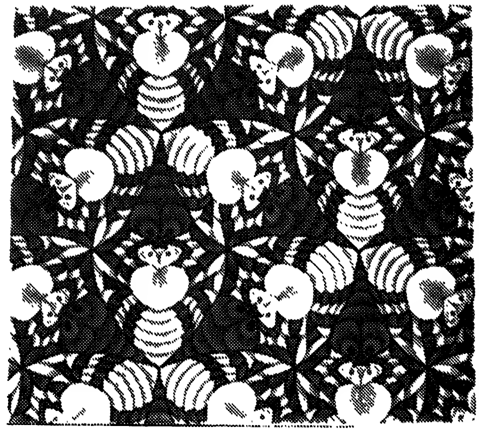
DORIS SCHATTSCHNEIDER

Introduction. Groups of transformations which leave invariant a specified item are familiar objects of study for students and researchers alike. Finite groups of plane isometries which leave invariant a regular polygon are elementary examples: C_n, the cyclic group of order n, can be realized as the group of rotations leaving invariant a regular n-gon, and D_n, the dihedral group of order $2n$, can be realized as the group of all isometries (rotations and reflections) leaving invariant the same polygon. A very interesting collection of discrete groups of plane isometries which are natural extensions of these examples exists, but is lacking in most introductory algebra texts. These are the groups of plane isometries which leave invariant a design or pattern in the plane. If the pattern is finite, such a group is necessarily a subgroup of some dihedral group. If the pattern is repeated regularly in one or in two directions, translations and glide-reflections are additional possible isometries of the pattern, and so the group leaving such a design invariant will be an infinite discrete group. Designs which are invariant under all multiples of just one translation are frieze, or border ornaments, and their associated groups are commonly called "frieze groups." Patterns which are invariant under linear combinations of two linearly independent translations repeat at regular intervals in two directions, and hence their groups are often termed "wallpaper groups."

The interweaving of elementary aspects of Euclidean transformation geometry and group theory makes these groups excellent ones for study—but there are several non-mathematical bonuses which make their study especially appealing. To analyze a repeating design to see what makes it "work," and to create original designs using the power of the mathematical "laws" which govern these designs, is a strong non-mathematical motive for studying these groups. (Suddenly, the word "symmetry" has true dual meaning; both its artistic and mathematical connotations are seen as inseparable.) Rudiments of elementary crystallography are part of the theory as well—another bonus.

A very specific incentive to learn about these groups is the opportunity to study examples of the imaginative interlocking patterns by the Dutch artist M. C. Escher (1898–1972). His work is perhaps the most concrete testament to the power gained in understanding these groups. He struggled for several years to produce animate interlocking designs, with very primitive results. When he became aware that these types of designs were governed by groups of isometries, he studied the mathematical literature available. In examining Escher's notebooks, this author discovered that he copied in full the

Doris Schattschneider received her Ph.D. from Yale University in 1966 (in the area of algebraic groups). After teaching at Northwestern University and the University of Illinois at Chicago Circle, she came to Moravian College in 1968. Here, her interest in art led her to create a January Term course, "Tessellations, the Mathematical Art," in which much of the information for this MONTHLY article was developed. A deepened interest in the work of M. C. Escher was a natural outgrowth of this course. Recently, she has collaborated with a graphic artist to produce a book and a collection of unique geometric models, *M. C. Escher Kaleidocycles*, published by Ballantine Books.—*Editors.*

Escher Foundation, Haags Gemeentemuseum, The Hague.

Two periodic drawings by M. C. Escher contrast his early effort at repeating design with his later masterful skill. The pattern of lions, dated "1926 or 1927," was done before he developed a system which grew out of his study of mathematical articles and periodic designs on the Alhambra. The pattern of bugs is dated 1942, one year after Escher recorded his codified system in notebooks.

paper by G. Pólya **[18]** which outlines the important properties of each of the groups and includes a chart of illustrative designs. (This chart is also reproduced in **[12]**, page 78.) Escher records that this visual information was of more importance to him than the written text. Another rich source of visual information for Escher was found in the Moorish tile patterns of the Alhambra, in Granada, Spain. He visited this site and carefully recorded in sketchbooks many of these periodic geometric designs. The designs he produced after he digested this information (and ultimately worked out his own system) are amazingly intricate, even mind boggling, to the innocent viewer.

The literature available on the plane symmetry groups is scattered, and often incomplete. Good descriptions of the frieze groups do exist (**[1]**, **[4]**, **[5]**, **[6]**, **[11]**, **[20]**). However, gathering complete and coherent information from the various sources on the "wallpaper groups" can be frustrating, since terminology is not standard and several different notations for the groups are used. In addition, a frequent error occurs—the notations for two of the groups are interchanged in several sources.

This article attempts to provide in compact form information to correct these problems and, in addition, provide useful visual references for readers of the literature on the plane symmetry groups. The sources used are listed in the references. Many other books and articles contain information on this topic; the remarks that follow pertain to these as well.

Terminology. Classification of periodic patterns. Part of the difficulty in reading from various sources on the plane symmetry groups is the variation in terminology used by authors. Not only are different terms used to identify the same object, but sometimes the same terms are employed (in different sources) to identify different objects. In this section we define terms as used in this presentation and indicate some other common terminology. In using any source, the reader should be especially careful to determine the definition of terms used by the author.

A "*periodic*" or "*repeating*" *pattern* in the plane is a design having the following property: There exist a finite region and two linearly independent translations such that the set of all images of the region when acted on by the group generated by these translations produces the original design. In addition (although rarely stated explicitly) it is assumed that there is a translation vector of minimum length that maps the pattern onto itself. This excludes a pattern of stripes from being termed periodic.

The *translation group* of a periodic pattern is the set of all translations which map the pattern onto itself. A smallest region of the plane having the property that the set of its images under this translation group covers the plane is called a *unit* of the pattern. All units have the same area, but

their outlines can have infinite variation. They are like tiles, all alike, which fill the plane without gaps or overlaps, and are laid in parallel rows. Some periodic designs incorporate part or all of the boundary of a unit as part of the design; others suppress this outline and one sees only a repeated figure against a blank background. For example, in the Escher lion design, four interlocked lions, each one facing in a different direction, form a unit of the pattern.

Every periodic pattern has naturally associated to it a *lattice* of points; choosing any point in the pattern, this lattice is the set of all images of that point when acted on by the translation group of the pattern. A *lattice unit* is a unit which is a parallelogram whose vertices are lattice points. The vectors which form the sides of a lattice unit generate the translation group of the pattern. (Crystallographers use the term *primitive cell* for a lattice unit; some authors use the term *unit cell*, or *cell*.)

In addition to translations, a periodic pattern may also be mapped onto itself by any of the other plane isometries: rotations, reflections or glide reflections. The *symmetry group* of the pattern is the set of all isometries which map the pattern onto itself. The classification of periodic patterns according to their symmetry groups is the two-dimensional counterpart of the system used by crystallographers to classify crystals. Hence, these groups are also termed the *two-dimensional crystallographic groups*.

The symmetry group of a periodic pattern necessarily maps a lattice associated to the pattern onto itself. Since centers of rotation of a pattern are mapped by translations to new centers of rotation (having the same order), only rotations of order 2, 3, 4, or 6 can occur as isometries of a periodic design. (This is often referred to as the crystallographic restriction.) If a pattern has no rotational symmetry, but reflections or glide reflections are in its symmetry group, then the lattice must have parallel rows of points at right angles to each other. These restrictions imply that there are five distinct types of lattice which can occur as the most general lattice possible for a plane symmetry group. For each lattice type there are conventionally chosen lattice units for purposes of classification. Chart 1 shows the five types of lattice, and for each a lattice unit.

LATTICES FOR PERIODIC PLANE PATTERNS

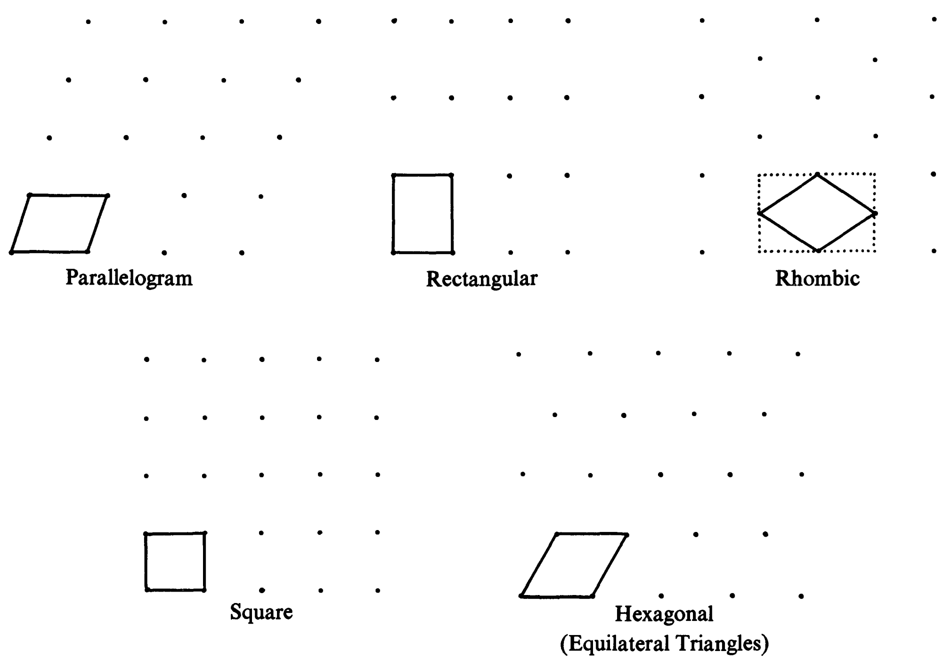


CHART 1. The lattice units outlined are those chosen by crystallographers for purposes of classification. The "centered cell," containing 2 units, is shown in dotted outline on the rhombic lattice.

LATTICE UNITS WITH SYMMETRIES OF PERIODIC PLANE PATTERNS

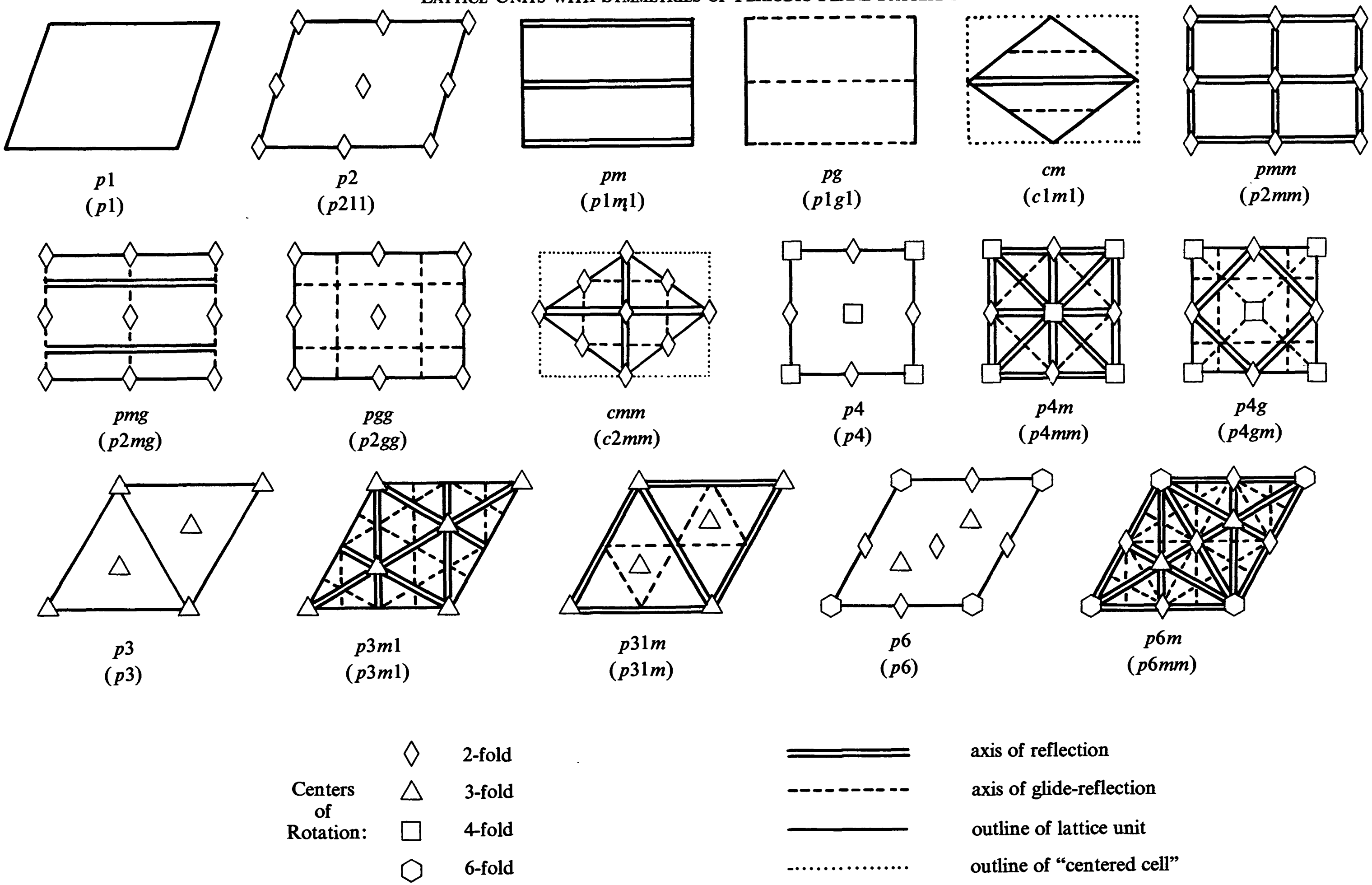


CHART 2. International notation identifies the seventeen two-dimensional crystallographic groups. The short form is given first, with the full notation in parentheses.

Arguing on the isometries possible for each of the five lattice types, it can be shown that there are seventeen distinct plane symmetry groups. In Chart 2, we show for each group a lattice unit and the placement of symmetry elements in the group relative to that lattice unit (i.e., centers of rotation, axes of reflection and glide reflection). This is an adaptation of the symbolism used in the International Tables for X-ray Crystallography [13]. Under each diagram is the crystallographic name or symbol for that group, both in the short and full form. Full explanation of this notation is found in [13] but is lacking in the mathematical literature, and so it might be helpful to include it here.

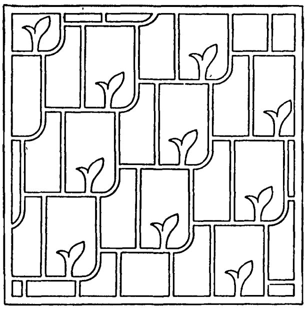
The crystallographic notation consists of four symbols which identify the conventionally chosen "cell," the highest order of rotation, and other fundamental symmetries. Usually a "primitive cell" (a lattice unit) is chosen with centers of highest order of rotation at the vertices. In two cases a "centered cell" is chosen so that reflection axes will be normal to one or both sides of the cell. The "x-axis" of the cell is the left edge of the cell (the vector directed downward). The interpretation of the full international symbol (read left to right) is as follows: (1) letter p or c denotes primitive or centered cell; (2) integer n denotes highest order of rotation; (3) symbol denotes a symmetry axis normal to the x-axis: m (mirror) indicates a reflection axis, g indicates no reflection, but a glide-reflection axis, 1 indicates no symmetry axis; (4) symbol denotes a symmetry axis at angle α to x-axis, with α dependent on n, the highest order of rotation: $\alpha=180°$ for $n=1$ or 2, $\alpha=45°$ for $n=4$, $\alpha=60°$ for $n=3$ or 6; the symbols m,g, 1 are interpreted as in (3). No symbols in the third and fourth position indicate that the group contains no reflections or glide-reflections. The many symmetry axes you see in the diagrams on Chart 2 result from the combination of translations or rotations with the symmetries indicated in the third and fourth position of the international symbol. Except in the case

RECOGNITION CHART FOR PLANE PERIODIC PATTERNS

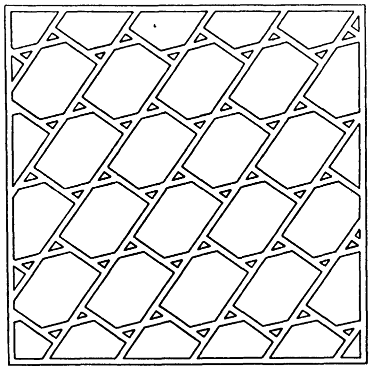
Type	Lattice	Highest Order of Rotation	Reflections	Non-Trivial Glide Reflections	Generating Region	Helpful Distinguishing Properties
$p1$	parallelogram	1	no	no	1 unit	
$p2$	parallelogram	2	no	no	1/2 unit	
pm	rectangular	1	yes	no	1/2 unit	
pg	rectangular	1	no	yes	1/2 unit	
cm	rhombic	1	yes	yes	1/2 unit	
pmm	rectangular	2	yes	no	1/4 unit	
pmg	rectangular	2	yes	yes	1/4 unit	parallel reflection axes
pgg	rectangular	2	no	yes	1/4 unit	
cmm	rhombic	2	yes	yes	1/4 unit	perpendicular reflection axes
$p4$	square	4	no	no	1/4 unit	
$p4m$	square	4	yes	yes	1/8 unit	4-fold centers on reflection axes
$p4g$	square	4	yes	yes	1/8 unit	4-fold centers not on reflection axes
$p3$	hexagonal	3	no	no	1/3 unit	
$p3m1$	hexagonal	3	yes	yes	1/6 unit	all 3-fold centers on reflection axes
$p31m$	hexagonal	3	yes	yes	1/6 unit	not all 3-fold centers on reflection axes
$p6$	hexagonal	6	no	no	1/6 unit	
$p6m$	hexagonal	6	yes	yes	1/12 unit	

CHART 3. A rotation through an angle of $360°/n$ is said to have order n. A glide-reflection is non-trivial if its component translation and reflection are not symmetries of the pattern.

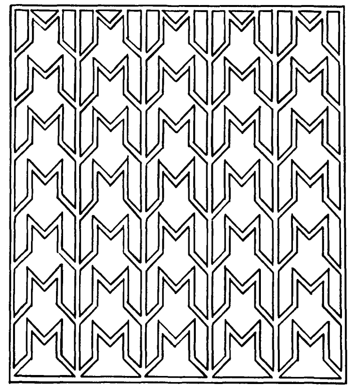
REPRESENTATIVE PATTERNS FOR



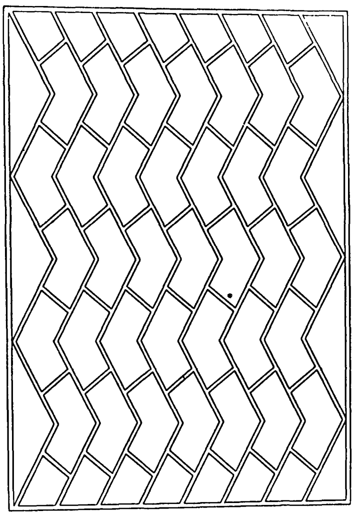
$p1$



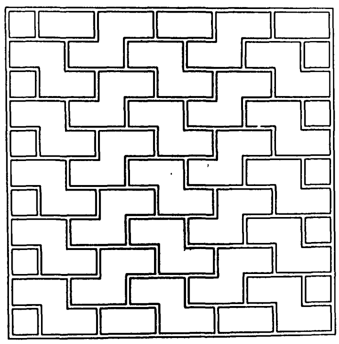
$p2$



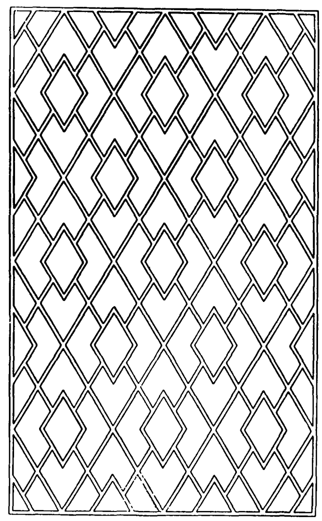
pm



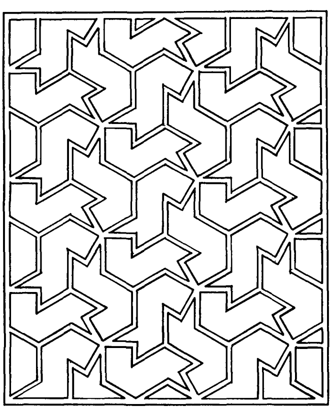
pmg



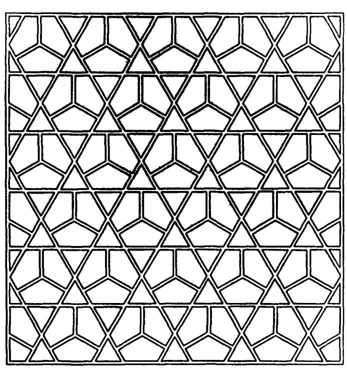
pgg



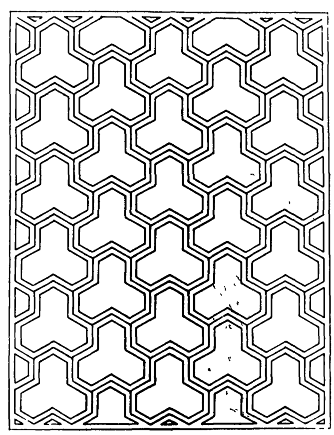
cmm



$p3$



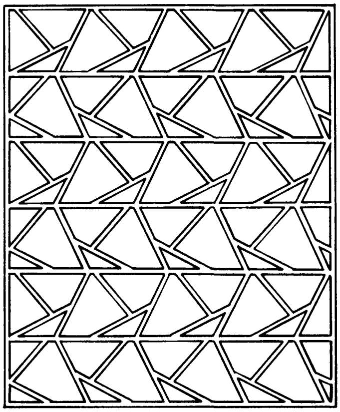
$p3m1$



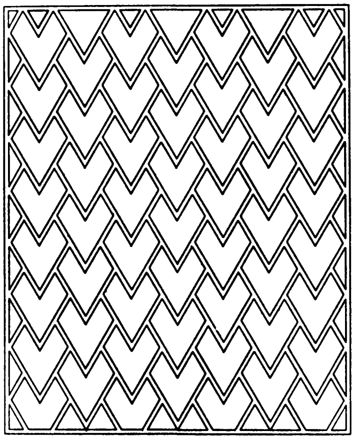
$p31m$

CHART 4. All designs except *pm*, *p*3, *pg* are found in **[10]**. The designs for *p*3 and *pg* are based on elements of Chinese lattice desi found in this book; the design for *pm* is based on a weaving pattern from the Sandwich Islands, found in **[14]**.

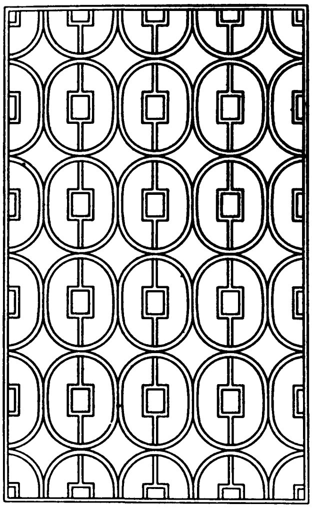
THE PLANE SYMMETRY GROUPS



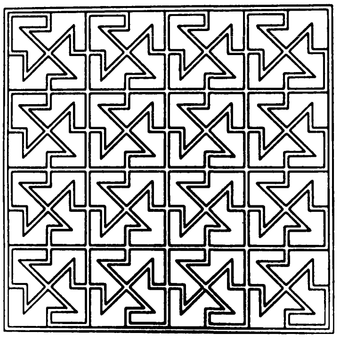
pg



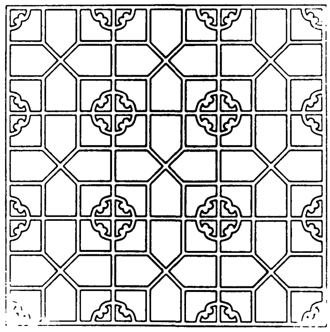
cm



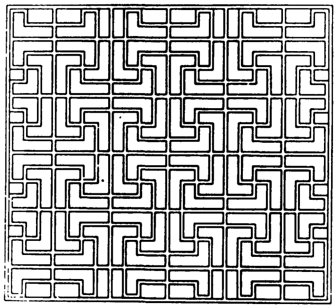
pmm



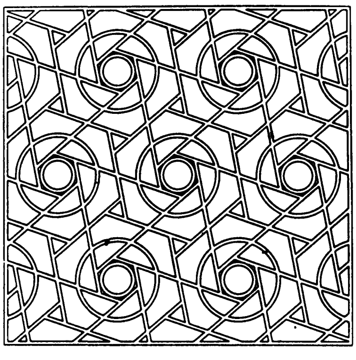
*p*4



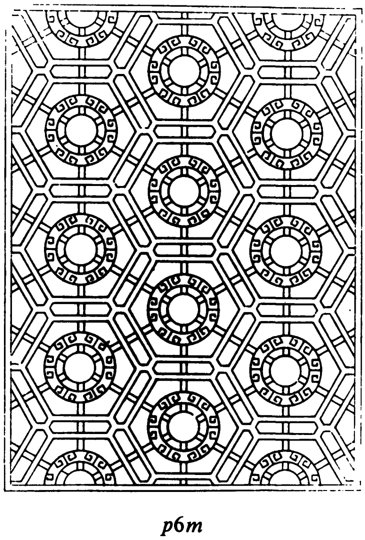
*p*4*m*



*p*4*g*



*p*6



*p*6*m*

ART 4. All designs except *pm*, *p*3, *pg* are found in [10]. The designs for *p*3 and *pg* are based on elements of Chinese lattice designs nd in this book; the design for *pm* is based on a weaving pattern from the Sandwich Islands, found in [14].

of $p3m1$ and $p31m$, the four-place symbols can be shortened without loss of identification and the shortened form of notation is in most common usage.

Recognition and classification of periodic patterns can be fun; in fact once you begin looking for them, you become aware of how surrounded we are by these ornamental designs. (There is a slight warning to those who engage in this pastime. While staring at a stranger's printed dress or gazing intently at a carpet design, you may find yourself the object of curious stares!) To classify a periodic design as one of the seventeen types, it is not necessary to obtain all the information indicated on Chart 2. A check list for recognition of patterns is provided in Chart 3. This was the result of several attempts by the author and students to reduce to a minimum the information necessary to distinguish between designs. Using this you can classify any design as to its symmetry group. For example, the Escher design of lions has only 2-fold rotations (centered where the paws meet) and has glide reflections, but no reflections; hence it is type *pgg*. The reader should assign the correct pattern type to the design of bugs.

In addition to pattern books for wallpaper, tiles, floor coverings and fabrics, collections of decorative art also provide rich sources of patterns. The collection of M. C. Escher's designs **[15]** is a delightful source for analyzing patterns and contains commentary by C. H. MacGillavry, a crystallographer, aimed at helping the beginner discover the symmetries of the patterns. Three other widely differing collections currently in print appear in the bibliography: [**2**], [**10**], [**14**]. Journal articles [**8**], [**9**], [**19**] are also of interest. Museum collections often contain a wide variety of sources of repeating patterns which attest to the timelessness and universality of their use as decorative art. See the article "Mathematics and Islamic Art," by John Niman and Jane Norman, this MONTHLY, pp. 489–490.

We provide representative patterns in Chart 4 to test the reader's ability to recognize the various types of designs associated to the seventeen groups. For each of these patterns, an instructive exercise is to find a lattice unit of the type shown in Chart 2. (Hint: Begin by looking for a center of rotation of highest order for the pattern; next find axes of reflection or glide reflection.)

Group generators. Creation of periodic patterns. Finding generators for a group is a standard task. In the case of the plane symmetry groups, however, it has more than algebraic importance. Not only will a few isometries generate the symmetry group of a periodic design, but the same isometries, acting on a small portion of the design, will produce replicas of this region, and create the total plane-covering design. We call a *generating region* of a periodic pattern a smallest region of the plane whose images under the full symmetry group of the pattern cover the plane. (Crystallographers use the term *asymmetric unit* for a generating region; several mathematicians use the term *fundamental region* or *fundamental domain*.) The area of a generating region will always be a rational part of a unit, and, as with units, all generating regions of a pattern will have the same area, even though their outlines can vary greatly. In the Escher design of lions, one lion is a generating region. Often the term *motif* is used to denote the smallest portion of a design which generates the whole periodic design when acted on by the symmetry group of the design; in this usage the motif is a symbol which can be located within a generating region.

For algebraic (and geometric) analysis of these groups, a minimal set of generators is the most desirable choice. However, if we wish to use a set of generators to create a design by having it act on a generating region, a different choice of generators may be better suited to the task. In Chart 5 we show for each group two sets of generators and their location relative to a lattice unit containing a generating region of the pattern. The choice of a minimal set of generators is adapted from [**6**, Table 1] (other minimal sets of generators are indicated in [**11**, p. 40]). For each group, the second set of generators given includes the translation vectors which form the sides of the lattice unit. These generators may be preferred for producing a tile or printing a pattern by hand or computer. The rotations, reflections, or glide reflections shown fill out a unit with images of the generating region; then the translations repeat this unit to cover the plane. (A lattice unit is filled out for the first 12 types; a unit in the shape of a regular hexagon is filled out for the last 5 types.)

GENERATORS FOR THE PLANE SYMMETRY GROUPS

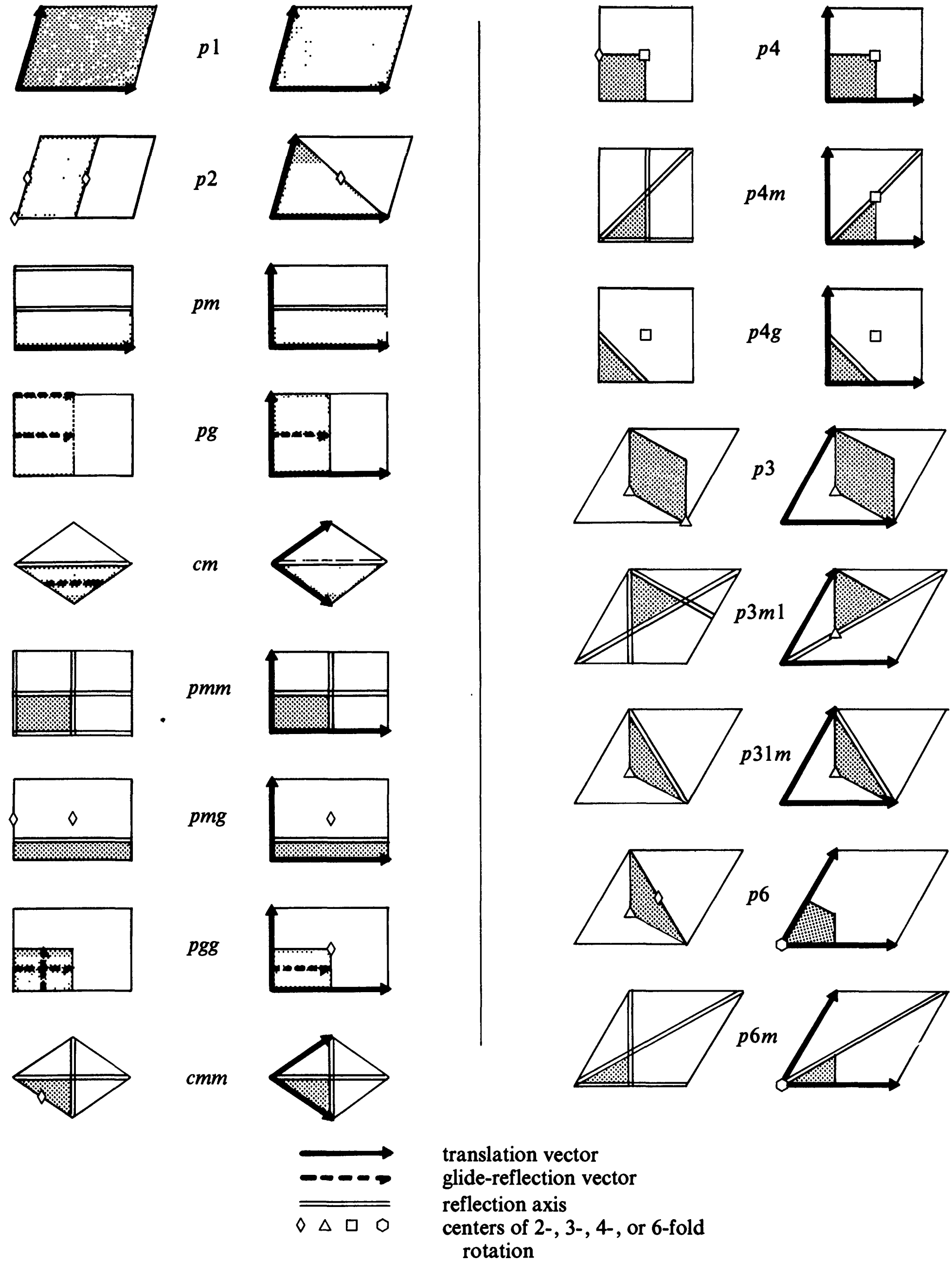


CHART 5. For each group, two sets of generators are indicated relative to a lattice unit containing a shaded generating region. A minimal set of generators is shown at the left, while a set of generators which includes the lattice unit translation vectors is shown at the right.

Since patterns of types $p3m1$ and $p31m$ are often confused, we demonstrate in Figure 1 how to use Chart 5 by creating a pattern of each type generated from the same motif. In each case, we begin with a single "hockey stick" motif, placed in a shaded generating region of each pattern type, and having its endpoints at centers of three-fold rotation. Each pattern is then produced by acting on the generating region by the isometries indicated on Chart 5, in the sequence shown.

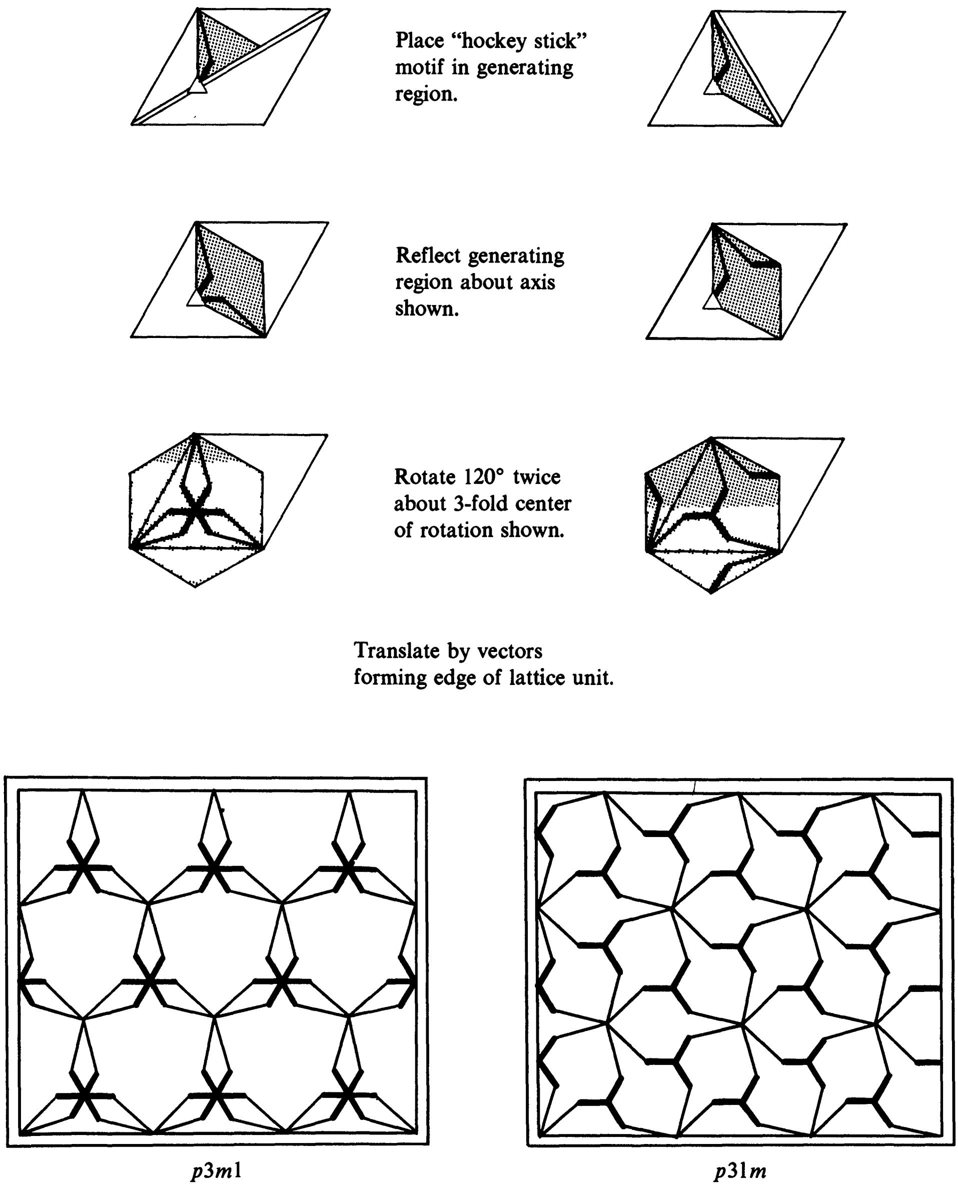


FIG. 1. Generating patterns of types $p3m1$ and $p31m$, beginning with the same motif. Note that in the completed $p31m$ pattern, a "natural" generating region is half of the arrow-shaped tile, while a "natural" unit is 3 of these interlocked tiles whose outline looks like a rotor.

Visually, the differences in the two patterns in Figure 1 are striking—there is no possibility that one pattern could be mistaken for the other. Charts which show each of the seventeen patterns which arise when the same motif (properly placed in a generating region) is acted on by each of the symmetry groups provide a clear visual demonstration of their differences. Such charts can be found in [4], [5], [11], [17], [20]. (Note the remarks in the next section concerning $p3m1$ and $p31m$.)

Notations for the groups. Interchange of p3ml and p31m. The history of the classification of the two-dimensional crystallographic groups dates back to the late nineteenth century. Both [7] and [11] give a brief account of this history. An extensive discussion and comparison of the early literature is

given in [3]. Many mathematicians have produced varying notations for the groups, and the variety of notations continues, even in books published within the last twenty years. Thus it is difficult to read the literature without some crossreference chart of notation. The notation adopted by the International Union of Crystallography in 1952 is in most widespread use, and this was used as the "norm" in preparing our crossreference table of notation, Chart 6.

Comparison of Notation for the Plane Symmetry Groups

Internat'l (short)	Pólya; Guggenheimer	Niggli	Speiser	Fejes Tóth; Cadwell	Shubnikov-Koptsik	Wells Bell & Fletcher
$p1$	C_1	$C_1{}^{\mathrm{I}}$	C_1, Abb. 17	W_1	$(b/a)1$	1
$p2$	C_2	$C_2{}^{\mathrm{I}}$	C_2, Abb. 18	W_2	$(b/a):2$	2
pm	D_1kk	$C_s{}^{\mathrm{I}}$	C_s^{I}, Abb. 19	W_1^2	$(b:a):m$	3
pg	D_1gg	$C_s{}^{\mathrm{II}}$	C_s^{II}, Abb. 20	W_1^3	$(b:a):\tilde{b}$	4
cm	D_1kg	$C_s{}^{\mathrm{III}}$	C_s^{III}, Abb. 21	W_1^1	$(a/a)/m$	8
pmm	D_2kkkk	$C_{2v}{}^{\mathrm{I}}$	C_{2v}^{I}, Abb. 22	W_2^2	$(b:a):2\cdot m$	5
pmg	D_2kkgg	$C_{2v}{}^{\mathrm{III}}$	C_{2v}^{III}, Abb. 24	W_2^3	$(b:a):m:\tilde{a}$	6
pgg	D_2gggg	$C_{2v}{}^{\mathrm{II}}$	C_{2v}^{II}, Abb. 23	W_2^4	$(b:a):\tilde{b}:\tilde{a}$	7
cmm	D_2kgkg	$C_{2v}{}^{\mathrm{IV}}$	C_{2v}^{IV}, Abb. 25	W_2^1	$(a/a):2\cdot m$	9
$p4$	C_4	$C_4{}^{\mathrm{I}}$	C_4, Abb. 26	W_4	$(a:a):4$	10
$p4m$	D_4^*	$C_{4v}{}^{\mathrm{I}}$	C_{4v}^{I}, Abb. 27	W_4^1	$(a:a):4\cdot m$	11
$p4g$	D_4°	$C_{4v}{}^{\mathrm{II}}$	C_{4v}^{II}, Abb. 28	W_4^2	$(a:a):4\odot\tilde{a}$	12
$p3$	C_3	$C_3{}^{\mathrm{I}}$	C_3, Abb. 29	W_3	$(a/a):3$	13
$p3m1$	D_3^*	$C_{3v}{}^{\mathrm{I}}$	C_{3v}^{II}, Abb. 31	W_3^1	$(a/a):m\cdot 3$	15
$p31m$	D_3°	$C_{3v}{}^{\mathrm{II}}$	C_{3v}^{I}, Abb. 30	W_3^2	$(a/a)\cdot m\cdot 3$	14
$p6$	C_6	$C_6{}^{\mathrm{I}}$	C_6, Abb. 32	W_6	$(a/a):6$	16
$p6m$	D_6	$C_{6v}{}^{\mathrm{I}}$	C_{6v}, Abb. 33	W_6^1	$(a/a):m\cdot 6$ (some alternatives exist)	17

Chart 6. Sources referred to in the table are listed in the References. The groups are listed in consecutive order as they appear in the International Tables of X-ray Crystallography, [13]. Note that Speiser interchanges the Niggli notations of $C_{3v}{}^{\mathrm{I}}$ and C_{3v}^{II} (figure numbers in the Speiser column are for the 2nd, 3rd, and 4th editions of his book).

In the preparation of this chart, it became apparent that the notation for the two groups $p3m1$ and $p31m$ was frequently interchanged in the literature, and so other crossreference charts could not be assumed to be accurate. The earliest occurrence of this interchange which was noted occurs in Speiser's book, [21]. He uses the notation of the paper by Niggli, [16], but interchanges Niggli's notation for these two groups. Since it is natural to assume these notations are the same, we include information from both sources on our chart. Other books which include this interchange of notation are: Bell and Fletcher [1], Budden, [4], Coxeter, [6], and Coxeter and Moser, [7]. It is quite likely that this notational error has been perpetuated in other works referring to these sources. (The crossreference Table 3 in [7] is correct if in the left column $p31m$ and $p3m1$ are interchanged.)

If the interpretation of the crystallographic notation explained earlier is understood, then it is always possible to determine the correct name for the symmetry group of a periodic design. This, together with the other information provided here, should enable the reader to make any necessary corrections of inaccurate identification in the literature.

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EULER'S FORMULA FOR *n*th DIFFERENCES OF POWERS

Dedicated to Professor L. Carlitz on his seventieth birthday.

H. W. GOULD

1. Introduction. Write down the sequence of fourth powers of the non-negative integers. Below these, write the first differences. Below these, write differences again. Repeat this process as long as you wish, and you obtain the following array of numbers:

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$$
\begin{array}{cccccccccccccccccccc}
0, & & 1, & & 16, & & 81, & & 256, & & 625, & & 1296, & & 2401, & & 4096, & & 6561, & \dots \\
& 1, & & 15, & & 65, & & 175, & & 369, & & 671, & & 1105, & & 1695, & & 2465, & \dots \\
& & 14, & & 50, & & 110, & & 194, & & 302, & & 434, & & 590, & & 770, & \dots \\
& & & 36, & & 60, & & 84, & & 108, & & 132, & & 156, & & 180, & \dots \\
& & & & 24, & & 24, & & 24, & & 24, & & 24, & & 24, & \dots \\
& & & & & 0, & & 0, & & 0, & & 0, & & 0, & \dots
\end{array}
$$

Notice that on the fourth row, the fourth differences all turn out to be $24=4!=4\cdot 3\cdot 2\cdot 1$, and higher differences are all zero. Try this for any other positive integral power n and you find that row n consists of the number $n!$ repeated, and all higher differences are zero. This property of differences of powers of whole numbers, although known before Euler, was first studied intensively by him [**A3**], [**A4**]. In [**A5**] he made use of the property in part of his proof that every prime of the form $4n+1$ is the sum of two squares. Euler's theorem about nth differences of nth powers being $n!$ is one of the easiest theorems to discover and students of mathematics who doodle with powers of numbers often rediscover the formula in high school and college and wonder why the theorem is true. Indeed, I remember discovering the formula myself when I was still in high school, but I was unhappy with my proof. The formula was one of the motivations for my further study of number theory and combinatorics. Just recently, a letter from Linda Church, a senior mathematics major at the University of Washington, gave a rediscovery made in junior high school. There seems to be no handy account of the formula which is not cryptic, written for specialists, or located in an ancient book or journal. Even where the formula is proved or discussed, one usually finds little background or leisure in the discussion. Milne-Thomson [**A10**, p. 36] dispenses with the formula by saying (on the basis of preceding theorems) "Clearly $\Delta^n 0^m=0$ if $n>m$, $\Delta^n 0^n=n!$" But no history is given there. Tepper [**B1-5**] is the most recent popular rediscovery of the formula. The formula was used by Lagrange [**A2**, p. 62 for references] to prove (as we shall see later) that Fermat's congruence

$$a^p\equiv a \pmod p \qquad \text{for any integer } a \text{ and prime } p \tag{1.1}$$

implies Wilson's congruence

$$(p-1)!\equiv -1 \pmod p \qquad \text{for any prime } p. \tag{1.2}$$

The most recent appearance of the formula for $n!$ as an nth difference of powers was as a problem in the 1976 Putnam Competition [**D5**]. In view of the frequent rediscovery of the formula, its importance in certain theorems in the finite difference calculus (which is still not taught widely at an elementary level), and its sheer esthetic appeal as an elegant property of the natural numbers, it seems that a detailed expository paper is in order to set forth proofs and references and which might reach a wide audience of readers. That is the reason for this paper.

Since Euler's formula is related to the Stirling numbers of the second kind, we give some elementary facts and history concerning these also. We recount some recent appearances of the formula of Euler and show its use in manufacturing endless variations on a basic theme. All of this should be common knowledge to students and teachers of mathematics, but it isn't. I hope that the reader will find Euler's formula a useful tool to have at his or her command. Finally, we give five sets of references to sixty works of importance concerning Stirling numbers.

2. Basic formulas and proof by induction. To see how the array with which we started leads to compact formulas, observe the following computations made in forming the array:

$$\begin{cases}(256-81)-(81-65)=256-2(81)+65=175-65=110,\\(81-16)-(16-1)=81-2(16)+1=65-15=50,\\(16-1)-(1-0)=16-2(1)+0=14;\end{cases}$$

$$\begin{cases}256-3(81)+3(16)-1=110-50=60,\\81-3(16)+3(1)-0=50-14=36;\end{cases}$$

$$256-4(81)+6(16)-4(1)+0=60-36=24;$$
$$625-5(256)+10(81)-10(16)+5(1)-0=0.$$

The binomial coefficients $\binom{n}{k}=n!/k!(n-k)!=n(n-1)(n-2)\cdots(n-k+1)/k!$ make their appearance naturally as we form successive differences. The last two computations can therefore be made to read as below:

$$\sum_{k=0}^{4}(-1)^k\binom{4}{k}(4-k)^4=\sum_{k=0}^{4}(-1)^{4-k}\binom{4}{k}k^4=4! \quad \text{and} \quad \sum_{k=0}^{5}(-1)^{5-k}\binom{5}{k}k^4=0.$$

The general formula therefore appears to be

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^j=\begin{cases}0, & \text{if } 0\leqslant j<n,\\ n! & \text{if } j=n.\end{cases} \tag{2.1}$$

This is Euler's basic formula which we shall now prove.

To prove that the higher differences are zero, we may proceed as follows. We need to know that the binomial coefficients satisfy the recurrence relation

$$\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1} \tag{2.2}$$

and that

$$\binom{n}{k}=0 \quad \text{for} \quad 0\leqslant n<k \quad \text{or for} \quad k<0. \tag{2.3}$$

Now call the series in (2.1) $f(n,j)$, and we find that an inductive proof for $f(n,j)=0$ for $0\leqslant j<n$ can be developed as follows:

$$\begin{aligned} f(n+1,j)&=\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}k^j=\sum_{k=0}^{n+1}(-1)^{n+1-k}\left\{\binom{n}{k}+\binom{n}{k-1}\right\}k^j\\ &=-\sum_{k=0}^{n+1}(-1)^{n-k}\binom{n}{k}k^j+\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(k+1)^j\\ &=-f(n,j)+\sum_{r=0}^{j}\binom{j}{r}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^r, \end{aligned}$$

where we also used the binomial theorem to expand $(k+1)^j$. Thus we find

$$f(n+1,j)=-f(n,j)+\sum_{r=0}^{j}\binom{j}{r}f(n,r), \qquad 0\leqslant j\leqslant n, \tag{2.4}$$

or

$$f(n+1,j)=\sum_{r=0}^{j-1}\binom{j}{r}f(n,r). \tag{2.5}$$

Formula (2.4) is all we need to see that if $f(n,j)=0$ for $0\leqslant j<n$, then $f(n+1,j)=0$ for $0\leqslant j<n+1$. Since $f(1,j)=0$ for $0\leqslant j<1$, i.e., $f(1,0)=0$, then $f(2,j)=0$ for $0\leqslant j<2$, etc. and the induction is complete.

Put $g(n)=f(n,n)$. We wish next to prove that $g(n)=n!$ to complete our proof of (2.1). An inductive proof, using the fact that $f(n,j)=0$ for $0\leqslant j<n$, can be set down as follows:

$$g(n+1)=\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}k^{n+1}=(n+1)\sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n}{k-1}k^n,$$

since it is easy to see that

$$k\binom{n+1}{k}=(n+1)\binom{n}{k-1}.$$

Now changing the dummy summation index k to $k+1$ we have

$$g(n+1)=(n+1)\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(k+1)^n.$$

Now use the binomial theorem on $(k+1)^n$ and interchange summation order. So

$$g(n+1)=(n+1)\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^j$$

$$=(n+1)\sum_{j=0}^{n}\binom{n}{j}f(n,j)=(n+1)g(n),$$

whence we have the recurrence relation $g(n+1)=(n+1)g(n)$. This is the recurrence relation for factorials, and since clearly $g(0)=1$, we have an inductive proof that $g(n)=n!$ as desired.

A proof of (2.1) very much like ours may be found in Schwatt [**A14**, pp. 100–101], a little book that can be a very valuable companion to anyone interested in learning the operations with series. Schwatt, however, does not give many references to the literature and ignores the history of the formulas. Nevertheless, it is an indispensable treatise for the wealth of detail and formulas.

3. Recurrence relation for $f(n,j)$. The array of values of $f(n,j)$ is as follows:

	0	1	2	3	4	5	6	7	8	(n)
0	1	0	0	0	0	0	0	0	0	
1	0	1	0	0	0	0	0	0	0	
2	0	1	2	0	0	0	0	0	0	
3	0	1	6	6	0	0	0	0	0	
4	0	1	14	36	24	0	0	0	0	
5	0	1	30	150	240	120	0	0	0	
6	0	1	62	540	1560	1800	720	0	0	
7	0	1	126	1806	8400	16800	15120	5040	0	
8	0	1	254	5796	40824	126000	191520	141120	40320	
(j)										

It is easy to verify for a few values that the f's satisfy the recurrence

$$f(n,j+1)=nf(n-1,j)+nf(n,j). \tag{3.1}$$

Proof.

$$nf(n,j)=n\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^j \quad\text{and}\quad nf(n-1,j)=n\sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}k^j,$$

so that

$$nf(n-1,j)+nf(n,j)=n\sum_{k=0}^{n}(-1)^{n-k}\left\{\binom{n}{k}-\binom{n-1}{k}\right\}k^j$$

$$=n\sum_{k=0}^{n}(-1)^{n-k}\binom{n-1}{k-1}k^j$$

$$=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{j+1}=f(n,j+1),$$

as desired.

4. Relation to Stirling numbers of the second kind. It is an interesting fact that $f(n,j)$ is divisible by $n!$ and the numbers defined by

$$S(n,k)=\frac{f(k,n)}{k!}, \tag{4.1}$$

in the notation of Riordan [**A13**], are called Stirling numbers of the second kind. They form the array

$(n) \backslash (k)$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0
3	0	1	3	1	0	0	0	0	0	0
4	0	1	7	6	1	0	0	0	0	0
5	0	1	15	25	10	1	0	0	0	0
6	0	1	31	90	65	15	1	0	0	0
7	0	1	63	301	350	140	21	1	0	0
8	0	1	127	966	1701	1050	266	28	1	0
9	0	1	255	3025	7770	6951	2646	462	36	1

This array of numbers evidently was first published in the book by Jacob Stirling [**A15**, p. 8] and the numbers $S(n,k)$ are known universally as Stirling numbers of the second kind. The notation $S(n,k)$ is that popularized by John Riordan [**A13**], although the study of the literature is hampered by the use of at least 30 different notations for the numbers. This may also explain why authors often fail to recognize the Stirling numbers; for they are constantly being denoted by different writers with different letters and with shifted indices. In [**E8**], [**E9**], and [**E10**] I discuss notations.

There are also Stirling numbers of the first kind; we have to say a little about them. Information on them can be found in Riordan [**A13**] and Jordan [**A9**]. A complete bibliography is out of the question here, for the author has a file of hundreds of items on both kinds of Stirling numbers. In References section E we append some useful references.

Using the notation (4.1), the recurrence relation (3.1) translates into

$$S(n+1,k)=S(n,k-1)+kS(n,k), \tag{4.2}$$

which, together with the obvious initial values, proves that $S(n,k)$ is indeed an integer.

The Stirling number $s(n,k)$ of the first kind is usually defined as follows. Put $(x)_n=x(x-1)(x-2)\cdots(x-n+1)$ with $(x)_0=1$. This is the so-called "descending factorial". Note that the binomial coefficients are related by $(x)_n=\binom{x}{n}n!$. Then the expansion

$$(x)_n=\sum_{k=0}^{n} s(n,k)x^k, \qquad n\geqslant 0, \quad \text{all complex } x, \tag{4.3}$$

defines the Stirling numbers of the first kind. They have the recurrence relation

$$s(n+1,k)=s(n,k-1)-ns(n,k). \tag{4.4}$$

Incidentally, (4.3) is inverse to the expansion

$$x^n=\sum_{k=0}^{n} S(n,k)(x)_k, \qquad n\geqslant 0, \quad \text{all complex } x, \tag{4.5}$$

which we shall prove below and which is another way of defining the Stirling numbers of the second kind. As a matter of fact (4.3) and (4.5) may be substituted the one into the other and used to prove the two orthogonality relations

$$\sum_{k=j}^{n} S(n,k)s(k,j)=\delta_j^n=\sum_{k=j}^{n} s(n,k)S(k,j), \tag{4.6}$$

where δ_j^n is the Kronecker delta and is 0 for $n\neq j$ and 1 for $n=j$. Relation (4.6) also says that matrices of the two kinds of Stirling number are multiplicative inverses of each other.

The array of Stirling numbers of the first kind begins as follows:

$(n) \backslash (k)$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	−1	1	0	0	0	0	0	0
3	0	2	−3	1	0	0	0	0	0
4	0	−6	11	−6	1	0	0	0	0
5	0	24	−50	35	−10	1	0	0	0
6	0	−120	274	−225	85	−15	1	0	0
7	0	720	−1764	1624	−735	175	−21	1	0
8	0	−5040	13068	−13132	6769	−1960	322	−28	1

To illustrate (4.6) by matrices, the reader is invited to verify that

$$\begin{pmatrix}1&0&0\\1&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\2&-3&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\-1&1&0\\2&-3&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\1&3&1\end{pmatrix}.$$

As a matter of fact (4.6) also yields the valuable inversion theorem:

$$F(n)=\sum_{k=0}^{n} S(n,k)G(k) \quad \text{if and only if} \quad G(n)=\sum_{k=0}^{n} s(n,k)F(k). \tag{4.7}$$

Here is an inductive proof of formula (4.5). Suppose it be true for exponents $0,1,\ldots,n$. Then

$$x^{n+1}=xx^n=x\sum_{k=0}^{n}\binom{x}{k}k!S(n,k).$$

Now, $\binom{x}{k}=\binom{x-1}{k}+\binom{x-1}{k-1}$, so that $x\binom{x}{k}=(k+1)\binom{x}{k+1}+k\binom{x}{k}$, and thus

$$\begin{aligned}x^{n+1}&=\sum_{k=0}^{n}\left\{(k+1)\binom{x}{k+1}+k\binom{x}{k}\right\}k!S(n,k)\\&=\sum_{k=0}^{n}(k+1)!\binom{x}{k+1}S(n,k)+\sum_{k=0}^{n}kk!\binom{x}{k}S(n,k)\\&=\sum_{k=1}^{n+1}k!\binom{x}{k}S(n,k-1)+\sum_{k=0}^{n}kk!\binom{x}{k}S(n,k)\\&=\sum_{k=0}^{n+1}k!\binom{x}{k}\{S(n,k-1)+kS(n,k)\}=\sum_{k=0}^{n+1}\binom{x}{k}k!S(n+1,k),\end{aligned}$$

by (4.2), so the induction goes through. The formula is correct for $n=0,1,2$, and hence is correct for all integers $n\geqslant 0$.

The Stirling numbers of the second kind are very useful in summing certain kinds of series. This is due to the fact that they occur in a valuable operational expansion. Consider the operator $\theta=xD$ where we differentiate and then multiply by x. Iterations of this operator can be found if we know higher derivatives of a function, and the link is $S(n,k)$. In fact

$$\theta^n f(x)=\sum_{k=0}^{n} S(n,k)x^kD^kf(x), \qquad \text{for all integers } n\geqslant 0. \tag{4.8}$$

The proof by induction is safely left to the reader who merely has to note that $\theta^{n+1}f=xD(\theta^n f)$ and after differentiating make use of (4.2) to reconstitute the terms.

Observing that $\theta^p x^k=k^p x^k$, we see that

$$\sum_{k=0}^{n} k^p x^k=\theta^p\sum_{k=0}^{n} x^k=\theta^p\left\{\frac{x^{n+1}-1}{x-1}\right\}, \tag{4.9}$$

and by use of (4.8) it is then possible to get a well-known Stirling number expansion for sums of powers of the natural numbers when $x \to 1$.

A simple case of (4.9) to exhibit here is that when $n \to \infty$. The higher derivatives of $(1-x)^{-1}$ are easily found. We obtain the known formula

$$\sum_{k=0}^{\infty} k^p x^k = \sum_{k=0}^{p} S(p,k) k! \frac{x^k}{(1-x)^{k+1}}, \qquad |x|<1. \tag{4.10}$$

The formula is developed this way in Schwatt [**A14**, p. 85], but it is a very ancient formula. Grunert [**E13**] gives an extensive account of the uses of the operator θ to sum series. For example, he gives, just as Schwatt [**A14**, p. 84], the formula

$$\sum_{k=0}^{\infty} \frac{k^p}{k!} x^k = \theta^p e^x = e^x \sum_{k=0}^{p} S(p,k) x^k. \tag{4.11}$$

In particular then, the fact that the so-called Bell or exponential numbers $B(n)$ are given by

$$B(n) = \sum_{k=0}^{n} S(n,k) = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!} \tag{4.12}$$

is a very old result. The Bell numbers $1,1,2,5,15,52,203,\ldots$ enumerate the partitions of a set of n elements and have other magic properties. See Gould [**E12**] for a detailed bibliography. (Somehow I failed to cite Grunert's paper in my Bell bibliography of 172 items!)

From (4.9) we could obtain the well-known formula [**A14**, p. 86]

$$\sum_{k=0}^{n} k^p = \sum_{k=0}^{p} \binom{n+1}{k+1} k! S(p,k), \qquad n \geqslant 0, \quad p \geqslant 0, \tag{4.13}$$

but we shall prove this formula by an appeal to (4.5) and the binomial identity

$$\sum_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1}, \tag{4.14}$$

which is an easy one to prove by induction and is formula (1.52) in my book [**A6**]. Assuming the truth of (4.5) we find

$$k^p = \sum_{j=0}^{p} \binom{k}{j} j! S(p,j),$$

whence

$$\begin{aligned} \sum_{k=0}^{n} k^p &= \sum_{k=0}^{n} \sum_{j=0}^{p} \binom{k}{j} j! S(p,j) = \sum_{j=0}^{p} j! S(p,j) \sum_{k=0}^{n} \binom{k}{j} \\ &= \sum_{j=0}^{p} j! S(p,j) \sum_{k=j}^{n} \binom{k}{j} = \sum_{j=0}^{p} \binom{n+1}{j+1} j! S(p,j), \end{aligned}$$

as desired.

Formula (4.13) is an ancient result and probably one of the easiest to derive, despite the fact that the sum was determined widely by Bernoulli number expansions. Some of the formulas in [**E11**] use Stirling numbers.

With this much of a digression into what can be done with Stirling numbers (as an extension of Euler's basic formula), we will return to the use of Euler's formula to develop one of the useful basic theorems of the calculus of finite differences and exhibit still other consequences.

5. The Calculus of finite differences. In the calculus of finite differences as expounded in most of the standard treatises, we begin with the definition of a finite difference quotient of a function, with increment h,

$$\underset{x,h}{\Delta} f(x) = \Delta f = \frac{f(x+h)-f(x)}{h}. \tag{5.1}$$

The abbreviated notation Δf should be used only when it is clear that the increment h is fixed and that the operator is with respect to x. Caution is advised since in a large part of the older literature Δf is used only when $h=1$. Higher order differences may next be defined by iteration using

$$\underset{x,h}{\Delta^{n+1}} f(x) = \underset{x,h}{\Delta}\ \underset{x,h}{\Delta^n} f(x). \tag{5.2}$$

It is then easy to prove by induction that in fact

$$\underset{x,h}{\Delta^n} f(x) = \frac{1}{h^n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+kh), \qquad n \geqslant 0. \tag{5.3}$$

The proof uses only formula (2.2) and we leave this to the reader.

It is now clear what this has to do with Euler's formula. Choosing $f(x)=x^j$, we see that

$$\left.\underset{x,1}{\Delta^n} x^j\right|_{x=0} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^j, \tag{5.4}$$

so that another way of writing the original formula is

$$\left.\underset{x,1}{\Delta^n} x^j\right|_{x=0} = \Delta^n 0^j = \begin{cases} 0, & 0 \leqslant j < n \\ n!, & j = n \end{cases}. \tag{5.5}$$

The fact that $h=1$ occurs frequently in work with difference equations and recurrence relations led to the abbreviated notation $\Delta^n 0^j$, and the description of Euler's formula as "Differences of Zero" arose from formula (5.5). Before the Stirling number notations became widely used, the notation $\Delta^n 0^j$ was very common. See for example references **[A1]**, **[A8]**, **[A10]**, and much other older actuarial literature.

The first thing we wish to do now is to remove the restriction in Euler's formula (5.5) that $h=1$. In fact it is easy to see that

$$\left.\underset{x,h}{\Delta^n} x^j\right|_{x=0} = h^{j-n}\left.\underset{x,1}{\Delta^n} x^j\right|_{x=0}, \tag{5.6}$$

so that for non-zero h we have merely to multiply both sides of (5.5) by h^{j-n} in order to obtain

$$\left.\underset{x,h}{\Delta^n} x^j\right|_{x=0} = \begin{cases} 0, & 0 \leqslant j < n, \\ n!, & j = n, \end{cases} \tag{5.7}$$

a fact also known to Euler.

Now we may go yet a step further. What happens when $x \neq 0$? We use the binomial theorem and find

$$\begin{aligned}
\underset{x,h}{\overset{n}{\Delta}} x^j &= \frac{1}{h^n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(x+kh)^j \\
&= \sum_{r=0}^{j}\binom{j}{r}x^{j-r}h^r\frac{1}{h^n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^r \\
&= \sum_{r=0}^{j}\binom{j}{r}x^{j-r}h^{r-n}\left.\underset{y,1}{\Delta^n} y^r\right|_{y=0}
\end{aligned}$$

which, in virtue of (5.5) reduces to 0 when $0 \leqslant j < n$, and $n!$ when $j=n$. So we have proved in general that

$$\underset{x,h}{\Delta^n} x^j = \begin{cases} 0, & 0 \leqslant j < n, \\ n!, & j = n. \end{cases} \tag{5.8}$$

This is the general lemma needed to establish the main theorem: Let

$$f(x)=\sum_{j=0}^{p} A_j(p)x^j$$

be an arbitrary polynomial of degree p in x. Then

$$\Delta_{x,h}^{n} f(x)=\begin{cases}0, & 0\leqslant p<n,\\ n!A_n(n), & p=n.\end{cases} \tag{5.9}$$

This is formula (Z.8) in my book **[A6]** and is a central fact in the elementary calculus of finite differences. As for the proof, note that the summation or difference operator is a linear operator. (A linear operator L is an operator such that $L(f+g)=Lf+Lg$ and $L(cf)=cLf$ for any constant c and two functions f,g; or we may say all at once that L is characterized by $L(af+bg)=aLf+bLg$ for functions f,g and constants a,b.) Since we have a linear operator, we have only to see that (5.8) is true and apply this fact to each term in the polynomial to establish (5.9).

The corollaries of this well-known theorem are enormous. We may set down strange-looking identities which are essentially just manifestations of Euler's formula.

For example, Butler **[D1]**, **[D2]** used enumerative techniques on semigroups to obtain "the following rather extraordinary result"

$$n!=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2^{n+1}+n-1-k)^n, \tag{5.10}$$

and

$$n!=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left\{\frac{n^2+5n+2}{2}-k\right\}^n. \tag{5.11}$$

Nothing extraordinary here! We can just as easily say more generally

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(A+Bk)^p=\begin{cases}0, & 0\leqslant p<n,\\ B^n n!, & p=n,\end{cases} \tag{5.12}$$

and A and B may be any weird constants we desire.

Since we may treat the binomial coefficients as polynomials when we write

$$\binom{x}{n}=\frac{x(x-1)(x-2)\cdots(x-n+1)}{n!},\qquad \binom{x}{0}=1, \tag{5.13}$$

so that $\binom{x}{n}$ is a polynomial of degree n in x, then we can manufacture all kinds of peculiar looking binomial identities just as easily. Thus it is obvious from this and (5.9) that

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{x+kz}{p}=\begin{cases}0, & 0\leqslant p<n,\\ z^n, & p=n,\end{cases} \tag{5.14}$$

for any complex x,z; this is formula (3.150) in my book **[A6]**.

Here is another less transparent example: Let $p\geqslant 0$. Then

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left[\binom{\binom{k}{2}}{p}\right]=\begin{cases}0, & n>2p,\\ 1\cdot3\cdot5\cdot7\cdots(2p-1) & \text{for } n=2p.\end{cases} \tag{5.15}$$

The zero part is easy since the iterated binomial coefficient

$$\binom{\binom{x}{2}}{p}=\frac{(x^2-x)(x^2-x-2)(x^2-x-4)\cdots(x^2-x-2p+2)}{2^p p!}$$

is a polynomial of degree $2p$ in x so the sum is zero whenever $n>2p$. What is more, it is easy to see

that the coefficient of the highest power of x is precisely $1/2^p p!$ so that in the case $n=2p$, we have that the sum equals $(2p)!/2^p p! = 1\cdot 3\cdot 5\cdot 7\cdots(2p-1)$ as desired.

To obtain weird formulas, it is only necessary to devise strange-looking polynomials for which we can easily find the coefficient of the highest power of x. Formula (5.15) generalizes easily to

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{\binom{k}{p}}{q}=\begin{cases}0, & n>pq,\\ (pq)!/(p!)^q q!, & \text{for } n=pq,\end{cases} \tag{5.16}$$

for all integers $p,q\geqslant 0$. There is no remarkable simplification or tidy closed formula when $n<pq$ but the series can be expressed as a sum involving Stirling numbers of the second kind; but as this has no elegance, we omit the details. Formula (5.15) was posed as a problem **[D4]** in 1963, and a solution (equivalent to what we have above) appeared 8 years later. (We may remark that the author set down (5.16) in 1962 to illustrate how easy it is to generate strange-looking formulas.)

Here is another peculiar formula:

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{2}^{n/2}=\frac{n!}{2^{n/2}}\qquad\text{for even } n\geqslant 2. \tag{5.17}$$

This was posed to me in a letter in April 1973 from Thomas Osler who got it from Herbert Wilf. As I showed in a letter a few weeks later, the formula is a consequence of Euler's theorem. For we have the polynomial $f(x)=\binom{x}{2}^{n/2}=$ a polynomial of degree n in x when n is an even number. It follows that the sum equals $n!$ times the leading coefficient which is just $2^{-n/2}$, so the formula follows.

Three very wicked-looking binomial sums are tabulated in my book **[A6]** as formulas (X.16), (X.17), (X.18) to illustrate Euler's theorem. One of these, (X.16) says that for any complex a_i, b_i, $1\leqslant i\leqslant r$,

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\prod_{i=1}^{r}\binom{a_i+b_ik}{m_i}=0, \tag{5.18}$$

provided $0\leqslant m_1+m_2+\cdots+m_r<n$. The proof is trivial from (5.9). Formula (X.17) extends part of (5.16) above and says that for any complex a_i

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\prod_{i=1}^{r}\binom{\binom{a_ik}{b_i}}{c_i}=0,\qquad b_i,c_i \text{ integers}\geqslant 0, \tag{5.19}$$

provided $b_1c_1+b_2c_2+\cdots+b_rc_r<n$.

The reader may now be able to provide any number of similar examples.

The 1976 Putnam Competition **[D5]** has the formula in the form

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(x-k)^n=n!. \tag{5.20}$$

The sum is not given, but any student who knows his finite difference calculus will have spent no more than ten seconds to write down $n!$ as the answer. Those who missed the problem will relish learning Euler's theorem.

Besides (2.1) I think the most frequent rediscovery recently has been the form (5.20) or other special case of (5.12). Tepper **[B1-5]** got the formula in the form (5.20) but only for $x=$ integer. Menon **[D3]** got a special case of (5.12) and rediscovered combinatorial properties not seeming to relate his work to known properties of Stirling numbers. Pérez de Varga and Mariano Quirós et al **[C1-5]** rediscovered the formula in the original form (2.1). The fact that $f(n,j)\neq 0$ for $n<j$ is a known

ancient property of the Stirling numbers again. Problem E 1837 [**A19**] was a rediscovery of the solution to the difference equation (3.1). Problem 3108 [**A17**] was precisely the special case of (5.12) when $B=-1$. It was cited in the editorial comment [**B3**] on Tepper's formula.

Some further applications of Euler's formula in the calculus of finite differences can be made. The formula helps us make the similarity between some results in finite differences and ordinary differential calculus more plausible.

Recall formula (5.3). We would like to show that if the limit exists (and in fact it does for functions that possess derivatives up through the nth) then

$$\lim_{h\to 0} \underset{x,h}{\Delta}{}^n f(x) = D_x^n f(x). \tag{5.21}$$

By formula (5.3), what we want to prove is that

$$\lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} f(x+kh) = D_x^n f(x), \tag{5.22}$$

when the limit exists.

By Euler's formula we see that by just setting $h=0$ in the indicated expression, we have an indeterminate form of type $0/0$. Applying l'Hospital's rule j times, we find that we must investigate the limit

$$\lim_{h\to 0} \frac{1}{D_h^j h^n} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} k^j f^{(j)}(x+kh),$$

which, by Euler's formula is of the indeterminate form $0/0$ for every j such that $0 \leqslant j < n$. But for $j=n$ we get by Euler's formula that our limit is precisely

$$\frac{f^{(n)}(x)}{n!} \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} k^n = f^{(n)}(x),$$

as desired to show. This proof is that given by M. S. Klamkin in his solution to a problem in this Monthly [**A18**]. The analogy between the finite calculus and the infinitesimal calculus is very neat. We have

$$\underset{x,h}{\Delta}{}^n x^j = D_x^n x^j = \begin{cases} 0, & 0 \leqslant j < n, \\ n!, & j = n. \end{cases} \tag{5.23}$$

The same holds for any polynomial of degree j in x, i.e.,

$$\underset{x,h}{\Delta}{}^n f(x) = D_x^n f(x) = \begin{cases} 0, & 0 \leqslant j < n, \\ An!, & j = n, \end{cases} \tag{5.24}$$

where A is the leading coefficient (coefficient of the highest power) of the polynomial.

Incidentally, other analogies abound. We saw that except for a factorial factor the "differences of zero" are Stirling numbers of the second kind, i.e.,

$$k!S(n,k) = \Delta^k 0^n = \underset{x,1}{\Delta}{}^k x^n \Big|_{x=0} \tag{5.25}$$

For the Stirling numbers of the first kind, we see from Taylor's formula and (4.3) that with $(x)^n = x(x-1)(x-2)\cdots(2)(1)$, $(x)^0 = 1$, then

$$k!s(n,k) = D^k\,(0)^n = D_x^k\,(x)^n|_{x=0}, \tag{5.26}$$

so that the analogy is exact. Relation (5.25) arises from the Newton-Gregory interpolation expansion which is analogous to the Taylor series. Just as the Taylor series for a polynomial of degree n in x says that

$$f(x+y) = \sum_{k=0}^{n} \frac{y^k}{k!} D_x^k f(x), \tag{5.27}$$

so in the finite difference calculus one can prove the Newton-Gregory expansion

$$f(x+y)=\sum_{k=0}^{n}\binom{y/h}{k}h^k \underset{x,h}{\Delta}{}^k f(x). \tag{5.28}$$

When $h \to 0$, (5.28) passes over into (5.27).

In view of the binomial coefficient inversion theorem that

$$F(n)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}G(k) \tag{5.29}$$

if and only if

$$G(n)=\sum_{k=0}^{n}\binom{n}{k}F(k), \tag{5.30}$$

formula (5.3) inverts at once to yield

$$f(x+nh)=\sum_{k=0}^{n}\binom{n}{k}h^k \underset{x,h}{\Delta}{}^k f(x), \tag{5.31}$$

which is a special case of (5.28).

In the same way that we obtained (4.13) from (4.9) by using (4.14), the reader may verify that (4.14) used with (5.31) yields

$$\sum_{j=0}^{n} f(x+jh)=\sum_{k=0}^{n}\binom{n+1}{k+1}h^k \underset{x,h}{\Delta}{}^k f(x) \tag{5.32}$$

for any polynomial of degree n in x. An analogous formula from (5.27) would be

$$\int_0^m f(x+y)\,dy=\sum_{k=0}^{m}\frac{m^k}{(k+1)!}D_x^k f(x). \tag{5.33}$$

We end our remarks on the finite difference calculus by noting that higher order differences and derivatives of a polynomial may be expressed in terms of one another using the Stirling numbers. The formulas for these expressions are as follows:

$$\frac{1}{m!}h^m \underset{x,h}{\Delta}{}^m f(x)=\sum_{j=0}^{n} S(j,m)\frac{1}{j!}h^j D_x^j f(x) \tag{5.34}$$

and

$$\frac{1}{m!}h^m D_x^m f(x)=\sum_{j=0}^{n} s(j,m)\frac{1}{j!}h^j \underset{x,h}{\Delta}{}^j f(x), \tag{5.35}$$

where f is a polynomial of degree n in x. In view of (4.6) the proof of one of these would immediately give the other.

6. Proof by generating functions. The proof we gave in Section 2 for Euler's formula was a simple proof by induction. It may be of interest, however, to recount a standard proof of (2.1) by the method of generating functions. This is the proof given in Riordan [**A13**, p. 13] and is a very old method also. The idea is to consider the function $f(x)=(e^x-1)^n$ and to expand it in two seemingly different ways and equate the coefficients of powers of x and so obtain an identity.

On the one hand we have

$$(e^x-1)^n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}e^{kx}=\sum_{j=0}^{\infty}\frac{x^j}{j!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^j,$$

and on the other hand $(e^x-1)^n=(x+x^2/2+\cdots)^n$, whence the lowest power that can occur in the expansion is x^n with coefficient 1, and all lower powers have coefficient 0 and this proves (2.1). The beauty of this generating function method is evident when compared to the long inductive proofs we gave earlier. But alternative proofs shed different light.

The generating function proof was suggested by an anonymous writer in comments **[B5]** on Papp's proof **[B4]** of the formula. Schwatt **[A14**, pp. 18–19] used the generating function method with $(e^x-1)^n$. By this method we can easily figure out the coefficients in the expansion of $(e^x-1)^n$ for a few steps and find the following:

$$(e^x-1)^n=x^n\left\{1+\frac{n}{2}x+\frac{n(3n+1)}{24}x^2+\frac{n^2(n+1)}{48}x^3 + \frac{n(15n^3+30n^2+5n-1)}{1152}x^4+\cdots\right\},\tag{6.1}$$

from which we have the known formulas

$$\begin{aligned}S(j,n)&=\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^j\\&=0,\quad 0\leqslant j<n,\\&=1,\quad j=n,\\&=n/2,\quad j=n+1,\\&=\frac{n(3n+1)}{24},\quad j=n+2,\\&=\frac{n^2(n+1)}{48},\quad j=n+3,\\&=\frac{n(15n^3+30n^2+5n-1)}{1152},\quad j=n+4,\\&\quad ,\ldots,\end{aligned}\tag{6.2}$$

and special cases and general formulas of this sort are tabulated in Schwatt **[A14**, p. 102]. The formulas found in this way are very old and we cite Grunert **[A7]**. The first few of Grunert's formulas are given in Dickson **[A2**, p. 66]. Numerous older references may be cited. Paasche **[E22]** has corrected errors in some of the published values of (6.2). Harborth **[E14]** has proved that every prime divides almost all Stirling numbers of the second kind.

7. Fermat's congruence implies Wilson's congruence. Lagrange, as we have remarked in the introduction, used Euler's formula to show that the Wilson congruence could be proved from the Fermat congruence. He did this as follows. First of all, it is trivial that $(2-1)!=1\equiv-1 \pmod 2$, so we suppose that $p\geqslant 3$ is an odd prime. By Euler's formula, then

$$\begin{aligned}(p-1)!&=\sum_{k=0}^{p-1}(-1)^{p-1-k}\binom{p-1}{k}k^{p-1}=\sum_{k=1}^{p-1}(-1)^k\binom{p-1}{k}k^{p-1}\\&\equiv\sum_{k=1}^{p-1}(-1)^k\binom{p-1}{k}=\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}-1\\&=0-1=-1 \pmod p,\end{aligned}$$

as desired to show.

I do not know any really simple way to use Euler's formula to go the other way, i.e., deduce Fermat's congruence out of Wilson's. Some suitable formula inverse to Euler's would be useful if any such existed.

It is possible to get some other kinds of number-theoretic information out of Euler's formula. Starting with Euler's formula we have

$$(p-1)! = \sum_{k=1}^{p-1} (-1)^{p-1-k}\binom{p-1}{k}k^{p-1} \text{ for any prime } p \geqslant 2.$$

Now, Lucas [**A2**, p. 272 for references] showed that

$$\binom{p-1}{k} \equiv (-1)^k \text{ for all } 0 \leqslant k \leqslant p-1, p = \text{prime}, \tag{7.1}$$

so we have

$$(p-1)! \equiv \sum_{k=1}^{p-1} (-1)^{p-1-k}(-1)^k k^{p-1} = (-1)^{p-1}\sum_{k=1}^{p-1} k^{p-1} \pmod p$$

and applying Wilson's congruence we get

$$\sum_{k=1}^{p-1} k^{p-1} \equiv (-1)^p \pmod p.$$

The number $(-1)^p = -1$ for odd primes p, and for $p=2$ the congruence is trivially true for $1 \equiv -1 \pmod 2$ so we have proved that

$$\sum_{k=1}^{p-1} k^{p-1} \equiv -1 \pmod p \tag{7.2}$$

for all primes p. Much stronger results are known for general sums.

Explicit formulas play a valuable role in passing from one form of a congruence to another. The binomial theorem, of course, is a valuable formula in this regard. Euler [**A2**, p. 60 for references] proved the Fermat congruence by induction using the binomial theorem. We may state such an argument as follows. For any integer a

$$\begin{aligned}(a+1)^p-(a+1) &= \sum_{k=0}^{p}\binom{p}{k}a^k - a - 1 \\ &= \sum_{k=1}^{p-1}\binom{p}{k}a^k + (a^p - a).\end{aligned} \tag{7.3}$$

But it is easy to see that

$$\binom{p}{k} \equiv 0 \pmod p \qquad \text{for all } 1 \leqslant k \leqslant p-1 \text{ and prime } p, \tag{7.4}$$

so that (7.3) yields

$$(a+1)^p - (a+1) \equiv a^p - a \pmod p. \tag{7.5}$$

Since $1^p - 1 \equiv 0 \pmod p$ we have a proof by induction that $a^p \equiv a \pmod p$ for all integers $a \geqslant 1$ and prime moduli p.

Dickson [**A2**, p. 59] cites G. Peano's *Formulaire Mathématique* (1901) for evidence that the Chinese knew in 500 B.C. that $2^p \equiv 2 \pmod p$, and this of course was rediscovered by Pierre Fermat more than 2000 years later in 1640 A.D.

8. A modified Euler formula implies Fermat's congruence. By formula (5.14) we have

$$\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}\binom{pa-pk}{p}=(-a)^p,$$

whence, for $p \geqslant 2$,

$$\sum_{k=1}^{p-1}(-1)^k\binom{p}{k}\binom{pa-pk}{p}=a^p-\binom{pa}{p}=\{a^p-a\}-\left\{\binom{pa}{p}-a\right\}. \tag{8.1}$$

Since again by (7.4) $p\,|\binom{p}{k}$ for all k such that $1 \leqslant k \leqslant p-1$, then our formula shows that

$$a^p-a\equiv\binom{pa}{p}-a(\mathrm{mod}\,p) \tag{8.2}$$

for any integer a.

It is easy to see that $\binom{pa}{p}-a\equiv 0(\mathrm{mod}\,p)$ and here is a short inductive proof of this. We use the Vandermonde addition formula

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n}, \tag{8.3}$$

formula (3.1) in my book [**A6**], and which itself follows from a comparison of the coefficients of t^n in the identity $(1+t)^x(1+t)^y=(1+t)^{x+y}$. Since (8.3) is a polynomial relation in x,y it holds for arbitrary complex values of x,y. Thus we find

$$\binom{p(a+1)}{p}=\binom{pa+p}{p}=\sum_{k=0}^{p}\binom{pa}{p-k}\binom{p}{k}$$

$$=\binom{pa}{p}+1+\sum_{k=1}^{p-1}\binom{pa}{p-k}\binom{p}{k},$$

and again using $p\,|\binom{p}{k}$ for $1 \leqslant k \leqslant p-1$, we get

$$\binom{p(a+1)}{p}\equiv\binom{pa}{p}+1(\mathrm{mod}\,p)$$

or

$$\binom{p(a+1)}{p}-(a+1)\equiv\binom{pa}{p}-a(\mathrm{mod}\,p) \tag{8.4}$$

for all integers a and prime p. Since $\binom{pa}{p}\equiv a(\mathrm{mod}\,p)$ trivially when $a=0$, we have an inductive proof that

$$\binom{pa}{p}\equiv a(\mathrm{mod}\,p) \tag{8.5}$$

for any integer a and any prime p.

This result with (8.2) then proves the Fermat congruence.

Congruence (8.5) is interesting in itself. Since $\binom{pa}{p}=a\binom{pa-1}{p-1}$, we have

$$\binom{pa}{p}-a=a\left\{\binom{pa-1}{p-1}-1\right\}. \tag{8.6}$$

According to Dickson [**A2**, p. 271], Catalan noted in 1874 that in fact

$$\binom{pa-1}{p-1} \equiv 1 (\mathrm{mod} p) \tag{8.7}$$

for any prime p and integer a. This with (8.6) is another way to get (8.5). Much more is known about such matters. Dickson [**A2**, p. 275] reports that Guerin found in 1916 that

$$\binom{pa-1}{p-1} \equiv a-1 (\mathrm{mod} p^3) \tag{8.8}$$

for prime $p > 3$. On the right side, $a-1$ should be 1.

9. Comparison of formulas. We wish to make some remarks about how various formulas compare to one another. Euler's formula for $n!$ may be stated as

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n = n! \tag{9.1}$$

and the binomial theorem gives us the formula

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} n^{n-k} = (n-1)^n \tag{9.2}$$

which differs from Euler's in that $(n-k)^n$ has been turned into n^{n-k}. One wonders then if there exist formulas which are somehow not quite either (9.1) or (9.2) or rather somehow are both. There are such formulas. They arise from what is known as Abel's generalization of the binomial theorem. One simple example [**A6**, formula (1.117)] is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^{n-k} (k+1)^{k-1} = (x-1)^n. \tag{9.3}$$

Here instead of having a fixed base raised to a variable power, or the reverse, we have a variable base $x+k$ raised to a variable power $n-k$. But the formula does not involve factorials on the right side. It is always interesting to speculate on the form that identities may take and hope to write formulas in as elegant a way as possible when looking for some clever proof. Really, the reader must excuse the author if the present paper reads like something straight out of Euler, but there is a place for old-fashioned formulas. Some famous mathematician is supposed to have said that if he had the right formula he could prove anything. And that is the use of formulas, to find clever proofs. Of formulas alone there is no end, but a good formula can be the key to a proof.

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RATIONAL RECIPROCITY LAWS

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Abstract. It is well known that the famous Legendre law of quadratic reciprocity, of which over 150 proofs are in print, has been generalized over the years to algebraic fields by a number of famous mathematicians from Gauss to Artin to the extent that it has become virtually unrecognizable. On the other hand, it seems to have escaped notice that in the past decade there were developed rational reciprocity laws for higher power residues which are more direct and easily recognizable generalizations of the Legendre law. These recent developments will be the subject of this report.

1. Introduction. Euler appears to have been the first to ask for what primes p is a given number a (prime to p) a quadratic residue of p. He had already obtained what is now known as Euler's criterion which can be written

$$a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \equiv x^2 \pmod{p} \\ -1 & \text{if } a \not\equiv x^2 \pmod{p} \end{cases} \tag{1}$$

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from which it is quite obvious that -1 is a quadratic residue of p if and only if $p=4n+1$. He found that 2 and 3 are quadratic residues of primes p if and only if $p=8n\pm1$ and $12n\pm1$ respectively. He also made the conjecture that if p and q are distinct odd primes then q is a quadratic residue of p if and only if $-p$ is a residue of q.

Legendre extended Euler's criteria for 2 and 3 to be residues and gave for $q<100$ the arithmetical progressions for primes p having q as a quadratic residue, namely:

$$p=4qn\pm r_i, \text{ where } r_i\equiv 1(\operatorname{mod}4) \text{ and } (r_i/q)=1. \tag{2}$$

He is also responsible for what is now known as the Legendre symbol used in (1), in terms of which he wrote down the reciprocity law which now bears his name:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{(p-1)(q-1)/4}. \tag{3}$$

Neither Euler nor Legendre succeeded in giving a proof of either (2) or (3). A good account of the early history and a proof of the equivalence of (2) and (3) will be found in the recent book by W. J. LeVeque [17]. While the elegance of (3) speaks for itself, (2) shows that the character of a fixed prime q to an arbitrary prime p depends on the infinite class of primes to which p belongs.

Gauss rediscovered the reciprocity law before his eighteenth birthday and was "tormented" by it for a whole year before he produced the first of his seven proofs. About a hundred years later Bachmann collected 50 proofs, and 15 years ago Gerstenhaber [8] published "The 152nd Proof of the Law of Quadratic Reciprocity" in this MONTHLY. In all likelihood there are another dozen proofs in existence by now.

Gauss was the first to consider extending the quadratic reciprocity law to higher power residues. If we let $p=kn+1$, then (1) becomes

$$a^{(p-1)/k}=\left(\frac{a}{p}\right)_k\equiv\zeta^{\operatorname{ind}a}(\operatorname{mod}p), \tag{4}$$

where ζ is a primitive kth root of unity and where the index of a is taken with respect to some primitive root g of p.

For $k=4$ we have $p=4n+1=a^2+b^2=(a+ib)(a-ib)=p_1p_2$. This led Gauss to the study of what is now known as the Gaussian primes $p_1=a+ib$ and to the discovery of a quartic reciprocity law for these complex primes. This law was proved by Eisenstein who wrote it in the elegant form which parallels (3), namely

$$\left(\frac{p_1}{q_1}\right)_4\left(\frac{q_1}{p_1}\right)_4=(-1)^{(p_1-1)(q_1-1)/16}. \tag{5}$$

This statement should be supplemented by the fact that -1 is a quartic residue of p if and only if $p=8n+1$ and that 2 is a quartic residue of p if and only if $p=a^2+64b_1^2$. Gauss was not only aware of this, but gave conditions for all primes $q\leqslant 19$ to be quartic residues of p in terms of the permissible ratios of a/b modulo q.

Kummer considered the problem for prime k and developed the theory of cyclotomic fields in order to prove a reciprocity law in such fields. Hilbert reinterpreted the reciprocity law in terms of the norm residue symbol and generalized it to arbitrary algebraic number fields. In his ninth problem, Hilbert asks for "the most general law of reciprocity in an arbitrary algebraic number field." In his 1969 account of Hilbert's ninth problem Faddeev [7] credits Šafarevič with the solution of the problem in 1949. On the other hand, in the 1976 AMS volume on Hilbert's problems Tate [20] does not even mention Šafarevič, but credits Artin with the solution in 1927, although he goes on to discuss further generalizations. It should be noted that in recent years the reciprocity problem has been restated in terms of the splitting of a general polynomial into factors modulo p. This is a generalization of the obvious fact that the quadratic equation splits into two distinct linear factors if and only if its discriminant is a quadratic residue of p. In an expository paper in this MONTHLY with the fetching

title "What is a reciprocity law?" Wyman [26] discusses the reciprocity problem from this point of view and concludes the paper with the following remark:

"Finally I have to confess that I still do not know what a reciprocity law is or what it should be. The reciprocity problem like many other number-theory problems can be stated in a fairly simple and concrete way. However the simply stated problems are often the hardest and a complete solution seems to be very far out of reach. In fact, we probably will not know what we are looking for until we have found it."

It is the purpose of this report to speak for those number-theorists who believe that they know what a reciprocity law is and not only know what they are looking for but have actually been discovering new rational reciprocity laws in the past decade.

2. Rational quartic reciprocity laws. We have already seen that Legendre's reciprocity law can be interpreted in at least three different ways and that none of the generalizations led to a rational reciprocity law. For our purposes we will simply define a reciprocity law as a reciprocal relation between the characters of two odd primes, or more generally between the characters of some function of such primes. To obtain a rational reciprocity law, we must put some conditions on the primes to insure that the product of these characters is ± 1.

In the quartic case, $p=q=1 \pmod 4$ and so the assumption that $(p/q)=1$ would insure that $(p/q)_4$ and $(q/p)_4$, and hence their product, are ± 1. The problem of determining these signs would give a generalization of either (2) or (3) or both. Some 20 years ago I combed the literature and asked my co-workers whether such a rational law was known to them with negative results.

Because Gauss and others have found binary quadratic forms representing p in terms of q and μ, where $\mu \equiv a/b \pmod q$, it seemed reasonable to try to find a general expression for q to be a residue of p in terms of these forms. This was done in 1958 [9] thus giving a generalization of (2). Unfortunately, the reciprocity law which was equivalent to these criteria did not appear to be rational and was not even stated in [9]. It reads:

$$\left(\frac{p}{q}\right)_4\left(\frac{q}{p}\right)_4=\left(\frac{2}{q}\right)\left(\frac{a+\sqrt{p}}{q}\right). \tag{6}$$

It was a decade later that Burde [6] gave a very elegant rational reciprocity law which is as follows:

Let $p=a^2+b^2$ and $q=A^2+B^2$ with $a\equiv A\equiv 1 \pmod 4$ and let $(p/q)=1$, then

$$\left(\frac{p}{q}\right)_4\left(\frac{q}{p}\right)_4=(-1)^{(q-1)/4}\left(\frac{aB-bA}{q}\right). \tag{7}$$

Although (6) was not recognized at first as a rational reciprocity law, it is not hard to show that (6) and (7) are equivalent. This fact proved in [11] provides a totally different proof of (7) and shows that the rational quartic reciprocity law is equivalent to the fact that the quartic character $(q/p)_4$ depends on the class of binary quadratic forms of discriminant $-pq$ which represent p and not on p itself.

Another rational form of the quartic reciprocity law for those primes which are represented by the form $p=c^2+qd^2$ was given independently and by entirely different methods by Ezra Brown [3] and myself [11] as follows:

If $p\equiv q\equiv 1 \pmod 4$ are such that $p=c^2+qd^2$, then

$$\left(\frac{p}{q}\right)_4\left(\frac{q}{p}\right)_4=\begin{cases} 1 & \text{if } q\equiv 1 \pmod 8 \\ (-1)^d & \text{if } q\equiv 5 \pmod 8 \end{cases}. \tag{8}$$

For $q=5$, 13, and 37 every prime p is represented as $p=c^2+qd^2$ if $(p/q)=1$. Brown [4, 5] also gave similar reciprocity laws for primes represented by other quadratic forms.

Meanwhile, another kind of reciprocity law which is very closely connected with the quartic law has been resurrected and is now known as the Scholz reciprocity law. The history of this law is as follows:

In 1969 Barrucand and Cohn [1] proved that the quadratic unit $\epsilon_2 = 1+\sqrt{2}$ is a quadratic residue of p if and only if $p=c^2+32d^2$. Jacob Brandler [2] showed that $\epsilon_5=(1+\sqrt{5})/2$ and $\epsilon_{13}=(3+\sqrt{13})/2$ are quadratic residues of $p=c^2+qd^2$ if and only if d is even and that $\epsilon_{17}=4+\sqrt{17}$ is always a quadratic residue of p.

Comparing this with (8) it was not hard to conjecture that

$$\left(\frac{\epsilon_q}{p}\right)=\left(\frac{p}{q}\right)_4\left(\frac{q}{p}\right)_4 \quad \text{if} \quad \left(\frac{p}{q}\right)=1 \tag{9}$$

from which it would immediately follow from symmetry that

$$\left(\frac{\epsilon_q}{p}\right)=\left(\frac{\epsilon_p}{q}\right). \tag{10}$$

I again combed the literature and asked my friends, but nobody knew whether this elegant result was true or false. Only after I devised a cyclotomic proof of (9) and therefore of (10) in [**10**] did I discover a verbal statement of (10) as part four of a complicated five-part theorem in class field theory in Scholz [**19**]. Since then, a completely elementary proof has been devised by Williams [**25**].

3. Rational octic and higher reciprocity laws. Results and conjectures about quartic characters of quadratic units will be found in my paper [**12**]. Recently Leonard and Williams [**15, 16**] proved some of these conjectures. They have made a detailed study of the quartic character of these units in these papers, but have not yet proved a quartic analogue of Scholtz' reciprocity law (10). We would expect that such a law would be intimately connected with an octic reciprocity law which was obtained independently by K. S. Williams [**24**] and P. Y. Wu [**27**] as follows:

Let $p=a^2+b^2=c^2+2d^2\equiv 1 (\mathrm{mod}\, 8)$ and $q=A^2+B^2=C^2+2D^2\equiv 1(\mathrm{mod}\, 8)$, $a\equiv c\equiv A\equiv C\equiv 1(\mathrm{mod}\, 4)$, with $(p/q)_4=(q/p)_4=1$; then

$$\left(\frac{p}{q}\right)_8\left(\frac{q}{p}\right)_8=\left(\frac{aB-bA}{q}\right)_4\left(\frac{cD-dC}{q}\right). \tag{11}$$

This should be supplemented by the well-known fact that

$$\left(\frac{2}{p}\right)_8=1 \quad \text{if and only if} \quad \begin{cases} p=a^2+256b^2 & \equiv 1(\mathrm{mod}\, 16)\\ p=a^2+\ \ 64b^2 & \equiv 9(\mathrm{mod}\, 16), b \text{ odd}. \end{cases} \tag{12}$$

Similar criteria have been worked out for all primes $q\leqslant 47$ in terms of $p=a^2+b^2=c^2+2d^2\equiv 1(\mathrm{mod}\, 8)$ by von Lienen [**19**], but no explicit conditions were given for a general q to be an octic residue of p in terms of the corresponding sets of binary quadratic forms, although obviously such criteria must exist and be equivalent to (11).

Leonard and Williams now have just published 16th power reciprocity law [**14**] which involves representation of p and q by a quaternary quadratic form, so that the work on kth power reciprocity laws is still in progress.

In this connection we must mention another form of the $2k$th power law which was derived from a recent generalization of the Gauss Lemma [**13**], which was used by Gauss in his third proof of the quadratic reciprocity law. This generalization states that if λ is a kth power residue of p, and if the product of the first half of the residues of p by λ taken modulo p contains $\mu_p(\lambda)$ residues which exceed $p/2$ then

$$(-1)^{\mu_p(\lambda)}=\left(\frac{\lambda}{p}\right)_{2k}. \tag{13}$$

Applying this lemma with $\lambda=q$, and then with $\lambda=p$ and p replaced by q, we obtain:

$$\left(\frac{p}{q}\right)_{2k}\left(\frac{q}{p}\right)_{2k}=(-1)^{\mu_p(q)+\mu_q(p)}, \qquad \left(\frac{p}{q}\right)_k=\left(\frac{q}{p}\right)_k=1. \tag{14}$$

This reduces to (3) for $k=1$, since $\mu_p(q)+\mu_q(p)\equiv(p-1)(q-1)/4 (\bmod 2)$. For $k=2$ and $k=4$ comparison of (14) with (7) and (11) relates the parity of the numbers $\mu_p(q)$ and $\mu_q(p)$ with a,b,c,d in the quadratic partitions of p. No direct proof of this fact has so far been obtained.

4. Rational reciprocity laws for odd powers. Because the character $(a/p)_k$ is never -1, we can no longer expect rational reciprocity laws, but we can still obtain criteria for q to be a kth power residue of p in terms of quadratic forms. This has been done by Jacobi for $k=3$ and $q\leqslant 37$, in terms of the partition of $4p=L^2+27M^2$. General conditions and quadratic forms for p in terms of q and the ratios $\mu\equiv L/M (\bmod q)$ similar to those obtained for $k=4$ will be found in [9]. Recently K. S. Williams [23] separated the remaining cases of $L/M (\bmod q)$ to correspond with the two non-residue classes.

For quintic residues the case is complicated by the fact that the criteria depend on the representation

$$16p=x^2+50u^2+50v^2+125w^2 \quad \text{with} \quad 4xw=v^2-u^2-4uv. \tag{15}$$

Recently K. S. Williams extended the known criteria to $q\leqslant 19$ in terms of the ratios of u/w and v/w [21].

He also returned to Euler's criterion and gave rational expressions for $a^{(p-1)/3}$ in terms of L and M in [22] and for $a^{(p-1)/5}$ in terms of the x,u,v,w in [23], and so we have come full circle back to Euler. For arbitrary k there appears to be no better way of finding out whether a given number is a kth power residue of a large prime p than by raising it to the $(p-1)/k$th power $(\bmod p)$ and asking whether it is one or not, especially with the advent of high-speed computing.

5. Applications. Far from concluding that our more elaborate criteria are of no value, we can turn the tables around and use Euler's criterion to obtain conditions on the variables

$$a,b,c,d \quad \text{in} \quad p=a^2+b^2=c^2+qd^2.$$

These conditions are useful, for example, in proving a number to be a prime by representing it uniquely by one of these quadratic forms.

Another application of the criteria for kth power residuacity is to the divisibility by p of the $(p-1)/k$th term of a second order recurring series [10].

The connection between the criteria for the quadratic unit and the solvability of $u^2-Du^2=-4$ was established in [19].

Beginning with [1], in [3], [11], [15], [16] and others the kth power residuacity of units was connected with the parity of class numbers in various quadratic fields both real and imaginary.

Finally, a relation was recently established between the $2k$th character of λ and the parity of the permutation arising from multiplying the kth power residues by λ [13]. We hope that many other connections will come to light in the future.

Presented to the annual meeting of the Northern Section of the MAA in San Francisco on Feb. 26, 1977.

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CORRECTIONS TO
"The Rational Cuboid Revisited"

(This MONTHLY, 84 (1977) 518-533)

JOHN LEECH

J. Lagrange has pointed out the following correction to my article. At the top of p. 523, for $x_i = 550{,}576$ read $x_i = 520{,}576$. See also his article [17], in which he gives another parametric solution of (3.2) and announces a complete proof of impossibility for the case Spohn [16] left incomplete. Two minor misprints: in the middle of p. 524, for $(x_2x_3)^3$ read $(x_2x_3)^2$; in the middle of p. 530 for $\alpha + b$ read $\alpha + \beta$.

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MISCELLANEA

10. The efforts of computer engineers have already produced a mechanized Briggs (who spent his lifetime computing logarithms) and a mechanized Barlow (whose famous Tables were his life's work) but no one has ever conceived of a mechanized Napier (for he *invented* logarithms).

B. V. Bowden, *Faster Than Thought*, London, 1953, p. 321

PROGRESS REPORTS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

EXTREMAL PROBLEMS FOR POLYNOMIALS

R. P. BOAS

One of the most romantic of all story plots is the one about the inventor who patiently works for years in obscurity and finally achieves resounding success. Such stories are rare in mathematics; this is a report on one of them.

So—once upon a time (just about a century ago) the chemist Mendeleev (the one who invented the periodic table of the elements) was interested in the relationship between the specific gravity of a solution and the concentration of the dissolved substance. This question seems no longer to be of much interest to physical chemists, but a service station still uses the relationship to check the concentration of antifreeze in the cooling system of your automobile, and a brewer or vintner uses it to test the concentration of alcohol in beer or wine. Mendeleev made accurate graphs (they agree with modern tables to three significant figures) for many substances, and found that the curves exhibited small kinks, so that no very simple formulas would represent them. He proceeded to fit the curves by a succession of parabolic arcs, and found that the arcs did not join smoothly. He then wondered whether the corners corresponded to genuine phenomena or were just the result of errors of measurement. He seems to have reasoned somewhat as follows. Suppose I have two quadratic polynomials $P_2(x)$ and $Q_2(x)$ on adjacent intervals, with graphs that join continuously but at an angle. Suppose I know that the slope of P_2 on its interval cannot exceed a number S, whereas the slope of Q_2 is distinctly larger than S; then I cannot expect to replace P_2 and Q_2 by a single quadratic on the union of the intervals. Hence Mendeleev was led to ask, if we know how large a quadratic polynomial P_2 is on a given interval $[a,b]$, how large can the derivative P_2' be on the same interval?

Mendeleev was enough of a mathematician to answer the question for himself, the answer being that if $|P_2(x)| \leqslant M$ on an interval of length $2L$, then $|P_2'(x)| \leqslant 4ML$ on the same interval, 4 being the best (that is, smallest) number that will always work. (Incidentally this led Mendeleev to conclude

that the corners were genuine.) If you were a chemist and found a pretty theorem like this, you would want to tell it to a mathematician, and that is just what Mendeleev did—he told it to A. A. Markov. If you are a mathematician and a chemist tells you a nice result about quadratic polynomials, you naturally want to generalize it to polynomials of degree n, and that is just what Markov did: he proved that if $P_n(x)$ is of degree n and $|P_n(x)| \leqslant M$ on an interval of length $2L$, then $|P_n'(x)| \leqslant n^2ML$ on the same interval, and that n^2 is the best constant. This has become famous as Markov's theorem; it is not only elegant but has many applications very different from the one that interested Mendeleev.

As it happens, equality can be attained in Markov's theorem only at the end-points of the interval in question. This suggests the problem of finding the point-by-point maximum for $|P_n'(x)|$: to find $M_n(x) = \max|P_n'(x)|$, where the maximum is taken over all polynomials of degree n such that $|P_n(x)| \leqslant M$ on $[a,b]$. Until recently not much was known about $M_n(x)$ except for what S. Bernstein proved in 1912. This is most easily stated for the interval $[-1,1]$: if $|P_n(x)| \leqslant M$ on $[-1,1]$ then $|P_n'(x)| \leqslant n(1-x^2)^{-1/2}$ on the same interval. This is useful in many applications, but is of course much weaker than Markov's theorem for x near ± 1; on the other hand it is much better for x near 0. Very many papers have been written about Bernstein's and Markov's theorems and similar extremal problems, but the problem of finding $M_n(x)$ remained untouched for many years.

The last statement is not quite true. Markov's paper (written in Russian in a not very accessible journal) must be one of the most often cited papers—everybody who writes about this kind of problem feels compelled to cite it—and one of the least read. If you go to the trouble to read it you will find that Markov not only raised the problem that I just mentioned, but took a considerable step toward solving it, and did solve it for $n=2$.

With this much background we can proceed to the story. Around 1930 E. V. Voronovskaya began to publish a series of short articles dealing with the Hausdorff moment problem, which is the study of sequences $\mu_n = \int_0^1 t^n\,dH(t)$, where H is a function of bounded variation. These articles are quite technical, use a rather forbidding special terminology, give few proofs, and attracted almost no attention. Very likely nobody who read them realized what they were leading up to. But—a quarter-century later Voronovskaya's work really paid off. She could solve not only the point-by-point Markov problem, but also a great variety of other extremal problems that had previously seemed too difficult for anyone to do anything with. Her work showed in particular that the kind of solution that the world was used to seeing was too simple to be possible for anything but the simplest problems. The only full account of her work appeared in 1963 in a book published by the Leningrad Institute for Electrotechnical Communication (and not easy to come by), although some of the details, including the solution of Markov's problem, had come out in journal articles in the late 1950's; an English translation [1] of the book was published in 1970.

How does the Hausdorff moment problem connect with extremal problems for polynomials? Answer: via the Riesz representation theorem for continuous linear functions on the space C of continuous functions $X(t)$ on $[0,1]$. This theorem says that these functionals are of the form $F(X) = \int_0^1 X(t)\,dH(t)$ with H of bounded variation, and moreover the norm of the functional F is the total variation of H. Now since polynomials form a dense set in C, a linear functional F is completely determined by its values on the polynomials or, since F is linear, by its values on $X_n(t) = t^n$, $n = 0,1,2,\ldots$; that is, by the moments $\int_0^1 t^n\,dH(t)$. Furthermore, $|F(P_n(t))| \leqslant \|F\| \max_t |P_n(t)|$, with $\|F\| = \operatorname{var} H = \sup|F(P_n(t))|$ for $\|P_n(t)\| \leqslant 1$. Now let $F(P_n) = P_n'(x)$ for a particular value of x. This functional corresponds to the moment sequence $0,1,2x,3x^2,\ldots$; the norm of F is $\max|P_n'(x)|$ over all polynomials of degree n for which $\max_t|P_n(t)| \leqslant 1$, and Markov's general problem is seen to be the problem of determining the norm of this particular functional. This reinterpretation does not, of course, solve the problem, but it does suggest a systematic attack of a wide variety of such problems in place of the many ad hoc methods that have been devised for special classes of extremal problems.

The actual solution of Markov's problem is not easy to describe. The extremals of the original Markov problem—to find $\max|P_n'(x)|$ given $\max|P_n(x)|$, on (say) the interval $[0,1]$—are the Čebyšev polynomials $T_n(x) = \cos n \cos^{-1}(2x-1)$. It turns out that the interval $[0,1]$ breaks up into a succession

of three kinds of intervals. In intervals of the first kind, $|P_n'(x)|$ is maximized by $|T_n'(x)|$. In another set of intervals the extremal polynomials have the form $\pm T_n(\lambda x)$ or $\pm T_n(\lambda(1-x))$. But in the remaining intervals the extremal polynomials are the much less well known Zolotarev polynomials, indeed different ones at each point, and the maximal function $M_n(x)$ has only been exhibited parametrically, although it can be evaluated numerically by computer. Reference [1] shows graphs for $n=2$ (where the Zolotarev polynomials do not occur), 3, 4, and 5. For $n>2$ the function $M_n(x)$ is, perhaps surprisingly, neither monotonic nor convex; it is, however, continuously differentiable (which shows that Markov's $M_3(x)$ cannot be correct for all values of x). Perhaps one should not be too surprised at finding so much complexity in an apparently simple problem, especially since one of Voronovskaya's theorems (discovered independently by Rogosinski) is that *every* polynomial is extremal for *some* extremal problem.

Reference

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MATHEMATICAL NOTES

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THE TOEPLITZ-HAUSDORFF THEOREM AND ELLIPTICITY CONDITIONS

ALAN MCINTOSH

Let A be a linear operator in C^n with matrix (a_{ij}). The numerical range $W(A)$ and the set $V(A)$ are defined by

$$W(A)=\{(A\zeta,\zeta)\mid \zeta\in C^n, \|\zeta\|=1\}, \quad \text{and}$$
$$V(A)=\{(A\xi,\xi)\mid \xi\in R^n, \|\xi\|=1\},$$

where $(\zeta',\zeta)=\Sigma\zeta_i'\bar{\zeta}_i$ and $\|\zeta\|=(\zeta,\zeta)^{1/2}$. Clearly, $V(A)\subset W(A)$.

Toeplitz [5] and Hausdorff [3] proved that $W(A)$ is a convex subset of C if $n\geqslant 2$. It is shown in Theorem 1(ii) that $V(A)$ is also convex if $n\geqslant 3$. This is done by a simple geometric argument. It is then shown in Theorem 1(iii) that the convexity of $W(A)$ is a consequence of the convexity of $V(A)$, thus providing a new proof of the Toeplitz-Hausdorff theorem which is more transparent than the usual ones. (See, however, the paper of Chandler Davis [2].)

The set $V(A)$ is of interest for the following reason. A partial differential operator with constant complex coefficients,

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial}{\partial x_i}+c,$$

is elliptic if $0\notin V(A)$, and strongly elliptic if $\operatorname{Re} V(A)>0$. In relating these conditions, it is useful to know that $V(A)$ is a convex subset of C for $n\geqslant 3$. See [4] for comments about this and a characterization of proper ellipticity using $V(A)$ when $n=2$.

Some further results on $V(A)$ and its relationship to $W(A)$ are contained in Theorems 3 and 4.

THEOREM 1. (i) *If* $n=2$, *then* $V(A)$ *is an ellipse (possibly degenerate).* (ii) *If* $n\geqslant 3$, *then* $V(A)$ *is a convex subset of* C. (iii) *If* $n\geqslant 2$, *then* $W(A)$ *is a convex subset of* C.

Proof. (i) If $n=2$, then

$$\begin{aligned} V(A)&=\{a_{11}\cos^2\theta+(a_{12}+a_{21})\cos\theta\sin\theta+a_{22}\sin^2\theta\,|\,0\leqslant\theta<2\pi\}\\ &=\{\tfrac{1}{2}(a_{11}-a_{22})\cos 2\theta+\tfrac{1}{2}(a_{12}+a_{21})\sin 2\theta+\tfrac{1}{2}(a_{11}+a_{22})\,|\,0\leqslant\theta<2\pi\}, \end{aligned}$$

which is an ellipse with centre $\frac{1}{2}\mathrm{tr}(A)$.

(ii) Suppose $n\geqslant 3$. Note that $V(A)=f(S^{n-1})$, where $S^{n-1}=\{\xi\in R^n\,|\,\|\xi\|=1\}$ and $f:S^{n-1}\to C$ is defined by $f(\xi)=(A\xi,\xi)$. Let $z=f(\xi)$ and $z'=f(\xi')$ be two points in $V(A)$. Let X be the two-dimensional subspace spanned by ξ and ξ' and let P be the orthogonal projection onto X. Then $V(A)\supset f(S^{n-1}\cap X)=V(PA)|_X)$, which, by (i), is an ellipse passing through z and z'. Moreover, if $\gamma:[0,2\pi]\to S^{n-1}\cap X$ is a parametrization which makes the circle $S^{n-1}\cap X$ into a simple closed curve, then we see from the proof of (i), that $f_o\gamma$ is a parametrization which traces out the ellipse $f(S^{n-1}\cap X)$ twice in the same direction. Now γ is null-homotopic in S^{n-1}, so $f_o\gamma$ is null-homotopic in $V(A)$. Hence the interior of the ellipse $f(S^{n-1}\cap X)$ is contained in $V(A)$, and so, in particular, is the line segment joining z to z'. We conclude that $V(A)$ is convex.

(iii) $$W(A)=V\left(\begin{bmatrix} A & iA\\ -iA & A\end{bmatrix}\right).$$ Q.E.D.

THEOREM 2. (Toeplitz-Hausdorff). *If A is a linear operator with domain D in a complex Hilbert space, then the numerical range of A, namely* $\{(Au,u)\,|\,u\in D,\|u\|=1\}$, *is a convex set.*

Proof. Reduce to two dimensions (as was done in proving Theorem 1(ii)), and apply Theorem 1(iii).

Denote the transpose of A by A^t, and let $A^s=\frac{1}{2}(A+A^t)$, the symmetric part of A. The convex hull of a set S is denoted by $\mathrm{co}\,S$.

THEOREM 3. (i) $V(A)=V(A^s)$. (ii) *If* $A=A^t$, *then* $W(A)=\mathrm{co}\,V(A)$. *In particular, if* $n\geqslant 3$ *and* $A=A^t$, *then* $W(A)=V(A)$.

Proof. Part (i) is obvious. If $A=A^t$, then

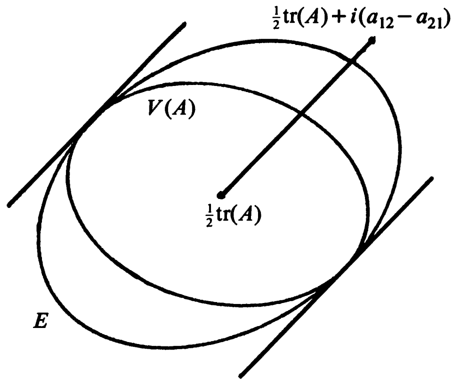
$$\begin{aligned} W(A)&=V\left(\begin{bmatrix} A & iA\\ -iA & A\end{bmatrix}\right)=V\left(\begin{bmatrix} A & 0\\ 0 & A\end{bmatrix}\right)\quad\text{(by (i))}\\ &=\{\|\xi\|^2(A(\|\xi\|^{-1}\xi),\|\xi\|^{-1}\xi)\\ &\quad+\|\eta\|^2(A(\|\eta\|^{-1}\eta),\|\eta\|^{-1}\eta)\,|\,\xi,\eta\in R^n,\|\xi\|^2+\|\eta\|^2=1\}\\ &=\mathrm{co}\,V(A). \end{aligned}$$ Q.E.D.

THEOREM 4. *If* $n=2$, *then* (i) $W(A)=\mathrm{co}\,E$ *for some ellipse E (possibly degenerate) with the same centre as the ellipse $V(A)$, namely* $\frac{1}{2}\mathrm{tr}(A)$, *and* (ii) $W(A)\subset\mathrm{co}\,V(A)+i(a_{12}-a_{21})[-\frac{1}{2},\frac{1}{2}]$.

Proof. (i) It will be shown in the lemma that there exists a unitary operator U such that UAU^* is symmetric. For such an operator U, let $E=V(UAU^*)$. Then E is an ellipse with centre $\frac{1}{2}\mathrm{tr}(UAU^*)$ $=\frac{1}{2}\mathrm{tr}(A)$, and $W(A)=W(UAU^*)=\mathrm{co}\,V(UAU^*)=\mathrm{co}\,E$.

(ii) If ζ is a unit vector in C^2, then

$$\begin{aligned} (A\zeta,\zeta)&=(A^s\zeta,\zeta)+\tfrac{1}{2}(a_{12}-a_{21})(\zeta_2\bar\zeta_1-\zeta_1\bar\zeta_2)\in W(A^s)+i(a_{12}-a_{21})[-\tfrac{1}{2},\tfrac{1}{2}]\\ &=\mathrm{co}\,V(A)+i(a_{12}-a_{21})[-\tfrac{1}{2},\tfrac{1}{2}]. \end{aligned}$$ Q.E.D.



LEMMA. *If $n=2$ there is a unitary operator U such that UAU^* is symmetric.*

Proof. Let $SU(2)$ denote the space of 2×2 unitary matrices with determinant 1, and define $g:SU(2)\to C$ by $g(U)=(UAU^*)_{12}-(UAU^*)_{21}$. It suffices to show that $0\in g(SU(2))$. Consider the loop in $SU(2)$ defined by

$$\gamma(\theta)=\begin{bmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{bmatrix},\qquad 0\leqslant\theta\leqslant 2\pi.$$

Computation shows that $g(\gamma(\theta))=(a_{12}-a_{21})\cos 2\theta+i(a_{22}-a_{11})\sin 2\theta$, so, as θ varies from 0 to 2π, $g(\gamma(\theta))$ traces out an ellipse with centre 0 twice in the same direction. Now $SU(2)$ is simply connected [1, p. 60], so γ is null-homotopic in $SU(2)$, and consequently $g_0\gamma$ is null-homotopic in $g(SU(2))$. Hence $0\in g(SU(2))$. Q.E.D.

NOTE ADDED IN PROOFS. It has been brought to my attention that Louis Brickman has proved Theorems 1(ii) and 3 by different techniques. See *Proc. Amer. Math. Soc.*, 12 (1961) 61–66.

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1. Claude Chevalley, Theory of Lie Groups, I, Princeton University Press, Princeton, NJ, 1946.
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THE PLANES OBTAINABLE BY GLUING REGULAR TETRAHEDRA

S. K. STEIN

J. H. Mason [1] shows that it is impossible to glue regular tetrahedra together face to face to form a ring. We use the four matrices he introduced to show:

(a) In a chain of regular tetrahedra no two faces are parallel.

(b) In a chain of regular tetrahedra no edge is parallel to a face (unless it trivially lies on the face).

Clearly (b) implies (a), which, in turn, implies Mason's result.

To review Mason's approach, consider a regular tetrahedron with vertices A,B,C,D. Let R_1 be the isometry "reflection in the plane of BCD." This is "the reflection of A, with B,C,D fixed," and records the gluing of two tetrahedra along the common face BCD. Then

$$A'=R_1(A)=-A+2/3B+2/3C+2/3D,\quad B'=R_1(B)=B,\quad C'=R_1(C)=C,\quad D'=R_1(D)=D.$$

This is recorded in matrix multiplication as

$$(A',B',C',D')=(A,B,C,D)\begin{bmatrix}-1&0&0&0\\2/3&1&0&0\\2/3&0&1&0\\2/3&0&0&1\end{bmatrix}.$$

The four reflections (gluings) are described by the matrices

$$R_1=\begin{bmatrix}-1&0&0&0\\2/3&1&0&0\\2/3&0&1&0\\2/3&0&0&1\end{bmatrix},\quad R_2=\begin{bmatrix}1&2/3&0&0\\0&-1&0&0\\0&2/3&1&0\\0&2/3&0&1\end{bmatrix},$$

$$R_3=\begin{bmatrix}1&0&2/3&0\\0&1&2/3&0\\0&0&-1&0\\0&0&2/3&1\end{bmatrix},\quad R_4=\begin{bmatrix}1&0&0&2/3\\0&1&0&2/3\\0&0&1&2/3\\0&0&0&-1\end{bmatrix}.$$

A chain of $d+1$ tetrahedra, starting with the tetrahedron $ABCD$, corresponds to a product

$$(A,B,C,D)M_1M_2\cdots M_d$$

where each M_i is one of R_1,R_2,R_3,R_4. Moreover, in such a product, consecutive M_i's may be assumed to be distinct. This assumption excludes a trivial ring of length two, but does not restrict the set of tetrahedra obtainable from $ABCD$ by gluing.

Now replace the fraction "2/3" throughout R_1,R_2,R_3,R_4 by the symbol a. The following lemma, implicit in [1], is the key to the argument.

LEMMA. *Let $M_1,M_2,\ldots,M_d$ be a sequence of the matrices R_i, where $2/3$ is replaced by a. Assume $M_i\neq M_{i+1}$, $i=1,2,\ldots,d-1$. Then each of the sixteen terms in the product matrix $M_1M_2\cdots M_d$ is a polynomial in a with integer coefficients and lead coefficient 1 or -1 (or else the 0-polynomial). Moreover, if $M_1=R_i$ and $M_d=R_j$, then*:

(i) *The entry in the ith row and jth column of $M_1M_2\cdots M_d$ has degree $d-1$. Other entries in the ith row have degree less than $d-1$.*

(ii) *The other three entries in the jth column have degree d. The entries not on the ith row and not on the jth column have degree less than d.*

The inductive proof is straightforward. (The 0-polynomial is taken to have degree $-\infty$, say.)

With the aid of this lemma, (b) is easy to establish. For instance, let us show that no edge obtainable from the edge BD is parallel to a face obtainable from the face ABC.

Consider, therefore, two sequences of gluings resulting in

$$(A',B',C',D')=(A,B,C,D)M_1M_2\cdots M_d \quad\text{and}\quad (A'',B'',C'',D'')=(A,B,C,D)N_1N_2\cdots N_e$$

We will show that the edge $B'D'$ is not parallel to the face $A''B''C''$.

If $B'D'$ were parallel to $A''B''C''$ then

$$B'''D'''=BDM_1M_2\cdots M_dN_eN_{e-1}\cdots N_1 \tag{1}$$

would be parallel to ABC. (The notation $BDM_1M_2\cdots M_dN_eN_{e-1}\cdots N_1$ denotes the image of the edge BD under the product of the indicated isometries, applying M_1 first, and so on.) In (1) delete any consecutive pairs of identical matrices. After deletion, we would have a nonempty sequence $P_1,P_2,\ldots,P_f$ of matrices satisfying the hypothesis of the lemma and such that

$$B'''D'''=BDP_1P_2\cdots P_f \tag{2}$$

is parallel to ABC.

To show that (2) cannot hold, introduce an xyz coordinate system such that the face ABC lies in the xy-plane. Then the z-component of BD is not 0; call it z_0. We wish to show that the z-component of $B'''D'''$ is not 0.

The final reflection in (2), namely P_f, can be assumed to be a reflection of either the vertex $BP_1P_2\cdots P_{f-1}$ or the vertex $DP_1P_2\cdots P_{f-1}$. Otherwise, the edge $B'''D'''$ would be present earlier in the chain, and we could consider the shortest chain that reaches the edge $B'''D'''$. Let us assume that B''' is obtained by final reflection in $BP_1P_2\cdots P_{f-1}$, that is, $P_f = R_2$.

By the lemma, the second column of the matrix $P_1P_2\cdots P_f$ has three terms of degree f and one term of degree $f-1$. For concreteness, assume that the term of degree $f-1$ occurs, say, in the fourth column. Denote the four polynomials in the 2nd column in order, $p_f^{(1)}, p_f^{(2)}, p_f^{(3)}, p_{f-1}$, the subscripts recording degrees. Then

$$B''' = p_f^{(1)}A + p_f^{(2)}B + p_f^{(3)}C + p_{f-1}D.$$

Similarly, by the lemma,

$$D''' = q^{(1)}A + q^{(2)}B + q^{(3)}C + qD,$$

where $q^{(1)}, q^{(2)}, q^{(3)}$ have degree $\leqslant f-1$ and q has degree $\leqslant f-2$.

The z-component of $B'''D'''$ is simply

$$(p_{f-1} - q)z_0. \tag{3}$$

If (3) were 0, $p_{f-1} - q$ would be 0. Thus $2/3$ would be a root of a polynomial with integer coefficients and lead coefficient 1 or -1, in contradiction with the rational-root theorem. This shows that no edge obtainable from BD is parallel to a face obtainable from ABC.

A similar argument shows that no edge obtainable from AB is parallel to a face obtainable from ABC, other than the trivial case when the sequence of reflections leave AB fixed. This establishes (b), hence (a).

Other conclusions can be drawn in a similar way. For instance, under repeated reflections, the vertex A is never carried to a point lying in the plane of BCD, nor to a point lying in the plane through A parallel to BCD.

Reference

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ON THE INCLUSION $L^p(\mu) \subset L^q(\mu)$

B. SUBRAMANIAN

It is well known that $l^p(X) \subset l^q(X)$, if $0 < p < q$, where X is a non-empty set. We wish to investigate under what conditions $L^p(\mu) \subset L^q(\mu)$ for general measure spaces.

Throughout this discussion X is a non-empty set, $\mathscr{Q}$ a σ-algebra of subsets of X, μ a positive measure; p, q are positive real numbers, such that $p < q$. We use the symbols $L^p(\mu)$, $L^\infty(\mu)$ in the sense of Royden [**1**, p. 93].

LEMMA. *If $L^1(\mu) \subset L^2(\mu)$, then $L^1(\mu)$ must be contained in $L^\infty(\mu)$.*

Proof. Suppose $f \in L^1(\mu)$ and $f \notin L^\infty(\mu)$. We show there exists a function $h \in L^1(\mu)$ such that $h \notin L^2(\mu)$ and $h \notin L^\infty(\mu)$.

We may assume $f \geqslant 0$. Let g: $X \to R$ be a function defined as follows:

$$g(x)=\begin{cases} f(x)^{\frac{1}{2}} & \text{when } f(x)\geqslant 1\\ 0 & \text{when } 0\leqslant f(x)<1.\end{cases}$$

It is clear $g\notin L^\infty(\mu)$ and $g\in L^1(\mu)\cap L^2(\mu)$ since $g\leqslant g^2\leqslant f$.

Let $A_n=\{x\colon (n-1)\leqslant g(x)<n\}$. Without loss of generality, we may assume $c_n=\mu A_n>0$ ($n=1,2,\ldots$). We define a function $h\colon X\to R$ by $h(x)=(n-1)+1/\sqrt{c_n}$ ($x\in X$ and $x\in A_n$).

Since $g\in L^1(\mu)\cap L^2(\mu)$, we have,

$$\infty>\int g\,d\mu=\sum_{n=1}^{\infty}\int_{A_n} g\,d\mu\geqslant\sum_{n=1}^{\infty}(n-1)c_n \tag{1}$$

and

$$\infty>\int g^2 d\mu=\sum_{n=1}^{\infty}\int_{A_n} g^2 d\mu\geqslant\sum_{n=1}^{\infty}(n-1)^2c_n. \tag{2}$$

Since $\Sigma_{n=1}^{\infty}c_n\leqslant c_1+\Sigma_{n=2}^{\infty}(n-1)c_n$, it follows from (1) that $\Sigma_1^\infty c_n$ is convergent. By the arithmetic-geometric mean inequality $2\sqrt{c_n}\leqslant(n-1)^2c_n+1/(n-1)^2$ ($n=2,3,\ldots$). Hence, it follows from (2) and the convergence of $\Sigma_{n=2}^{\infty}1/(n-1)^2$ that $\Sigma_{n=1}^{\infty}\sqrt{c_n}$ converges. Now

$$\int h\,d\mu=\sum_{n=1}^{\infty}(n-1)c_n+\sum_{1}^{\infty}\sqrt{c_n}<\infty$$

and thus, $h\in L^1(\mu)$. On the other hand

$$\int h^2 d\mu=\sum_{1}^{\infty}\left[(n-1)^2c_n+2(n-1)\sqrt{c_n}+1\right]=\infty,$$

so that $h\notin L^2(\mu)$. Q.E.D.

Now we give a set of necessary and sufficient conditions on the measure space $(X,\mathcal{Q},\mu)$ which ensures $L^p(\mu)\subset L^q(\mu)$ whenever $0<p<q$.

PROPOSITION. (i) *If $\mu(X)<\infty$, then $L^p(\mu)\subset L^q(\mu)$ whenever $0<p<q$, iff any collection of disjoint measurable sets of positive measure is finite.*

(ii) *If $\mu(X)=\infty$, then $L^p(\mu)\subset L^q(\mu)$ iff for any sequence $\{E_n\}$ of disjoint measurable sets of positive measure, the sequence $\{\mu E_n\}$ is bounded away from zero.*

Proof of (i). *Necessity*: Let $\mu(X)<\infty$. Suppose $\{E_n\}$ is a sequence of disjoint measurable sets such that for infinitely many $n,\mu(E_n)\neq 0$.

Since $\mu(\cup_1^\infty E_n)=\Sigma_1^\infty\mu E_n<\infty$, we must have $\mu E_n\to 0$ as $n\to\infty$. Let $\{E_{n_k}\}$ be a subsequence such that $0<\mu E_{n_k}<1/k^3$.

Take $h\colon X\to R$ to be the function defined as follows:

$$\begin{aligned} h(x)&=k \quad \text{if } x\in E_{n_k}\\ &=0 \quad \text{if } x\notin\cup\{E_{n_k}\colon k\in N\}.\end{aligned}$$

Then $h\notin L^\infty(\mu)$ and

$$\int h\,d\mu=\Sigma_{k=1}^{\infty}k\cdot\mu(E_{n_k})<\Sigma_1^\infty\frac{1}{k^2}<\infty.$$

Hence $h\in L^1(\mu)$. In view of the lemma, this implies $L^p(\mu)\subset L^q(\mu)$ is not always true. Thus the condition is necessary.

Sufficiency: Let f be a measurable function. Then the condition implies $f\in L^\infty(\mu)$. Since $\mu(X)<\infty$, it follows that $f\in L^p(\mu)$, for each $p>0$. Hence $L^p(\mu)=L^q(\mu)=L^\infty(\mu)$.

Proof of (ii). *Necessity*: Suppose $\mu(X)=\infty$. If $\{E_n\}$ is a sequence of disjoint measurable sets such

that $\mu E_n > 0$ and $\mu E_n \to 0$ as $n \to \infty$, then as in case (i) we can construct $h \in L^1(\mu)$ with $h \notin L^\infty(\mu)$ and hence $L^p(\mu) \subset L^q(\mu)$ is not always true. Therefore the condition is necessary.

Sufficiency: Suppose $f \in L^p(\mu)$. Set $A_n = \{x\colon\ n < |f(x)| \leqslant n+1\}$ $n = 0, 1, 2, \ldots$, so that $\{A_n\}$ is a disjoint collection of measurable sets. If $\mu A_n \neq 0$ for infinitely many n, then $\mu A_n \geqslant c > 0$ for these n, by hypothesis, and hence we must have

$$\infty = \sum_{n=1}^{\infty} n^p \mu(A_n) \leqslant \sum_{n=0}^{\infty} \int_{A_n} |f|^p \, d\mu = \int |f|^p \, d\mu < \infty$$

a contradiction.

Hence for some non-negative integer n_0, $\mu A_n = 0$ whenever $n > n_0$. Let $q > p$. Since on A_0, $|f|^q \leqslant |f|^p$ and since for $1 \leqslant n \leqslant n_0$,

$$(n+1)^q = \left(1 + \frac{1}{n}\right)^p (n+1)^{q-p} n^p \leqslant 2^p (n_0+1)^{q-p} n^p,$$

$$\begin{aligned}
\int |f|^q \, d\mu &= \int_{A_0} |f|^q \, d\mu + \sum_{n=1}^{n_0} \int_{A_n} |f|^q \, d\mu \\
&\leqslant \int_{A_0} |f|^p \, d\mu + \sum_{n=1}^{n_0} (n+1)^q \mu(A_n) \\
&\leqslant \int_{A_0} |f|^p \, d\mu + 2^p (n_0+1)^{q-p} \sum_{n=1}^{n_0} n^p \mu(A_n) \\
&\leqslant \int_{A_0} |f|^p \, d\mu + 2^p (n_0+1)^{q-p} \sum_{n=1}^{n_0} \int_{A_n} |f|^p \, d\mu \\
&\leqslant k \int |f|^p \, d\mu < \infty,
\end{aligned}$$

where $k = 1 + 2^p (n_0+1)^{q-p}$. Hence $f \in L^q(\mu)$. Q.E.D.

I would like to thank the referee and Professor Martin Helling for their helpful comments on the preparation of the note.

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APPLICATION OF A MEAN VALUE THEOREM FOR INTEGRALS TO SERIES SUMMATION

EBERHARD L. STARK

This note will give an answer to a query by M. Spivak [1, p. 237]. In his popular *Calculus*, Problem 13*29 reads: "Suppose that g is increasing on $[a,b]$ and that f is integrable on $[a,b]$. Prove that there is a number ξ in $[a,b]$ such that

$$\int_a^b f(t) g(t) dt = g(a) \int_a^\xi f(t) dt + g(b) \int_\xi^b f(t) dt." \tag{1}$$

It is accompanied by the remark: "The result of the problem, which I have cribbed from several older

calculus texts, is called The Third Mean Value Theorem for Integrals. To be quite frank, I haven't the slightest idea what it is good for."

Indeed, (1) is good for a rather elementary proof of

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{2}$$

which would avoid, e.g., convergence theory for Fourier series ([**2**, p. 381, 410; **3**, p. 531; **4**, p. 480]). For the Dirichlet kernel the two representations

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(2n+1)(x/2)}{2\sin(x/2)} \qquad (x \in \mathbf{R}, n \in \mathbf{N}) \tag{3}$$

are well known. Set $M_n = \int_0^{\pi} tD_n(t)\,dt$; by employing the polynomial representation of (3), partial integration results in

$$M_{2m-1} = 2\left\{\frac{\pi^2}{8} - \sum_{k=1}^{m} \frac{1}{(2k-1)^2}\right\} \qquad (m \in \mathbf{N}). \tag{4}$$

On the other hand, the closed representation of (3) leads to

$$\begin{aligned} M_{2m-1} &= \int_0^{\pi} \left(\frac{t/2}{\sin(t/2)}\right) \sin(4m-1)\frac{t}{2}\,dt \\ &= 2\left\{1 + \left(\frac{\pi}{2} - 1\right)\cos(4m-1)\frac{\xi}{2}\right\}\frac{1}{4m-1} \qquad (0 \leqslant \xi \leqslant \pi) \\ &= O(1/m) \qquad (m \to \infty), \end{aligned} \tag{5}$$

where, in the second step, (1) has been applied for the functions $f(x) = \sin((4m-1)x/2) \in C[0,\pi]$, $g(x) = (x/2)/\sin(x/2)$, with $g(0) = 1$, $g(\pi) = \pi/2$. A combination of (5) and (4) yields

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8};$$

the value of the sum (2) is now immediate.

For the first part of the proof, i.e., (3) implies (4), see [**5**]; however, the representation ([**5**, (2)]) for $D_n(x)$, though being "easily verified by induction and routine trigonometric identities," is wrong throughout that note. (For a simple proof, without induction, see, e.g., [**2**, p. 405; **3**, p. 538; **4**, p. 470].)

As a matter of fact, it should be mentioned that in textbooks (1) or slightly altered versions are generally called *The Second Mean Value Theorem* (e.g., [**3**, p. 130, 153; **4**, p. 95]) or, more precisely, *The Weierstrass Form of Bonnet's Theorem* ([**2**, p. 163]). In addition, there are even sharper versions of Bonnet's theorem ([**2**, p. 163, Cor. 4.2; **3**, p. 153, Ex. 26]), for which the first integral on the right-hand side of (1) may be cancelled, thus reducing the actual calculations in (5) to a minimum. Further useful references are, e.g., [**6**, p. 189, 217 (giving an application to the proof of Taylor's formula); **7**, p. 304; **8**, p. 214; **9**, p. 325; and **10**, p. 618, 666 (pointing out some historical sources)]. For a somewhat related proof, compare [**11**], and for further elementary proofs of (2), see [**12**] and the literature cited there.

Editor's note: The second mean value theorem is a bread-and-butter lemma in theories like trigonometric series, Laplace transforms, etc. See, for example, *Zygmund Trigonometric Series*, 2nd ed., vol. 1, p. 58 (Cambridge, 1959), Widder, *The Laplace Transform* (Princeton, 1941), p. 65.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

IS THERE AN OCTIC RECIPROCITY LAW OF SCHOLZ TYPE?

DUNCAN A. BUELL AND KENNETH S. WILLIAMS

Recently there has been a revival of interest in the determination of rational reciprocity laws. The first law of this kind to be discovered was the famous Legendre-Gauss law of quadratic reciprocity

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{(p-1)(q-1)/4},$$

where p and q are odd primes and (p/q) is the Legendre symbol, which is plus or minus one according as p is or is not a quadratic residue of q.

If one assumes that $p \equiv q \equiv 1 \pmod 4$ and that $(p/q) = +1$, then the symbol $(p/q)_4$ is plus or minus one according as p is or is not a quartic residue of q. Under these assumptions Scholz's reciprocity law can be stated

$$\left(\frac{p}{q}\right)_4\left(\frac{q}{p}\right)_4=\left(\frac{\varepsilon_p}{q}\right)=\left(\frac{\varepsilon_q}{p}\right), \tag{1}$$

where ε_p denotes the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{p})$ and is an integer modulo q if p is a square modulo q. This law was proved by Scholz [5] in 1934 using class field theory. Since then, several more elementary proofs have been published [2], [3], [6], and the expository article by Emma Lehmer [4] gives an account of recent developments.

Our aim is to formulate an octic analogue of the Scholz reciprocity law under the assumption that

p and q are primes such that $p \equiv q \equiv 1 \pmod 8$, $(p/q)_4 = (q/p)_4 = +1$, so that the symbols $(p/q)_8$ and $(q/p)_8$ are defined. Then, by (1), we have $(\varepsilon_p/q) = (\varepsilon_q/p) = +1$, so that $(\varepsilon_p/q)_4$ is plus or minus one according as ε_p is or is not a quartic residue of q. When the norm of ε_{pq}, denoted by $N(\varepsilon_{pq})$, is equal to $+1$, we have to introduce the class number $h(pq)$ of the real field $\mathbf{Q}(\sqrt{pq})$. It is known [1] that if $p \equiv q \equiv 1 \pmod 8$, then

$$h(pq) \equiv \begin{cases} 0 \pmod 8, & \text{if } N(\varepsilon_{pq}) = -1, \\ 0 \pmod 4, & \text{if } N(\varepsilon_{pq}) = +1. \end{cases}$$

Armed with this information, we can state the following conjecture. Let p and q be primes such that $p \equiv q \equiv 1 \pmod 8$ and $(p/q)_4 = (q/p)_4 = +1$, then

$$\left(\frac{p}{q}\right)_8 \left(\frac{q}{p}\right)_8 = \begin{cases} \left(\dfrac{\varepsilon_p}{q}\right)_4 \left(\dfrac{\varepsilon_q}{p}\right)_4, & \text{if } N(\varepsilon_{pq}) = -1, \\ (-1)^{h(pq)/4} \left(\dfrac{\varepsilon_p}{q}\right)_4 \left(\dfrac{\varepsilon_q}{p}\right)_4, & \text{if } N(\varepsilon_{pq}) = +1. \end{cases}$$

This conjecture is based on numerical evidence alone and has been verified by machine computations for all 272 prime pairs (p,q) with $p < q < 2000$, $p \equiv q \equiv 1 \pmod 8$ and $(p/q)_4 = (q/p)_4 = +1$.

We acknowledge with thanks the help of Hugh C. Williams of the University of Manitoba, who kindly computed a number of values of $h(pq)$ for us, and we thank the referee for suggestions which led to the improvement of the format of this paper.

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CLASSROOM NOTES

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INDETERMINATE FORMS OF EXPONENTIAL TYPE

JOHN V. BAXLEY AND ELMER K. HAYASHI

When $\lim f(x)^{g(x)}$ yields an indeterminate of the form 0^0, ∞^0, or 1^∞, the usual procedure is to consider $g(x)\log f(x)$ and apply L'Hospital's Rule (see, for instance [1], [2], [4]). However, this method

requires that f and g be differentiable and even so it may fail as in the examples

$$\lim_{x\to 0+}\left(2\sin\sqrt{x}+\sqrt{x}\,\sin(1/x)\right)^{x}$$

and

$$\lim_{x\to 0}\left(1+x\sin(1/x^{4})e^{-1/x^{2}}\right)e^{1/x^{2}}.$$

See [3]. Using the results in this paper, it is easy to see at a glance that both of the above limits are 1; in fact it will be clear why 0^0 and ∞^0 almost always have limit 1. These results are simple and make no direct use of L'Hospital's rule; in fact no smoothness conditions are required of the functions f and g. Only their orders of magnitude are significant. All results will be stated for x approaching 0. The corresponding results for x approaching a or $\pm\infty$ are obtained by using the appropriate change of variables.

THEOREM 1. *If* $\lim_{x\to 0+} g(x)=0$, *if there exists a real number* α *such that* $b(x)=f(x)/x^{\alpha}$ *is positive, bounded, and bounded away from* 0 *as* $x\to 0+$, *and if* $\gamma=\alpha\lim_{x\to 0+} g(x)\log x$ *exists or is* $\pm\infty$, *then*

$$\lim_{x\to 0+} f(x)^{g(x)}=e^{\gamma},$$

where $e^{\infty}=\infty$ *and* $e^{-\infty}=0$.

Proof. Using the hypotheses on f, it follows that $g(x)\log f(x)=g(x)\log(x^{\alpha}b(x))=\alpha g(x)\log x+g(x)\log b(x)$. But $\log b(x)$ is bounded and $g(x)\to 0$ as $x\to 0+$. Thus

$$\lim_{x\to 0+} g(x)\log f(x)=\lim_{x\to 0+}\alpha g(x)\log x=\gamma.$$

The theorem follows by exponentiation.

Note that when $\alpha>0$ the indeterminate form is 0^0, and when $\alpha<0$ the indeterminate form is ∞^0. Further, if $f(x)^{g(x)}\to k\neq 0,\infty$, then $g(x)\log f(x)\to\log k$ so the rate of growth of $\log f(x)$ is closely tied to the rate at which $g(x)\to 0$; in fact, the reader may easily develop a converse to the theorem.

If the order of magnitude of f is the same as x^{α}, then according to Theorem 1 $\lim_{x\to 0+} f(x)^{g(x)}=1$ whenever $\lim_{x\to 0} g(x)\log x=0$. More generally, the battle between 0 and 1 as the rightful value of 0^0 and between 1 and ∞ as the value of ∞^0 is determined by the rate at which $g(x)\to 0$. If (i) $g(x)\to 0$ faster than $1/\log x$, then $f(x)^{g(x)}\to 1$; if (ii) $g(x)\to 0$ slower than $1/\log x$, then $f(x)^{g(x)}\to 0$ if $\alpha>0$, and $f(x)^{g(x)}\to\infty$ if $\alpha<0$; if (iii) $g(x)\to 0$ at the same rate as $1/\log x$, then $\lim_{x\to 0} f(x)^{g(x)}$ is ambiguous. Since case (i) almost always obtains, the indeterminate form 0^0 or ∞^0 is almost always 1. The following corollary is a particular case.

COROLLARY 1. *If* f *is as in Theorem* 1, *and if* $g(x)=x^{\beta}c(x)$ *where* $c(x)$ *is bounded and* $\beta>0$, *then*

$$\lim_{x\to 0+} f(x)^{g(x)}=1.$$

In using Theorem 1 or Corollary 1 to evaluate an indeterminate form, one has only to identify the orders of magnitude of f and g. This technique is easily used in the first example, where $\alpha=1/2$, $\beta=1$, $b(x)=2x^{-\frac{1}{2}}\sin\sqrt{x}+\sin(1/x)$, and $c(x)=1$. Since $b(x)$ and $c(x)$ are bounded and $b(x)$ is bounded away from 0, the limit in this case is 1 by Corollary 1.

The technique is further illustrated in the following simplification of the work of previous authors.

COROLLARY 2. *If* $f(0)=g(0)=0$, *if* f *is analytic at* 0 *and positive as* $x\to 0+$, *and if* g *is differentiable at* 0, *then*

$$\lim_{x\to 0+} f(x)^{g(x)}=1.$$

Proof. Since $g'(0)=\lim_{x\to 0} g(x)/x$, $g(x)=xc(x)$ where $c(x)$ is bounded as $x\to 0$. Since f is analytic at 0 and $f(0)=0$, there is a positive integer α such that $f(x)=x^{\alpha}b(x)$ where b is analytic at 0 and $b(0)\neq 0$. Thus b is bounded and bounded away from 0 as $x\to 0$, and by Corollary 1, $\lim_{x\to 0+} f(x)^{g(x)}=1$.

The next theorem concerns the case 1^∞.

THEOREM 2. *If* $\lim_{x\to 0} f(x)=1$, *if* $\lim_{x\to 0} g(x)=\infty$, *and if* $\lim_{x\to 0} g(x)(f(x)-1)=\gamma$, *then* $\lim_{x\to 0} f(x)^{g(x)}=e^\gamma$.

Proof. The technique we use here is motivated by a common proof of the chain rule. Let $y=f(x)^{g(x)}$. Then $\log y=g(x)\log f(x)=g(x)\log(f(x)-1+1)$. Now recall that

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=\log'(1)=1.$$

So define

$$H(x)=\begin{cases}(1/x)\log(1+x), & \text{if } x\neq 0,\\ 1, & \text{if } x=0.\end{cases}$$

Then H is continuous and $\log(1+x)=xH(x)$ for all $x>-1$. Thus

$$\lim_{x\to 0} g(x)\log f(x)=\lim_{x\to 0} g(x)(f(x)-1)\lim_{x\to 0} H(f(x)-1)=\gamma\cdot 1=\gamma,$$

and by exponentiation the proof is complete.

In the second example given at the beginning of this paper, $g(x)(f(x)-1)=x\sin(1/x^4)\to 0$ as $x\to 0$. Hence $\lim_{x\to 0} f(x)^{g(x)}=e^0=1$. If instead,

$$f(x)=1+e^{-1/x^2}\operatorname{Arctan}(1/x^2)+x\sin(1/x^4)e^{-1/x^2},$$

then $g(x)(f(x)-1)\to\pi/2$ as $x\to 0$ and $\lim_{x\to 0} f(x)^{g(x)}=e^{\pi/2}$.

Note that if $f(x)\to 1$ and $f(x)^{g(x)}\to k\neq 1,\infty$, then $g(x)\log f(x)\to\log k$. Hence $g(x)\log(f(x)-1+1)\to\log k$; therefore $g(x)(f(x)-1)\to\log k$. Thus a converse of Theorem 2 holds and the rate of growth of $g(x)$ is closely tied to the rate at which $f(x)-1\to 0$.

A general case when 1^∞ has the value 1 is expressed in the following corollary, whose proof is left to the reader.

COROLLARY 3. *If there exist* $\alpha>\beta>0$ *such that* $f(x)=1+x^\alpha b(x)$ *and* $g(x)=x^{-\beta}c(x)$ *where* $b(x)$ *and* $c(x)$ *are bounded as* $x\to 0+$, *then*

$$\lim_{x\to 0+} f(x)^{g(x)}=1.$$

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ON CAUCHY'S INEQUALITIES FOR HERMITIAN MATRICES

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The purpose of this note is to give a simple and elementary proof for the following theorem, whose proof is usually accomplished by an application of the Courant-Fischer theorem [**1**, **2**].

THEOREM. *Let A be an n-square Hermitian matrix with eigenvalues*

$$\lambda_1\geqslant\cdots\geqslant\lambda_n,$$

and let B be a k-square principal submatrix of A with eigenvalues $\mu_1\geqslant\cdots\geqslant\mu_k$. *Then*

$$\lambda_{n-k+s}\leqslant\mu_s\leqslant\lambda_s,\qquad s=1,\dots,k.$$

Proof. It is sufficient to prove the theorem for $k=n-1$, i.e., that the eigenvalues of an $(n-1)$-square principal submatrix of A interlace with the eigenvalues of A. The general case follows by applying the result to a chain of matrices $A, B_1, B_2, \ldots, B_{n-k-1}, B$, where B_1 is $(n-1)$-square, B_2 is $(n-2)$-square,$\ldots, B_{n-k-1}$ is $(k+1)$-square and each is a principal submatrix of the preceding one. Consequently, let B be an $(n-1)$-square principal submatrix of A, obtained by the deletion of the qth row and the qth column of A. Let $A=UDU^*$, where $U=(u_{ij})$ is a unitary n-square matrix and $D=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$. We have

$$[\det(\lambda I-A)]^{-1}\operatorname{adj}(\lambda I-A)=(\lambda I-A)^{-1}=U(\lambda I-D)^{-1}U^*,$$

whence, taking the (q,q) entry in both sides, we obtain

$$\frac{\det(\lambda I-B)}{\det(\lambda I-A)}=\sum_{j=1}^{n}\frac{|u_{qj}|^2}{\lambda-\lambda_j}.$$

The above function is clearly monotonically decreasing at all points of continuity. It follows that it has precisely one zero between two successive poles and such a zero is necessarily an eigenvalue of B. If A has only simple eigenvalues and all $u_{qj}\neq 0$, then the $n-1$ eigenvalues of B interlace strictly with the n eigenvalues of A. It may, of course, happen that A has multiple eigenvalues or that some of the $u_{qj}=0$. In that case some of the eigenvalues of B will coincide with some of those of A, but the interlacing property is preserved.

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SECOND COUNTABLE AND SEPARABLE FUNCTION SPACES

R. A. McCoy

A well-known theorem which appears in several texts on general topology says that if X is a locally compact space, then the space of continuous functions from X into Y with the compact-open topology has a countable base whenever X and Y have countable bases. We show here how this theorem, its converse, and a theorem characterizing separability of function spaces, can be conveniently presented in a topology class. These two theorems have generalizations and interesting corollaries, which one can ask the students to find. Here we shall restrict our discussion to functions into the space R of real numbers.

The notation $C(X)$ will be used to denote the space of continuous real-valued functions on X with the compact-open topology. A subbase for this topology is

$$\{[C,V]\,|\,C \text{ is a compact subset of } X \text{ and } V \text{ is an open subset of } R\},$$

where $[C,V]=\{f\in C(X)\,|\,f(C)\subseteq V\}$. The topology of pointwise convergence, to which we will occasionally refer, is defined in a similar manner, except that singleton subsets of the domain are used instead of compact subsets. When τ is a topology on $C(X)$ other than the compact-open topology, we will use $C_\tau(X)$ to denote the function space with this topology. The letter I will denote the closed unit interval in R, and, for convenience, we will assume that our spaces are all Hausdorff spaces.

A useful tool for working with function spaces is the induced function between function spaces. If $f\colon X\to Y$ is continuous, then the induced function $f^*\colon C(Y)\to C(X)$ may be defined by $f^*(g)=g\circ f$ for every $g\in C(Y)$. Properties of this function make interesting exercises. For example: f^* is always

continuous, but when is it an open map? One might observe the following striking duality for f^*. We will say that a function is *almost surjective* if its image is dense in the range.

1. (a) For all X, if f is almost surjective, then f^* is injective.
   (b) For Urysohn space X, if f^* is almost surjective, then f is injective.
2. (a) For normal space Y, if f is injective, then f^* is almost surjective.
   (b) For normal space Y, if f^* is injective, then f is almost surjective.

Since we will use 2(a) in the sequel, we give the following proof:

Let $g \in C(X)$, and let $[C_1, V_1] \cap \cdots \cap [C_n, V_n]$ be a basic open subset of $C(X)$ containing g. Let $C = C_1 \cup \cdots \cup C_n$, and let $D = f(C)$. Since each C_i is compact and Y is Hausdorff, then $f|_C$ is a homeomorphism onto D. Define $h_D \in C(D)$ by $h_D(y) = g(f^{-1}(y))$ for each $y \in D$. By Tietze's extension theorem, h_D has an extension $h \in C(Y)$. Since for each $x \in C$, $f^*(h)(x) = h(f(x)) = h_D(f(x)) = g(x)$, then $f^*(h) \in [C_1, V_1] \cap \cdots \cap [C_n, V_n]$.

Before establishing the next theorem, one should prove the standard result that if X is locally compact and second countable, then $C(X)$ is second countable (see for example [2] page 265).

THEOREM 1. *Let X be a Urysohn space, and let τ be a topology on $C(X)$ which is larger than or equal to the topology of pointwise convergence and smaller than or equal to the compact-open topology. Then $C_\tau(X)$ is separable if and only if there exists a continuous injection from X into the Hilbert cube I^ω.*

Proof. We know that $C(I^\omega)$ is second countable by the remark preceding the theorem. Let f: $X \to I^\omega$ be a continuous injection. Then the induced function f^*: $C(I^\omega) \to C(X)$ is almost surjective by 2(a), and hence $C_\tau(X)$ must be separable.

Conversely, if $\{f_n\}$ is a countable dense subset of $C_\tau(X)$, then define f: $X \to R^\omega$ by $f(x) = \langle f_n(x) \rangle$ for every $x \in X$. Now f is continuous since its composition with each projection map is continuous. To see that f is injective, let x and y be distinct elements of X. Since X is Urysohn, there exists a $g \in C(X)$ such that $g(x) \neq g(y)$. Let V and W be disjoint open subsets of R containing $g(x)$ and $g(y)$, respectively. Then $[\{x\}, V] \cap [\{y\}, W] \neq \varnothing$, so that there exists a positive integer k with $f_k(x) \in V$ and $f_k(y) \in W$. But then $f(x) \neq f(y)$, so that f is injective. The desired continuous injection from X into I^ω can now be defined by taking the composition of f with a continuous injection from R^ω into I^ω.

Theorem 1 appears in [4] and a generalization can be found in [3]. We may now use Theorem 1 to prove the following theorem.

THEOREM 2. *Let X be a locally compact space. Then $C(X)$ is second countable if and only if X is second countable.*

Proof. Let $C(X)$ be second countable, and let $\hat{X} = X \cup \{p\}$ be the one-point compactification of X. If j: $X \to \hat{X}$ is the inclusion map, then the induced function j^*: $C(\hat{X}) \to C(X)$ is continuous (and injective). Let $\mathfrak{B}$ and $\mathcal{C}$ be countable bases for R and $C(X)$, respectively. The countable family $\{j^{*-1}(W) | W \in \mathcal{C}\} \cup \{[\{p\}, V] | V \in \mathfrak{B}\}$ generates some topology τ on $C(\hat{X})$. Clearly τ is smaller than or equal to the compact-open topology. To see that τ is larger than or equal to the topology of pointwise convergence, let $x \in \hat{X}$, let U be open in R, and let $g \in [\{x\}, U]$ in $C(\hat{X})$. If $x = p$, let $V \in \mathfrak{B}$ with $g(x) \in V \subseteq U$, so that $g \in [\{p\}, V] \subseteq [\{x\}, U]$. If $x \in X$, let $W \in \mathcal{C}$ with $g|_X \in W \subseteq [\{x\}, U]$ in $C(X)$, so that $g \in j^{*-1}(W) \subseteq [\{x\}, U]$ in $C(\hat{X})$. Therefore, since $C_\tau(\hat{X})$ is separable, Theorem 1 tells us that there exists a continuous injection f: $\hat{X} \to I^\omega$. Since $\hat{X}$ is compact, f is an embedding. Thus X may be embedded in I^ω and must then be second countable.

Theorem 2 may be generalized by replacing the range with any second countable space containing an arc as a retract. One might compare Theorem 2 with a characterization of first countability of $C(X)$ given in [1]. A special case is that *if X is locally compact, then $C(X)$ is first countable if and only if X is Lindelöf.*

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4. S. Warner, The topology of compact convergence on continuous function spaces, Duke Math. J., 25 (1958) 265-282.

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MATHEMATICAL EDUCATION

EDITED BY PAUL T. MIELKE AND W. E. MASTROCOLA

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MATHEMATICS AND ISLAMIC ART

JOHN NIMAN AND JANE NORMAN

In 1976 the Department of High School Programs of the Metropolitan Museum of Art in conjunction with Hunter College (CUNY) originated a project on the teaching of secondary school mathematics through works of art selected from various civilizations and periods. The first part of the project, which draws exclusively on Islamic art, has been completed. It comprises the development of teaching materials as well as lectures and workshops given at the museum in a course for junior and senior high school teachers of mathematics.

The use of two-dimensional Euclidean geometry in artistic modes of expression was most pronounced in Islamic art. It constituted the basis for the formation of rhythmic and intertwining patterns, elegant calligraphic inscriptions, and blossoming two-dimensional crystallographic patterns. This feature of Islamic art, together with abstractness, inherent logic, and universality, makes it a particularly valuable means of teaching mathematical topics such as tessellation, algebra, and symmetry. The subject matter becomes alive when presented through this medium.

Using the collection at the Museum, we were able to show artistic representations corresponding to the various types of tessellation. The theorem of regular tessellation was developed through analysis of patterns from Persian carpets, Spanish fabrics, Syrian incense burners, and architectural fragments from all parts of the Islamic world. The investigations led to the proof that the only regular polygons of one type which cover the plane, leaving no gaps, are the equilateral triangles, the squares, and the hexagons. Students proceeded from dividing the angle measure of a point by the interior angle of a polygon of sides n; i.e.,

$$\frac{360}{(n-2)180/n},$$

and then searching for those values of n that render the expression a positive integer.

The study of semi-regular tessellation or tiling the plane with a combination of regular polygons was based on analysis of paintings and Islamic objects. A most striking example is the Persian miniature painting of the beheading of St. John the Baptist, showing the (4, 6, 12) (i.e., a square, a hexagon, and a dodecagon meet at every vertex) and the (3, 4, 6, 4) tessellation. Another rich source is the eighteenth century *Nur-ad-din* room, brought from Damascus and set up in the Museum. The

fountain illustrates the (3,6,3,6) and (3,3,3,3,6) tessellation and the stained glass window the (3,4,6,4) tessellation. Among the various geometric patterns decorating the riser between the entrance and the sitting room there is also a good illustration of the (3,3,4,3,4) semi-regular tessellation.

Tessellation with irregular convex and concave polygons can be found in works dating from eighth century Egypt to sixteenth century Persia. These art materials were used to investigate various mathematical questions such as: What type of pentagons with equal sides but without congruent angles tessellate? What type of irregular hexagons tessellate? Do all quadrilaterals and triangles tessellate?

Work with tessellation led to the study of the geometrical concepts relating to arcs, circles, polygons, angles, proofs, congruence, similarity, ratio and proportion, lengths and areas. Workshop sessions included learning how to construct with a compass and a straightedge the square, triangular, and hexagonal grids, as well as the eight semi-regular tessellation grids. Experiments with the overlaying of two or more different grids yielded striking patterns, one of which illustrated the Pythagorean theorem.

Islamic art proved to be an equally rich source for the study of symmetry. Animal images of a Persian carpet are woven in perfect mirror symmetry. Elements of translation and rotation symmetry abound in a calligraphic inscription depicting the word "happiness." The arabesque in an ivory plaque exhibits several types of band ornaments. The following topics in symmetry and their correspondences in the works of art have been explored: translation, rotation, and reflection symmetry, the seven classes of one-sided bands, and the seventeen classes of the plane space groups.

The potential of teaching mathematical concepts through artistic forms is being recognized by some mathematicians. Schattschneider developed at Moravian College a course in analyzing periodic designs. In her article on the plane symmetry groups appearing in this issue she emphasizes the importance of integrating concepts in group theory and transformation geometry with art: "To analyze a repeating design to see what makes it work, and to create original designs using the power of mathematical 'laws' which govern these designs, is a strong non-mathematical motive for studying these groups." The interested reader should refer to the bibliography in Schattschneider's article and to her forthcoming paper on "Tiling the plane with congruent pentagons" (to appear in MATHEMATICS MAGAZINE). It should also be mentioned that many institutions in the U.S. besides the Metropolitan Museum of Art have significant collections of Islamic art. They include the Freer Gallery in Washington, the Walters Art Gallery in Baltimore, the Fine Arts Museum in Boston, and the Art Museums in Chicago, Cincinnati, Cleveland, and Philadelphia. We encourage teachers of mathematics to take advantage of these available possibilities. Materials are currently being compiled that include texts, slides, overhead projector transparencies, and worksheets. They will be available for distribution at cost in the near future.

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A MATHEMATICS FILM FESTIVAL

PIERRE J. MALRAISON AND PAUL J. CAMPBELL

During the years 1973–74 and 1974–75 we conducted an experiment in communicating mathematical ideas to a primarily non-mathematical college audience. A "Mathematics Film Festival" was presented at Carleton and Saint Olaf Colleges in the spring of each year. The festivals were ten weeks in length, with about one hour of films each week. The type of films ranged from Norman McLaren's *Spheres* to the filmed lecture of Gian-Carlo Rota in *The Marriage Theorem—Part II*. Advertised for "general audiences," the festivals attracted a wide spectrum of people. The size of the audiences ranged from a combined attendance high of 230 to a low of 11, averaging about 30 at each college.

1. The Films. In the course of the two years we screened about 40 films. We cannot say, "These are the 40 best mathematical films." Since we were running a festival and not trying to teach material, we steered away from the many course-specific instructional films. We have listed in an appendix all of the films we showed, along with references to published reviews. Here we shall confine our remarks to our choice of the "Top Ten" (within which group we hazard no further ranking).

The ideal mathematics film would express and convey accurately some significant mathematics in a manner making full use of the film medium. Such a marriage of accessible and inspiring mathematical content with technical proficiency and creative artistic vision is hard to find.

Still, we tried to balance the two dimensions, the mathematical and the cinematic, in our evaluation of the films we saw. Films such as *Maurits Escher, Painter of Fantasies* and *Spheres* do not explicitly convey any mathematics. But they do treat geometrical objects and their symmetries, and they do so in an outstandingly beautiful way that enhances their mathematical subject matter.

It is much rarer for the intrinsic interest of the mathematics to succeed in "carrying" a film of poor cinematic quality. The abstract nature of much of mathematics renders treatment of it on film a task demanding utmost creativity and imagination. Some mathematics, however, finds its most natural expression in film. A sterling example is *Regular Homotopies in the Plane—Part I*, in which color, motion, and animation combine with a laconic script to permit an extremely effective treatment of a theorem of topology in Euclidean 3-space. Another example, without animation, is *Dihedral Kaleidoscopes*, which offers the added fascination of showing the viewer how the film was made.

A filmed chalkboard lecture by a famous mathematician often fails to make the most effective use of the film medium, although such a film may be valuable as an historical artifact. *Let Us Teach Guessing!* is in a class by itself among filmed lectures; rather than the mathematics, Professor Pólya's method of teaching is the message. His sprightly personality and lively interaction with the class leap off the screen.

The right film can save thousands of words and dozens of dusty chalk diagrams. *The Seven Bridges of Königsberg* embodies a single mathematical idea, economically presented in elegant animation in just four minutes. Everything is there, paced just right. Its close cousin, *Flatland*, is another outstanding example of a short but every effective mathematics film.

Surprisingly, the least common kind of mathematics film is one concerning applications of mathematics. We were favorably impressed by *Weather by the Numbers*, and wished there were more films like it available.

Mathematics of the Honeycomb features outstanding natural history photography and effective use of stills of diagrams and bar graphs. A brief film guide available from the producer, Moody Institute of Science, offers free reprints of relevant articles from the *Mathematics Teacher*.

Apart from *The Seven Bridges of Königsberg*, the film *Newton's Equal Areas* comes the closest of our "Top Ten" to being an instructional film usable in the usual college mathematics curriculum. We enjoyed its use of sound: just the right amount of repetition and vectors that thunked their way across the screen.

2. Funding. Films must be rented or purchased, so the first consideration in sponsoring a film festival is money. Sources tend to vary widely from institution to institution (we received a grant from the presidents of the two colleges in the interest of cooperation), so our concern here will not be *where* to get money, but *how much* money to get.

Purchase of films runs from 11 to 20 times the cost of a single rental showing; so unless a very large amount of money is available and frequent reshowings are planned, rental is the way to go. The average cost for a week's screenings was \$18.50 the first year and \$23.50 the second; the average cost per film was \$10.50.

Other costs were return postage (including insurance) and the printing of posters; those totalled about \$35 a year. The cost of a student projectionist was absorbed by the mathematics departments, and campus newspapers and daily calendars provided some free publicity. Our two-year budget was

$500, with $220 spent the first year and $280 the second.

The major advantage of joint sponsorship was a sharing of costs: $200 is a lot of money for a department to consider spending on a mathematics film festival, but $100 is not so bad. The standard three-day minimum rental period enabled us to show films at both colleges on different days and allowed for occasional preview and classroom showings. The next rental period is seven days, at approximately double the three-day rental. Thus, a number of colleges in a metropolitan area could conceivably have separate screenings at each institution at a cost per school of much less than it would cost any one of them alone to show a film.

The most comprehensive list of films (and other audiovisual aids) available in the United States is *Audiovisual Materials in Mathematics* by Joseph A. Raab (National Council of Teachers of Mathematics, 1971). The Association of Teachers of Mathematics in Great Britain has a similar list for the United Kingdom.

For some films we were able to obtain faster service and lower rates from a nearby film library (University of Minnesota) than from the national film-rental companies. The quality of service by all of our distributors was uniformly good, however. International Film Bureau was especially good about handling orders on short notice. In most cases we ordered films two to three months in advance; therefore, where films were not available from one source, we had enough lead time to reorder from somewhere else. All of the distributors gladly send out catalogs.

3. Conclusions. At the end of two years, we have learned something about running a mathematics film festival. Ten weeks is too long. Never have a show consisting solely of beautiful (but difficult) mathematics. Use your colleagues for publicity—a recommendation to a class to attend a film is worth many posters. One thing we didn't try, which may add to the effectiveness of a film festival, is to have discussion after each show.

A film festival is an effective way to communicate to a non-mathematical public and give them some idea of what mathematics is about. It is also a valid cultural and aesthetic event in its own right if films are chosen properly. Five weeks of films showing people talking and writing on a blackboard would not be very enjoyable visually, even for a mathematician. Films exist now and continue to be made which combine significant mathematical content with artistic and technical excellence.

Appendix: Films Shown, with References to Published Reviews

	Distributor	*Amer. Math. Monthly*	*Math. Teacher*
Caroms†	IFB		66 (1973) 51
Challenge in the Classroom	MFR		
Cosmic Zoom	McG-H		
Dance Squared	IFB		64 (1971) 627
*Dihedral Kaleidoscopes†	IFB		66 (1973) 51
Donald in Mathmagicland	U of M		
The Dot and the Line	U of I		
Dr. Posin's Giants: Isaac Newton	U of I		
Equidecomposable Polygons†	IFB	82 (1975) 687	65 (1972) 734
*Flatland	McG-H		64 (1971) 44–45
Geometric Vectors—Addition†	IFB	82 (1975) 420	
Göttingen and New York	MFR		
An Historical Introduction to Algebra	MFR		
Infinity	AIMS	83 (1976) 71–72	
Isaac Newton	U of I		
*Let Us Teach Guessing!	MFR	75 (1968) 219	
Look Again	AIMS	83 (1976) 72	64 (1971) 525
The Marriage Theorem, Parts I & II	MFR		
*Mathematics of the Honeycomb	Moody		64 (1971) 334
Matrioska	U of I		
*Maurits Escher, Painter of Fantasies	MFR		

Mr. Simplex Saves the Aspidistra	MFR		
*Newton's Equal Areas	IFB	79 (1972) 1054	63 (1970) 449
Nim and Other Oriented Graph Games	MFR		
Notes on a Triangle	IFB		63 (1970) 363
The Perfection of Matter	McG-H		
Plateau's Problem	R.E.		
Possibly So, Pythagoras	IFB		
Projective Generation of Conics	IFB	82 (1975) 538–539	66 (1973) 51
*Regular Homotopies in the Plane, Part I	IFB		
*The Seven Bridges of Königsberg	U of M		
Space Filling Curves	IFB		
*Spheres	IFB		
Topology	MFR	75 (1968) 790	
Unsolved Problems: Two and Three Dimensions (2 films)	MFR		
*Weather by the Numbers	U of I		

*: "Top Ten"
†: Also reviewed in *Mathematics Teaching*, 67 (June 1974) 46–47.

AIMS: A.I.M.S., 5420 Melrose, Los Angeles, CA 90038
IFB: International Film Bureau, 332 So. Michigan Ave., Chicago, IL 60604
McG-H: McGraw-Hill Films, McGraw-Hill Book Co., 330 West 42nd St., New York, NY 10036
MFR: Modern Film Rentals, 2323 New Hyde Park Road, New Hyde Park, NY 11040
Moody: Moody Institute of Science, 12000 East Washington Blvd., Whittier, CA 90606
R.E.: A film by Sr. Rita Ehrmann—not available commercially
U of I: University of Indiana Audiovisual Library, Bloomington, IN
U of M: University of Minnesota Audiovisual Library Service, 3300 University Ave., Minneapolis, MN

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"PROOFS" TO GRADE

Richard J. St. André and Douglas D. Smith

In most post-calculus courses, the writing of a correct proof is *the* demonstration of mathematical reasoning and understanding of abstract content. It is expected that students become increasingly able to write and recognize correct proofs as they progress through mathematics courses. In early courses students may be asked to reproduce proofs or write proofs which follow a definite pattern (e.g., mathematical induction); in advanced courses definitions and theorems seem the only acceptable forms of communication.

We are concerned here with that stage in mathematical education at which students begin to write original proofs. At Central Michigan University this occurs in our linear algebra course, usually taken in the sophomore year or in the second semester of the freshman year. Linear algebra at this level, with matrix algebra and an introduction to axiomatic study of vector spaces, has the advantages of being accessible to most students and of providing excellent examples of mathematical reasoning. Furthermore, it is a course in which students generally recognize the need for proofs, in contrast to the situation in which theorems are considered obvious (e.g., theorems about sums and products of limits or that the graph of a linear equation is a straight line).

We present here one method of supplementing the assignment of theorems to be proved which we have found valuable in teaching students to write and recognize correct proofs. In our course we do not formally present propositional or quantification logic, but rather assign lists of purported theorems with alleged proofs which the students grade for correctness and clarity. The exercises,

entitled *"Proofs" to Grade*, are closely related to current topics in the course and illustrate errors in logic or understanding of the subject matter, or, in a few instances, present unusual correct proofs. *"Proofs" to Grade* exercises are not intended to replace those for which students must produce their own proofs. In fact, many of our "proofs" are presented as follow-up exercises to these more traditional assignments.

The instructions for all the problems are the same:

Assign a letter grade to each alleged proof. Assign a grade of A (excellent) if the proof is correct, even if it is not the simplest proof or the proof you would have given. Assign an E (failure) if the claimed theorem is incorrect, if the main idea of the proof is incorrect, or if most of the statements in it are incorrect. Assign a grade of C (partial credit) for a proof which is largely correct, but contains one or two incorrect statements or justifications. Whenever the proof is incorrect, explain your grade, telling what is incorrect and why it is incorrect.

Normally several "proofs" of the same "theorem" are given. Bogus proofs are usually built around a single type of error. This error may involve an elementary mistake in logic, such as misuse of quantifiers or a confusion between proving a statement and its converse. Some students initially feel that statements like "If $P^2=P$, then P is the identity matrix" can be proved by "checking" the converse, analogously to checking subtraction by adding. Other errors involve more or less subtle use of properties that either are untrue or unjustified (e.g., every matrix has an inverse, all matrices commute) or visually appealing symbolic arguments (symbol pushing), such as $(AB)^{-1}=A^{-1}B^{-1}$. Grading proofs such as these reduce the misconceptions a student may have about the properties being studied.

Summary. Our success with these exercises may be explained in at least two ways. First, they are a substitute for the well-known method of learning by teaching. We are all aware of the value of the experience of grading papers, becoming aware of the variety of possible errors, and, most importantly, explaining the errors. Secondly, we have found some students say they need *examples* when they begin to write proofs. What this startling complaint probably means is that the students have not been directly involved in the many proofs they have seen in classes and in textbooks. *"Proofs" to Grade* provides students with experience in working with proof techniques and reading proofs critically.

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Editorial note. The following (unintentional) example of a "proof to grade" was published by a famous mathematician. The reader is invited to guess who wrote it (originally in French). Answer next month.

THEOREM. *Let $F(x)$ and $f(x)$ be arbitrary functions; we have, for all x and h,*

$$\frac{F(x+h)-F(x)}{f(x+h)-f(x)}=\varphi(k),$$

where φ is a well-defined function and k is between x and $x+h$.

Proof. Put

$$\frac{F(x+h)-F(x)}{f(x+h)-f(x)}=P;$$

then

$$F(x+h)-Pf(x+h)=F(x)-Pf(x),$$

from which we see that the function $F(x)-Pf(x)$ does not change when we replace x by $x+h$. It follows that if it is not constant between these limits, which can happen only in special cases, this function will have one or more maxima and minima between x and $x+h$. Let k be the value of x

corresponding to one of these; evidently $k=\psi(P)$, ψ being a well-defined function; then we must also have $P=\varphi(k)$, φ being an equally well-defined function; which proves the theorem.

From this one may deduce as a corollary that

$$\lim \frac{F(x+h)-F(x)}{f(x+h)-f(x)}=\varphi(x), \qquad h \to 0,$$

is necessarily a function of x, which proves, *a priori*, the existence of derivatives.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to A. P. Hillman, 709 Solano Drive, S.E., Albuquerque, NM 87108. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Hillman.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before September 30, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2719. *Proposed by John S. Lew, IBM Watson Research Center, Yorktown Heights, New York*

For a fixed positive integer m let S_m be the sum of the series

$$\pm 1 \pm \frac{1}{3} \pm \frac{1}{5} \pm \frac{1}{7} \pm \frac{1}{9} \pm \cdots$$

where the first m terms have sign $+$, the next m terms have sign $-$, then the succeeding m terms have sign $+$, etc.

Gregory and Leibniz independently found $S_1=\pi/4$. On learning of this result, Newton (24 October 1676) wrote to Oldenburg [D. T. Whiteside (ed.): *The Mathematical Papers of Isaac Newton*, vol. 4, Cambridge Univ. Press, London 1971, p. 532] a letter intended also for Leibniz, asking for the more difficult sum S_2. Leibniz apparently did not solve the problem.

Evaluate S_2 and S_3.

E 2720. *Proposed by Ralph P. Boas, Northwestern University*

Show that $\sin^2 x<\sin(x^2)$ for $0<x\leqslant(\pi/2)^{1/2}$.

E 2721. *Proposed by Allen Emerson, Austin, Texas*

Let $a_0, a_1 > 0$ and define a_n $(n \geqslant 2)$ recursively by

$$a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}\,.$$

Show that (a_n) is convergent and compute its limit.

E 2722*. *Proposed by Clark Kimberling, University of Evansville, Indiana*

A ball is drawn from an urn containing one red ball and one green ball. If it is red it is returned to the urn with one additional red ball and one additional green ball, but if it is green no balls are put into the urn. After the first drawing, subsequent drawings take place following the same rules. Find the probability that the urn contains all the time at least one green ball.

E 2723. *Proposed by Allen Moy, University of Illinois, Chicago*

For a fixed $t > 0$ find

$$\lim_{n\to\infty}\left[e^{-nt}\sum_{k=0}^{n-1}\frac{(nt)^k}{k!}\right].$$

E 2724. *Proposed by Harry Lass, Jet Propulsion Laboratory, Pasadena, California*

An urn contains k_1 white balls, k_2 red balls and k_3 blue balls. The balls are withdrawn one at a time at random without replacements until all balls of one color (either red or white or blue) have been removed. (i) Determine the probability that all white balls are removed first. (ii) Determine the mean number of trials until all balls of some one color have been removed.

ELEMENTARY SOLUTIONS

A Class of Convex Polygons

E 2641 [1977, 216]. *Proposed by Philip Straffin, Beloit College*

Given a convex polygon and a point p inside it, define $D(p)$ to be the sum of perpendicular distances from p to the sides of the polygon (extended if necessary). Characterize those convex polygons for which $D(p)$ is independent of p.

Solution by F. B. Strauss, University of Texas at El Paso. Let K be our polygon and let q be a fixed point in the interior of K. Let $a_1, a_2, \ldots, a_n$ be the distances from q to the sides of K and let $u_1, u_2, \ldots, u_n$ be the unit vectors directed from q perpendicularly to the sides of K.

If $v = \overrightarrow{pq}$ then the distance d_i from p to the ith side of K is given by $d_i = a_i + (u_i, v)$. Hence

$$D(p) = D(q) + \left(\sum_{i=1}^{n} u_i, v\right)$$

and so $D(p)$ is independent of p if and only if $\sum_{i=1}^{n} u_i = 0$.

Note that this characterization and the proof extend readily to convex polyhedra in $\mathbf{R}^m$.

Also solved by the Bennett College Team, Howard Eves, Leon Gerber, O. P. Lossers (Netherlands), L. E. Mattics, T. Sekiguchi, and the proposer. Partially solved by Michael Goldberg, and Victor Pàmbuccian (Romania).

An Application of Gaussian Integers

E 2642 [1977, 216]. *Proposed by Antonio Rocha, Belo Horizonte, Brazil*

Let x, y, z be integers such that

$$x^2 + y^2 = z^{2m}, \qquad (x, y) = 1,$$

where m is a positive integer. If $4m - 1 = p$ is a prime, show that p divides xy.

Solution by K. Inkeri, University of Turku, Finland. Since $(x,y)=1$, one of x and y is odd. They are not both odd since in that case $x^2+y^2\equiv 2(\text{mod}\,4)$ and x^2+y^2 would not be a square. Thus z is odd.

Now we shall use the fact that the ring $\mathbf{Z}[i]$ of Gaussian integers is a unique factorization domain. Let t be a Gaussian integer dividing both $x+iy$ and $x-iy$. Then $t|2x$ and $t|2y$. Since $(x,y)=1$ we obtain $t|2$. On the other hand, since

$$(x+iy)(x-iy)=z^{2m}, \tag{1}$$

t divides the odd integer z^{2m}. It follows from these two observations that t is a unit, i.e., $x+iy$ and $x-iy$ are relatively prime. Therefore (1) implies that

$$x+iy=i^k(a+ib)^{2m} \tag{2}$$

for some k. By Fermat's theorem

$$\begin{aligned}(a+ib)^{4m}&=(a+ib)(a+ib)^p\\ &\equiv(a+ib)(a^p-ib^p)\\ &\equiv(a+ib)(a-ib)=a^2+b^2 \qquad (\text{mod}\,p).\end{aligned}$$

Using (2) we obtain

$$(x^2-y^2)+2ixy\equiv(-1)^k(a^2+b^2) \qquad (\text{mod}\,p)$$

and so $p|xy$.

Also solved by D. Borwein (Canada), Paul Bruckman, N. J. Fine, F. J. Flanigan, Lorraine Foster, M. G. Greening (Australia), Douglas Hensley, A. A. Jagers (Netherlands), Allan Johnson, Jr., J. C. Lagarias, Eric Lander, L. E. Mattics, Stephen Riley, A. E. Stratton & R. W. K. Odoni (England), Ernst Trost (Switzerland), and Daniel Weisser.

Comment. Inkeri proves that the same assertion is valid for the equation $x^2+dy^2=z^{2m}$, $(x,y)=1$ under the following hypotheses: d is a square-free integer $\not\equiv -1(\text{mod}\,8)$, $p=4m-1$ is a prime, d is not a square mod$p$, and $(2m,h)=1$ where h is the class number of the quadratic field $Q(\sqrt{-d}\,)$.

An Application of Quadratic Reciprocity

E 2643 [1977, 217]. *Proposed by Harry D. Ruderman, Hunter College Campus School, New York*

Show that for no integer $n>1$, 2^n-1 divides 3^n-1.

Solution by M. G. Greening (University of New South Wales, Australia), N. J. Fine (Pennsylvania State University), and K. Inkeri (University of Turku, Finland), independently. Assume that $A_n=2^n-1$ divides $B_n=3^n-1$ for some $n>1$. For even n we have $3|A_n$ and $3\nmid B_n$. Thus n is odd, say $n=2m-1$. Since $2^4\equiv 2^2$ (mod 12) we have $A_n\equiv -5$ (mod 12). Since every prime >3 is congruent to ± 1 or ± 5 (mod 12), there exists at least one prime divisor p of A_n such that $p\equiv\pm 5$ (mod 12). Since $p|3B_n$ we have $3\equiv 3^{n+1}=3^{2m}$ (mod$p$), i.e., 3 is a quadratic residue mod p. This contradicts the Quadratic Reciprocity Law because $p\equiv\pm 5$ (mod 12).

Also solved by D. Borwein (Canada), G. H. Chambers (England), Jesse Deutsch, J. C. Lagarias, L. E. Mattics, Victor Pambuccian (Romania), Harwood Rosser, James Slifker, Blair Spearman, Walter Stromquist, University of Wyoming Problem Group, and the proposer.

A Theorem of Polya

E 2644 [1977, 217]. *Proposed by Solomon W. Golomb and Lloyd R. Welch, University of Southern California*

Let $A_n=ru^n+sv^n$ $(n\geqslant 0)$ where r,s,u,v are integers, $u\neq\pm v$ and let P_n be the set of prime divisors of A_n. Show that the union P of all P_n is infinite.

Solution by L. E. Mattics, University of South Alabama. We must assume that $rsuv\neq 0$ (which was omitted by mistake). Without loss of generality we can also assume that $(ru,sv)=1$. Hence, if $p\in P$ then $p\nmid rsuv$.

Let $p_1, p_2, \ldots, p_m$ be distinct primes belonging to P. It suffices to show that there exists $p \in P$ distinct from $p_1, p_2, \ldots, p_m$.

Since $u \neq \pm v$ we have $u^i \neq v^i$ for $i \geqslant 1$. Hence we can choose a positive integer a such that $u^i \not\equiv v^i$ $(\bmod p_k^a)$ for $1 \leqslant i \leqslant m+1$ and $1 \leqslant k \leqslant m$. If $1 \leqslant i < j \leqslant m+1$ we have

$$A_j - v^{j-i} A_i = r u^j (u^{j-i} - v^{j-i})$$

and so we cannot have $A_i \equiv A_j \equiv 0$ $(\bmod p_k^a)$ for any k $(1 \leqslant k \leqslant m)$. Consequently, there exists a t $(1 \leqslant t \leqslant m+1)$ such that $A_t \not\equiv 0$ $(\bmod p_k^a)$ for $1 \leqslant k \leqslant m$. Choose a positive integer b such that $u^b \equiv v^b \equiv 1$ $(\bmod p_k^a)$ for $1 \leqslant k \leqslant m$. For instance, one can take $b = \varphi(c^a)$ where $c = p_1 p_2 \cdots p_m$. Choose n large enough so that $|A_{t+nb}| > c^a$ (this is possible because $u \neq \pm v$). Since $A_{t+nb} \equiv A_t \not\equiv 0$ $(\bmod p_k^a)$ for $1 \leqslant k \leqslant m$, A_{t+nb} must have a prime divisor different from $p_1, p_2, \ldots, p_m$.

Also solved by David Cantor, L. Kuipers (Switzerland), A. E. Stratton & D. Rees & R. W. K. Odoni (England), and the proposers.

Editor's comment. Cantor refers to G. Pólya, *Arithmetische Eigenschaften der Reihenentwicklungen rationaler Funktionen*, J. Reine Angew Math., 151 (1921), p. 1–31. Our problem is a very special case of Satz II′, p. 17, of that important paper.

Shuffling Along a Row

E 2645 [1977, 217]. *Proposed by Jerrold W. Grossman, Oakland University, Michigan*

A deck of N cards is shuffled according to the following scheme: The cards, labeled 1 through N, are placed in order in a row. Independent random integers $r_1, \ldots, r_N$ are chosen successively, $1 \leqslant r_i \leqslant N$, and after the choice of each r_i the card then in position i is interchanged with the card then in position r_i. What is the probability that the card s ends up in position t after the shuffle is complete?

Solution by the proposer (revised by the editor). Let $C(s;i)$ denote the card occupying sth place after i interchanges. For $i<j$ we let $p(s,t;i,j)$ be the probability of $C(s;i) = C(t;j)$.

LEMMA. *If $i<j$ then $p(i,t:i-1,j) = 1/N$ for all t.*

Proof. Since

$$p(i,t;i-1,j) = \sum_{s=1}^{N} p(i,s:i-1,i)\, p(s,t;i,j)$$

it suffices to notice that $p(i,s;i-1,i) = 1/N$ for all s and

$$\sum_{s=1}^{N} p(s,t;i,j) = 1.$$

Let first $s > t$. The event $C(s;0) = C(t;N)$ can be realized in two mutually exclusive ways: either
(i) $C(s;0) = C(t;t)$ and $r_{t+1}, \ldots, r_N$ are different from t, or
(ii) $r_1, \ldots, r_{s-1}$ are different from s and then $C(s;s-1) = C(t;N)$.
The probabilities of (i) and (ii) are

$$\frac{1}{N}(1-1/N)^{N-t} \quad \text{and} \tag{1}$$

$$\frac{1}{N}(1-1/N)^{s-1}, \tag{2}$$

respectively, because $p(s,t;0,t) = 1/N$ and $p(s,t;s-1,N) = 1/N$ (by the Lemma). Hence $p(s,t;0,N)$ is the sum of expressions (1) and (2).

Now let $s \leqslant t$. Then the event $C(s,0) = C(t;N)$ can be realized either via (ii) or

(iii) at least one of $r_1,\dots,r_{s-1}$ is equal to s, $C(s;0)=C(t;t)$, and $r_{t+1},\dots,r_N$ are different from t. The probability of (iii) is

$$\left[1-(1-1/N)^{s-1}\right]\cdot\frac{1}{N}\cdot(1-1/N)^{N-t}. \tag{3}$$

Since (ii) and (iii) are mutually exclusive, $p(s,t;0,N)$ is in this case equal to the sum of expressions (2) and (3).

Also solved by L. E. Mattics, Eric Rosenthal, and Gillian Valk. Partially solved by Dana Kamerud.

Alternating Sum of Certain Chords

E 2646 [1977, 217]. *Proposed by William Wernick, City College, New York*

Let $A_1,\dots,A_n$ be vertices of a regular n-gon inscribed in a circle with center O. Let B be a point on arc A_1A_n and $\theta=\angle A_nOB$. If a_k is the length of the chord BA_k express $\Sigma_{k=1}^n(-1)^k a_k$ as a function of θ.

Solution by M. G. Greening, University of New South Wales, Australia. We have $a_k=2r\sin\left(\frac{k\pi}{n}-\frac{\theta}{2}\right)$ where r is the radius of the circle, and thus

$$\sum_{k=1}^{n}(-1)^k a_k\cdot\cos\frac{\pi}{2n}=\sum_{k=1}^{n}(-1)^k r\left\{\sin\left(\frac{(2k-1)\pi}{2n}-\frac{\theta}{2}\right)+\sin\left(\frac{(2k+1)\pi}{2n}-\frac{\theta}{2}\right)\right\}$$
$$=\left(-1+(-1)^{n+1}\right)r\sin\left(\frac{\pi}{2n}-\frac{\theta}{2}\right).$$

Hence,

$$\sum_{k=1}^{n}(-1)^k a_k=2r\sin\left(\frac{\theta}{2}-\frac{\pi}{2n}\right)\sec\frac{\pi}{2n},\qquad n\text{ even}$$
$$=0,\qquad n\text{ odd}.$$

Also solved by Gordon Bennett, M. T. Bird, Paul Bruckman, Ricardo Diaz, Ragnar Dybrik (Norway), Howard Eves, Clark Givens, Michael Goldberg, Allan Johnson, Jr., Hans Kappus (Germany), Hagop Ketchedjian, L. Kuipers (Switzerland), Deborah Lockhart, O. P. Lossers, Jr. (Netherlands), Kishore Marathe, L. E. Mattics, Lawrence Ringenberg, Zoran Taborin (Yugoslavia), Paul Zwier, and the proposer. Partially solved by M. R. Gopal.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before September 30, 1978.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6216*. *Proposed by M. J. Pelling, Balliol College, Oxford, England*

For what fields A do we have $A=A_1+A_2$ qua Abelian groups where A_1 and A_2 are proper subfields of A?

6217*. *Proposed by M. J. Pelling, Balliol College, Oxford, England*

Let B be a subset of the nonnegative integers having positive density. Is it always true that there is an infinite subset X of B and an infinite sequence $k_1<k_2<\cdots$ of integers such that all the translates $X+k_i\subseteq B$?

6218*. *Proposed by M. J. Pelling, Balliol College, Oxford, England*

Let S be a subset of the real line R having cardinality of the continuum. Is there always a monotonic $f: R\to R$ such that $m^*f(S)>0$ where m^* is outer Lebesgue measure?

6219. *Proposed by M. J. Pelling, Balliol College, Oxford, England*

Construct an uncountable class of real numbers not normal in the scales of 3 and 5.

6220*. *Proposed by Mohammad Ismail, Auburn University, Alabama*

A collection K of sets is called a chain (resp. antichain) if for any A, $B\in K$, either $A\subseteq B$ or $B\subseteq A$ (resp. for any A, $B\in K$, $A\not\subseteq B$ and $B\not\subseteq A$). Let ω_1 be the first uncountable ordinal. Does there exist a family $\mathcal{P}=\{K_\alpha:\alpha<\omega_1\}$ of collections of subsets of a set X satisfying the following conditions:

(1) Each K_α is an infinite countable antichain.

(2) If $\alpha<\beta<\omega_1$, then every member of K_β is contained in some member of K_α and no member of K_α is contained in any member of K_β.

(3) If $\mathcal{P}^*=\cup_{\alpha<\omega_1}K_\alpha$, then every chain, and every antichain in $\mathcal{P}^*$ is countable.

6221. *Proposed by F. David Hammer, University of California, Davis*

Recently, Shelah found a group of cardinality $\aleph_1$ with no proper subgroups of that cardinality. Prove that such cannot happen with abelian groups. In fact every uncountable abelian group has a proper subgroup of the same cardinality.

ADVANCED SOLUTIONS

Evaluation of an Integral

5608 [1968, 686]. *Proposed by R. E. Shafer, Lawrence Radiation Laboratory, University of California, Livermore*

Evaluate

$$F(a)=\int_0^\infty e^{-u^2}I_0(u^2)K_0(au)du \tag{1}$$

as a convergent series in a for $|\arg a|\leqslant\frac{1}{2}\pi$.

Solution by M. L. Glasser, Battelle Memorial Institute, and Peter Ungar, Courant Institute, New York University. The following asymptotic formulas show the convergence of (1) ([2], p. 139):

$$K_0(z)=\left(\frac{\pi}{2}\right)^{1/2}z^{-1/2}e^{-z}\left(1+O\left(|z|^{-1}\right)\right)\quad(\mathrm{Re}\,z\geqslant 0) \tag{2}$$

$$I_0(u)=\frac{1}{\sqrt{2\pi}}u^{-1/2}e^{u}\left(1+O\left(u^{-1}\right)\right)\quad u>0. \tag{3}$$

At $z=0$, $I_0(z)$ is regular and $K_0(z)$ has a logarithmic singularity. The asymptotic estimates show that, in any compact subset of $\mathrm{Re}\,a\geqslant 0$ which does not contain the origin, the integral (1) converges uniformly and *hence $F(a)$ is continuous on the set* $\{a|\mathrm{Re}\,a\geqslant 0$ and $a\neq 0\}$ *and analytic in its interior.*

We shall first obtain a closed formula for the Mellin transform of F, which is

$$g(s)=\int_0^\infty F(a)a^{s-1}da \tag{4}$$

and the required series will be obtained by changing the integral in the Mellin inversion formula into a sum of residues. The inversion formula is

$$F(a)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g(s)a^{-s}ds, \tag{5}$$

where the integral is taken along a vertical line in the strip where (4) converges.

We find ([1, p. 331, (26)])

$$\int_0^\infty K_0(au)a^{s-1}da = u^{-s}2^{s-2}\Gamma^2\left(\frac{s}{2}\right), \qquad \mathrm{Re}\, s>0, \qquad u>0. \tag{6}$$

Multiply by $e^{-u^2}I_0(u^2)$ and integrate. Since $I_0(x)$ and $K_0(x)$ are positive for $x>0$, we may change the order of integration when $s>0$:

$$g(s)=\int_0^\infty F(a)a^{s-1}da=\int_0^\infty\int_0^\infty e^{-u^2}I_0(u^2)K_0(au)a^{s-1}du\,da=\frac{1}{4}2^s\Gamma^2\left(\frac{s}{2}\right)\int_0^\infty e^{-u^2}I_0(u^2)u^{-s}du. \tag{7}$$

Since the integrals converge absolutely for $s>0$, (7) holds in the whole halfplane $\mathrm{Re}\, s>0$. By [1, p. 330, (22)],

$$\int_0^\infty e^{-x}I_0(x)x^{t-1}dx=\frac{\Gamma\left(\frac{1}{2}-t\right)\Gamma(t)}{2^t\sqrt{\pi}\,\Gamma(1-t)}, \qquad 0<\mathrm{Re}\, t<\frac{1}{2}. \tag{8}$$

Substitute $x=u^2$ and $t=\frac{1}{2}-\frac{1}{2}s$ in this formula to evaluate the last integral in (7). We get

$$g(s)=\frac{1}{8\sqrt{2\pi}}(2\sqrt{2}\,)^s\Gamma^3\left(\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)/\Gamma\left(\frac{1}{2}+\frac{s}{2}\right), \qquad 0<\mathrm{Re}\, s<1. \tag{9}$$

Next we evaluate (5) with this $g(s)$, and with $c=\frac{1}{2}$, say, as a sum of residues.

Stirling's formula tells us that in any fixed vertical strip

$$\ln|\Gamma(z)|\sim-\tfrac{1}{2}\pi|z| \qquad \text{for} \quad |z| \quad \text{large.} \tag{10}$$

Hence in such a strip

$$\ln|g(s)|\sim-\frac{3}{4}\pi|s| \tag{11}$$

and thus (5) certainly converges for $|\arg a|\leqslant\frac{1}{2}\pi$.

The function $1/\Gamma(z)$ is entire and has simple zeros at $z=0,\ -1,\ -2,\ldots$. Thus $g(s)a^{-s}$ is meromorphic and its poles in the halfplane $\mathrm{Re}\, s<\frac{1}{2}$ are at $s=-2n$, $n=0,\ 1,\ldots$. From the estimate (11) it follows easily that the integral of $g(s)a^{-s}$ along $\mathrm{Re}\, s=\frac{1}{2}$ is equal to the integral taken along $\mathrm{Re}\, s=\frac{1}{2}-2N$ (N any positive integer) plus the sum of the residues in the strip between the two lines.

The integral

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}g(s)a^{-s}ds \tag{12}$$

converges absolutely by (11). From $\Gamma(z-1)=\Gamma(z)/(z-1)$ we get that if we replace s by $s-2$ then $g(s)a^{-s}$ is divided by $-4(s-2)^3/a^2(s-1)^2$. From this we can conclude that the integral (12) taken along $\mathrm{Re}\, s=\frac{1}{2}-2N$ will go to 0 as the positive integer N goes to ∞. Hence $F(a)$ is equal to the sum of the residues of $g(s)a^{-s}$ at the poles at $s=-2n$, $n=0,\ 1,\ 2,\ldots$.

Using $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$, we rewrite $g(s)$ in a form in which the arguments of the Γ-functions are all positive when $s=-2n$. We get

$$g(s)a^{-s}=\frac{\pi^{3/2}}{8\sqrt{2}}\left(\frac{a}{2\sqrt{2}}\right)^{-s}\frac{(\cos\frac{1}{2}\pi s)\Gamma^2(\frac{1}{2}-\frac{1}{2}s)}{(\sin^3\frac{1}{2}\pi s)\Gamma^3(1-\frac{1}{2}s)}.$$

To find the residue at the pole $s=-2n$ we have to expand (12) in terms of $t=s+2n$. Rewriting (12) in terms of t and rearranging we get

$$g(s)a^{-s}=\frac{1}{\sqrt{2}\,\pi^{3/2}}\left(\frac{a}{2\sqrt{2}}\right)^{2n}\frac{1}{t^3}\left(\frac{\sin\frac{1}{2}\pi t}{\frac{1}{2}\pi t}\right)^{-3}$$

$$\cos\frac{\pi t}{2}\Gamma^2\left(\frac{1}{2}+n-\frac{1}{2}t\right)\left(\Gamma\left(1+n-\frac{1}{2}t\right)\right)^{-3}\left(\frac{a}{2\sqrt{2}}\right)^{-t}. \tag{13}$$

Here the last five factors are regular and different from 0 at $t=0$. To obtain the power series expansion of $f_1(t)\cdots f_5(t)$ we use

$$\prod_{j=1}^{5} f_j(t)=\prod_{j=1}^{5} f_j(0)\quad \exp\left\{\sum_{j=1}^{5}\left(\ln f_j(t)-\ln f_j(0)\right)\right\}. \tag{14}$$

To get the residue we shall need only the coefficient of t^2 in $f_1\cdots f_5$ since there is a $1/t^3$ in front. Using

$$\begin{aligned}\ln\left(1+a_1x+a_2x^2+\cdots\right)&=\left(a_1x+a_2x^2+\cdots\right)-\tfrac{1}{2}\left(a_1x+\cdots\right)^2\\ &=a_1x+\left(a_2-\tfrac{1}{2}a_1^2\right)x^2+\cdots\end{aligned} \tag{15}$$

we get

$$\ln\frac{\sin\frac{1}{2}\pi t}{\frac{1}{2}\pi t}=-\frac{1}{24}\pi^2t^2+\cdots, \tag{16}$$

$$\ln\cos\tfrac{1}{2}\pi t=-\frac{1}{8}\pi^2t^2, \tag{17}$$

$$\ln\Gamma\left(\tfrac{1}{2}+n-\tfrac{1}{2}t\right)-\ln\Gamma\left(\tfrac{1}{2}+n\right)=-\tfrac{1}{2}\psi\left(\tfrac{1}{2}+n\right)t+\frac{1}{8}\psi'\left(\tfrac{1}{2}+n\right)t^2 \tag{18}$$

where $\psi(z)=(\ln\Gamma(z))'$,

$$\ln\Gamma\left(1+n-\tfrac{1}{2}t\right)-\ln\Gamma(1+n)=-\tfrac{1}{2}\psi(1+n)t+\frac{1}{8}\psi'(1+n)t^2. \tag{19}$$

Thus the exponent in (14) is

$$\begin{aligned}&\left(-\psi\left(\frac{1}{2}+n\right)+\frac{3}{2}\psi(1+n)-\ln\frac{a}{2\sqrt{2}}\right)t\\ &\qquad+\left(\frac{1}{4}\psi'\left(\frac{1}{2}+n\right)-\frac{3}{8}\psi'(1+n)\right)t^2+\cdots=bt+ct^2+\cdots.\end{aligned} \tag{20}$$

Using that $e^{bt+ct^2}+\cdots=1+bt+(\frac{1}{2}b^2+c)t^2+\cdots$ as we substitute (20) in (14), we get that the residue of (13) at $s=-2n$ is

$$\frac{1}{\sqrt{2}\,\pi^{3/2}}\frac{\Gamma^2(\frac{1}{2}+n)}{\Gamma^3(1+n)}\left(\frac{1}{2}b^2+c\right).$$

Evaluating the Γ-functions and summing we finally get

$$\begin{aligned}F(a)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{\left(\frac{1}{2}\frac{3}{2}\cdots\left(n-\frac{1}{2}\right)\right)^2}{(n!)^3}\left(\frac{a^2}{8}\right)^n&\left\{\frac{1}{2}\left(\psi\left(\frac{1}{2}+n\right)-\frac{3}{2}\psi(1+n)+\ln\frac{a}{2\sqrt{2}}\right)^2\right.\\ &\left.+\frac{1}{4}\psi'\left(\frac{1}{2}+n\right)-\frac{3}{8}\psi'(1+n)\right\}.\end{aligned} \tag{21}$$

Also solved by the proposer.

References

1. Bateman Manuscript Project: Tables of Integral Transforms, Vol. 1.
2. Magnus, Oberhettinger and Soni: Special Functions of Mathematical Physics.

Power Series in a Closed Disk

6080* [1976, 205]. *Proposed by R. N. Hevener, Jr., University of South Carolina*

A theorem of Abel states that if $\Sigma_{n=0}^{\infty}a_nz^n$ converges on the closed interval A, then (i) convergence

is uniform on A, whence (ii) it determines a continuous function on A. Is either part of this theorem true if A denotes a closed disk instead of an interval? If we impose the additional hypothesis, trivially satisfied in Abel's theorem, that the function be continuous on the boundary of A, is either part true?

I. *Partial solution by Paul G. Chauveheid, University of Liege, Belgium.* We give here a negative answer to two of the questions stated in the problem. This is a consequence of the following two theorems, which are proved in A. Zygmund, *Trigonometrical Series* (Cambridge U. Press, 2nd ed., 1959, vol. I, Thm. 1.14 and 1.17, pp. 300–301).

There is a power series $c_0+c_1z+\cdots$ regular for $|z|<1$, continuous for $|z|\leqslant 1$ and divergent for $z=1$.

There is a power series $\Phi(z)=\Sigma c_n z^n$ regular for $|z|<1$, continuous for $|z|\leqslant 1$, convergent on $|z|=1$ but nonuniformly on every arc of $|z|=1$.

It is noteworthy that these facts led Banach (*Théorie des opérations linéaires*) to seek a sequence of analytic functions replacing the monomials z^n of the Taylor series, for which this phenomenon does not occur. This problem has been solved rather recently in the affirmative (see S. V. Bočkarev, *Suščestvovanie bazisa v prostranstve…*, Mat. Sbornik 137 (1974), 3–18).

It is easily proved, however (by means of the Abel decomposition formula), that *if a series $\Sigma a_n z^n$ converges uniformly on a set E_0, it converges uniformly on the union E of the straight lines with origin 0 ending at points of E_0* (cf. J. Favard, *Cours d'analyse de l'École Polytechnique*, vol. II, ex. 4, p. 376; Gauthier-Villars, Paris, 1960).

II. *Partial solution by the proposer.* If $f(z)=\Sigma a_n z^n$ is continuous for $|z|=1$, then by a theorem of de la Vallée-Poussin (Zygmund, *Trigonometric Series*, vol. 1, 1968, p. 326) the given power series is a Fourier series, so by the solution to Dirichlet's problem the function is continuous on A.

Also partially solved by Tom Boehme (Egypt).

Editor's note. As for the case $\Sigma a_n z^n$ convergent for $|z|\leqslant 1$, there are examples such that $f(z)$ is not continuous on $|z|=1$.

Rational Function Solutions of $x^n-y^2=1$

6082 [1976, 205]. *Proposed by Thomas C. Craven, University of Hawaii*

Let $K(t)$ be the rational function field in one variable over a field K of arbitrary characteristic. Does the equation $x^n-y^2=1$ have a nonconstant solution in $K(t)$ when $n>2$?

Solution by William C. Waterhouse, Pennsylvania State University. In special cases there are solutions as follows:

char $K=2$		$x=t^2$	$y=t^n-1$
char $K=p>2$,	$n=p^r$	$x=t^2+1$	$y=t^{p^r}$
char $K=p>2$,	$n=2p^r$	$x=(t^2+1)/2t$	$y=[(t^2-1)/2t]^{p^r}$.

Otherwise there are none. Write $n=p^rm$ with $m>2$ not divisible by char (K). If y^2 is in $K(t)^{p^r}$, so is y, say $y=z^{p^r}$; then $z^2=x^m-1$ would have a nonconstant solution. This would embed the field $L=K[x,z]/(x^m-z^2-1)$ in $K(t)$, so by Lüroth's theorem L would be pure transcendental over K. But this is not so, since, e.g., $(dx)/z$ is a differential with no poles; actually L has genus $[(m-1)/2]$.

Also solved by R. W. L. Odoni (England), L. E. Mattics, J. C. Lagarias, and the proposer.

Editor's note. The solutions by Mattics and Lagarias use a reduction to either $A^p+B^p=C^p$ (p odd, p prime to char K) or $A^4-B^2=C^4$ (char K odd), where A, B and C are in $K[t]$. These equations have no nontrivial solutions by an argument similar to that in N. Greenleaf, *On Fermat's equation in $C(t)$*, this MONTHLY, Sept. 1969, pp. 808–809.

Some Rotations of $\mathbf{R}^3$

6102 [1976, 572]. *Proposed by Barbara Osofsky, Rutgers University*

Let A and B be nontrivial rotations of $\mathbf{R}^3$ about l_1 and l_2, respectively, which are axes through $(0,0,0)$ such that $A^2=B^3=\mathrm{Id}$. Hausdorff has shown that if $\cos 2\theta$ is transcendental, where θ is the angle between l_1 and l_2, then all relations between A and B are generated by $A^2=\mathrm{Id}$ and $B^3=\mathrm{Id}$. Show that the same is true for $\theta=\frac{1}{4}\pi$.

Solution by Scot Adams, Cornell University. We prove the following theorem:

THEOREM. *Let Ψ and Φ be two lines passing through a given point making an angle of $\pi/4$ with one another. Define ψ to be a rotation of $2\pi/3$ about Ψ and define φ to be a rotation of π about Φ. Then all relations between ψ and φ are generated by $\psi^3=1$ and $\varphi^2=1$.*

Proof. Form a coordinate system with origin at the intersection of Ψ and Φ, with z-axis Ψ, and such that Φ is contained in the xz-plane. Then, identifying a rotation with its matrix, we have

$$\psi^{\pm 1}=\begin{bmatrix} -\frac{1}{2} & \frac{\mp\sqrt{3}}{2} & 0 \\ \frac{\pm\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \varphi=\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and so

$$\psi^{\pm 1}\varphi=\frac{1}{2}\begin{bmatrix} 0 & \pm\sqrt{3} & -1 \\ 0 & 1 & \pm\sqrt{3} \\ 2 & 0 & 0 \end{bmatrix}.$$

Suppose n is a positive integer, and η is a rotation of the form $\eta=\psi^{k_1}\varphi\psi^{k_2}\varphi\ldots\psi^{k_n}\varphi$, where $k_1,\ldots,k_n\in\{1,-1\}$. Let

$$\eta=\frac{1}{2^n}\begin{bmatrix} m_{11} & m_{12}\sqrt{3} & m_{13} \\ m_{21}\sqrt{3} & m_{22} & m_{23}\sqrt{3} \\ m_{31} & m_{32}\sqrt{3} & m_{33} \end{bmatrix}$$

for some even integers $m_{11},m_{21},m_{31},m_{32},m_{33}$ and some odd integers $m_{12},m_{22},m_{13},m_{23}$. The theorem follows by seeing that $\eta\neq 1$. Now

$$2\psi^{\pm 1}\equiv\alpha\equiv\begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(modulo 2)}$$

while

$$\varphi\equiv\beta\equiv\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \text{whence } y=\alpha\beta=\begin{bmatrix} 0 & \sqrt{3} & 1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

and $y^2\equiv y$ (modulo 2). Thus $2^n\eta\equiv y^n\equiv y$, and it follows that $\eta\neq 1$.

Also solved by Roger Lyndon, D. L. Wang & P. Y. Wang, and the proposer.

Editor's note. Lyndon proves a stronger result, that the group generated by three rotations of order 3 about three orthogonal axes in 3-space is the free product of the cyclic groups of order 3 generated by the three rotations separately. By a classical result [Cayley: Philos. Mag. 26 (1845) 141–145] this follows from the following theorem.

THEOREM. *Let $A=(1+\sqrt{3}\,i)/2$, $B=(1+\sqrt{3}\,j)/2$, $C=(1+\sqrt{3}\,k)/2$, quaternions, and let $W=X_1\cdots X_t$, $t\geqslant 1$, where each X_i is one of $A^{\pm 1}$, $B^{\pm 1}$, $C^{\pm 1}$ and no $X_{i+1}=X_i^{\pm 1}$. Then $W\neq\pm 1$.*

Sketch of proof. Write $W_i = X_1 \cdots X_i = w_i/2^{n_i}$ where $w_i \in Z[\sqrt{3}, i, j, k]$ with n_i maximal. It suffices to show that $n_{i+1} \geqslant n_i$ for all i, $1 \leqslant i < t$. In fact it is shown that $n_{i+1} = n_i$ or $n_{i+1} = n_i + 1$, with at most three successive n_i equal. The proof is by induction, assuming w_{i-1} given of the required form and calculating $w_i, \ldots, w_{i+3}$ modulo successive powers of 2 up through $2^3 = 8$. The argument is complicated by the necessity of considering the various admissible possibilities for $X_i, \ldots, X_{i+3}$.

Principal Ideal Domains

6116 [1976, 748]. *Proposed by S. H. Cox, Jr., Universidad de Puerto Rico*

Let A be an integral domain satisfying the condition: For every nonzero ideal I of A there is an epimorphism $A \to A'$ of rings such that I and A' are isomorphic A-modules. For example, a principal ideal domain satisfies the condition with $A \to A'$ the identity $A = A'$. Show that each domain satisfying the condition is a principal ideal domain. (Warning: ring epimorphisms need not be surjective.)

Solution by Leonard Scott and Douglas Costa, University of Virginia. The assertion is false. To see this, let R be an integral domain with quotient field K. Let $A = R + XK[X] \subseteq K[X]$. If I is an ideal of A, it extends to a principal ideal of $K[X]$ generated by some $f(X) \in K[X]$. Set $L = \{a \in K \mid af(X) \in I\}$. It is easy to check that $I = f(X)LA$. Now let us add the assumption that R is a discrete valuation ring. Since L is an R-submodule of K, either $L = K$ or L is cyclic. If L is cyclic, I is principal and $I \cong A$. Otherwise, $I = f(X)KA = f(X)K[X] \cong K[X]$. Now $K[X]$ is the localization of A with respect to the nonzero elements of R and is therefore an epimorphic extension of A. Thus the hypotheses of the assertion are satisfied. The ideal $XK[X]$ however is not even finitely generated in A.

The assertion is true if A is assumed to be either noetherian or one-dimensional. By exercise 10, p. 8, of Kaplansky, *Commutative Rings*, A will be a PID if all of its prime ideals are principal. In either of the two present cases it is therefore sufficient, by the *Hauptidealsatz*, to show that every maximal ideal is principal.

Let M be a maximal ideal of A and let $A \to A'$ be a ring epimorphism with $A' \cong M$ as A-modules. Since A' is torsion-free of rank one as an A-module, it is isomorphic to a subring of the quotient field of A containing A. Replace A' by this ring. Then the A-module isomorphism $A' \cong M$ may be extended to an A'-module isomorphism, whence $M = A'x$ for some $x \in M$. Tensoring the epimorphism $A \to A'$ by A/M gives an epimorphism $A/M \to A'/M$, which must be surjective since A/M is a field. Thus $A' = A$ and $M = Ax$ is principal.

Linear Compositions of Two Entire Functions

6117 [1976, 748]. *Proposed by M. J. Pelling, Watford, Hertfordshire, England*

A well-known theorem asserts that given entire functions $f(z), g(z)$ with no common zero, then there exist entire functions $a(z), b(z)$ such that $af + bg = 1$ identically.

(i) Show that it is always possible to choose $a(z)$ zero-free.

(ii) Is it always possible to choose both $a(z)$ and $b(z)$ to be zero-free?

Solution by Lee A. Rubel, University of Illinois. (i) We want $af + bg = 1$ with $a \neq 0$ supposing that f, g, have no common zeros. By Helmer's theorem, there are entire functions A, B so that $Af + Bg = 1$ and hence for any entire function λ, $(A + \lambda g)f + (B - \lambda f)g = 1$, so we may choose $a = A + \lambda g$ and want $A + \lambda g = e^h$. At a zero z_n of g we need $e^{h(z_n)} = A(z_n)$ (with appropriate multiplicity) and by the interpolation theorem for entire functions, this is certainly possible since $A(z_n) \neq 0$ since $A(z_n)f(z_n) = 1$.

(ii) We prove the following theorem.

THEOREM. *Let f and g be nonconstant polynomials. Then there do not exist nonconstant entire functions a and b, having no zeros, such that $af + bg = 1$.*

Hence, if we choose f and g to be nonconstant polynomials so that $af+bg=1$ for no constants a and b, which is certainly the case if f and g have different degrees, then $af+bg=1$ implies that a or b must have some zeros.

Proof of theorem. We may of course suppose that f and g have no common zeros. Then there exist polynomials A and B, neither identically zero, such that $Af+Bg=1$. Then $a=A-\lambda g$ and $b=B+\lambda f$ also work. In fact, any a and b, with $af+bg=1$ must have this form. For then $(A-a)f+(B-b)g=0$. Hence the zero set $Z(A-a)$ of $A-a$ contains $Z(g)$. Thus, there is an entire function λ such that $A-a=\lambda g$. Similarly there is an entire function μ such that $B-b=\mu f$. Now $\lambda fg+\mu fg=0$ so that $\mu=-\lambda$ and hence $a=A-\lambda g$, $b=B+\lambda f$ as required.

If now a and b have no zeros, then the meromorphic function $F=1/\lambda$ must nowhere agree with any of the three rational functions a_1, a_2, a_3, where

$$a_1(z)=0,\ a_2(z)=g(z)/A(z) \quad \text{and} \quad a_3(z)=-f(z)/B(z).$$

Consider now the meromorphic function

$$G=\frac{F-a_1}{F-a_2}\,\frac{a_3-a_2}{a_3-a_1}.$$

The only points for which we could have $G(z)=0$ or $G(z)=1$ or $G(z)=\infty$ are among the points where $a_1(z)=a_2(z)$ or $a_1(z)=a_3(z)$ or $a_2(z)=a_3(z)$, and there are at most finitely many of them. By Picard's great theorem, G cannot then have an essential singularity at ∞, and hence G is a rational function. Therefore F is a rational function, so that $\lambda=1/F$ is a polynomial and hence a and b must both be polynomials. Since a and b have no zeros, they must be constants, and so the problem is solved.

Also solved by James Magliano, and by Adam Riese.

Note. Magliano shows that $a(z)\cdot z+b(z)\cdot e^z=1$ is not possible with $a(z), b(z)$ entire and zero-free.

Linear Combinations of Entire Functions without Zeros

6118 [1976, 748]. *Proposed by M. J. Pelling, Balliol College, Oxford, England*

(i) Show that there is no nonconstant solution to $e^{f(z)}+e^{g(z)}=1$ in entire functions $f(z)$ and $g(z)$.
(ii) Is there a nonconstant solution to $e^{f(z)}+e^{g(z)}+e^{h(z)}=1$ in entire functions f, g, h?

Solution by Chen-Han Sung, University of Wisconsin, Milwaukee. In 1897, E. Borel proved the following theorem which is named after him as "Borel's Lemma."

THEOREM. *Let $f^1,\dots,f^m$ be entire functions such that* $\exp(f^1(z))+\cdots+\exp(f^m(z))=1$. *Then some f^j is constant.*

For $m=2$, which is (i) of our problem, this is equivalent to the classical small Picard theorem. If $\exp f^1,\dots,\exp f^m$, are linearly independent, then the above theorem actually says that all the f^j are constant. But this is not true if $\exp f^1,\dots,\exp f^m$ are linearly dependent. In fact we have

$$\exp(z)+\exp(z+i\pi)+\exp(0)=1.$$

So the answer for (ii) is that there are solutions except when it is given that $\exp(f(z))$, $\exp(g(z))$ and $\exp(h(z))$ are linearly independent.

Reference: E. Borel, *Sur les zéros des fonctions entières*, Acta Math. 20 (1897), 357–396.

Also solved by R. Goldstein (England), Gary Gunderson, O. P. Lossers (Netherlands), James Peters, and Frank Wadleigh. Partial solutions by E. L. Isaacson and Q. G. Mohammed (India).

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

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Ordinary and Delay Differential Equations. By R. D. Driver. Springer-Verlag, New York, 1977. ix + 501 pp. $16.80. (Telegraphic Review, May 1977.)

The author has achieved his objective of writing an intermediate level textbook. The book is not sufficiently sophisticated or complete to be a graduate level textbook but, on the other hand, students will be more comfortable if they have had a course in advanced calculus. A previous course in differential equations is not a prerequisite. It also serves very well as a supplementary book for either a beginning course or for a graduate course in differential equations. Although the author uses the claim of "real world" applications monotonously and does, in fact, motivate equations from a variety of fields, the book is not a suitable candidate for courses intended for sophomore students of engineering.

Four of the eight chapters of the book are devoted to delay differential equations and they constitute an outstanding contribution to undergraduate literature. For decades some elementary differential equations books have contained a few sections on delay equations but in the 160 pages in this text devoted to this subject the author treats these equations in some depth and does so lucidly. These four chapters are the best part of the book. These chapters begin with several well-chosen examples, establish existence theorems for systems with bounded delays, treat linear systems (including variation of parameters) and consider the stability of delay equations.

As for the part of the book devoted to differential equations, *per se*, the author begins *ab initio* and proceeds very carefully. The background of potential students is kept in mind so that the Jordan canonical form is not employed, the Poincaré-Bendixson theorem is stated but not proved, etc. Existence and uniqueness theorems are carefully proved. In early chapters the author is content with proving a (Lipschitz condition) uniqueness theorem since an existence theorem is artificial in chapters where one exhibits solutions of the differential equations considered.

An appendix of theorems needed from the calculus is provided, as are answers to most of the generous number of problems in the book. The answers are sometimes rather helpful. For example, in the case of one question as to whether a certain condition can be found, the answer given is "'No' is probably the correct answer." It is only fair to say that the answer continues by citing a paper of the author's in which methods unavailable in the text give an affirmative answer.

W. R. UTZ, University of Missouri, Columbia

TELEGRAPHIC REVIEWS

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GENERAL, S**, P*, L**. *What is the Name of This Book? The Riddle of Dracula and Other Logical Puzzles.* Raymond M. Smullyan. P-H, 1978, 241 pp, $8.95. [ISBN: 0-13-955088-7] An extraordinary collection of logical puzzles and anecdotes, ranging from silly old chestnuts to intricate concatenations of liar enigmas engagingly expressed by a cast of history's most illogical characters. Concludes with a series of puzzles that express the essence of Gödel's Incompleteness Theorem. A full prepublication review appeared in Martin Gardner's March 1978 column in *Scientific American.* LAS

GENERAL, L. *More Number Games, Mathematics Made Easy Through Play.* Abraham B. Hurwitz, Arthur Goddard, David T. Epstein. Funk & Wagnalls, 1976, ix + 294 pp, $9.95. [ISBN: 0-308-10255-X] Collection of games and number tricks for parents (and teachers) to play with children during elementary and junior high school years. Games require paper and pencil, dice, dominoes, cards or other prepared materials. Topics from arithmetic through geometry, plus a chapter of humor (some good, some bad).PJ

BASIC, S(13)? *Basic Geometry.* Bernard Feldman. Wadsworth, 1977, ix + 144 pp, $5.95 (P). [ISBN: 0-534-00510-1] A self-study oriented supplemental volume to Wadsworth's Precalculus Series covering basic properties of lines, angles, triangles, area and volume. Consists of lists of definitions, notation and properties, examples and exercises (a complete solution key is included). No discussion or explanations. JNC

BASIC, S(13). *Practical Mathematics, Sixth Edition.* Claude Irwin Palmer, *et al.* McGraw, 1977, xviii + 557 pp, $11.95; $9.50 (P). [ISBN: 0-07-048253-5] A complete guide to precalculus mathematics, written for vocational students. Each section does a single concept, so it is quite useful as a reference text. Sections end with an abundance of drill problems and, occasionally, practical problems. The revisions to use metric measures and stress use of calculators is not significant. (*Fifth Edition,* TR, December 1970.) TLS

BASIC, T(13: 1). *Elementary Mathematics, Second Edition.* Robert E. Willcutt, Donald D. Paige. Prindle, 1978, xi + 303 pp, $12.95. [ISBN: 0-87150-242-9] Number systems and a bit of geometry with a problem-solving approach. FLW

BASIC, S. *Arithmetic and Calculators: How to Deal with Arithmetic in the Calculator Age.* William G. Chinn, Richard A. Dean, Theodore N. Tracewell. Freeman, 1978, vii + 488 pp, $9.95 (P); $17. [ISBN: 0-7167-0015-8] After 300 pages of basic arithmetic, a few more "exotic" topics are included: averages, rates, ratios, primes, Fibonacci numbers, roots, and the Pythagorean theorem. LLK

PRECALCULUS, T(13: 1, 2). *Fundamentals of Algebra and Trigonometry, Fourth Edition.* Earl W. Swokowski. Prindle, 1978, vi + 570 pp, $13.50. [ISBN: 0-87150-252-4] This *Fourth Edition* represents an effort to make the book less formal, with more emphasis on graphing. Many new exercises were added. (*Second Edition,* TR, February 1972; *Third Edition,* TR, August-September 1975.) MU

PRECALCULUS, T(13: 1). *College Algebra and Trigonometry.* J.S. Ratti. Macmillan, 1978, x + 412 pp, $13.95. [ISBN: 0-02-398530-5] All the usual topics. Shorter than some, it looks manageable for a one-term precalculus course. LLK

PRECALCULUS, T?(13), S. *Advanced Trigonometry.* Kenneth S. Miller, John B. Walsh. Krieger, 1977, viii + 108 pp, $8.50. [ISBN: 0-88275-391-6] Engineering mathematics with minimal recourse to calculus: DeMoivre's Theorem, power series for sine and cosine, Euler's formula, hyperbolic functions, Fourier series, Tschebycheff polynomials. LCL

EDUCATION, P, L. *Weeding and Sowing: Preface to a Science of Mathematical Education.* Hans Freudenthal. Reidel, 1978, ix + 314 pp, $34. [ISBN: 90-227-0789-8] A series of personal reflections about the possible shape of a science "before it comes into being," written "in order to prevent anything from being raised to the level of a science when it is not one." Written five years ago (in German), it has been put to the test (but not revised) since then. Contains 75 references, all to works by the author. LAS

EDUCATION, S(15-16), P. *Jean Piaget: Psychologist of the Real.* Brian Rotman. Cornell U Pr, 1977, 200 pp, $15. [ISBN: 0-8014-1139-4] A two-part study of Piaget's theories: a concise exegesis followed by an equal quantity of critical analysis. The first part appears sympathetic while the second part raises several serious objections based on a broader view than Piaget's of the development of mathematics in the twentieth century. The presentation is definitely non-technical but unfortunately the writing appears hurried and carelessly edited. JAS

EDUCATION, T(15-17: 2, 3). *Elementary College Mathematics.* Sol Weiss. Prindle, 1977, x + 533 pp, $12.95. [ISBN: 87150-217-8] For prospective elementary teachers. Each chapter includes informal introduction, content, enrichment material, basics revisited and puzzles. Topics include sets, relations and functions; numeration; development of the reals; number theory; geometry; probability and statistics. Appendices on logic and computation. Extensive coverage with insufficient explanatory material for intended audience. PJ

EDUCATION, T(14-16: 1, 2). *Basic Concepts of Mathematics for Elementary Teachers.* Roy Dubisch. A-W, 1977, xi + 463 pp, $13.95. [ISBN: 0-201-01167-0] Essential mathematics for elementary teachers including geometry, probability and statistics, number theory, functions. Excellent use of material taken directly from elementary school texts to illustrate how topics will be taught to children. Emphasizes discovery. Includes games, puzzles and manipulative aids. Good programmed exercise sets. Journal reprints and bibliography at chapter end to encourage professional reading. Deserves consideration. PJ

EDUCATION, S*(15-17), P*, L. *Mathematical Questions from the Classroom.* Richard J. Crouse, Clifford W. Sloyer. Prindle, 1977, ix + 308 pp, $8.95 (P). [ISBN: 0-87150-219-4] Compilation of hundreds of mathematical questions raised by students in the secondary classroom; Junior High, Algebra I and II, Geometry and Advanced courses. Answers, provided for every question, discuss student difficulties and misconceptions, possible teacher responses, suggested reading and reprints of appropriate journal articles. Excellent for secondary methods. PJ

EDUCATION, S(15-17), P, L. *Elementary School Mathematics, A Guide to Current Research, Fourth Edition.* Leroy G. Callahan, Vincent J. Glennon. ASCD, 1975, xi + 188 pp, $5 (P). [ISBN: 0-87120-076-7] Well-balanced summaries of and commentary on selected research in four aspects of elementary school mathematics (K-7); curriculum, child, learning environment and teaching methods. Extensive bibliography. PJ

EDUCATION, S(16-17), P, L. *The Mathematics Laboratory, Readings from the Arithmetic Teacher.* Ed: W. George Cathcart. NCTM, 1977, vi + 226 pp, $4.50 (P). Forty articles about the what, why, when and how of the mathematics laboratory in the elementary school. Considers developing, selecting and organizing materials and activities. Includes teacher education and evaluation. Noteworthy introductory and concluding statements that present serious cautions concerning the use and misuse of the mathematics laboratory. PJ

EDUCATION, P, L*. *Organizing for Mathematics Instruction, 1977 Yearbook.* F. Joe Crosswhite, Robert E. Reys. NCTM, 1977, xi + 240 pp, $8.50. Twelve essays on specific alternative teaching approaches (elementary through college) such as individualization, goal-referenced instruction, learning cooperatives, simulations, mastery learning, alternative schools, and teacher-centered. One non-thematic essay on hand-held calculators. PJ

EDUCATION, S(16), P, L. *Developing Computational Skills, 1978 Yearbook.* Marilyn N. Suydam, Robert E. Reys. NCTM, 1978, ix + 245 pp, $10.66. [ISBN: 0-87353-121-3] Fourteen articles dealing with the teaching and assessment of computational skills. Includes specific activities, instructional sequences, teaching suggestions for learning-disabled students, discussions of the role of computation, using calculators to teach and much more. CEC

EDUCATION, T(16-17: 1). *Teaching Elementary School Mathematics for Understanding, Fourth Edition.* John L. Marks, *et al.* McGraw, 1975, viii + 406 pp, $14.50. This new edition adds lists of experiments for pupils; appendices on games, activities and materials; consideration of the mathematics laboratory; brief mention of functions; and lists of projects for the teacher-in-training. Updated bibliography. PJ

EDUCATION, S(16), P. *Results from the First Mathematics Assessment of the National Assessment of Educational Progress.* Thomas Carpenter, *et al.* NCTM, 1978, iii + 140 pp, $5.50 (P). [ISBN: 0-87353-123-X] Results of the first National Assessment of Mathematics conducted during the 1972-73 school year. Interpretations and overviews of the results are included. CEC

HISTORY, S, P, L*. *In at the Beginnings: A Physicist's Life.* Philip M. Morse. MIT Pr, 1977, vii + 375 pp, $14.95. [ISBN: 0-262-13124-2] Autobiography of one of the twentieth century's most versatile scientists. Mathematicians may be especially interested in his role in the development of operations research: Morse was one of the founders of ORSA, and did much to promote OR as a valuable new discipline. "To me the pleasure coming from understanding how traffic behaves is as great as that coming from understanding how two atoms combine." LAS

HISTORY, P. *Zeno and the Discovery of Incommensurables in Greek Mathematics.* Helmut Hasse, Heinrich Scholz. Arno Pr, 1976, 473 pp, $9. [ISBN: 0-405-07311-9] A reprint of a monograph and two papers on the title topic "Die Grundlagenkrisis der Griechischen Mathematik" by Helmut Hasse and Heinrich Scholz (1928); "Sur la Constitution des Livres Arithmétiques des Éléments d'Euclide et Leur Rapport à la Question de l'Irrationalité" by H.G. Zeuthen (1910); and "Sur les Connaissances Géométriques des Grecs Avant la Réforme Platonicienne" by H.G. Zeuthen (1913). JAS

HISTORY, P, L. *The Mathematical Work of Charles Babbage.* J.M. Dubbey. Cambridge U Pr, 1978, viii + 235 pp, $26.50. [ISBN: 0-521-21649-4] Babbage's concern for notation formed the intellectual roots of the Analytical Engine, as well as for pioneering work in what we now call operations research and systems science. "...His mind was extraordinarily aligned to middle-twentieth century thinking, without his being in any way responsible for the later developments in mathematical science...his immediate influence on his contemporaries was almost negligible." LAS

HISTORY, P. *The Correspondence of Isaac Newton, V. VII, 1718-1727.* Ed: A. Rupert Hall, Laura Tilling. Cambridge U Pr, 1977, xlv + 522 pp, $65. [ISBN: 0-521-08722-8] Final volume, containing scientific correspondence concerning new editions of Newton's works and his quarrel with Bernoulli, together with personal and business letters. Concludes with a partial Newton genealogy to help the reader keep track of the pleas from impecunious relatives. LAS

FOUNDATIONS, P. *Two Essays on Entropy.* Rudolf Carnap. U of Calif Pr, 1977, xxii + 115 pp, $12.75. [ISBN: 0-520-02715-9] Two previously unpublished manuscripts (written in 1952-54) examining the role of entropy in classical physics and its use in inductive logic, preceded by a detailed introduction relating Carnap's position to those of Boltzmann and Gibbs. LAS

FOUNDATIONS, S(17-18), P*, L. *Handbook of Mathematical Logic.* Ed: Jon Barwise. Stud. in Logic and Found. of Math., V. 90. North-Holland, 1977, xi + 1165 pp, $75. [ISBN: 0-7204-2285-X] Outstanding collection of thirty-one survey articles covering in some detail every major area of current research in mathematical logic. Each article written by a prominent logician specifically for this volume. Breadth and depth of treatment is generally suitable for the professional mathematician who is not a specialist in logic, though sophistication may be required for some articles. A very inviting book in which to browse as well as to obtain considerable detail on specific topics, considering the breadth of coverage. The handbook should be welcomed by the entire mathematical community. (Comprehensive name and subject indexes, separate bibliographies. Excellent typesetting.) GHM

FOUNDATIONS, T(14-18: 1), S. *The Foundations of Mathematics.* Ian Stewart, David Tall. Oxford U Pr, 1977, xi + 263 pp, $17.50; $8.95 (P). [ISBN: 0-19-853164-8; 0-19-853165-6] Introduction to mathematical rigor for students in transition from 'school mathematics' to university standards. Develops naive set theory, the logic of actual mathematical practice, and properties of natural, real and complex numbers. Stresses relation between intuition and rigor, with good discussions of actual mathematical practice in light of formal logic. A nice book with a small scope. GHM

FOUNDATIONS, T(17-18), S, P*, L. *Boolean-Valued Models and Independence Proofs in Set Theory.* J.L. Bell. Clarendon Pr, 1977, xviii + 126 pp, $14.50. [ISBN: 0-19-853168-0] Systematic account of the theory of Boolean-valued models together with applications to independence results. Includes many problems, with hints for solutions. Interesting historical perspectives provided by Dana Scott in the Foreword. LCL

FOUNDATIONS, S(16-18), P, L. *Elements of Intuitionism.* Michael Dummett. Clarendon Pr, 1977, xii + 467 pp, $19.75. [ISBN: 0-19-853158-3] Introduction to intuitionistic logic for persons with knowledge of classical formal logic and general awareness of history of intuitionism. Intended as an informal, intelligible introduction to the recent technical literature. Presents considerable detail on choice sequences and spreads, and formal proof theory and semantics; little on intuitionistic reconstruction of mathematics. Concludes with philosophical remarks. A very welcome and valuable book. GHM

FOUNDATIONS, S(17-18), P. *A Course in Mathematical Logic.* Yu. I. Manin. Trans: Neal Koblitz. Grad. Texts in Math., V. 53. Springer-Verlag, 1977, xiii + 286 pp, $19.80. [ISBN: 0-387-90243-0; 3-540-90243-0] Textbook of recent advances in mathematical logic (chiefly continuum problem, diophantine equations, recursion theory and algebra), aimed at professional mathematicians. Background of author outside logic gives book a wider perspective. Includes discussion of quantum logic, and linguistic and philosophical questions. Excellent translation. No exercises. GHM

COMBINATORICS, T(17: 1-3), S, P, L. *Combinatorics with Emphasis on the Theory of Graphs.* Jack E. Graver, Mark E. Watkins. Grad. Texts in Math., V. 54. Springer-Verlag, 1977, xv + 351 pp, $24.80. [ISBN: 0-387-90245-7; 3-540-90245-7] Multigraphs, networks, matchings, chromatic theory, designs, matroids, enumeration theory. Sophisticated, algebraically based, self-contained theory written within a general framework provided by the concept of "system." A valuable addition to the literature. LCL

COMBINATORICS, T(13-18: 1, 2), S, L. *Introductory Combinatorics.* Richard A. Brualdi. North-Holland, 1977, x + 374 pp, $18.50. [ISBN: 0-7204-8610-6] A beginning level text that includes a wide variety of topics such as the pigeon-hole principle, permutations and combinations, binomial coefficients, inclusion-exclusion principle, recurrence relations, generating functions, combinatorial designs, graph theory, chromatic numbers, connectivity, optimization. The presentation is clear and complete and assumes only the mathematical maturity of two semesters of calculus. Chapter exercises, some with solutions. Bibliography. Index. RJA

COMBINATORICS, P. *Lecture Notes in Mathematics-622: Combinatorial Mathematics V.* Ed: C.H.C. Little. Springer-Verlag, 1977, vii + 213 pp, $11.50 (P). [ISBN: 0-387-08524-6; 3-540-08524-6] The proceedings of the Fifth Australian Conference, held at the Royal Melbourne Institute of Technology in August, 1976. CEC

COMBINATORICS, T(18: 2), P. *Edge-colourings of Graphs.* S. Fiorini, R.J. Wilson. Research Notes in Math., No. 16. Fearon-Pitman, 1977, 154 pp, $12.50 (P). [ISBN: 0-273-01129-4] An extensive treatment of edge-coloring, including some applications to science and social sciences. Especially good bibliography and guide to the literature. Many exercises. Important book for graph theorists. SS

NUMBER THEORY, T*(15-17: 1), S, L*. *Fundamentals of Number Theory.* William J. LeVeque. A-W, 1977, vii + 280 pp, $13.95. [ISBN: 0-201-04287-8] This text is similar to the first volume of the author's *Topics in Number Theory,* but there are substantial and significant differences. The language of abstract algebra is used and there is a new emphasis on the history of the subject. There are also several new topics and the number of exercises has been increased. CEC

NUMBER THEORY, S(17), P. *Addition Theorems: The Addition Theorems of Group Theory and Number Theory.* Henry B. Mann. Krieger, 1976, xi + 114 pp, $10.50. [ISBN: 0-88275-418-1] A reprint, with corrections, of the author's 1965 monograph which obtains inequalities for the measure of the sum of two sets of numbers or of group elements in terms of the measures of the summands. CEC

NUMBER THEORY, S(18), P. *Lecture Notes in Mathematics-626: Number Theory Day.* Ed: M.B. Nathanson. Springer-Verlag, 1977, 241 pp, $11.50 (P). [ISBN: 0-387-08529-7; 3-540-08529-7] Five papers in number theory given at Rockefeller University on March 4, 1976. The participants are S. Chowla, P. Erdös, C.J. Moreno, M.B. Nathanson, and A. Selberg. CEC

NUMBER THEORY, P*. *Number Theory and Algebra.* Ed: Hans Zassenhaus. Acad Pr, 1977, xlvi + 393 pp, $37.50. [ISBN: 0-12-776350-3] A collection of thirty-two research papers dedicated to Henry B. Mann, Arnold E. Ross and Olga Taussky-Todd. The variety of topics extends over a wide range. Biographical sketches of the three honored number theorists are also included. CEC

NUMBER THEORY, T?, S(15), P, L*. *Lectures on Elementary Number Theory*. Hans Rademacher. Krieger, 1977, ix + 146 pp, $9.50. [ISBN: 0-88275-499-8] A reprint of the 1964 edition. There is no question that this engaging monograph deserves to remain in print. CEC

NUMBER THEORY, T(18), P. *Modular Forms and Functions*. Robert A. Rankin. Cambridge U Pr, 1977, xiii + 384 pp, $34. [ISBN: 0-521-21212-X] An introduction (modulo complex analysis) to elliptic modular functions and forms. Deals with forms both of integral and arbitrary real weight. Various constructions of modular forms are discussed, along with functions belonging to the full modular group, forms of level N, and Hecke operators on congruence groups. SG

NUMBER THEORY, P. *Transcendence Theory: Advances and Applications*. Ed: A. Baker, D.W. Masser. Acad Pr, 1977, x + 236 pp, $21.50. [ISBN: 0-12-074350-7] Proceedings of a 1976 conference held in Cambridge, England on transcendental number theory. The 16 papers deal with such topics as linear forms in logarithms (real and p-adic), elliptic and abelian functions, and algebraic independence of functions. SG

ALGEBRA, P. *Groups with Steinberg Relations and Coordinatization of Polygonal Geometries*. John R. Faulkner. Memoirs No. 185. AMS, 1977, iii + 135 pp, $8 (P). [ISBN: 0-8218-2185-7] "Groups which satisfy commutator relations of the type satisfied by Chevalley groups (i.e., groups with Steinberg relations) are shown to have certain non-associative division algebras as parameters. This, in turn, allows an introduction of coordinates into certain polygonal geometries." JAS

ALGEBRA, P. *Homological Localization Towers for Groups and Π-Modules*. A.K. Bousfield. Memoirs No. 186. AMS, 1977, vii + 68 pp, $7.20 (P). [ISBN: 0-8218-2186-5] A development of the algebraic localizations which arose in topological localization theory. The investigation is based on a construction of natural transfinite towers which eventually stabilize to the desired homological localizations. JAS

ALGEBRA, T(15-16), L. *Applied Modern Algebra*. Larry L. Dornhoff, Franz E. Hohn. Macmillan, 1978, xi + 500 pp, $15.95. [ISBN: 0-02-329980-0] An undergraduate text covering topics in algebra that are useful in computer science, electrical engineering, applied mathematics, and mathematics: graphs, rings, Boolean algebras, semigroups, groups, lattices, fields, linear algebra, linear machines, and coding theory. Includes many applications and a large number of exercises. Deserves serious consideration for use in an applied algebra course. SG

ALGEBRA, T*(15-16: 1, 2), S. *A First Undergraduate Course in Abstract Algebra, Second Edition*. Abraham P. Hillman, Gerald L. Alexanderson. Wadsworth, 1978, xv + 428 pp, $16.95. [ISBN: 0-534-00525-X] New section on conjugacy classes and the class equation, plus a number of new examples, remarks, and exercises. An extended review of the first edition appeared in August-September 1974. LCL

ALGEBRA, S(15-18), P, L. *Introduction to Group Characters*. Walter Ledermann. Cambridge U Pr, 1977, viii + 174 pp, $6.95 (P); $19.95. [ISBN: 0-521-29170-4; 0-521-21486-6] Well written, straightforward introduction to the characters of a finite group over the complex field. Stresses computational aspects. Discusses representation theory via matrices as well as module theory. Includes exercises. Possible outside reading for capable undergraduates in abstract algebra. GHM

ALGEBRA, T(17), P. *Algebra, V. 2*. P.M. Cohn. Wiley, 1977, xiii + 483 pp, $19.50. [ISBN: 0-471-01823-6] This graduate-level text provides an introduction to a broad range of topics in modern algebra: lattices, tensor products, homological algebra, Galois theory, real fields, quadratic forms, valuation theory, Artinian rings, commutative rings, Noetherian rings and polynomial identities. Because of the clear exposition and many fine exercises, the book is valuable both as a reference and as a text. SG

ALGEBRA, P. *Factorizations in Local Subgroups of Finite Groups*. G. Glauberman. CBMS Reg. Conf. in Math., No. 33. AMS, 1977, ix + 74 pp, $7.20 (P). [ISBN: 0-8218-1683-7] Proceedings of a conference held in 1976 at the University of Minnesota-Duluth. The author deals with the relationship between the structure of a finite group and the structure of and embedding of a given Sylow p-subgroup. SG

ALGEBRA, S(17-18), P, L. *Finite Embedding Theorems for Partial Designs and Algebras*. Charles C. Lindner, Trevor Evans. Pr U Montreal, 1977, 196 pp, $7 (P). [ISBN: 0-8405-0353-9] A partial Latin square is an n x n array such that in each row and column each of the integers 1, 2,...,n occurs at most once. This monograph examines various conditions for embedding such partial designs into larger arrays. Relationships to quasigroups and finite algebra approximations are discussed. JEG

FINITE MATHEMATICS, T(14-15: 1), S, L. *Discrete Mathematics in Computer Science*. Donald F. Stanat, David F. McAllister. P-H, 1977, xiii + 401 pp, $15.95. [ISBN: 0-13-216150-8] Sets, functions, binary relations, counting techniques and algebras. Intended as an introduction to discrete mathematics for computer science students. Some examples and problems presume programming, and the proofs presume some prior exposure to mathematics. RWN

FINITE MATHEMATICS, T(13-14: 1). *Applied Finite Mathematics, Second Edition*. Howard Anton, Bernard Kolman. Acad Pr, 1978, xv + 558 pp, $12.95. [ISBN: 0-12-059565-6] New material on mathematics of finance and stochastic processes; tree diagrams used more extensively in probability problems; additional exercises. Basic replaces Fortran in computer chapter. (First Edition, TR, January 1975.) LCL

FINITE MATHEMATICS, T(13-15). *Principles of Finite Mathematics*. William C. Swift, David E. Wilson. P-H, 1977, x + 455 pp, $13.95. [ISBN: 0-13-701359-0] Well-motivated treatment of standard topics (probability and statistics, game theory, vectors, linear programming). Emphasis on concreteness, "down-to-earth logic," but with no loss of accuracy in supporting theory. Many comments help the student sort out the relation between theory and application. The style is honest (as opposed to pedantic). Solutions to odd-numbered exercises. GHM

FINITE MATHEMATICS, T(13-15: 1-3). *College Mathematics with Applications to the Business and Social Sciences*. Bodh R. Gulati. Har-Row, 1978, xiv + 727 pp, $14.95. [ISBN: 0-06-042538-5] Functions, matrix algebra, linear programming, probability, an example from decision theory, matrix games,

exponential and logarithmic functions, sequences, the mathematics of finance, and an introduction to the calculus. FLW

FINITE MATHEMATICS, T(13-14: 1-3). *Fundamental Mathematics for the Management and Social Sciences, Second Edition.* Lloyd S. Emerson, Laurence R. Paquette. Allyn, 1978, xiii + 634 pp, $15.95. [ISBN: 0-205-06000-5] Functions, matrix algebra, determinants, linear programming, the mathematics of finance, probability, descriptive statistics, decisions under uncertainty, and an introduction to the calculus. (First Edition, TR, June-July 1975.) FLW

FINITE MATHEMATICS, T(13: 1). *Linear Mathematics, A Practical Approach.* Patricia Clark Kenschaft. Worth, 1978, xvii + 392 pp, $7.95. [ISBN: 0-87901-084-3]; *Instructor's Manual,* 70 pp, (P). Most of the emphasis is on linear programming, with some probability and game theory. A long list of applications by topic exhibits the emphasis on modelling the real world. LLK

CALCULUS, T*(13: 1). *Elementary Applied Calculus, A Short Course, Second Edition.* Raymond F. Coughlin. Allyn, 1978, xi + 452 pp, $13.95. [ISBN: 0-205-05965-1] Sections on the construction of a mathematical model, implicit differentiation, related rates, numerical integration, least squares and more applications are added to a first edition already distinguished by a wealth of applications (TR, October 1974). The new answer section includes extensive discussion of many problems. JNC

CALCULUS, T(13: 1), S. *Calculus, A Practical Approach, Second Edition.* Kenneth Kalmanson, Patricia C. Kenschaft. Worth, 1978, xviii + 429 pp, $13.95. [ISBN: 0-87901-083-5]; *Instructor's Manual,* 91 pp, (P). Several additions: enlarged review chapter, Newton's method, L'Hospital's rule, new chapter on trigonometry (including calculus), additional applications and exercises (some for hand calculator). (TR, May 1976.) LCL

CALCULUS, T(13-14: 3). *Calculus.* Jack G. Ceder, David L. Outcalt. Allyn, 1977, xiii + 1050 pp, $19.95. [ISBN: 0-205-05560-5] Standard fare, except that the authors motivate and define limits of functions, derivatives and integrals all in terms of sequences (after presenting an early chapter on limits of these). Intuitive and geometrically oriented; includes double and triple integration using various coordinate systems. Appears to be readable and carefully written. DFA

CALCULUS, T(13-14: 1, 2). *Calculus for Management, Life, and Social Sciences.* Raymond A. Barnett. Dellen Pub, 1978, xi + 390 pp, $15.95. [ISBN: 0-89517-003-5] Emphasizes the concepts most useful in the three areas mentioned and supplies examples from these areas. FLW

CALCULUS, T(13: 2). *A First Course in Calculus.* Serge Lang. A-W, 1978, xv + 572 pp, $16.95. [ISBN: 0-201-04149-9] The major change from previous editions is the inclusion of four chapters on functions of several variables covering vectors, curves (differentiation and length), partial derivatives and gradient. However the book is intended only for a two-semester course in one-variable calculus, and is not useable without supplementation for a full three-semester sequence. SG

CALCULUS, T(13: 1). *Basic Calculus with Applications.* Burton Rodin. Goodyear Pub, 1978, viii + 424 pp, $15.95. [ISBN: 0-87620-097-8] Good emphasis on applications with an applications index (one for Life Science, another for Social Science) inside front cover. Omits trigonometric functions completely. LLK

CALCULUS, T(13: 2). *Calculus for the Life Sciences.* Rodolfo De Sapio. Freeman, 1978, xv + 740 pp, $17. [ISBN: 0-7167-0371-8] Good applications; manageable two term course in calculus (including partial derivatives and double integrals). Taylor polynomials included, but no series. LLK

CALCULUS, S(13). *Algebraic Methods in Business, Economics, and the Social Sciences: A Short Course, Preliminary Edition.* Gerald Freilich, Frederick P. Greenleaf. Freeman, 1977, x + 398 pp, $4.95 (P). [ISBN: 0-7167-0470-6] Designed as a supplement to the authors' textbook *Calculus,* but is self-contained. Topics include inequalities and linear programming, growth and decay problems, matrices and solutions to systems of equations, and trigonometric functions. LLK

CALCULUS, T(13: 2). *Calculus.* Gilbert and Benjamin Baumslag. Quantum Pub, 1976, 427 pp, $5.95 (P). A lightweight, inexpensive, single variable calculus text. Includes series but no differential equations. Format a bit like a Schaum Outline, with a set of solved problems before each problem set. LLK

CALCULUS, T(13:2). *Calculus with Analytic Geometry.* Bevan K. Youse, F. Lane Hardy. HR&W, 1978, xiv + 847 pp, $19.95. [ISBN: 0-03-019856-9]; *Instructor's Manual,* 45 pp. [ISBN: 0-03-039361-7] Makes use of computer graphics for diagrams; includes an optional appendix of computer programs in Basic; format of orange margins and highlighting may not appeal to many users. LLK

CALCULUS, T(13-14: 3). *Calculus with Analytic Geometry, Second Edition.* Howard E. Campbell, Paul F. Dierker. Prindle, 1978, xvi + 878 pp, $21.50. [ISBN: 0-87150-244-5] Many changes from first edition (TR, May 1975). New format, more diagrams, chapter reviews on light blue pages, challenging problems added in dark blue. Early sections on trigonometric functions added. Chapter 10 has become 10 and 11 (with series a separate chapter); parametric equations, differentials, moments and centroids, hyperbolic functions also added. LLK

CALCULUS, T(13-14: 4). *Calculus: One and Several Variables with Analytic Geometry, Third Edition.* S.L. Salas, Einar Hille. Wiley, 1978, xiv + 948 pp, $21.50. [ISBN: 0-471-74983-4] Differs from the second edition (TR, May 1975) chiefly in having more vector analysis (curl; divergence; Green's, Divergence and Stokes's Theorems) and treating general as well as lower and upper Riemann sums. JD-B

REAL ANALYSIS, P. *On the Theory of Vector Measures.* William H. Graves. Memoirs No. 195. AMS, 1977, iv + 72 pp, $7.20 (P). [ISBN: 0-8218-2195-4] A study of the isomorphism between vector-valued measures on a ring of subsets of a set and the continuous linear maps on a locally convex space. LAS

REAL ANALYSIS, T(16-18: 2), S, P. *Integrals and Measures*. Washek F. Pfeffer. Pure and Appl. Math., V. 42. Dekker, 1977, ix + 259 pp, $19.75. [ISBN: 0-8247-6530-3] This book develops the theory of measure and abstract Lebesgue integral by the Daniell method of extending certain linear functionals. It contains convincing heuristic motivation and abundant exercises. The text is pleasantly readable. MU

REAL ANALYSIS, T?(15-17: 2, 3), S, L. *A Course of Mathematical Analysis*. S.M. Nikolsky. Trans: V.M. Volosov. MIR (US Rep: Four Continent Book Corp., 156 Fifth Ave., New York, NY 10010), 1977. *V. 1*, 460 pp,; *V. 2*, 441 pp. Nineteen concise chapters from sets and real numbers to generalized functions, differential forms and the Lebesgue integral, intended to "help the reader to proceed more easily to the study of mathematical physics." A "reading course" from the Moscow Physico-Technical Institute, its lack of exercises (and encyclopedic nature) makes it unsuitable as a text for typical American courses. But it is a valuable reference and supplement. LAS

COMPLEX ANALYSIS, T(18: 1, 2), S, P. *Lectures on Theory of Functions in Multiply Connected Domains*. Helmut Grunsky. Vandenhoeck & Ruprecht, 1978, 253 pp, 32DM (P). [ISBN: 3-525-40142-6] These are not typical lecture notes. The book is quite readable, and careful attention to detail is obvious. Although the text is limited to multiply-connected domains, the bibliography is more extensive. MU

COMPLEX ANALYSIS, P. *Univalent Functions and Orthonormal Systems*. I.M. Milin. Transl. of Math. Mono., V. 49. AMS, 1977, iv + 202 pp, $27.20. [ISBN: 0-8218-1599-7] Devoted to a method of investigating univalent functions (i.e., one-to-one meromorphic functions) by means of orthonormal systems. Many "recent" (1970) results of Soviet and other mathematicians on the coefficient problem are presented. Translation of 1971 Russian edition. GHM

DIFFERENTIAL EQUATIONS, P. *Ecuatti Diferentiale Asociate unor Operatori Neliniari pe Spatti Banach*. Nicolae Pavel. Editura Academiei (Romania), 1977, 283 pp, Lei 15,50 (P). "A presentation of the basic results (many due to the author) on nonlinear evolution equations associated to accretive (monotone, dissipative) operators, especially on general Banach spaces." JAS

DIFFERENTIAL EQUATIONS, T**(15-17). *Differential Equations and Their Applications, Second Edition*. Martin Braun. Appl. Math. Sci., V. 15. Springer-Verlag, 1978, xiii + 518 pp, $16.80 (P). [ISBN: 0-387-90266-X; 3-540-90266-X]; *Short Version*, 1978, viii + 319 pp, $10. [ISBN: 0-387-90289-9; 3-540-90289-9] Reprint of highly acclaimed 1975 book (TR, December 1975; ER, October 1977), now also available in an abridged edition in which most of the more advanced material (qualitative theory, Fourier series) is omitted. LAS

DIFFERENTIAL EQUATIONS, T(14: 2), S, L. *Differential Equations and the Calculus of Variations*. L. Elsgolts. Trans: George Yankovsky. MIR (US Distr: Four Continent Book Corp., 156 Fifth Ave., New York, NY 10010), 1977, 440 pp. The first section of this book gives an introduction to differential equations which includes stability and first order partial differential equations. The second section surveys the methods for solving variational problems. The style is verbose. There are lots of examples and an adequate number of exercises. CEC

DIFFERENTIAL EQUATIONS, T(15-18: 1, 2), S, L. *Computational Methods in Engineering and Science, with Applications to Fluid Dynamics and Nuclear Systems*. Shoichiro Nakamura. Wiley, 1977, xii + 457 pp, $25. [ISBN: 0-471-01800-7] Text on computational methods for differential equations. Begins with a summary of numerical methods. Contains finite difference methods for S-L eigenvalue problems, elliptic and parabolic partial differential equations, fluid flow, and transonic aerodynamic analysis. Also weighted residual methods, finite element method, coarse-mesh rebalancing method, and Monte Carlo method. Chapter problems and references. Five appendices. Index. RJA

DIFFERENTIAL EQUATIONS, P. *Advances in Computer Methods for Partial Differential Equations-II*. Ed: R. Vichnevetsky. IMACS (AICA), 1977, vii + 392 pp, $36. Proceedings of the Second IMACS (AICA) International Symposium on Computer Methods for Partial Differential Equations held at Lehigh University, Bethlehem, Pennsylvania, June 22-24, 1977. Author index. RJA

DIFFERENTIAL EQUATIONS, P. *Symmetry and Separation of Variables*. Willard Miller, Jr. Ency. Math. Appl. A-W (Adv. Bk. Prog.), 1977, xxx + 285 pp, $21.50. [ISBN: 0-201-13503-5] This series' first volume dealing with special functions provides a systematic development of the method of separation of variables in the context of Lie algebras. The forward (by R. Askey) is a fascinating, historical essay on special functions. TRS

DIFFERENTIAL EQUATIONS, T(16-17). *Nonlinear Ordinary Differential Equations*. D.W. Jordan, P. Smith. Clarendon Pr, 1977, viii + 360 pp, $27.50; $14.95 (P). [ISBN: 0-19-859620-0; 0-19-859621-9] A treatment of the qualitative theory of nonlinear equations with a variety of application. Topics: phase plane, linear approximation, nonlinear damping, geometrical and computational aspects, small-parameter expansion and singular perturbation, harmonics, subharmonics, entrainment and jump effects, and system stability. SG

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-568: Coincidence Degree, and Nonlinear Differential Equations*. Robert E. Gaines, Jean L. Mawhin. Springer-Verlag, 1977, 262 pp, $11 (P). [ISBN: 0-387-08067-8; 3-540-08067-8] An exposition based on lecture series given by the two authors in 1974 and 1975 together with an introduction and additional chapters on more recent material. Emphasis is on so-called "alternative problems." LLK

DIFFERENTIAL EQUATIONS, T**, S, L**. *Fourier Series and Boundary Value Problems, Third Edition*. Ruel V. Churchill, James Ward Brown. McGraw, 1978, viii + 271 pp, $16.50. [ISBN: 0-07-010843-9] Widely used since 1941, this is an introduction to orthogonal sets of functions and their applications to partial differential equations in engineering and physics. Changes from 1963 edition include: more motivation, new exercises, some reorganization of chapters, more on Bessel and Legendre functions of the second kind, updated bibliography. An excellent text, assuming calculus only. TRS

DIFFERENTIAL EQUATIONS, S(17-18), P. *Methoden der Potentialtheorie für Elliptische Differentialgleichungen Beliebiger Ordnung.* Bert-Wolfgang Schulze, Günther Wildenhain. Math. Reihe, B. 60. Birkhäuser, 1977, xv + 408 pp, sFr. 108. [ISBN: 3-7643-0944-X]

NUMERICAL ANALYSIS, T(15-18: 1, 2), S, P, L. *The Numerical Treatment of Integral Equations.* Christopher T.H. Baker. Clarendon Pr, 1977, xiv + 1034 pp, $49.50. [ISBN: 0-19-853406-X] Emphasis is on practical numerical methods for the approximate solution of integral equations although abstract theory is developed when needed. Contains many examples and test calculations. Begins with a survey of the theory of integral equations and selected topics in numerical analysis; continues with eigenvalue problems, linear equations of the second kind, Fredholm and Volterra integral equations. References and index. RJA

NUMERICAL ANALYSIS, S(17-18), P. *The State of the Art in Numerical Analysis.* Ed: D. Jacobs. Acad Pr, 1977, xix + 978 pp, $39. [ISBN: 0-12-378650-9] Proceedings of the conference held at the University of York, England, in April 1976. Text is divided into seven parts, each of which comprises a survey of a particular branch of numerical analysis: linear algebra, error analysis, optimization and non-linear systems, ordinary differential equations and quadrature, approximation theory, parabolic and hyperbolic problems, elliptic problems and integral equations. Subject index. RJA

NUMERICAL ANALYSIS, P. *Mathematical Software III.* Ed: John R. Rice. Acad Pr, 1977, ix + 388 pp, $15. [ISBN: 0-12-587260-7] Proceedings of the Symposium on Mathematical Software, Mathematics Research Center, University of Wisconsin, Madison, March 28-30, 1977. Fourteen papers. Index. RJA

NUMERICAL ANALYSIS, P. *Solutions of Ill-Posed Problems.* Andrey N. Tikhonov, Vasiliy Y. Arsenin. Trans: Fritz John. V.H. Winston (Distr: Halsted Pr), 1977, xiii + 258 pp, $19.75. [ISBN: 0-470-99124-0] On the numerical solution to problems which have solutions that are unstable with respect to small changes in the initial data. Discusses the selection and regularization methods and their application to integral equations, linear systems, summing Fourier series, functional minimization and linear programming. RWN

NUMERICAL ANALYSIS, P. *Error Propagation for Difference Methods.* Peter Henrici. Krieger, 1977, vii + 73 pp, $5.95. [ISBN: 0-88275-448-3] Reprinting of the 1963 edition on the analysis of multistep methods; a sequel to the author's well-known *Discrete Variable Methods in Ordinary Differential Equations.* RWN

NUMERICAL ANALYSIS, T(14-15). *Introduction à L'Analyse Numérique.* Jacques Baranger. Hermann (US Distr: SMPF, 14 E. 60th St., NY 10022), 1977, 131 pp, 36F (P). [ISBN: 2-7056-5855-6] Introductory numerical methods for nonlinear equations, linear systems, integrals, least squares, splines, ordinary differential equations and Dirichlet problems. Some theorems on errors. Programming problems. RWN

FUNCTIONAL ANALYSIS, T*(17-18: 1, 2). *Introduction to Operator Theory I: Elements of Functional Analysis.* Arlen Brown, Carl Pearcy. Grad. Texts in Math., V. 55. Springer-Verlag, 1977, xiv + 474 pp, $24.80. [ISBN: 0-398-90257-0; 3-540-90257-0] An excellent introductory graduate text on functional analysis. The first half includes the essentials from topology, linear algebra, analysis and measure theory and serves as a reference for the second half, which contains the basics on normed linear spaces. Carefully organized and well-written, with lots of good examples and problems. TRS

FUNCTIONAL ANALYSIS, T??(16-18: 1). *Functional Analysis in Modern Applied Mathematics.* Ruth F. Curtain, A.J. Pritchard. Math. in Sci. and Eng., V. 132. Acad Pr, 1977, ix + 339 pp, $21.15. [ISBN: 0-12-196250-4] The main part of the book concerns applications of functional analysis to stability theory, linear systems theory, optimization and numerical methods. Includes an outline of background material and a development of spectral theory with applications to differential equations. Text suffers from poor editing, careless writing, and mathematical errors. No exercises. TRS

FUNCTIONAL ANALYSIS, T(18), P. *Function Spaces.* Alois Kufner, Oldřich John, Svatopluk Fučik. Noordhoff Intern, 1977, xv + 454 pp, Dfl. 120. [ISBN: 90-286-0015-9] A carefully organized monograph on the functional-analytic approach to the solution of (partial) differential equations. Focusses on spaces of functions whose generalized derivatives (in the sense of distributions) have certain properties (e.g., Sobolev and Orlicz spaces). Should be of interest to workers in functional analysis, differential equations and approximation theory. TRS

FUNCTIONAL ANALYSIS, P. *Nonlinearity and Functional Analysis, Lectures on Nonlinear Problems in Mathematical Analysis.* Melvin S. Berger. Pure and Appl. Math., V. 74. Acad Pr, 1977, xix + 417 pp, $24.50. [ISBN: 0-12-090350-4] The main goal of these lectures is the application of abstract results to diverse, concrete problems in geometry and physics. Main topics: classification of nonlinear operators, inverse function theorems, parameter dependent perturbation phenomena, mapping degree, critical point theory. Excellent motivation and organization. No exercises. TRS

OPTIMIZATION. *Optimization Methods in Operations Research and Systems Analysis.* K.V. Mital. Halsted Pr, 1976, xiii + 259 pp, $9.75. [ISBN: 0-470-99056-2] A broad, mathematical introduction to linear and nonlinear programming. Includes the transportation problem, network flow, dynamic and geometric programming and game theory. Computational considerations. Exercises. RWN

OPTIMIZATION, T(17-18: 1, 2), P. *Dynamic Programming and Stochastic Control.* Dimitri P. Bertsekas. Math. in Sci. and Eng., V. 125. Acad Pr, 1976, xv + 397 pp, $22.50. [ISBN: 0-12-093250-4] Develops system control over finite horizons, emphasizing risk, feedback, sufficient statistics, adaptivity, contraction maps and the principle of optimality, and over infinite horizons including undiscounted, discounted and average cost problems. Assumes real analysis, linear algebra, probability and optimization. Applications, problems and references. RWN

ANALYSIS, P. *Lecture Notes in Mathematics-621: The Deficiency Index Problem for Powers of Ordinary Differential Expressions.* Robert M. Kauffman, Thomas T. Read, Anton Zettl. Springer-Verlag, 1977, vi + 112 pp, $8.30 (P). [ISBN: 0-387-08523-8; 3-540-08523-8]

ANALYSIS, P. *Transference Methods in Analysis.* Ronald R. Coifman, Guido Weiss. CBMS Reg. Conf. in Math., No. 31. AMS, 1977, 59 pp, $7.60 (P). [ISBN: 0-8218-1681-0] Expository lectures from the 1976 conference held at the University of Nebraska. Examples and applications chosen for their accessibility. JAS

STATISTICS, T(13: 1, 2), S. *Basic Statistics: A Primer for the Biomedical Sciences, Second Edition.* Olive Jean Dunn. Wiley, 1977, xi + 218 pp, $12.95. [ISBN: 0-471-22744-7] A non-mathematical approach presupposing only high school algebra. Emphasizes topics needed in health science and includes chapters on demography and life tables. FLW

STATISTICS, T(15-16: 1), S. *Stochastische Methoden.* K. Krickeberg, H. Ziezold. Springer-Verlag, 1977, viii + 201 pp, $12.90 (P). [ISBN: 0-387-08541-6; 3-540-08541-6] An elementary, compactly written and fairly sophisticated text on probability and statistics. Problems. JD-B

STATISTICS, T(16-17: 1, 2). *Parameter Estimation in Engineering and Science.* James V. Beck, Kenneth J. Arnold. Wiley, 1977, xix + 501 pp, $24.95. [ISBN: 0-471-06118-2] In the Wiley Series in Probability and Mathematical Statistics. Relatively new field (also called nonlinear estimation, nonlinear regression, identification,...) concerned with methods for estimating parameters in mathematical models (which may involve algebraic, ordinary and partial differential, and integral equations, and which are not necessarily nonlinear), with estimates of the accuracy of the estimated values, and with the development of improved models. First half provides an algebraic approach to linear estimation, including a concise review of statistical topics, while the last gives a matrix approach to linear and nonlinear estimation and the design of optimal experiments. Strong emphasis on assumptions. Many examples from the field of heat transfer. RSK

STATISTICS, S, P. *Applied Sampling.* Seymour Sudman. Acad Pr, 1976, x + 249 pp, $11.50. [ISBN: 0-12-675750-X] A practical guide to sampling procedures. Introduces a sampling idea and then illustrates it with real-world examples. Intended for researchers with limited resources and modest statistical backgrounds. RSK

SYSTEMS THEORY, P. *System Identification, Advances and Case Studies.* Ed: Raman K. Mehra, Dimitri G. Lainiotis. Math. in Sci. and Eng., V. 126. Acad Pr, 1976, xi + 593 pp, $19. [ISBN: 0-12-487950-0] A state-of-the-art interdisciplinary survey organized around Box and Jenkins' stages of system identification: model structure determination, parameter estimation, experimental design. Assumes background in statistical estimation and time series analysis. LAS

APPLICATIONS (ARCHAEOLOGY). *Raisonnement et Méthodes Mathématiques en Archéologie.* M. Borillo, *et al.* Ed. du Cent. Nat. Rech. Scient., 1977, ii + 221 pp, $10.40 (P). [ISBN: 2-222-02034-4] An interesting collection of essays dealing with mathematical methods problems arising from archeological research. Among the broad topics considered are multivariate analysis, classification to seriation, seriation algorithms, and the role of mathematics in archaeological analysis. SG

APPLICATIONS (BIOLOGY), S(15-17), P, L. *Lecture Notes in Biomathematics-6: Modeling and Control in the Biomedical Sciences.* H.T. Banks. Springer-Verlag, 1975, v + 114 pp, $7.40 (P). [ISBN: 0-387-07395-7; 3-540-07395-7] Survey lectures, with extensive bibliography, treating current work in enzyme kinetics, epidemics, glucose homeostasis, tumor growth, and (briefly) other topics. A useful overview of a major part of contemporary biomathematics. LAS

APPLICATIONS (BIOLOGY), T(13: 1), S. *Mathematical and Biological Interrelations.* Brian A.C. Dudley. Wiley, 1977, x + 319 pp, $13.95 (P). [ISBN: 0-471-99484-7] Elements of mathematics (measurement, proportion, scales, permutations, graphs, etc.) arising in the biology "commonly taught at school and college level." Mathematically-minded readers will find the progress "slow, even tortuous," but biology teachers and students may not. Could serve as a good source of real examples for precalculus courses. LAS

APPLICATIONS (BIOLOGY), S, P*, L. *Mathematical Methods of Population Biology.* F.C. Hoppensteadt. New York U, 1977, v + 167 pp, $5.25 (P). This set of lecture notes includes chapters on population dynamics, mathematical ecology, contagion, bacterial genetics, age structure, population genetics, effects of random sampling in genetic structure, diffusion approximation and biogeography. Deterministic and statistical models considered. No introduction, index, or exercises. TRS

APPLICATIONS (BIOLOGY), T(16-17: 1), S, P, L. *Lecture Notes in Biomathematics-9: A Stochastic Model for Immunological Feedback in Carcinogenesis: Analysis and Approximations.* Neil Dubin. Springer-Verlag, 1976, xi + 163 pp, $8.20 (P). [ISBN: 0-387-07786-3; 3-540-07786-3] A detailed exploration (complete with extensive computer-generated graphs and simulation) of approximations to the intractable differential equations of the nonlinear birth and death process of the immunological feedback model. The only effective methods are those of stochastic linearization and linearized transition probabilities. A good resource for a seminar in modelling, mixing differential equations, numerical methods, stochastic processes, and simulation in the context of an important current scientific problem. LAS

APPLICATIONS (BIOLOGY), P. *Theoretical Immunology.* Ed: George I. Bell, Alan S. Perelson, George H. Pimbley, Jr. Dekker, 1978, xi + 646 pp, $45. [ISBN: 0-8247-6618-0] 21 invited papers, including several dealing with mathematical models. LAS

APPLICATIONS (CHEMISTRY), P. *Three-Dimensional Nets and Polyhedra.* A.F. Wells. Wiley, 1977, xii + 268 pp, $29.95. [ISBN: 0-471-02151-2] A systematic study, trying to arrive at a topological classification theorem. Many individual cases are studied in varying detail, with helpful drawings and pictures. Many references are made to crystalline structures, the main motivation for the study. TLS

APPLICATIONS (DATA PROCESSING), S(15-16). *Datenbanksysteme: Konzepte und Modelle.* Gunter Schlageter, Wolffried Stucky. Teubner, Stuttgart, 1977, 261 pp, (P). [ISBN: 3-519-02339-3] A self-contained survey of recent developments in data systems. The presentation offers a middle ground: neither a "here is what to do" approach nor a state-of-the-art study of data structures. JAS

APPLICATIONS (ECONOMICS), P. *Lecture Notes in Economics and Mathematical Systems-138: Expectations and Stability in Oligopoly Models.* Koji Okuguchi. Springer-Verlag, 1976, vi + 103 pp, $8 (P). [ISBN: 0-387-08056-2; 3-540-08056-2] Theory of equilibrium (based on the contraction mapping theorem) for industries with few firms, derived and elaborated from the original 1838 model of A.C. Cournot. Early chapters are devoted to duopolies,later ones to analysis in the presence of likely uncertainties. LAS

APPLICATIONS (ECONOMICS), P, L. *Lecture Notes in Economics and Mathematical Systems-140: Theory of the Price Index: Fisher's Test Approach and Generalizations.* Wolfgang Eichhorn, Joachim Voeller. Springer-Verlag, 1976, 95 pp, $8 (P). [ISBN: 0-387-08059-7; 3-540-08059-7] A systematic investigation of the axiomatic theory of a price index designed to measure the "purchasing power" of money, based on the 1922 theory of I. Fisher and dozens of variations introduced in recent years. Primary focus is on consistency (rare) and inconsistency (common) theorems. It turns out that most of Fisher's desiderata are mutually inconsistent, and that the most commonly used form of his price index equations is incorrect. LAS

APPLICATIONS (ECONOMICS), P. *The Theory of Joint Maximization.* Peter B. Dixon. North-Holland, 1975, xii + 212 pp, $24.95. [ISBN: 0-7204-3194-8] A revised and expanded Ph.D. thesis (Harvard, 1972; advisor: Leontief) in which maximization principles are applied to problems involving multiple economic units. JAS

APPLICATIONS (ENGINEERING), *Stochastic Problems in Dynamics.* Ed: B.L. Clarkson. Fearon-Pitman, 1977, 566 pp, $22.50. [ISBN: 0-273-01104-9] Invited papers (with edited discussion) from a 1976 symposium at the University of Southampton designed to bring together mathematicians and engineers to address dynamical problems of a stochastic nature. LAS

APPLICATIONS (ENGINEERING), T(14-16: 3), S. *Engineering Mathematics.* A.J.M. Spencer, *et al.* Van N-Rein, 1977. *V. 1,* xii + 536 pp, $11.95 (P); $23.95. [ISBN: 0-442-30147-2]; *V. 2,* xi + 400 pp, $12.95 (P). [ISBN: 0-442-30208-8] Developed by mathematicians in the department of theoretical mechanics at the University of Nottingham, this two-volume work is an encyclopedia of basic engineering mathematics. Volume 1 contains the core material (e.g., ordinary differential equations and partial differential equations, vector calculus, finite differences, numerical analysis, probability and statistics), while Volume 2 contains largely optional topics (e.g., linear, nonlinear, and dynamic programming, variational problems, more statistics, more differential equations). A good synthesis of analytic and numerical methods with many applications and exercises. TRS

APPLICATIONS (ENGINEERING), P. *Control and Dynamic Systems: Advances in Theory and Applications, V. 13.* Ed: C.T. Leondes. Acad Pr, 1977, xviii + 365 pp, $25.50. [ISBN: 0-12-012713-X] The thirteenth collection of papers, this time focused on techniques of control and dynamic systems and their application to modern complex engineering, industrial, and other systems. JAS

APPLICATIONS (ENGINEERING), T(17: 1), S, P. *Lecture Notes in Physics-68: Energy Methods in Time-Varying System Stability and Instability Analyses.* Y.V. Venkatesh. Springer-Verlag, 1977, ix + 256 pp, $11.50 (P). [ISBN: 0-387-08430-4; 3-540-08430-4] This monograph treats stability and instability analyses of the vector differential equation which describes the generalized Aizerman-Lur'e-Postnikov nonlinear time varying feedback system and also more general systems described by integral equations having the same feedback structure. Includes exercises and lots of references. CEC

APPLICATIONS (PHYSICS), P. *Information Mechanics.* Frederick W. Kantor. Wiley, 1977, xiii + 397 pp, $21.95. [ISBN: 0-471-02968-8] This rather unique book looks at modern physics from the point of view of information theory. The author makes a careful and thorough presentation of his work, of open questions, and of potentially interesting experiments. The indexing, appendices, and other aids to the reader appear to offer reasonable access to this "report of work in progress." This work certainly represents an original and independent--yet not entirely eclectic--point of view. JAS

APPLICATIONS (PHYSICS), T(16-17). *Introduction to Special Relativity.* Herman M. Schwartz. Krieger, 1977, 458 pp, $19.50. [ISBN: 0-88275-478-5] Corrected reprint of the 1968 McGraw-Hill original edition (TR, August 1968). LAS

APPLICATIONS (SOCIAL SCIENCE), T**(15-16: 1, 2), S, L. *An Introduction to Mathematical Models in the Social and Life Sciences.* Michael Olinick. A-W, 1978, xiii + 466 pp, $16.95. [ISBN: 0-201-05448-5] A rich, diverse and interesting collection of applications of elementary mathematics (linear algebra, differential equations, probability, axiomatics) to such areas as ecology, utility and measurement theory, arms race, and epidemics. Each chapter has exercises, projects, references, and extensive biographical sketches (with illustrations) of principal investigators. An extraordinary volume that will entice many schools to try a course in modelling. LAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Jay E. Goldfeather, Carleton; Paul Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D. C. 20036.

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1225 Connecticut Avenue., N.W., Washington, D. C. 20036.

PERSONAL ITEMS

U.S. Naval Academy: Assistant Professor James P. Foti has become a member of the staff of the Electromagnetic Analysis Center. Associate Professor Mahlon F. Stilwell has retired. Dr. Anthony M. Gaglione, formerly Assistant Professor at CCNY, has been appointed Assistant Professor.

Assistant Professor Joseph H. Mayne, Loyola University of Chicago, has been promoted to Associate Professor.

Professor Rufus Isaacs, Johns Hopkins University, has been appointed Professor Emeritus of Mathematical Sciences in the Faculty of Arts and Sciences.

Assistant Professor Michael J. Tierney, Maryville College, St. Louis, Missouri, has been promoted to Associate Professor.

Associate Professor Victor E. Hill, Williams College, has been promoted to Professor.

Dr. Calvin J. Irons, Lecturer in Mathematics, Kelvin Grove College of Advanced Education, Brisbane, Australia, has been promoted to Senior Lecturer in Mathematics and Head of the Department of Mathematics.

Professor Louis Ross, University of Akron, has retired with the title of Professor Emeritus.

Professor Robert H. Newton, University College of North Wales, School of Electronic Engineering Science, died on October 19, 1977 at the age of 59. He was a member of the Association for eleven years.

Mr. Welby R. Stevens, Orlando, Florida, died on December 7, 1977 at the age of 75. He was a member of the Association for fifty-three years. He was well-known for his study of hurricanes and was consulted by the pioneers in establishing pumping from man-made stations in deep off-shore locations.

More complete information on deaths reported briefly in last month's *News and Notices* brought the following: Professor Edward G. Begle, who died on March 2, 1978, received the Distinguished Service Award from the Association (*The American Mathematical Monthly,* January, 1969). Professor Walter R. Talbot, retired from Morgan State College, died on December 26, 1977 at the age of 68. He was a member of the Association for forty-two years.

SHORT COURSE ON MATHEMATICAL MODELING

The Michigan Section of the MAA will co-sponsor a Summer Short Course on "Mathematical Modeling and Its Applications" to be held from July 31 to August 4 at Hope College in Holland, Michigan. The principal lecturer will be Dr. Maynard Thompson of Indiana University and the associate lecturer will be Dr. Melvin Nyman of Manchester College, Indiana.

NSF SUMMER HIGH SCHOOL PROGRAMS IN OHIO

The National Science Foundation is supporting three mathematics programs in Ohio for high school students through its Student Science Training program this year.

Hiram College will have a summer program in Mathematics and Computers for 30 students within a 150 mile radius of Hiram. Project Direector is Dr. Edward J. Smerek; Math Department; Hiram College; Hiram, Ohio 44234.

John Carroll University will have a summer program in Mathematics for 15 students from the Greater Cleveland area. Project Director is Dr. Leo J. Schneider; Math Department; John Carroll University; University Heights, Ohio 44118.

University of Cincinnati will have an academic year program in Mathematical Modeling in the Sciences for 28 students from the Greater Cincinnati area. Project Director is Dr. Raymond H. Rowling; Math Department; University of Cincinnati; Cincinnati, Ohio 45221.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifty-fifth annual meeting of the Louisiana-Mississippi Section was held at the Buena Vista Hotel, Biloxi, Mississippi, on February 17-18, 1978, hosted by the University of Mississippi. Registration totaled 129 persons attending the meeting held jointly with the Louisiana-Mississippi Branch of the NCTM. Professor Robert A. Shive, Jr., of Millsaps College presided as Chairman. Dr. Leonard Gillman, University of Texas and Treasurer, MAA, gave the addresses: *Our Friend the Harmonic Series* on Friday, and *How Many Roots Has a Quadratic Equation* on Saturday.

Contributed papers were given as follows:

General Solutions for $u_{xx} + u_{yy} = 0$ *and* $u_{xx} - u_{yy} = 0$, John L. Tilley, Mississippi State University.

An Application of the Jordan Canonical Form to the Epidemic Problem, Linda Gilbert, Louisiana Tech.

An Intuitive Proof of the Nonembedability of the Graph K_{313}, Allen Scholnick, McNeese State.

C - Groups, Ed Oxford and Gary Walls, University of Southern Mississippi.

Some Extremal Elements of the Convex Cone of Monotone Processes of Concave Type, Melvyn W. Jeter and W. C. Pye, University of Southern Mississippi.

Subsequential Limit Points of a Continuous Function, J. L. Solomon, Mississippi State University.

The Widths of Simple Trees and Their Subtroids, Frances O. McDonald, William Carey College.

A Characterization of Hilbert - Schmidt Operators, Johnny Gills, Jr., Jackson State University.

General Solutions to the Reverse Order Law $(AB)^+ = B^-_{mr} A^-_{1r}$, Jimmie Gilbert, Louisiana Tech.

T. M. Groups, Gary Walls, University of Southern Mississippi.

The Gossip Problem, K. B. Reid, Louisiana State University.

Primary Extensor Formulation of the Inclined Plane Equations of Motion, Gertrude C. Okhuysen, Mississippi State University.

A Pivot - Point Procedure for Solving Systems of Equations and Calculation of Determinants, Johnny Gills, Jr., Jackson State University.

A Study on the Inference of Group Membership on Achievement Test Scores Across IQ Scores, Rawiwan Thumchai, Jackson State University.

Some Geometrical Aspects of Dimension Theory, Ludvik Janos, Mississippi State University.

Some Multi-Tube Flow Problems, J. B. Garner, Louisiana Tech University.

Pythagorean Quadruples, Robert Heller, Mississippi State University.

On Circularly Correlated Samples from a Normal Distribution, T. A. Watkins, University of New Orleans.

Playing with Triangles in n-Dimensions, Glenda McBride, student at Nicholls State University.

An Inverse Eigenvalue Problem, Barry Piazza, student at Nicholls State University.

The Counting Numbers of Infinity, Ann Maria Tomberlin, Mississippi University for Women.

An Application of Boolean Algebra to Medical Science, Craig Bordelon, student at Nichlls State University.

A Problem on the Generation of Random Vectors, Al Feyerabend, Nichools State University.

Officers for 1978-79 are: Chairman, J. B. Garner, Louisiana Tech; Mississippi Vice Chairman, Ed Oxford, University of Southern Mississippi; Louisiana Vice Chairman, Rev. Tom Mulcrone, S.J., Loyola University; Secretary-Treasurer, J. R. Foote, University of New Orleans.

J. R. Foote, *Secretary-Treasurer*

MARCH MEETING OF THE FLORIDA SECTION

The Eleventh Annual Spring Meeting of the Florida Section of the MAA was held on March 3-4, 1978, at the Clearwater Campus of St. Petersburg Junior College; 166 registrants attended the meeting.

Six invited addresses were presented as follows: *There were Giants...*, Professor R. Creighton Buck, University of Wisconsin; *Mathematical Modeling in Ecology: Stream Pollution*, Professor W. J. Padgett, University of South Carolina; *Combinatorial Problems with Surprising Solutions*, Professor David P. Roselle, Secretary of the Mathematical Association of America; *Suppose there is no Derivative*, Professor Dwight Goodner, Florida State University; *Ramblings and Implications - the Mathematics Tragedy and What We Might Do*, Professor William Palow, Miami Dade Community College; *Invariance of Indeterminates in Polynomial and Power Series Rings*, Professor Robert Gilmer, Florida State University.

The meeting was held jointly with the Florida Chapter of the American Statistical Association and included talks from a State Articulation Conference. The following talks were presented: *The 'Straw' Poll on Flexible, Statewide, Standardized Syllabi on the Calculus Sequence and Related Courses*, Anthony Shershin, Florida International University; *State Mathematics Literacy Test*, Herbert L. Johnson, Supervisor of Secondary Mathematics, Pinellas County; *Multivariate Discriminate Analysis and Prediction of Loan Default*, Nick Belloit, Florida Eta Chapter, University of North Florida; *Progressive Operations of the Third Kind*, George R. Blanton, Jr., Florida Delta Chapter, University of Florida; *Poulet, Super-Poulet, and Related Numbers*, Jann Wise, Florida Eta Chapter, University of North Florida; *Mercator Projections*, Teresa Schultz, Florida Epsilon Chapter, University of South Florida.

The following papers were presented to the section:

Functional Design or Colorful Mathematics, E. P. Miles, Jr., Florida State University.

A Complete Residue System in the Integral Quaternions, J-S. Shiue, University of South Florida.

A Mathematician's View of the Earth Resources Program in the Shuttle Era, Willie H. Greene, John F. Kennedy Space Center.

Topological Structures in Neurobiology, Gerhard X. Ritter, University of Florida.

Integration Over k-Dimensional Subsets of R^n, Michael D. Taylor, Florida Technological University.

Student Contributions to Computer Oriented Curricular Projects, Scott E. Rimbey, Florida State.

A Property of a Defining Sequence for a Pseudo-arc, Phillip Bartick, University of Miami.

A Characterization of a Simple Arc, James Kell, III, University of Miami.

Crisis in Mathematics - An Alternative to Mathematics History, Dennis Kletzing, Stetson University.

On Making a Successful Transition from Manipulation to Mathematics, Edward Norman, Florida Tech.

Calculus with Computer - An Informal Approach, Bruce H. Edwards, University of Florida.

Existence Theorem for Perturbed [illegible] Unbounded Nonlinearities, Karen Singkofer, University of South Florida.

Bounds for Polynomials with a Multiple Root on the Unit Circle, Michaël Lachance, University of South Florida.

On Hamel Basis Ordering, Nancy J. Fordyce, Florida State University.

A Study of the Factors Which Contributed to the Superior Achievement in Mathematics of South Vietnamese Refugee Family Children Noted in the Greater Chattanooga Area Public Schools, Chattanooga, Tennessee, in 1975-76, in the Elementary Schools, Herschel Sellers, University of Tennessee at Chattanooga.

Methods of Numerical Integration and Their Computer Implementation - Report on a Self-Study Module Class-Tested in Calculus Courses, Wendell Motter, Florida A & M University.

Calculator College Calculus, Alan Wayne, Pasco-Hernando Community College.

A Proven Model for the Administration of a Centrally Coordinated Mass-Lecture Mathematics Course, Pat Scott and Kermit Sigmon, University of Florida.

Pedagogical Implications of a Subsuming Generalization, Herbert Wills, Florida State University.

The Florida Section has divided the State into seven areas and in the fall of 1976 and 1977 Mini-Sectional Meetings were held in six of the seven areas. Teachers from Junior High Schools, High Schools and Colleges were invited to attend. The programs dealt primarily with teaching and articulation. At the luncheon-business meeting committee reports were presented and Professor Frederic Hoffman of Florida Atlantic University was elected Chairman-elect; Professors Roy Raines of Manatee Junior College and John Leeson of University of North Florida, were elected as Vice-Chairmen.

Frank L. Cleaver, *Secretary*

APRIL MEETING OF THE MISSOURI SECTION

The 1978 Spring Meeting of the Missouri Section of the MAA was held at Central Missouri State University on April 7 and 8. There were 92 persons in attendance including approximately 65 members.

Professor R. D. Anderson of Louisiana State University represented the MAA and spoke on *Algorithmicly Defined Functions.* He also presided over a discussion of the MAA Placement Exams. Invited addresses were given by Robert E. Reys of the University of Missouri-Columbia on *What's Happening in Mathematics Education, Implication for All of Us,* and by our banquet speaker John K. Beem of the University of Missouri-Columbia on *Time, Space, and Cosmology.*

Section Governor Charles Stuth of Stephens College presided at a meeting of MAA Department Representatives. Dr. H. K. Stumpff of Central Missouri State University hosted a breakfast for Mathematics Department Chairpersons at which the CLEP Examination was discussed.

At the business meeting the following officers were elected: Chairman, Yudell Luke, University of Missouri-Kansas City; Vice-Chairman, Michael Z. Williams, Westminster College; Chairman of the High School Visiting Lecturers, Leonard Palmer, Southeast Missouri State University. Tim J. Steger of Washington University was awarded the prize for the top Missouri student on the Putnam examination and the Washington University team was recognized as the top team in the nation.

The following contributed papers were presented:

A Method for Solving $x^2 = A$ *in Matrices,* by W. R. Utz, University of Missouri-Columbia.

Common Problems Experiences in the Application of Regression Analysis to Economic Problems, by B. A. Brock, Central Missouri State University.

Diagnosis of College Students' Mathematics Errors, by T. Goodman, Central Missouri State.

Developmental Analysis, by L. Sherwood, Penn Valley Community College.

Niven Numbers, by R. Kennedy, Central Missouri State University.

Characterizing the G.C.D. and L.C.M., by M. Ahuja, Southeast Missouri State University.

Packing the Unit Interval with a Steinhaus Class, by K. W. Lee, Missouri Western State College.

A Comment on the Method of Proof by Contradiction or 'Who Killed Cock Robin', by H. Polowy, Lincoln University.

Dissection of a Square and the Fibonacci Series, by G. Ragland, Florissant Valley Community College.

Latin Squares, Finite Groups, and the Four-Color Problem, by R. Friedlander, University of Missouri-St. Louis.

An Ordering-Theoretic Method of Analyzing Sex-Differences on the Fraction Concept, by G. Austin-Martin, Stephens College.

J. D. Kubicek, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-eighth Summer Meeting, Brown University, Providence, August 8–10, 1978.

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 weeks before meeting.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, St. Peter's College, Englewood Cliffs, November 4, 1978.

NORTH CENTRAL, University of Saskatchewan, Saskatoon, October 20–21, 1978.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO

OKLAHOMA–ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 weeks before meeting.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 16–17, 1978.

PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.

ROCKY MOUNTAIN, University of Denver, Denver, spring 1979.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN, University of Tennessee, Chattanooga, spring 1979.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

Texas, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Chicago, January 3–8, 1979.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Stouffer's Greenway Plaza Hotel, Houston, Texas, October 10–14, 1978.

AMERICAN MATHEMATICAL SOCIETY, Brown University, Providence, Rhode Island, August 9–12, 1978.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of British Columbia, Vancouver, June 19–22, 1978.

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton Park Hotel, Washington, D.C., December 4–6, 1978.

ASSOCIATION FOR SYMBOLIC LOGIC, Madison, Wisconsin, June 18–24, 1978.

ASSOCIATION FOR WOMEN IN MATHEMATICS, Brown University, Providence, Rhode Island, August 8–12, 1978.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, University of Western Ontario, London, Ontario, June 1–2, 1978.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, Stevens Point, Wisconsin, August 6–9, 1978.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Bonadventure Hotel, Los Angeles, California, November 12–16, 1978.

PI MU EPSILON

SCHOOL, SCIENCE AND MATHEMATICS ASSOCIATION, Hotel Fort Des Moines, Des Moines, Iowa, November 2–4, 1978.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Hyatt Regency Hotel, Knoxville, Tennessee, October 30–November 1, 1978.

Logical choices.

HILBERT'S THIRD PROBLEM
Vladimir G. Boltyanskiy,
V.A. Steklov Institute of Mathematics, Moscow

Examines the present theory of equidecomposability, with numerous illustrations included. **CONTENTS:** The Measurement of Area and Volume. Equidecomposability of Polygons. Equidecomposability of Polyhedra. Conclusion. Appendix. Bibliography. (A V.H. Winston & Sons publication)

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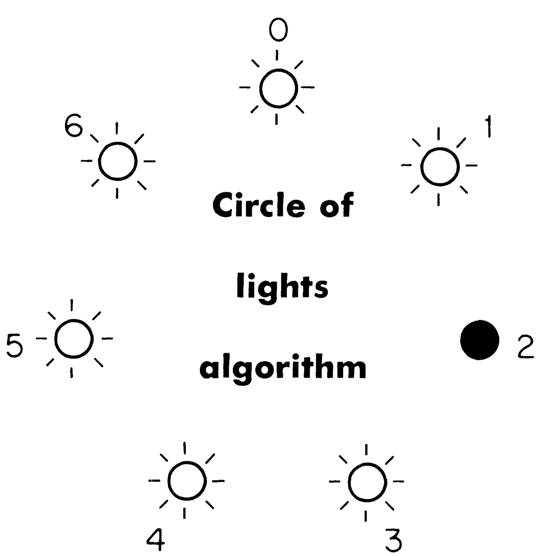

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ANALYTIC PROOFS OF THE "HAIRY BALL THEOREM" AND THE BROUWER FIXED POINT THEOREM

JOHN MILNOR

This note will present strange but quite elementary proofs of two classical theorems of topology, based on a volume computation in Euclidean space and the observation that the function $(1+t^2)^{n/2}$ is not a polynomial when n is odd. The argument was suggested by the methods of Asimov [1]. Familiar proofs of these theorems all use either combinatorial arguments, homology theory, differential forms, or methods from geometric topology. Compare [2], [3], [4], [6], [7].

Here is a preliminary version of the first one.

THEOREM 1. *An even-dimensional sphere does not possess any continuously differentiable field of unit tangent vectors.*

By definition, the *sphere* S^{n-1} is the set of all vectors $\mathbf{u}=(u_1,\ldots,u_n)$ in the Euclidean space $\mathbf{R}^n$ such that the Euclidean length $\|\mathbf{u}\|$ equals 1. A vector $\mathbf{v}(\mathbf{u})$ in $\mathbf{R}^n$ is *tangent* to S^{n-1} at $\mathbf{u}$ if the Euclidean inner product $\mathbf{u}\cdot\mathbf{v}(\mathbf{u})$ is zero.

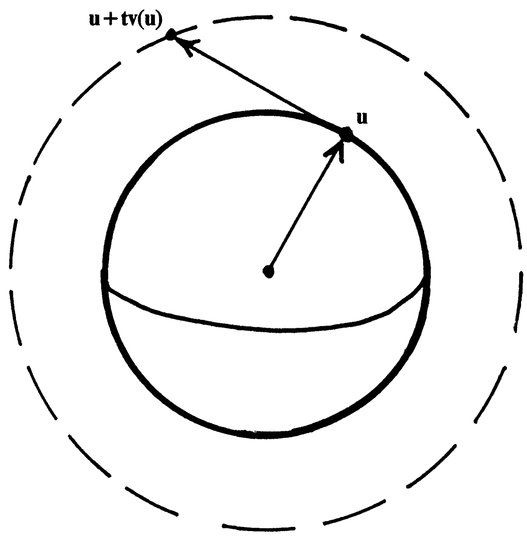

FIG. 1

The hypothesis that the dimension $n-1$ is even is essential. For if $n-1$ is odd then the formula

$$\mathbf{v}(u_1,\ldots,u_n) = (u_2,-u_1,\ldots,u_n,-u_{n-1})$$

defines a differentiable field of unit tangent vectors on S^{n-1}.

The proof of Theorem 1 will depend on two lemmas. The first involves a volume computation. Let A be a compact region in $\mathbf{R}^n$, and let $\mathbf{x}\mapsto\mathbf{v}(\mathbf{x})$ be a continuously differentiable vector field which is defined throughout a neighborhood of A. The values $\mathbf{v}(\mathbf{x})$ can be arbitrary vectors in $\mathbf{R}^n$. For each real number t, consider the function

$$\mathbf{f}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{v}(\mathbf{x})$$

which is defined for all x in A.

John Milnor received his Ph.D. from Princeton and taught there until 1967. After two years at M.I.T., he joined the Institute for Advanced Study; he has also been a visiting professor at Berkeley and at UCLA. He was awarded a Fields Medal in 1962 and the National Medal of Science in 1966; he is a member of the National Academy of Sciences. His principal research has been in the topology of manifolds.—*Editors*

LEMMA 1. *If the parameter t is sufficiently small, then this mapping $\mathbf{f}_t$ is one-to-one and transforms the region A onto a nearby region $\mathbf{f}_t(A)$ whose volume can be expressed as a polynomial function of t.*

Proof. Since A is compact, and since the function $\mathbf{x} \mapsto \mathbf{v}(\mathbf{x})$ is continuously differentiable, there exists a *Lipschitz constant* c so that

$$\|\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y})\| \leqslant c\|\mathbf{x}-\mathbf{y}\| \tag{$*$}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in A. [This is proved as follows. First consider the special case where A is a cube with edges parallel to the coordinate axes. Passing from $\mathbf{x}$ to $\mathbf{y}$ in n steps by changing one coordinate at a time, and applying the Mean Value Theorem of differential calculus, one sees that $|v_i(\mathbf{x})-v_i(\mathbf{y})| \leqslant \sum_j c_{ij}|x_j-y_j|$, where

$$c_{ij} = \sup_A |\partial v_i/\partial x_j|.$$

Therefore

$$\|\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{y})\| \leqslant \sum_i |v_i(\mathbf{x})-v_i(\mathbf{y})| \leqslant \sum_{i,j} c_{ij}|x_j-y_j| \leqslant \sum_{i,j} c_{ij}\|\mathbf{x}-\mathbf{y}\|,$$

as required. Now an arbitrary compact set A in $\mathbf{R}^n$ can be covered by finitely many open cubes I_α, chosen so that a Lipschitz condition $(*)$ holds whenever $\mathbf{x}$ and $\mathbf{y}$ belong to the same cube. But if $\mathbf{x}$ and $\mathbf{y}$ in A *do not* belong to any common cube I_α, then the distance $\|\mathbf{x}-\mathbf{y}\|$ is bounded away from zero. In fact this expression $\|\mathbf{x}-\mathbf{y}\|$ can be thought of as a continuous and nowhere zero function on the compact set $A \times A - \cup I_\alpha \times I_\alpha$. It is now easy to choose a constant c so that the Lipschitz condition $(*)$ holds uniformly for all $\mathbf{x}$ and $\mathbf{y}$ in A.]

Choose any t with $|t|<c^{-1}$. Then $\mathbf{f}_t$ is one-to-one; for if $\mathbf{f}_t(\mathbf{x})=\mathbf{f}_t(\mathbf{y})$ then $\mathbf{x}-\mathbf{y}=t(\mathbf{v}(\mathbf{y})-\mathbf{v}(\mathbf{x}))$, hence the inequality $\|\mathbf{x}-\mathbf{y}\| \leqslant |t|c\|\mathbf{x}-\mathbf{y}\|$ implies that $\mathbf{x}=\mathbf{y}$.

The matrix of first derivatives of $\mathbf{f}_t$ can be written as $I+t[\partial v_i/\partial x_j]$, where I is the identity matrix. Hence its determinant is a polynomial function of t, of the form $1+t\sigma_1(\mathbf{x})+\cdots+t^n\sigma_n(\mathbf{x})$, the coefficients being continuous functions of $\mathbf{x}$. This determinant is strictly positive for $|t|$ sufficiently small. Integrating over A, we see that the volume of the image region can be expressed as a polynomial function of t,

$$\text{volume } \mathbf{f}_t(A) = a_0 + a_1 t + \cdots + a_n t^n,$$

with coefficients $a_k = \int \cdots \int_A \sigma_k(\mathbf{x})\,dx_1 \cdots dx_n$. ■

Now suppose that the sphere S^{n-1} has a continuously differentiable field $\mathbf{u} \mapsto \mathbf{v}(\mathbf{u})$ of unit tangent vectors. For any real number t, note that the vector $\mathbf{u}+t\mathbf{v}(\mathbf{u})$ has length $\sqrt{1+t^2}$.

LEMMA 2. *If the parameter t is sufficiently small, then the transformation $\mathbf{u} \mapsto \mathbf{u}+t\mathbf{v}(\mathbf{u})$ maps the unit sphere in $\mathbf{R}^n$* **onto** *the sphere of radius $\sqrt{1+t^2}$.*

Assuming this Lemma for the moment, we can now prove Theorem 1. As region A we take the region between two concentric spheres, defined by the inequalities $a \leqslant \|\mathbf{x}\| \leqslant b$. We extend the vector field $\mathbf{v}$ throughout this region by setting $\mathbf{v}(r\mathbf{u}) = r\mathbf{v}(\mathbf{u})$ for $a \leqslant r \leqslant b$. It follows that the mapping $\mathbf{f}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{v}(\mathbf{x})$ is defined throughout the region A, and maps the sphere of radius r *onto* the sphere of radius $r\sqrt{1+t^2}$ providing that t is sufficiently small. (Note that $\mathbf{f}_t(r\mathbf{u}) = r\mathbf{f}_t(\mathbf{u})$.) Hence it maps A onto the region between spheres of radius $a\sqrt{1+t^2}$ and $b\sqrt{1+t^2}$. Evidently

$$\text{volume } \mathbf{f}_t(A) = \left(\sqrt{1+t^2}\right)^n \text{volume } (A).$$

Thus, if n is odd, this volume is not a polynomial function of t. Comparing Lemma 1, we obtain a contradiction which proves Theorem 1.

Now we must prove Lemma 2. Here are two alternative arguments, the first based on the "Shrinking Lemma" (see [5]), and the second based on elementary point set topology.

First Proof. Let the region A considered above be defined by the inequalities $1/2 \leqslant \|\mathbf{x}\| \leqslant 3/2$. Choose t small enough so that $|t| < 1/3$ and $|t| < c^{-1}$, where c is a Lipschitz constant for $\mathbf{v}$. Then for each fixed $\mathbf{u}_0$ in S^{n-1} the auxilliary mapping

$$\mathbf{x} \mapsto \mathbf{u}_0 - t\mathbf{v}(\mathbf{x})$$

carries the complete metric space A into itself (since $|t\mathbf{v}(\mathbf{x})| < 1/2$); and satisfies a Lipschitz condition with Lipschitz constant less than 1. Hence, by the Shrinking Lemma, this auxiliary map has a unique fixed point. In other words, the equation $\mathbf{f}_t(\mathbf{x}) = \mathbf{u}_0$ has a unique solution $\mathbf{x}$. Multiplying $\mathbf{x}$ and $\mathbf{u}_0$ by $\sqrt{1+t^2}$, Lemma 2 follows.

Second Proof. We may assume that $n \geqslant 2$. If t is sufficiently small, then the matrix of first derivatives of $\mathbf{f}_t$ is non-singular throughout the compact region A. (Compare the proof of Lemma 1.) Using the Inverse Function Theorem, it follows that $\mathbf{f}_t$ maps open sets in the interior of A to open sets. Hence the image $\mathbf{f}_t(S^{n-1})$ is a relatively open subset of the sphere of radius $\sqrt{1+t^2}$. But this image $\mathbf{f}_t(S^{n-1})$ is also compact and hence closed. Since S^{n-1} is connected, an open and closed subset must be the entire sphere, and the conclusion follows. ■

A slightly sharper version of Theorem 1, which does not mention differentiability or unit vectors, follows as an immediate corollary.

THEOREM 1′. *An even dimensional sphere does not admit any* **continuous** *field of* **non-zero** *tangent vectors.*

Proof. Suppose that the sphere S^{n-1} possesses a continuous field of non-zero tangent vectors $\mathbf{v}(\mathbf{u})$. Let $m > 0$ be the minimum of $\|\mathbf{v}(\mathbf{u})\|$. By the Weierstrass Approximation Theorem **[5]**, there exists a polynomial mapping $\mathbf{p}$ from S^{n-1} to $\mathbf{R}^n$ satisfying

$$\|\mathbf{p}(\mathbf{u}) - \mathbf{v}(\mathbf{u})\| < m/2$$

for all $\mathbf{u}$. Defining a differentiable vector field $\mathbf{w}(\mathbf{u})$ by the formula

$$\mathbf{w} = \mathbf{p} - (\mathbf{p}\cdot\mathbf{u})\mathbf{u}$$

for every $\mathbf{u}$, the computation $\mathbf{w}\cdot\mathbf{u} = 0$ shows that $\mathbf{w}(\mathbf{u})$ is tangent to S^{n-1} at $\mathbf{u}$, while the computation

$$\|\mathbf{w} - \mathbf{p}\| = |\mathbf{p}\cdot\mathbf{u}| < m/2,$$

together with the triangle inequality, shows that $\mathbf{w} \neq \mathbf{0}$. Therefore, the quotient $\mathbf{w}(\mathbf{u})/\|\mathbf{w}(\mathbf{u})\|$ is an infinitely differentiable field of unit tangent vectors on the sphere S^{n-1}. If $n-1$ is even, this is impossible by Theorem 1. ■

Starting with Theorem 1′, it is quite easy to prove the Brouwer Fixed Point Theorem:

THEOREM 2. *Every continuous mapping* $\mathbf{f}$ *from the disk* D^n *to itself possesses at least one fixed point.*

Here D^n is defined to be the set of all vectors $\mathbf{x}$ in $\mathbf{R}^n$ with $\|\mathbf{x}\| \leqslant 1$.

Proof. If $\mathbf{f}(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x}$ in D^n, then the formula $\mathbf{v}(\mathbf{x}) = \mathbf{x} - \mathbf{f}(\mathbf{x})$ would define a non-zero vector field $\mathbf{v}$ on D^n which points *outward* everywhere on the boundary, in the sense that $\mathbf{u}\cdot\mathbf{v}(\mathbf{u}) > 0$ for every point $\mathbf{u}$ in S^{n-1}.

With a little care, we can modify this definition to obtain a non-zero vector field $\mathbf{w}$ on D^n which points *directly outward* on the boundary, in the sense that $\mathbf{w}(\mathbf{u}) = \mathbf{u}$ for every $\mathbf{u}$ in S^{n-1}. For example, set

$$\mathbf{w}(\mathbf{x}) = \mathbf{x} - \mathbf{y}(1 - \mathbf{x}\cdot\mathbf{x})/(1 - \mathbf{x}\cdot\mathbf{y}),$$

where $\mathbf{y} = \mathbf{f}(\mathbf{x}) \neq \mathbf{x}$. Evidently $\mathbf{w}(\mathbf{x}) = \mathbf{x}$ whenever $\mathbf{x}\cdot\mathbf{x} = 1$. This expression depends continuously on $\mathbf{x}$, since the denominator never vanishes. It is clearly non-zero whenever $\mathbf{x}$ and $\mathbf{y}$ are linearly independent; while if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent the identity $(\mathbf{x}\cdot\mathbf{x})\mathbf{y} = (\mathbf{x}\cdot\mathbf{y})\mathbf{x}$ implies that $\mathbf{w}(\mathbf{x}) = (\mathbf{x} - \mathbf{y})/(1 - \mathbf{x}\cdot\mathbf{y}) \neq \mathbf{0}$.

Let us transplant this hypothetical vector field $\mathbf{w}(\mathbf{x})$ to the southern hemisphere of the unit sphere S^n in $\mathbf{R}^{n+1}$. Identifying $\mathbf{R}^n$ with the hyperplane $x_{n+1}=0$ which passes through the "equator" of S^n, we will use stereographic projection from the north pole $(0,\ldots,0,1)$ to map each point $\mathbf{x}$ of D^n to a point $\mathbf{s}(\mathbf{x})=\mathbf{u}$ of the southern hemisphere $u_{n+1}\leqslant 0$. The precise formula is

$$\mathbf{s}(\mathbf{x})=(2x_1,\ldots,2x_n,\mathbf{x}\cdot\mathbf{x}-1)/(\mathbf{x}\cdot\mathbf{x}+1).$$

Applying the derivative of the mapping $\mathbf{s}$ at $\mathbf{x}$ to the vector $\mathbf{w}(\mathbf{x})$, we obtain a corresponding tangent vector $\mathbf{W}(\mathbf{u})$ to S^n at the image point $\mathbf{s}(\mathbf{x})=\mathbf{u}$. (The vector $\mathbf{W}(\mathbf{u})$ can be described as the velocity vector $d\mathbf{s}(\mathbf{x}+t\mathbf{w}(\mathbf{x}))/dt$ of the spherical curve $t\mapsto\mathbf{s}(\mathbf{x}+t\mathbf{w}(\mathbf{x}))$, evaluated at $t=0$.) In this way, we obtain a non-zero tangent vector field $\mathbf{W}$ on the southern hemisphere. At every point $\mathbf{u}=\mathbf{s}(\mathbf{u})$ of the equator, since $\mathbf{w}(\mathbf{u})=\mathbf{u}$ points directly outward, computation shows that the corresponding vector $\mathbf{W}(\mathbf{u})=(0,\ldots,0,1)$ points due north (i.e., away from the southern hemisphere).

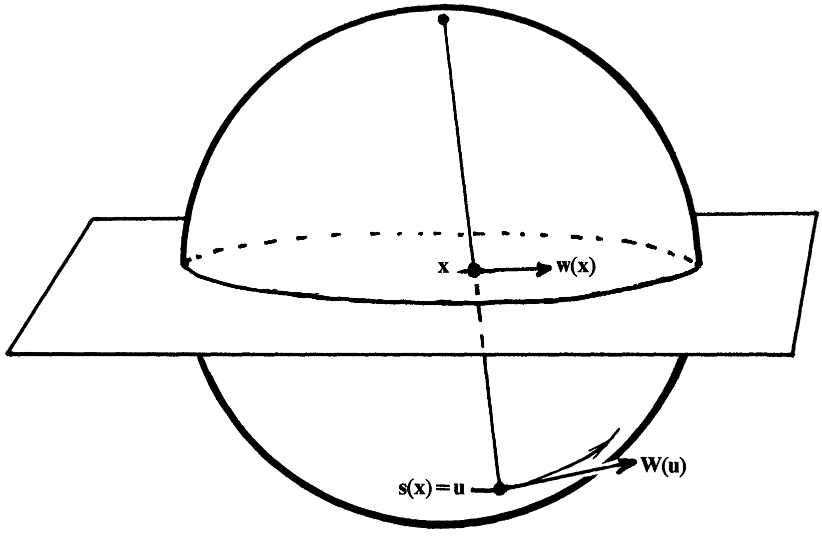

FIG. 2

Similarly, using stereographic projection from the south pole, the vector field $-\mathbf{w}(\mathbf{x})$ corresponds to a vector field on the northern hemisphere which also points due north on the equator. Piecing these two vector fields together, we obtain a non-zero tangent vector field $\mathbf{W}$ which is defined and continuous everywhere on S^n. If n is even, this is impossible by Theorem 1′.

This contradiction proves the Brouwer Fixed Point Theorem for even values of n. But this suffices to prove the Theorem for an odd value $n=2k-1$ also. For any map $\mathbf{f}$ from D^{2k-1} to itself without fixed point would give rise to a map $\mathbf{F}(x_1,\ldots,x_{2k}) = (\mathbf{f}(x_1,\ldots,x_{2k-1}),0)$ from D^{2k} to itself without fixed point. ■

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ON THE EVOLUTION OF NONCOMMUTATIVE HARMONIC ANALYSIS

KENNETH I. GROSS

Dedicated to my dear friend Ernst Snapper, with admiration and respect

Introduction. This paper is an outgrowth of my preparation for an invited address delivered on January 31, 1977, to the Mathematical Association of America at its annual meeting in St. Louis, Missouri. It is a short, elementary exposition of the main themes that lead to the current frontier in noncommutative harmonic analysis. As such, it represents a written elaboration of what I felt could reasonably be presented in one hour to a general mathematical audience.

Quite briefly, noncommutative harmonic analysis is the meeting ground of group theory, analysis, and geometry. It is an area of mathematics that has been flourishing for fifty years. For those who practice the art, its hybrid nature is a source of uncommon beauty, depth, and difficulty. Most significantly, noncommutative harmonic analysis has been inspirational for new points of view in a wide range of applications that includes not only group theory, analysis, and geometry, but number theory, probability, ergodic theory, and modern physics.

My goals here are to give the general reader a feeling for the historical flow of ideas and to display some of the more readily accessible connections with classical mathematics and nonclassical physics. Although one principally writes for the reader, this has been a marvelous personal experience. For it is not the typical task of a research mathematician to attempt a nontechnical commentary on a subject which seems foreordained by nature as intrinsically technical.

Finally, it is not possible to enumerate by name all the people who have contributed to my point of view. However, special thanks are extended to my colleague Jonathan Brezin for stimulating and enjoyable conversations that helped organize my thoughts at crucial stages of the exposition.

Occasionally, a comment came to mind that seemed worthy of mention, but was either parenthetical to the main theme or else a slightly more technical supplement to the main theme. To help preserve the continuity of discourse, such remarks are set aside in smaller print.

1. Classical harmonic analysis: Fourier series. By reason of its importance in almost all aspects of harmonic analysis, as well as its primary historical position, we begin with an account of Fourier series. The fundamental ideas originate in differential equations, in particular with solutions of the classical partial differential equations of mathematical physics. Illustrative is the *one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \tag{1.1}$$

first derived by d'Alembert in 1747 as the law governing the motion of a vibrating string. In this equation, $u = u(x,t)$ represents the displacement from equilibrium at time t of the point x on the string, and the constant a^2 is determined from the physical characteristics of tension T and mass density ρ, according to the formula $a^2 = T/\rho$. The general solution of (1.1), again due to d'Alembert at that early date, is given by the expression

$$u(x,t) = f_1(x+at) + f_2(x-at) \tag{1.2}$$

where f_1 and f_2 are arbitrary twice differentiable functions of a real variable. This solution can be viewed as the superposition of two "waves" f_1 and f_2 propagating with speed a in opposite directions along the string.

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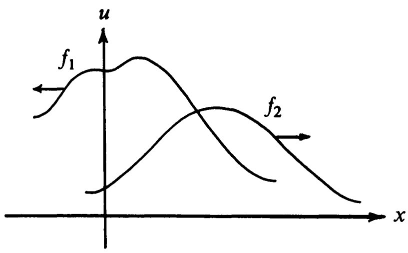

FIG. 1.

If, for example, one imposes upon a solution to (1.1) the *boundary conditions*

$$u(0,t)=u(L,t)=0 \tag{1.3}$$

for all t (with L fixed), as well as the *initial conditions*

$$u(x,0)=\phi(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0)=\psi(x)$$

for all x (with ϕ and ψ fixed), it is not difficult to see that the solution (1.2) then takes the form

$$u(x,t)=f(at+x)-f(at-x) \tag{1.4}$$

where the function f is periodic with period $2L$ and is uniquely determined, up to an arbitrary constant, as

$$f(x)=\frac{1}{2}\left(\phi(x)+\frac{1}{a}\int_0^x \psi(s)\,ds\right). \tag{1.5}$$

Of course, the initial data ϕ and ψ should be periodic with period $2L$. Physically, (1.4) and (1.5) describe the motion of a string of length L that is clamped at the end points and which at time $t=0$ is given initial displacement $\phi(x)$ and initial velocity $\psi(x)$.

The notion of *harmonic analysis* originates with the possibility of superimposing certain of the solutions (1.4); namely, the nth fundamental modes, or nth *harmonics*,

$$\sin(n\pi x/L)\cos(n\pi at/L) \quad \text{and} \quad \sin(n\pi x/L)\sin(n\pi at/L)$$

which correspond in (1.5) to $f(x)$ of the form $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$, respectively. As expressed most colorfully by Daniel Bernoulli in 1753, "all sonorous bodies contain potentially an infinity of corresponding ways of making their regular vibrations." Rephrased more mathematically, $f(x)$ should be of the form

$$f(x)=a_0+\sum_{n=1}^{\infty}\{a_n\cos(n\pi x/L)+b_n\sin(n\pi x/L)\}. \tag{1.6}$$

The study of such series representations and related generalizations is referred to as *Fourier analysis*. It would appear that this reference to Fourier is well deserved, for not only does Fourier's 1807 investigation of heat flow give the first broad examination of the series (1.6), including the determination of the coefficients in terms of f, but it marks the beginning of modern real analysis. In particular, many of our most decisive mathematical concepts—for example, the *modern notion of function, contemporary set theory*, both the *Riemann* and *Lebesgue integrals*, and, in more recent years, *distribution theory*—were generated by problems in Fourier analysis.

A modern treatment of Fourier series would recast (1.6) in complex notation (with $L=\pi$) as

$$f(\theta)=\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \tag{1.7}$$

where

$$c_n=\hat{f}(n)=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\,d\theta. \tag{1.8}$$

Here, $0 \leqslant \theta \leqslant 2\pi$ parametrizes the unit circle $\mathbf{T}$ of complex numbers $e^{i\theta}$ of absolute value 1. The right side of (1.7) is called the *Fourier series* of f, and $\{\hat{f}(n)\}$ is the sequence of *Fourier coefficients* of f.

There are many techniques or, in the technical jargon, methods of summability, for obtaining convergence in (1.7). The most common and elementary procedure involves the introduction of the inner product

$$\langle f_1 | f_2 \rangle = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\,\overline{f_2(\theta)}\,d\theta, \tag{1.9}$$

and applies to functions f on $\mathbf{T}$ which are square-integrable; that is, the norm

$$\|f\| = \langle f | f \rangle^{1/2} = \left\{ \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta \right\}^{1/2} \tag{1.10}$$

is finite. Thus, the functions e_n on $\mathbf{T}$ given by $e_n(\theta) = e^{in\theta}$ form a complete orthonormal system; (1.7) is quite simply the expansion of f in terms of this orthonormal basis (i.e., $\hat{f}(n) = \langle f | e_n \rangle$ for all n); and equality in (1.7) is interpreted in the mean-square sense as

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} c_n e_n\| = 0.$$

In other words,

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}\,(n)|^2. \tag{1.11}$$

Equation (1.11) is an example of a so-called *Parseval* or *Plancherel Formula*, the extension of which to more broad, general contexts will be a central theme of the sequel.

The previous paragraph describes what is known as the L^2-theory of Fourier series. In contrast to convergence in the mean-square sense, the general problem of *pointwise* convergence of Fourier Series is extraordinarily difficult. Thus, by an argument of Dirichlet (1837), it is not too hard to show that the Fourier series of a continuously differentiable function on the circle is absolutely convergent. Yet, it was not proved until 1966 that the Fourier series of a continuous function (more generally, square-integrable function) converges almost everywhere. The proof of this fact, due to L. Carleson, is very complicated (cf. **[38]**). A general heuristic principle can be stated: The smoother the function, the more rapidly its Fourier series converges, and vice-versa.

We shall give two interpretations of the Fourier series (1.7). The first illustrates the fundamental and intrinsic connection between Fourier series and differential equations on the circle. Let D denote differentiation with respect to θ. It is appropriate (although it may seem somewhat premature in this one-dimensional context) to call the operator $\Delta = D^2$ the *Laplacian* on $\mathbf{T}$. Upon the observation that $\Delta e_n = -n^2 e_n$ for all n, the Fourier series (1.7) takes on an entirely new appearance. For we see that *$-\Delta$ is a positive (and self-adjoint) differential operator on* $\mathbf{T}$, *and* (1.7) *is the expansion of f in eigenfunctions of the Laplacian.*

Moreover, from two integrations by parts,

$$(\Delta y)\hat{}\,(n) = -n^2\hat{y}(n) \tag{1.12}$$

for any twice differentiable function y on $\mathbf{T}$ and for all n. Now consider the differential equation on $\mathbf{T}$

$$\Delta y = -f + \alpha, \tag{1.13}$$

where f is any fixed continuous (more generally, square-integrable) function on $\mathbf{T}$ and α is the constant

$$\alpha = \hat{f}\,(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta. \tag{1.14}$$

By (1.12), the solution y is uniquely determined up to an additive constant by the equations

$$\hat{y}\,(n) = \frac{1}{n^2}\hat{f}\,(n), \qquad n \neq 0, \tag{1.15}$$

for its Fourier coefficients.

The constant α appears in (1.13) for the following technical reason. By (1.12) with $n=0$, the left side of (1.13) is a function with mean value zero over $\mathbf{T}$. Thus, (1.14) gives precisely the value of α for which the right side of (1.13) also has mean zero. Put another way, by integration of (1.13) one sees that this is the value of α such that $dy/d\theta$ is continuous on $\mathbf{T}$.

Of course, one really wants the function y, not its Fourier coefficients. Posed in more generality, given functions f_1 and f_2, we want that function, denoted $f_1 * f_2$, for which the Fourier coefficients are

$$(f_1 * f_2)^{\wedge}(n) = \hat{f}_1(n)\hat{f}_2(n). \tag{1.16}$$

By means of formula (1.8) for $\hat{f}_1(n)$ and $\hat{f}_2(n)$, it is easy to verify that the desired function is given by the *convolution*

$$(f_1 * f_2)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f_1(\phi) f_2(\theta-\phi)\,d\phi \tag{1.17}$$

of f_1 and f_2.

From the algebraic point of view, a vector space V of functions on $\mathbf{T}$, which is closed under convolution, has the structure of a commutative ring relative to convolution as multiplication. As examples of such spaces which arise in a variety of both abstract and concrete problems in analysis, we list the spaces $\mathcal{C}(\mathbf{T})$ of continuous functions, $\mathcal{C}^1(\mathbf{T})$ of continuously differentiable functions, $\mathcal{S}(\mathbf{T})$ of infinitely differentiable functions, $L^1(\mathbf{T})$ of Lebesgue integrable functions, and $L^2(\mathbf{T})$ of Lebesgue square-integrable functions, all of which are closed under convolution.

Thus, let g be that function on $\mathbf{T}$ such that

$$\hat{g}(n) = \begin{cases} 0 & \text{if } n=0 \\ 1/n^2 & \text{if } n \neq 0. \end{cases} \tag{1.18}$$

Then the complete solution of the differential equation (1.13) (*nonhomogeneous Laplace's equation* on $\mathbf{T}$, if you like) is given by

$$y = g * f + c \tag{1.19}$$

where c is an arbitrary constant. An elementary calculation shows that g is the function defined by $g(\theta) = (1/2)\theta^2 - \pi\theta + (1/3)\pi^2$ for $0 \leqslant \theta \leqslant 2\pi$.

The second interpretation of Fourier series makes explicit its group theoretic nature. In the usual way—that is, by means of addition modulo 2π in the parameter θ—the circle $\mathbf{T}$ is an abelian group. Moreover, the functions e_n are group homomorphisms, and these are all the continuous homomorphisms of the circle into the multiplicative group of nonzero complex numbers. Under pointwise multiplication as the law of composition, these homomorphisms themselves form a group, isomorphic to the additive group $\mathbf{Z}$ of integers, which is called the *dual group* of $\mathbf{T}$. From this perspective, *the Fourier series* (1.7) *is the expansion of f in terms of the dual group.*

Although there will be many variations, the theme throughout the sequel is as follows: *Try to analyze, or decompose, spaces of functions on a group, or a set on which a group acts, in terms of the most "elementary" functions which mirror the group operation*; that is to say, *the fundamental harmonics*. As we shall see, these functions are constructed from suitable homomorphisms, called *irreducible representations*, of the group in question.

2. Algebraic harmonic analysis: representations of finite groups. The concept of an abstract group, progenitive of the modern axiomatic point of view in algebra, developed in the latter third of the past century out of concrete manifestations—such as permutation groups, Galois groups, and Lie groups—that had arisen in the theory of equations, number theory, and geometry. In particular, the study of finite groups (in the modern sense) was already well under way, when in the 1890's Frobenius invented the notion of a representation of an abstract group by matrices, thereby introducing into the subject the techniques of linear algebra.

In modern terms, a *representation* T of a finite group G is a function which associates to each element x of G a linear transformation $T(x)$ on a nonzero finite-dimensional complex vector space $V=V_T$, such that two properties hold: (1) $T(xy)=T(x)T(y)$ for all $x,y\in G$; and (2) $T(e)=I$, where e is the identity in G and I is the identity transformation on V. Clearly, $T(xx^{-1})=T(x)T(x^{-1})=I$, or $T(x)^{-1}=T(x^{-1})$ for all $x\in G$; so a representation of G is simply a homomorphism of G into the general linear group $GL(V)$ of invertible linear transformations on V.

More in line with the original development by Frobenius (more accurately, the simplified treatment by Schur), it is often useful to think concretely of a representation as having matrix values. For if we fix a basis $e_1, e_2, \ldots, e_d$ of V where $d=\dim V$ then the equations $T(x)e_j=\sum_{i=1}^{d} t_{ij}(x)e_i$ define the matrix $t(x)=t_{ij}(x)$ of $T(x)$, and the mapping $x\to t(x)$ is the matrix-valued realization of T relative to the indicated basis. In short, the group $GL(V)$ is isomorphic to the *general linear group* $GL(d,\mathbf{C})$ of nonsingular $d\times d$ complex matrices. The number $d=d_T$ is called the *degree* of T, and the d^2 functions t_{ij} on G are the *matrix entries of* T *in the given basis.*

Within a decade, most notably through the work of Frobenius, Schur, and Burnside, the major framework was all in place, and representation theory had already been shown to be among the most powerful devices for penetrating the structure of a finite group.

More than amusing, it is in many ways instructive to contrast the change over a short period of time in Burnside's perception of the power of representation theory. The first edition of his famous book [5], written in 1897, completely omits any discussion of representation theory, for, in Burnside's words, "it would be difficult to find a result [in the pure theory of abstract groups] that could be most directly obtained by the consideration of groups of linear transformations." However, the 1911 second edition relies heavily upon the methods of representation theory; and in disavowing his previous opinion, Burnside comments that "it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions [i.e., transformations]." Indeed, seven years earlier, Burnside had successfully employed representation theory to obtain his celebrated result that a group of order $p^\alpha q^\beta$ (p,q prime) is solvable. Thus, one may safely conclude that, whatever the gastronomical discomfort, rich mathematical rewards accompanied the eating of his words.

Here, our look into the representation theory of a finite group will be necessarily brief, for limited space and the topic at hand preclude both a detailed treatment and a level of generality appropriate from the standpoint of algebra. What we are looking for is a glimpse of noncommutative harmonic analysis in its algebraic essence, shorn of complicating analytic considerations. For, as we will indicate in the next section, the role played by representation theory of a finite group is far more substantive in the genesis of modern harmonic analysis than in that of Fourier series, the group theoretic aspect of which is a relatively recent observation.

To be more precise concerning the degree of generality in which the results of this section are valid, we could replace the complex field of scalars by any field the characteristic of which is relatively prime to the order of the finite group in question. This yields the classical "ordinary theory" of representations of a finite group. There is a "modular theory," due largely to R. Brauer, in which there is no such restriction on the characteristic of the field. However, this theory is much too technical to be included here.

As a highly illustrative example, consider the symmetric group $\mathcal{G}$ on three objects, defined to be the group of all permutations of the set $S=\{1,2,3\}$. For $g\in\mathcal{G}$ and $s\in S$, let sg denote the image of s under the permutation g. To proceed to a richer structure, form the three-dimensional vector space $C(S)$ of all complex-valued functions on S. Then the set-theoretic action of $\mathcal{G}$ on S lifts to a linear action R of $\mathcal{G}$ on $C(S)$, called the *standard representation of* $\mathcal{G}$ *on* $C(S)$. Specifically, to each $g\in\mathcal{G}$ is associated the linear transformation of $C(S)$ given by

$$(R(g)f)(s)=f(sg). \tag{2.1}$$

Clearly, R is a representation of $\mathcal{G}$. The fundamental problem in harmonic analysis on S can now be formulated: *Decompose $C(S)$, or equivalently R, in terms of those functions f on S whose translates $R(g)f$ under $\mathcal{G}$ span as small a subspace of $C(S)$ as possible.*

Some notation is needed. Given a representation T of a finite group G on a space $V=V_T$, we say that a subspace W of V is *invariant* under T if $T(g)W\subset W$ for all $g\in G$. For a non-zero invariant subspace W, restriction of the transformations $T(g)$ to W defines a representation T_W of G on the space W, called the *subrepresentation* of T on W. A non-zero invariant subspace is said to be *irreducible* if it has no proper invariant subspace. If the representation space V_T itself is irreducible, then T is termed *irreducible*. Evidently, either a representation is irreducible, or else it has an irreducible subrepresentation.

Returning to the example, we shall say that a function f on S, not identically zero, is a *fundamental harmonic*—or more simply, an *S-harmonic*—if its translates $R(g)f$ under $\mathcal{G}$ span an irreducible subspace of $C(S)$. With the help of the natural inner product

$$\langle f_1|f_2\rangle=\sum_{s=1}^{3} f_1(s)\overline{f_2(s)} \tag{2.2}$$

on $C(S)$, it is a simple matter to find the spaces of S-harmonics. For it is clear that the constant functions form a one-dimensional space, say W_1, of S-harmonics. On the other hand, its orthogonal complement W_2, consisting of functions of mean zero (i.e., $\langle f|1\rangle=f(1)+f(2)+f(3)=0$), is a two-dimensional invariant subspace, and it is not difficult to see that W_2 is irreducible. Thus,

$$C(S)=W_1\oplus W_2 \tag{2.3}$$

is the decomposition of $C(S)$ *into spaces* W_1 *and* W_2 *of S-harmonics, and for all* $g\in\mathcal{G}$ *the transformation* $R(g)$ *decomposes as the direct sum*

$$R(g)=1\oplus\lambda(g) \tag{2.4}$$

where 1 *denotes the* 1*-dimensional subrepresentation* ($1(g)=1$ *for all* g) *on* W_1, *and* λ *is the two-dimensional subrepresentation on* W_2.

The standard representation R, the two-dimensional representation λ, and the decomposition (2.4) all have geometric interpretations. To see this, let e_1,e_2,e_3 be the usual orthonormal basis for 3-dimensional space. Viewed as functions in $C(S)$, $e_j(k)=1$ when $j=k$ and $e_j(k)=0$ otherwise ($j,k=1,2,3$). Relative to this basis

$$R(\tau)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad\text{and}\quad R(\sigma)=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

where τ is the transposition (12) and σ is the cyclic permutation (123). To obtain the matrix version of (2.4) change to a new orthonormal basis; namely, the basis E_1,E_2,E_3 where $E_1=(e_1-e_2)/\sqrt{2}$, $E_2=(e_1+e_2-2e_3)/\sqrt{6}$, and $E_3=(e_1+e_2+e_3)/\sqrt{3}$. Then E_3 spans the 1-dimensional subspace of $C(S)$ of constant functions, and E_2, E_3 span a 2-dimensional invariant subspace. Hence, with respect to this new basis of S-harmonics,

$$R(g)=\left[\begin{array}{cc:c}\multicolumn{2}{c:}{\multirow{2}{*}{$\lambda\ (g)$}}&0\\&&0\\\hdashline 0&0&1\end{array}\right]\quad\text{for all } g \text{ in } \mathcal{G}.$$

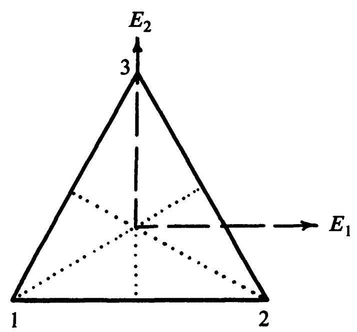

FIG. 2.

Moreover, λ is the familiar representation of $\mathcal{G}$ as symmetries of an equilateral triangle. For example, the matrices

$$\lambda(\tau)=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\quad\text{and}\quad\lambda(\sigma)=\frac{1}{2}\begin{pmatrix}-1&\sqrt{3}\\-\sqrt{3}&-1\end{pmatrix}$$

correspond geometrically to reflection about the direction E_2 and rotation through the angle $2\pi/3$, respectively.

As was indicated in its construction, R can be thought of as the linear version of the definition of $\mathcal{G}$. However, not all irreducible representations appear in R, and there is a larger representation $\mathfrak{R}$ of $\mathcal{G}$ which more fully captures the structure of $\mathcal{G}$. Thus, let $C(\mathcal{G})$ be the six-dimensional space of all

complex-valued functions F on $\mathcal{G}$, and for $g\in\mathcal{G}$ define

$$(\mathfrak{R}(g)F)(g_1)=F(g_1g) \tag{2.5}$$

for $F\in C(\mathcal{G})$. The representation $\mathfrak{R}$ is called the *right regular representation* of $\mathcal{G}$ on $C(\mathcal{G})$. Of course, it would be equivalent to work with the *left* regular representation $\mathfrak{L}$ defined on $C(\mathcal{G})$ by

$$(\mathfrak{L}(g)F)(g_1)=F(g^{-1}g_1). \tag{2.6}$$

We remark that the term "equivalent" has a technical meaning in the subject of group representations that is quite analogous to the notion of "similarity" in linear algebra. Two representations T_1 and T_2 of a finite group G are said to be *equivalent*, written $T_1\cong T_2$, if there exists an invertible linear transformation $A: V_{T_1}\to V_{T_2}$ such that $AT_1(g)A^{-1}=T_2(g)$ for all $g\in G$. Alternatively, $T_1\cong T_2$ if and only if there are bases of V_{T_1} and V_{T_2} relative to which the matrix-valued realizations of T_1 and T_2 are identical. For all intents and purposes, the abstract theory of group representations does not distinguish equivalent representations, and the basic concepts (e.g., invariant subspace, irreducibility, direct sum, etc.) apply to the equivalence classes as well as the representations themselves.

The harmonic analysis of $C(\mathcal{G})$ is easily accomplished with the help of the inner product

$$\langle F_1|F_2\rangle=\sum_{g\in\mathcal{G}}F_1(g)\overline{F_2(g)} \tag{2.7}$$

on $C(\mathcal{G})$, our knowledge of $C(S)$, and the following elementary but extremely powerful result known as SCHUR'S LEMMA. *Suppose T_1 and T_2 are irreducible representations of a group G and that A is a linear transformation from V_{T_1} to V_{T_2} such that $AT_1(g)=T_2(g)A$ for all $g\in G$. If T_1 and T_2 are inequivalent, then $A=0$. If $T_1\cong T_2$, then either $A=0$ or A is invertible and unique up to a constant factor. In particular, if $T_1=T_2$, then $A=cI$ for some complex number c.* It follows from Schur's Lemma that the matrix entries of inequivalent representations are orthogonal relative to (2.7).

The proof of Schur's Lemma follows immediately from the fact that the null space and range of A are invariant under T_1 and T_2, respectively.

To begin, note that the signature function, defined by $\operatorname{sgn}(g)=\pm1$ according to g even or odd, and the constant function 1 are both homomorphisms of $\mathcal{G}$ into the group $\{\pm1\}$. Each spans a one-dimensional invariant subspace of $C(\mathcal{G})$. It remains to find four other orthogonal functions.

Thus, bring λ into the picture in terms of the subspace $C_\lambda(\mathcal{G})$ of $C(\mathcal{G})$ spanned by the matrix entries $\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}$ relative to a choice of basis in W_2 (e.g., E_1 and E_2 in Fig. 2). It is not difficult to show (either by brute verification or a general argument based upon Schur's Lemma) that $\lambda_{11},\lambda_{12},\lambda_{21}$, and λ_{22} are mutually orthogonal, and $C_\lambda(\mathcal{G})$ is a four-dimensional invariant subspace. Moreover, although $C_\lambda(\mathcal{G})$ is not irreducible, the next best situation occurs. Namely, the subrepresentation of $\mathfrak{R}$ on $C_\lambda(\mathcal{G})$ is equivalent to $\lambda\oplus\lambda$. For if $C_\lambda^{(1)}(\mathcal{G})$ and $C_\lambda^{(2)}(\mathcal{G})$ are the two-dimensional subspaces of $C_\lambda(\mathcal{G})$ having bases $\{\lambda_{11},\lambda_{12}\}$ and $\{\lambda_{21},\lambda_{22}\}$, respectively, then these subspaces are invariant under $\mathfrak{R}$, $C_\lambda(\mathcal{G})=C_\lambda^{(1)}(\mathcal{G})\oplus C_\lambda^{(2)}(\mathcal{G})$, and by a straightforward calculation in the indicated bases, both subrepresentations on $C_\lambda^{(1)}(\mathcal{G})$ and $C_\lambda^{(2)}(\mathcal{G})$ are equivalent to λ.

Finally, we observe (again, either directly or from Schur's Lemma) that both 1 and sgn are orthogonal to $C_\lambda(\mathcal{G})$, and by a dimension count

$$C(\mathcal{G})=\mathbf{C}1\oplus\mathbf{C}\operatorname{sgn}\oplus C_\lambda(\mathcal{G}) \tag{2.8}$$

and

$$\mathfrak{R}\cong 1\oplus\operatorname{sgn}\oplus 2\lambda \tag{2.9}$$

where we have set $\lambda\oplus\lambda=2\lambda$.

In summary, *to within equivalence* 1, sgn, *and* λ *are the only irreducible representations of* $\mathcal{G}$, *and each appears in the right* (*or left*) *regular representation with multiplicity equal to its degree. In terms of* $C(\mathcal{G})$, *the matrix entries* 1, sgn, $\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}$ *of the irreducible representations are an orthogonal basis for* $C(\mathcal{G})$.

We have here the fundamental harmonic analysis—or, if you like, Fourier analysis—for the symmetric group $\mathcal{S}$. These results are specific examples from a general structure theory valid for any finite group G. The most prominent features are listed below.

1. *Unitarizability*. Given a representation T of G, there exists an inner product $\langle\cdot|\cdot\rangle$ on $V=V_T$ which is invariant under T; i.e., $\langle T(g)v|T(g)w\rangle=\langle v|w\rangle$ for all $g\in G$ and $v,w\in V$. To wit, if $(\cdot|\cdot)$ is any inner product on V, then the desired inner product is given by $\langle v|w\rangle=\Sigma_{g\in G}\,(T(g)v|T(g)w)$. In short, T can be assumed *unitary*; i.e., $T(g)$ is a unitary operator for each $g\in G$. In particular, $T(g)^*=T(g^{-1})$ for all $g\in G$. It follows that the orthogonal complement $W^\perp$ of an invariant subspace W is also invariant; and hence we have:

2. *Complete reducibility*. A representation T decomposes as an (orthogonal) direct sum of irreducible representations. Moreover, the irreducible representations that appear are unique to within equivalence.

3. *Uniqueness of decomposition*. Denote by $\hat{G}$ the collection of all equivalence classes $[\lambda]$ of irreducible representations λ of G. One calls $\hat{G}$ the *dual* or *dual object* of G. A class $[\lambda]\in\hat{G}$ is said to *occur in T* if T has a subrepresentation equivalent to λ, and the *multiplicity* n_λ of $[\lambda]$ in T is the number of times the class $[\lambda]$ appears in a complete reduction of T. Let $\lambda_1,\lambda_2,\ldots,\lambda_k$ be representatives for the distinct classes in $\hat{G}$ that appear in T. Then for each $j=1,\ldots,k$ there is a *unique* invariant subspace V_{λ_j} of $V=V_T$, on which T is equivalent to n_{λ_j} copies of λ_j; the spaces V_{λ_j} are mutually orthogonal; and

$$V=V_{\lambda_1}\oplus\cdots\oplus V_{\lambda_k}. \tag{2.10}$$

Equation (2.10) gives the *primary decomposition* of V, and the representation-theoretic version

$$T\cong n_1\lambda_1\oplus\cdots\oplus n_k\lambda_k \tag{2.11}$$

is the *primary decomposition* of T. The space V_{λ_j} is called the $[\lambda_j]$-*primary*, or $[\lambda_j]$-*isotypic, subspace* of V; or alternatively, *the subspace of T that transforms according to* λ_j.

4. *Schur orthogonality relations*. Let $\mathbf{C}(G)$ be the vector space of all functions $F:G\to\mathbf{C}$. The dimension of $\mathbf{C}(G)$ is n, where n is the order of G. Equip $\mathbf{C}(G)$ with the normalized inner product

$$\langle F_1|F_2\rangle=n^{-1}\sum_{g\in G}F_1(g)\overline{F_2(g)}\,. \tag{2.12}$$

For each class $[\lambda]\in\hat{G}$, choose a representation λ which is unitary, and let λ_{ij} denote the matrix entries relative to an orthonormal basis for V_λ. Then the Schur orthogonality relations (which follow directly from Schur's Lemma) state that the functions $e_{ij}^{(\lambda)}(g)=\sqrt{d_\lambda}\,\lambda_{ij}(g)$ on G, for $[\lambda]\in\hat{G}$ and $1\leqslant i$, $j\leqslant d_\lambda=\deg\lambda$, are an orthonormal subset of $\mathbf{C}(G)$. That this orthonormal set is complete is a consequence of the:

5. *Decomposition of the regular representation*. Let $\mathfrak{R}$ be the right regular representation of G on the space $C(G)$. Then each class $[\lambda]\in\hat{G}$ occurs in $\mathfrak{R}$ with multiplicity $d_\lambda=\deg\lambda$, and the $[\lambda]$-primary subspace of $C(G)$ is the space $C_\lambda(G)$ spanned by the matrix entries λ_{ij}. In short

$$C(G)=\sum_{[\lambda]\in\hat{G}}\oplus\,C_\lambda(G) \tag{2.13}$$

and

$$\mathfrak{R}\cong\mathfrak{L}\cong\sum_{[\lambda]\in\hat{G}}\oplus\,d_\lambda\lambda. \tag{2.14}$$

Note that by a dimension count, the order of G is related to the degrees of the irreducible representations by the formula

$$n=\sum_{[\lambda]\in\hat{G}}d_\lambda^2. \tag{2.15}$$

6. *The group algebra.* For $F_1, F_2 \in C(G)$, define their *convolution* by

$$(F_1 * F_2)(g) = \sum_{g_1 \in G} F_1(g_1) F_2(g_1^{-1} g). \tag{2.16}$$

With convolution as multiplication, $C(G)$ has the structure of an associative algebra, called the *group algebra* of G. From this point of view, the subspaces $C_\lambda(G)$, for $[\lambda] \in \hat{G}$, are two-sided ideals, and (2.13) is the decomposition of the group algebra into its minimal two-sided ideals. In particular, the representation theory of G is a special case of the Wedderburn theory of semi-simple rings.

In a common variant of the group algebra, the function $F \in C(G)$ is written as a formal sum, which in our case has the nonstandard form $\Sigma_{g \in G} F(g) g^{-1}$. By an elementary calculation, it is easy to see that in this formalism convolution is the "natural" multiplication

$$\Big(\sum_{g_1 \in G} F_1(g_1) g_1^{-1}\Big)\Big(\sum_{g_2 \in G} F_2(g_2) g_2^{-1}\Big) = \sum_{g_1 \in G} \sum_{g_2 \in G} (F_1(g_1) F_2(g_2)) g_1^{-1} g_2^{-1}.$$

7. *Character theory.* The character of a representation T is the complex-valued function $\chi_T(g) = tr(T(g))$ for $g \in G$. The importance of the notion of character, fundamental to the original work by Frobenius and Schur as well as to the modern point of view, lies in the fact that two representations are equivalent if and only if they have the same character. In other words, the trace is a complete invariant for the equivalence relation.

Finally, there is a Fourier analytic overview to the representation theory of G. That is the theme of the next section.

3. Compact harmonic analysis: The Peter–Weyl Theorem. Modern harmonic analysis begins in the 1920's with the confluence of two great streams of mathematical thought which had previously been developing concurrently but independently.

First, there is the role of group theory, rooted in large part in geometry through the philosophy (Klein's Erlanger program) of studying a space through its group of motions. In particular, the theory of geometric transformation groups, known today as *Lie groups* in honor of the Norwegian mathematician Sophus Lie, who created the subject over a century ago, arose from certain kinds of partial differential equations in rough analogy to the origin of Galois groups from algebraic equations.

On the other hand, a variety of purely analytic theories were developing in the late nineteenth century from very different aspects, such as boundary value problems, of differential equations. For example, if in the vibrating string problem described in Section 1 we do not assume that the tension and density are uniform across the string, then the wave equation (1.1) is replaced by the equation

$$\frac{\partial}{\partial x}\left(T(x)\frac{\partial u}{\partial x}\right) = \rho(x)\frac{\partial^2 u}{\partial t^2}. \tag{3.1}$$

The principle of superposition of solutions applied to more general equations such as (3.1) leads to eigenfunction expansions more general than Fourier series, to Sturm-Liouville theory, to spectral theory on Hilbert space, and ultimately to modern abstract functional analysis.

The genius for bringing together these two seemingly unrelated themes belongs to Hermann Weyl, who should be regarded as the father of modern harmonic analysis. The date of birth is 1927, and the official birth certificate is the remarkable paper **[14]** by Peter and Weyl, in which the structure theory for the representations of a finite group is carried over, essentially without change, to the context of compact Lie groups. For it turns out that it is not the finiteness of the group on which the validity of the properties (1)–(7) hinges, but rather on the existence of an averaging procedure over the group. That is to say, what is required is an *invariant integral* which assigns *finite volume* to the group. The decisive analytic idea, originating with Peter and Weyl and fundamental to essentially all the succeeding developments in noncommutative harmonic analysis, is the use of an *infinite dimensional*

representation and its decomposition by means of spectral theory for bounded operators on Hilbert space.

F. Peter was a schoolteacher who worked under Weyl for a short time. Upon completing this joint paper he apparently decided that the research life was not for him, and returned to high school teaching. Needless to say, it was a very good high school.

A *topological group* is a group G on which is given a topology with respect to which the group multiplication $(x,y)\to xy$ and inversion $x\to x^{-1}$ are continuous mappings. Clearly, each translation (e.g., right translation $x\to xa$ by a) is a homeomorphism of G, so the topology of G is completely determined by local behavior at the identity e. Thus, one says that G is *locally compact* if there exists a compact neighborhood of e; and still more restrictedly, G is *locally Euclidean* if there exists a neighborhood of e which is homeomorphic to an open subset of some Euclidean space $\mathbf{R}^n$. The most important topological groups are the *Lie groups*. These are locally Euclidean groups in which the group operations are infinitely differentiable mappings. For a long time it was an open question—*Hilbert's fifth problem*—as to whether a locally Euclidean group is necessarily a Lie group. That is to say, can continuity be replaced by differentiability? The affirmative resolution was given by Gleason, Montgomery, and Zippin in the 1950's.

As the terminology suggests, a topological group G, whether or not it is a Lie group, is called *compact* if G itself is a compact set. It is to the class of compact groups that the results of this section apply.

The first prerequisite for harmonic analysis is the availability of an *invariant integral* on the group. When dealing with noncommutative groups, an author is faced with the decision—completely arbitrary, of course—as to the use of left versus right translation. For reason of habit, we choose the side of right, and for the most part leave the sinister unsaid.

Throughout, we shall be extremely casual in our approach to integration. The reader familiar with measure theory will know what is meant by an "integral." Those less experienced will find it adequate to simply think of an invariant integral as a generalization to G of the ordinary Riemann integral on the real line.

An integral on the topological group G is said to be *right invariant* if

$$\int_G f(xa)dx=\int_G f(x)dx \tag{3.2}$$

for all $a\in G$. For a Lie group, it is rather easy to construct a right invariant integral. For locally compact groups, the existence of a right invariant integral was proved by Haar in 1933. Then von Neumann immediately established the uniqueness, up to a positive constant factor, of such an integral and obtained a number of important consequences.

Fix a right invariant integral on the locally compact group G. By its uniqueness, for each $a\in G$ there is a positive constant $\delta(a)$ defined by the identity $\int f(ax)dx=\delta(a)\int f(x)dx$. Then $\delta: G\to\mathbf{R}^+$ is a continuous homomorphism, called the *modular function* of G, which relates left and right invariance as well as inversion. Specifically, the formula $\int f(x)\delta(x)dx$ defines the corresponding left invariant integral, and $\int f(x)\delta(x)dx=\int f(x^{-1})dx$.

The group G is called *unimodular* if $\delta(a)=1$ for all a. Obviously, G is unimodular if and only if there exists an integral which is simultaneously left and right invariant, in which case it is also invariant under inversion. Evidently a compact group is unimodular, for the multiplicative group $\mathbf{R}^+$ has no compact subgroups other than $\{1\}$.

We note that in the same paper von Neumann used the existence of an invariant integral, together with the Peter–Weyl theorem, to give an elementary and short proof of Hilbert's fifth problem for compact groups.

Finally, in his celebrated book **[15]** in 1938, Weil provided a converse. Namely, local compactness is implied by the existence of a right invariant integral. In honor of its discoverer, an invariant integral is called a *Haar integral*.

Now fix a compact group G. From the compactness, it follows that G has *finite volume*. Therefore, we can assume that the invariant integral is so normalized that the volume of G is one; i.e.,

$$\operatorname{vol}(G)=\int_G 1\,dx=1. \tag{3.3}$$

For example, if $G=\mathbf{T}$, the circle group, then

$$\int_G f(x)dx=(2\pi)^{-1}\int_0^{2\pi} f(\theta)d\theta;$$

and if G is a finite group of order n, then

$$\int_G f(x)dx=n^{-1}\sum_{g\in G} f(g).$$

Armed with the invariant integral, motivated by the Fourier analysis of Section 1, and aware of the representation-theoretic facts in Section 2, we now describe the Peter–Weyl theory.

In view of the topological nature of G, we hereafter require all representations to be *continuous*; that is to say, the matrix entries are to be continuous functions on G. Let the dual object $\hat{G}$ be defined as in Section 2, and for each class $[\lambda]\in\hat{G}$ fix once and for all a unitary matrix-valued representation λ belonging to that class. Thus, for each x in G, $\lambda(x)$ is a $d_\lambda\times d_\lambda$ matrix, where $d_\lambda=\deg\lambda$.

THE PETER–WEYL THEOREM. *The normalized matrix entries* $e_{ij}^{(\lambda)}(x)=\sqrt{d_\lambda}\,\lambda_{ij}(x)$ for $[\lambda]\in\hat{G}$ and $1\leqslant i,j\leqslant d_\lambda$, *form a complete orthonormal system relative to the inner product*

$$\langle f_1|f_2\rangle=\int_G f_1(x)\,\overline{f_2(x)}\,dx. \tag{3.4}$$

The structure theory described for finite groups in Section 2 can now be transferred more or less directly to the context of compact groups. That is to say, there are complete analogs of properties (1) through (7), in which summation over a finite group is supplanted by integration over the compact group.

There are a few distinct features that ought to be mentioned when G is compact but not finite. The first concerns the introduction of infinite-dimensional spaces.

A *Hilbert space* is defined to be a (complex) vector space with inner product, which is complete relative to the norm determined from the inner product. In the case at hand, the vector space $\mathcal{C}(G)$ of all *continuous* functions on G with inner product (3.4) is not complete. Its completion is the Hilbert space $L^2(G)$ of all (Lebesgue) square-integrable functions on G. Thus, to be technically correct in the transition from finite groups to compact groups, one should replace $\mathcal{C}(G)$ by $L^2(G)$. Of course, $L^2(G)$ is infinite dimensional (if G is not finite), and the regular representations $\mathfrak{R}$ and $\mathfrak{L}$ are *infinite dimensional unitary representations* of G on the space $V=L^2(G)$. This is the highly original step taken by Peter and Weyl. Namely, they replaced the counting argument (cf. formula (2.15)) for a finite group by a spectral theoretic analysis of the infinite-dimensional representation space $L^2(G)$. Thus, in analogy to (2.14),

$$L^2(G)=\sum_{[\lambda]\in\hat{G}}\oplus\ \mathcal{C}_\lambda(G)$$

where each irreducible representation λ of G is finite dimensional, but in general $\hat{G}$ is an infinite set. Then (2.14) is valid as it stands; that is, each irreducible representation appears in the regular representation with multiplicity equal to its degree.

For the reader who has some experience with spectral theory on Hilbert space, we mention the simple idea at the heart of the proof of the Peter–Weyl Theorem. For f_1 and f_2 in $L^2(G)$, their convolution is given by

$$(f_1*f_2)(x)=\int_G f_1(y)f_2(y^{-1}x)dy$$

(cf. (2.16)). The operator $\mathfrak{L}(f)$ on $L^2(G)$ defined by $\mathfrak{L}(f)h=f*h$ for $h\in L^2(G)$, can be formally written as $\mathfrak{L}(f)=\int f(y)\mathfrak{L}(y)dy$. This operator $\mathfrak{L}(f)$ is called *left convolution* by f. Now, $\mathfrak{R}(a)\mathfrak{L}(f)=\mathfrak{L}(f)\mathfrak{R}(a)$ for all $a\in G$, so the primary subspaces of the right regular representation coincide with the eigenspaces of $\mathfrak{L}(f)$.

The Peter–Weyl Theorem is then readily reduced to the fact that the operators $\mathcal{L}(f)$ are completely continuous.

The Peter–Weyl Theorem can be rephrased in terms of *Fourier analysis*. The Fourier series of a function f on G is defined as

$$f(x)=\sum_{[\lambda]\in\hat{G}} d_\lambda \sum_{i,j=1}^{d_\lambda} \hat{f}_{ij}(\lambda)\lambda_{ij}(x) \tag{3.5}$$

where the numbers

$$\hat{f}_{ij}(\lambda)=\langle f|\lambda_{ij}\rangle=\int_G f(x)\,\overline{\lambda_{ij}(x)}\,dx \tag{3.6}$$

are the *Fourier coefficients* of f. The series (3.5) applies to those functions f which are square-integrable in that the norm

$$\|f\|=\left\{\int_G |f(x)|^2 dx\right\}^{1/2} \tag{3.7}$$

is finite; and equality in (3.5) is in the mean-square sense of

$$\|f\|^2=\sum_{[\lambda]\in\hat{G}} d_\lambda \sum_{ij=1}^{d_\lambda} |\hat{f}_{ij}(\lambda)|^2. \tag{3.8}$$

Formula (3.8) is called the *Plancherel formula* for G.

Of course, when $G=\mathbf{T}$ is the circle group, the dual object is just the dual group of functions e_n, and formulas (3.5), (3.6), (3.7), and (3.8) reduce to the Fourier series formulas (1.7), (1.8), (1.10), and (1.11), respectively.

More generally, let G be any compact abelian group. Then by Schur's Lemma the irreducible representations are all one-dimensional, and the dual object $\hat{G}$ is the *dual group of* all continuous homomorphisms λ of G into $\mathbf{T}$. Then the Fourier coefficients $\hat{f}(\lambda)=\int f(x)\lambda(x)dx$ are numbers, and the Fourier series (3.5) is more simply given by $f(x)=\Sigma_{\lambda\in\hat{G}}\hat{f}(\lambda)\lambda(x)$.

Since the Plancherel formula is exceedingly important in harmonic analysis, we offer some further elaboration. Let $L^2(G)$ denote the vector space of square integrable functions on G. Motivated by formula (3.6) we define the $d_\lambda\times d_\lambda$ matrix

$$\hat{f}(\lambda)=\int_G f(x)\,\overline{\lambda(x)}\,dx, \tag{3.9}$$

the matrix entries of which are the Fourier coefficients. Then the Fourier series (3.5) takes the form

$$f(x)=\sum_{[\lambda]\in\hat{G}} d_\lambda tr(\hat{f}(\lambda)\lambda(x)^t) \tag{3.10}$$

and the Plancherel formula (3.8) is more simply written

$$\|f\|^2=\sum_{[\lambda]\in\hat{G}} \|\hat{f}(\lambda)\|^2_{d_\lambda}, \tag{3.11}$$

where the space $\mathbf{C}^{d_\lambda x d_\lambda}$ of all $d_\lambda\times d_\lambda$ matrices is given the inner product

$$\langle A|B\rangle_{d_\lambda}=d_\lambda tr(AB^*)=d_\lambda\sum_{ij=1}^{d_\lambda} a_{ij}\overline{b_{ij}},$$

and the corresponding norm $\|A\|^2_{d_\lambda}=d_\lambda tr(AA^*)$. Form the orthogonal direct sum

$$L^2(\hat{G})=\sum_{[\lambda]\in\hat{G}}\oplus\mathbf{C}^{d_\lambda\times d_\lambda}. \tag{3.12}$$

Then (3.11) implies the noncommutative version of the PLANCHEREL THEOREM: *The mapping* $\mathfrak{F}: f \to \hat{f}$ *is a unitary transformation, the Plancherel transform, of* $L^2(G)$ *onto* $L^2(\hat{G})$. In particular, the Plancherel Theorem shows that noncommutativity in harmonic analysis resides in the noncommutativity of matrix multiplication.

The importance of the Plancherel transform lies in the fact that it quite explicitly gives the decomposition of the right (as well as left) regular representation $\mathfrak{R}$. Indeed, the equivalent representation $\hat{\mathfrak{R}}$ on $L^2(\hat{G})$, defined by $\hat{\mathfrak{R}}(a) = \mathfrak{F}\mathfrak{R}(a)\mathfrak{F}^{-1}$, is easily seen to be described in terms of right multiplication by $\lambda(a)^t$. Specifically, $(\hat{\mathfrak{R}}(a)\hat{f})(\lambda) = \hat{f}(\lambda)\lambda(a)^t$.

Examples: As with all abstract mathematical theories, it is important to analyze the examples to which the abstraction applies and to enrich the theory by any additional structures which may be present. The Peter–Weyl Theory is a case in point, for it can only be fully appreciated when combined with the rich algebraic, geometric, and combinatoric processes that collectively comprise the Cartan–Weyl representation theory of the compact *Lie groups*. Now, there exists a complete structure theory for the compact Lie groups. Very roughly, a connected compact Lie group splits as a product, the individual factors of which are either abelian or "simple," and these are listed below.

1. A connected compact *abelian* Lie group is necessarily a *torus*, by which is meant a direct product of circles.
2. The *special orthogonal group* $SO(n)$ consists of all $n \times n$ real matrices g of determinant 1 such that $gg^t = 1$. In other words, $SO(n)$ is the connected component of the identity matrix in the group of all matrices that preserve the natural inner product on $\mathbf{R}^n$.
3. The *special unitary group* $SU(n)$ is the analog over the complex field of $SO(n)$. It consists of all complex $n \times n$ matrices of determinant 1 such that $gg^* = 1$.
4. The *compact symplectic group* $Sp(n)$ is the analog over the quaternionic division algebra $\mathbf{H}$ of $SO(n)$. Specifically, $Sp(n)$ consists of all $n \times n$ matrices over $\mathbf{H}$ which preserve the natural (real) inner product on $\mathbf{H}^n$.
5. The groups $SO(n)$, $SU(n)$, and $Sp(n)$ form the three families of so-called *classical* compact simple groups. They are *simple* in the sense that modulo a finite group (the center) there are no normal subgroups. Aside from these classical groups there are only five other compact simple groups, denoted E_6, E_7, E_8, F_4, and G_2. They are referred to as the *exceptional* compact simple groups, and have realizations in terms of certain nonassociative algebras such as the Cayley numbers.

It is beyond the scope of this elementary introduction to go into the explicit nature of the harmonic analysis of the compact Lie groups (although we touch upon it in the next section). Suffice it to say that the irreducible representations were constructed algebraically, or "infinitesimally," by Cartan, and the formulas for the characters were calculated by Weyl, who also constructed the representations "globally." Although all of this was known over fifty years ago, it is still of importance today to obtain new realizations of the representations of compact Lie groups that bear upon various aspects of analysis, geometry, arithmetic, and mathematical physics.

Compact groups which are not Lie groups arise by suitable "limits" from Lie groups. In particular, there are a variety of p-adic analogs of the compact simple groups, but the representation theory and harmonic analysis of such groups is at present highly incomplete.

4. Harmonic analysis and special functions: Spherical harmonics. The classical treatises on the special functions of mathematical physics—for example, the five-volume Bateman manuscript or Watson's encyclopedic *Treatise on Bessel Functions*—provide imposing monuments to the cleverness of our forebears. To the purist they are often impenetrable, and their use may represent a turn of events born of desperation. For there one finds what seems to be a tangled chaotic jungle of series and integral representations, recurrence relations, difference and differential equations, and assorted variations for a myriad of higher transcendental functions, if ruled by reason, clearly not by order.

On the other hand, among physicists, engineers, and those with an applied bent, there is a familiarity that can effect a great uplifting of the spirit. A vivid illustration is given by my erudite colleague and friend W. J. Holman, III, who upon breaking a problem in complex analysis on a symmetric space through esoteric methods in special functions exclaimed: "Is it not sublime! Meditate it until you can feel its exquisite diathrodial ginglimus in every joint of your body." An uncommon group theoretical allusion, *diathrodial ginglimus* refers to that property of the jaw, unique among all the joints of the body, that it can be both rotated and translated.

Nonetheless, from the standpoint of harmonic analysis, there is an orderly overview to a sizable part of the literature on special functions. This approach is founded upon the Peter–Weyl Theory, and originates in 1929 with E. Cartan's determination of the special functions associated with compact Riemannian symmetric spaces. We give two examples illustrative of the general theory. The first, the *harmonics of the circle*, is simply a rephrasing of the Fourier series considerations from Section 1, and is presented for motivation. The second is a group-theoretical description of *spherical harmonics*.

EXAMPLE 1. We view the circle in two different ways. First, it is the 1-dimensional unit sphere S^1 of points $x=(x_1,x_2)$ in the Euclidean plane $\mathbf{R}^2$ having unit norm. Second, it is the rotation group $SO(2)$ of matrices

$$g=g(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \tag{4.1}$$

acting on $\mathbf{R}^2$ by the formula

$$xg=(x_1\cos\theta-x_2\sin\theta, x_1\sin\theta+x_2\cos\theta). \tag{4.2}$$

Of course, the action of $SO(2)$ on $\mathbf{R}^2$ restricts to the subset S^1, for S^1 is the *orbit* under $SO(2)$ of the "north pole" $\mathbf{1}=(1,0)$. That is to say, S^1 consists precisely of the points $\mathbf{1}g=(\cos\theta,\sin\theta)$. Let R denote the regular representation

$$(R(g)f)(x)=f(xg) \tag{4.3}$$

of $SO(2)$ on the Hilbert space $L^2(S^1)$ of square-integrable functions on S^1. More familiarly, $(R(\theta)f)(\phi)=f(\phi+\theta)$ where $x=(\cos\phi,\sin\phi)\in S^1$ and $g=g(\theta)\in SO(2)$.

Now, the two-dimensional Laplacian $\nabla^2=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2$ can be rewritten in polar coordinates as

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{4.4}$$

from which one sees the justification in referring to the operator $\Delta=d^2/d\theta^2$ as the Laplacian on the circle. Namely, it is the "circular part" of ∇^2. By the results in Section 1, the Hilbert space $L^2(S^1)$ decomposes as the orthogonal direct sum

$$L^2(S^1)=\sum_{n=0}^{\infty}\oplus\ H_n \tag{4.5}$$

of the eigenspaces H_n of the Laplacian. Recall that H_0 is the 1-dimensional null-space of Δ and contains the constant functions; whereas for $n\geqslant 2$, H_n is the 2-dimensional eigenspace corresponding to the eigenvalue $-n^2$ of Δ and having the functions $\cos n\theta$ and $\sin n\theta$ as basis. *From this point of view, H_n could be called the space of circular harmonics of order n, and the variant*

$$f(\theta)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta) \tag{4.6}$$

of the Fourier series (1.7) *is the orthogonal expansion of f in the special functions associated to the action of $SO(2)$ on S^1.* The precise relationship with the representation theory of $SO(2)$ is expressed by the commutation relation

$$R(g)\Delta=\Delta R(g) \tag{4.7}$$

for $g\in SO(2)$.

EXAMPLE 2. The preceding example has a higher dimensional generalization to the $(k-1)$-dimensional unit sphere S^{k-1} in the Euclidean space $\mathbf{R}^k$. For simplicity, we restrict to the case $k=3$.

The 2-sphere S^2 is not a group. However, it is a *homogeneous space*. In fact, $S^2 \cong SO(3)/SO(2)$. More precisely,

$$S^2 = K\backslash G \tag{4.8}$$

that is, the space of right cosets Kg, where $G=SO(3)$ is the rotation group on $\mathbf{R}^3$ and K is the subgroup, isomorphic to $SO(2)$, of matrices

$$k=k(\theta)=\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

In finer detail, $G=SO(3)$ is the group of proper isometries of $\mathbf{R}^3$; i.e., the 3×3 matrix g is in G if and only if $\|xg\|=\|x\|$ for all $x\in\mathbf{R}^3$ (alternatively, $gg^t=1$) and $\det g=1$. Here, $\|x\|=(x_1^2+x_2^2+x_3^2)^{1/2}$ is the norm of the point $x=(x_1,x_2,x_3)\in\mathbf{R}^3$. Then the unit sphere $S^2=\{x\in\mathbf{R}^3: \|x\|=1\}$ is the *orbit* of the north pole $\mathbf{1}=(0,0,1)$ under G; i.e., $S^2=\{\mathbf{1}g: g\in G\}$. The subgroup of G that leaves the point $\mathbf{1}$ fixed (alternatively, the subgroup of rotations about the x_3-axis) is easily seen to be the above group K, and the mapping $x=\mathbf{1}g\to Kg$ identifies S^2 with the space $K\backslash G$ of right cosets Kg in such a way that the action by rotation of G on S^2 corresponds to the natural right action of G on $K\backslash G$.

Next, the 3-dimensional Laplacian $\nabla^2=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial x_3^2$ can be rewritten in spherical coordinates as

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left[\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right] \tag{4.10}$$

where $r=\|x\|$ and ϕ and θ are latitudinal and longitudinal coordinates on S^2, respectively. In particular, *the Laplacian on the sphere* is

$$\Delta=\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}. \tag{4.11}$$

As in the example of the circle, Δ is a negative-definite self-adjoint differential operator on S^2, and the Hilbert space $L^2(S^2)$ of square-integrable functions on the sphere (relative to the usual surface integral) decomposes as the orthogonal direct sum

$$L^2(S^2)=\sum_{n=0}^{\infty}\oplus\ H_n \tag{4.12}$$

of the eigenspaces H_n of Δ. Here, the eigenvalue of Δ for H_n is $-n(n+1)$, and $\dim H_n=2n+1$.

The decomposition (4.12) also admits a group-theoretic interpretation, for Δ commutes with the right regular representation R of $SO(3)$ on $L^2(S^2)$ in complete analogy to formulas (4.3) and (4.7). This dual relationship can be summed up as follows:

Formula (4.12) *gives both the spectral decomposition of the Laplacian on the sphere as well as the primary decomposition of the regular representation of $SO(3)$ on $L^2(S^2)$. In particular, H_n is both an eigenspace of Δ, and an irreducible invariant subspace of $L^2(S^2)$. The functions in H_n are called the spherical harmonics of degree n, and they are the special functions associated to the action of $SO(3)$ on S^2.*

Denote by $\lambda^{(n)}$ the subrepresentation of R on the space H_n. Then $\lambda^{(n)}$ is an irreducible representation of $SO(3)$ of degree $2n+1$. These representations are all inequivalent and, in fact, exhaust the dual object $SO(3)^\wedge$. Now, $SO(3)$ is homeomorphic to 3-dimensional real projective space, so the matrix entries $(2n+1)^{1/2}\lambda_{ij}^{(n)}$ form the Peter–Weyl basis for the Hilbert space of square-integrable functions on the projective space. For the general case $k>3$, the analogous representations $\lambda^{(n)}$ of $SO(k)$ (i.e., those that appear in $L^2(S^{k-1})$) are irreducible and inequivalent, but they comprise only a small part of the full dual $SO(k)^\wedge$.

The deeper aspects of the decomposition (4.12), and the connection with classical special functions, are revealed by a characteristic and most striking feature. Namely, in each space H_n of spherical harmonics there exists a particular function f_n, unique up to a constant factor, which is *invariant* under K; that is to say, $R(k)f_n = f_n$ for all $k \in K$. In other words, f_n is left fixed by all rotations about the x_3-axis; or equivalently, f_n is a function on the sphere which is constant on parallels of latitude. Classically, f_n is called the *zonal spherical function of degree n*, but in modern terminology it is known simply as a *spherical function*. The spherical function uniquely determines the entire space H_n, for the translates $R(g)f_n$ under $SO(3)$ span H_n.

We can now reveal the classical identity of the spherical functions. For from (4.11) and the fact that f_n is independent of θ (it depends only upon latitude ϕ), f_n satisfies the differential equation

$$\frac{d^2y}{d\phi^2} + (\cot\phi)\frac{dy}{d\phi} = -n(n+1)y \tag{4.13}$$

which upon the subsitution $t = \cos\phi$ becomes *Legendre's equation*

$$(1-t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + n(n+1)y = 0. \tag{4.14}$$

Thus,

$$f_n(\theta,\phi) = P_n(\cos\phi) \tag{4.15}$$

where P_n is the *Legendre polynomial* of degree n. It follows that the Legendre polynomials have a purely group theoretical formulation, so it should not be surprising that the wealth of classical analysis relating to these special functions is put into place by means of harmonic analysis.

To hint at the general theory, the previous example is a special case of a phenomenon that applies to certain Riemannian manifolds (compact symmetric spaces) S for which G is a transitive group of isometries, K is the subgroup leaving fixed a base point, and Δ is the Laplace–Beltrami operator on S. However, the connection with special functions persists even more generally; and as one varies the group, space, and action through the more familiar examples, one accounts for the theory of Bessel functions, certain other cases of the hypergeometric function, and Jacobi, Gegenbauer, Hermite, and Laguerre polynomials. (The spherical functions for the action of $SO(k)$ on $S^{k-1} \cong SO(k)/SO(k-1)$ are Gegenbauer polynomials.) For more complicated groups, the associated special functions need not be classical. For example, for the collection of (non-compact!) real semi-simple Lie groups, Harish-Chandra has calculated the spherical functions. To be sure, the Harish-Chandra spherical functions form only a small part of the special functions which are crucial to his profound and impressive theory of harmonic analysis.

5. Noncompact classical analysis: Fourier integrals. The foregoing exposition has dealt exclusively with harmonic analysis on *compact* domains. However, major emphasis during the past four decades has been placed upon the *noncompact* noncommutative theory, a subject which is fundamentally more difficult than its compact counterpart and which should be viewed realistically as occupying an extensive expanse of the current frontier in harmonic analysis.

It is to noncompact problems that we now turn, beginning with the classical commutative case. Even in this well-known and comparatively elementary context, the first of the two major difficulties in noncompact harmonic analysis arises. Namely, *the fundamental harmonics do not occur discretely and the function spaces do not decompose as direct sums. Rather, the harmonics form a continuum, and the natural function spaces decompose as continuous smearings, called in the technical jargon "direct integrals."*

Historically, the noncompact theory originates, as with the compact theory, in the partial differential equations of mathematical physics; in particular, with the heat equation for an object, such as a metal rod, of infinite extent. Once more, the crucial idea is due to Fourier, who, in 1811, replaced the series representation of a solution by an integral representation, and thereby initiated the study of *Fourier integrals*.

Formally, the *Fourier integral*, or *Fourier transform*, $\hat{f}$ of a function f on the real line is defined for all real numbers λ by

$$\hat{f}(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx, \tag{5.1}$$

and f is recaptured from $\hat{f}$ by the *Fourier inversion formula*

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda. \tag{5.2}$$

Clearly, these formulas bear strong analogy to formulas (1.8) and (1.7), respectively, for Fourier coefficients and Fourier series of a function on the circle.

The constant $(2\pi)^{-1/2}$ in (5.1) appears as a matter of convenience, in that one avoids the appearance of a constant in the important formula (5.3) below. Group theoretically, this constant can be viewed as a suitable normalization of the Haar integral on $\mathbf{R}$.

The L^1-theory of the Fourier integral treats the case in which f is integrable over the real line. Then $\hat{f}$ is well defined. Nonetheless, the integrability of f does not imply the integrability of $\hat{f}$ (even in the Lebesgue sense), and generalized "methods of summability" are required to give rigorous sense to the integral in (5.2).

More to our purposes is the L^2-theory. For if f is both integrable and square-integrable (i.e., $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$), then $\hat{f}$ is square-integrable. Moreover, in complete analogy to the case of Fourier series, equality in (5.2) holds in the mean-square sense and the *Plancherel formula*

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda = \int_{-\infty}^{\infty} |f(x)|^2 dx \tag{5.3}$$

is valid. The L^2-theory is summarized by the PLANCHEREL THEOREM FOR THE REAL LINE: *The mapping $f \to \hat{f}$, originally defined on $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, extends uniquely to a unitary operator $\mathfrak{F}$, called the Plancherel transform, from the Hilbert space $L^2(\mathbf{R})$ to itself.*

Let us view the L^2-theory group-theoretically. The real line $\mathbf{R}$ under addition is a locally compact abelian group, and the regular representation R of $\mathbf{R}$ on $L^2(\mathbf{R})$, defined by

$$(R(a)f)(x) = f(x+a), \tag{5.4}$$

is unitary. Furthermore, the functions $e_\lambda(x) = e^{i\lambda x}$ exhaust the *dual group* of all continuous homomorphisms of $\mathbf{R}$ into the circle group $\mathbf{T}$. To facilitate the "spectral analysis" of R, we consider the equivalent Plancherel-transformed representation $\hat{R}$, given by $\hat{R}(a) = \mathfrak{F}R(a)\mathfrak{F}^{-1}$. A straightforward calculation shows that for any $a \in \mathbf{R}$

$$(\hat{R}(a)\hat{f})(\lambda) = e^{i\lambda a}\hat{f}(\lambda) \tag{5.5}$$

for all $\hat{f} \in L^2(\mathbf{R})$. Notice, in contrast to the compact case, that λ is a *continuous* variable. Consequently, formula (5.5) represents a *continuous decomposition* of $\hat{R}$ into its irreducible constituents e_λ. In symbols, we write

$$\hat{R} = \int_{\mathbf{R}}^{\oplus} e_\lambda d\lambda \tag{5.6}$$

(in analogy to the notation for a direct sum) and say that $\hat{R}$ is the *direct integral* of the one-dimensional representations e_λ. In particular, each specific value of λ is measure-theoretically irrelevant, so e_λ is not a subrepresentation of $\hat{R}$ in the usual sense. At best, it is an "infinitesimal" component. As a consequence, *we are led to conclude that $\hat{R}$, or equivalently the regular representation of* $\mathbf{R}$, *has no irreducible subrepresentations. Alternatively, the Hilbert space $L^2(\mathbf{R})$ of square-integrable functions on* $\mathbf{R}$ *possesses no irreducible invariant subspaces.*

As one might expect from previous considerations, the above group theory can be replaced by spectral theory of the Laplacian $\Delta = d^2/dx^2$ on $\mathbf{R}$. For Δ is negative-definite (and self-adjoint) and commutes with the regular representation. In this context, e_λ is an eigenfunction (specifically,

$\Delta e_\lambda = -\lambda^2 e_\lambda$), but these eigenfunctions are not square-integrable. Thus, we see that the noncompactness of $\mathbf{R}$ is manifested in a continuous spectrum for Δ, and the direct integral decomposition

$$L^2(\mathbf{R}) = \int_{\mathbf{R}}^{\oplus} \mathbf{C} e_\lambda \, d\lambda, \tag{5.7}$$

underlies the spectral decomposition

$$\Delta = \int_{\mathbf{R}}^{\oplus} (-\lambda^2) \, d\lambda \tag{5.8}$$

as well as the representation-theoretic formulation (5.6).

The theory of Fourier series and integrals has a generalization, due to Weil [**15**], to an arbitrary locally compact abelian group G. The *dual group* $\hat{G}$ consists of the continuous homomorphisms of G into the circle group $\mathbf{T}$, and $\hat{G}$ is itself a locally compact abelian group relative to pointwise multiplication as the law of composition and the compact-open topology. Then the Fourier transform

$$\hat{f}(\lambda) = \int_G f(x) \overline{\lambda(x)} \, dx \tag{5.9}$$

is defined for all $\lambda \in \hat{G}$, and the *inversion formula* takes the form

$$f(x) = \int_{\hat{G}} \hat{f}(\lambda) \lambda(x) \, d\lambda \tag{5.10}$$

where dx and $d\lambda$ are suitably normalized Haar integrals on G and $\hat{G}$, respectively. The classical theory of Fourier analysis—exclusive of the connection with differentiation—has an extension to this general context. Of course, when $G = \mathbf{T}$, the mapping $e_n \to n$ identifies $\hat{G}$ with the group $\mathbf{Z}$; when $G = \mathbf{R}$, the mapping $e_\lambda \to \lambda$ identifies $\hat{G}$ with $\mathbf{R}$ (so $\mathbf{R}$ is *self-dual*); and the formulas (5.9) and (5.10) reduce to those for Fourier series and Fourier integrals, respectively.

Finally, we can at least mention the *second* major difficulty in noncompact harmonic analysis, a complication that does *not* arise in the commutative theory. As we have indicated, the irreducible representations of a locally compact *abelian* group are all one-dimensional. For *compact* groups we have seen that the irreducible representations are finite-dimensional, and if the group is non-abelian they need not be of dimension 1. However, *for a group which is locally compact, but neither compact nor abelian, one is forced to deal with irreducible representations which are infinite-dimensional.*

6. Quantum mechanics and infinite-dimensional harmonic analysis. In our discussion of classical harmonic analysis, we have hinted at the interaction with classical physics. The most striking rapport is obtained with noncommutative harmonic analysis and quantum physics. For within a few years of the appearance in 1925 of Heisenberg's tour de force on "matrix mechanics," with which the floodgates were opened for mathematical quantum theory, the operator-theoretic basis for quantum mechanics was well established. Moreover, it had already been shown that the quantum numbers were parameters for representations of appropriate symmetry groups, and Weyl's classic treatment [**13**] of the connection with group theory had been published.

For our purpose of introducing infinite-dimensional representation theory, we shall examine the intimate relationship between the HEISENBERG UNCERTAINTY PRINCIPLE

$$PQ - QP = -iI \tag{6.1}$$

and the noncompact noncommutative group G of all 3×3 real matrices of the form

$$g(r_1, r_2, r_3) = \begin{bmatrix} 1 & 0 & 0 \\ r_2 & 1 & 0 \\ r_3 & r_1 & 1 \end{bmatrix}. \tag{6.2}$$

By reason of this association, G is known as the HEISENBERG GROUP.

Formula (6.1) can be given a brief mathematical description as follows. The physical states of a quantum mechanical system are to be viewed as vectors in an infinite-dimensional Hilbert space H,

and the observables are Hermitian linear transformations, typically unbounded, on H. In particular, for a dynamical system with one degree of freedom, there is the position observable Q and the momentum observable P which satisfy the fundamental commutation relation (6.1). This relation is interpreted physically as an expression of the uncertainty (we have normalized Planck's constant $\hbar$ to 1) inherent in simultaneous measurement of position and momentum. Now, Heisenberg realized P and Q as infinite matrices, the entries of which represented transitions between energy states. Shortly thereafter, Schrödinger, working with wave mechanics, realized the state space H (in modern terminology) as $L^2(\mathbf{R})$ and the operators P and Q as $-id/dx$ and multiplication by x, respectively. These competing mathematical models for quantum theory generated the obvious question: What are all possible solutions to the Heisenberg relation (6.1)? To this question Weyl, Stone, and von Neumann (independently) gave a *group theoretical* answer: Up to multiplicity and equivalence, Schrödinger's solution is unique. The group theory enters as follows.

In the above-mentioned book, Weyl observes that (6.1) can be converted into a group theoretical form if one replaces P and Q by the one-parameter groups of unitary operators that they generate, namely, $U(\sigma)=e^{i\sigma P}$ and $V(\tau)=e^{i\tau Q}$, respectively. Then *U and V are unitary representations of the additive group* $\mathbf{R}$ *on the Hilbert space H such that*

$$U(\sigma)V(\tau)U(\sigma)^{-1}V(\tau)^{-1}=e^{i\sigma\tau}I \tag{6.3}$$

for all $\sigma,\tau\in\mathbf{R}$. Formula (6.3), which is the "global version" of the Heisenberg relation, is called the *Weyl equation*. If we use script letters for Schrödinger's solution, so $(\mathcal{P}f)(x)=-if'(x)$ and $(\mathcal{Q}f)(x)=xf(x)$ for sufficiently smooth functions in $L^2(\mathbf{R})$, the corresponding solution of (6.3) is given by

$$(\mathcal{U}(\sigma)f)(x)=f(x+\sigma) \quad \text{and} \quad (\mathcal{V}(\tau)f)(x)=e^{i\tau x}f(x). \tag{6.4}$$

The term *global* is used in contrast to the term *infinitesimal*, the latter being referenced to differentiation. Thus, P and Q are recaptured from U and V by differentiation; specifically, $P=U'(0)$ and $Q=V'(0)$.

The uniqueness theorem mentioned above can now be stated with precision.

Let U and V be unitary representations of $\mathbf{R}$ *on a Hilbert space H such that* (6.3) *holds. Then H is the orthogonal direct sum of subspaces, say H_α, and there is a unitary mapping A from H_α to* $L^2(\mathbf{R})$ *such that*

$$AU(\sigma)A^{-1}=\mathcal{U}(\sigma) \quad \text{and} \quad AV(\tau)A^{-1}=\mathcal{V}(\tau) \tag{6.5}$$

for all $\sigma,\tau\in\mathbf{R}$, where $\mathcal{U}$ and $\mathcal{V}$ are the specific representations given by (6.4).

This result is sometimes called the Uniqueness Theorem for the Heisenberg Relations, but more often is referred to as the Stone–von Neumann Theorem. In short, the Schrödinger formulas (6.4) give the only representations of $\mathbf{R}$ that satisfy the Weyl equation, up to multiplicity and (unitary) equivalence.

We turn to the Heisenberg group G and its relationship to the previous theory. Notice from (6.2) that there are three *one-parameter subgroups* of G consisting of the matrices $u(r_1)=g(r_1,0,0)$, $v(r_2)=g(0,r_2,0)$, and $w(r_3)=g(0,0,r_3)$, respectively. Moreover, the former two subgroups generate all of G, for

$$g=w(r_3)v(r_2)u(r_1) \tag{6.6}$$

for all $g=g(r_1,r_2,r_3)\in G$, and

$$u(\sigma)v(\tau)u(\sigma)^{-1}v(\tau)^{-1}=w(-\sigma\tau) \tag{6.7}$$

for all $\sigma,\tau\in\mathbf{R}$.

A *one-parameter subgroup* of a topological group G is a homomorphism of $\mathbf{R}$ into G. Note that the one-parameter subgroup w is the center of G; i.e., $w(r)g=gw(r)$ for all $r\in\mathbf{R}$ and $g\in G$.

Evidently, the Weyl equation mirrors in operators the commutation relation (6.7) in G. In view of (6.6) it is natural to define

$$T(g)=e^{ir_3}\mathcal{V}(r_2)\mathcal{U}(r_1) \tag{6.8}$$

for $g \in G$, where $\mathcal{V}$ and $\mathcal{U}$ are given by (6.4). Now, from (6.2) and (6.4) one can easily verify that equation (6.8) defines a unitary representation T of G on the Hilbert space $L^2(\mathbf{R})$. Furthermore, *this representation illustrates the complication in noncompact noncommutative harmonic analysis alluded to at the end of the preceding section. For T is infinite-dimensional, but also irreducible.*

If T is an infinite-dimensional representation (i.e., dim $V=\infty$, where $V=V_T$), then we must refine the notion of irreducibility. Namely, T is irreducible if there are no proper *closed* invariant subspaces. In the situation at hand, although there are a number of *dense* subspaces of $L^2(\mathbf{R})$ that are invariant under T (e.g., the space of all continuous functions on $\mathbf{R}$ which vanish off some finite interval), $\{0\}$ and $L^2(\mathbf{R})$ itself are the only closed invariant subspaces.

From the basic representation T, one can construct a family of representations T_λ of G parametrized by all real numbers $\lambda \neq 0$. These are defined by the formula $T_\lambda(g)=T(g_\lambda)$ where $g_\lambda(r_1,r_2,r_3)=g(r_1,\lambda r_2,\lambda r_3)$ for all $g \in G$. In more detail,

$$(T_\lambda(g)f)(x)=e^{i\lambda(r_3+xr_2)}f(x+r_1) \tag{6.9}$$

for all $g \in G$ and $f \in L^2(\mathbf{R})$.

The mapping $g \to g_\lambda$ is an automorphism of G for each $\lambda \neq 0$. Alternatively, T_λ is that version of T which arises from the normalization $\hbar=\lambda$ of Planck's constant.

We can now restate the Stone–von Neumann Theorem as a characterization of the dual object $\hat{G}$ of the Heisenberg group: *With the exception of a set of* 1*-dimensional representations* (*which are substantially uninteresting in the harmonic analysis of G*; *cf. formulas* (6.10) *through* (6.13)), *all the irreducible representations of G are infinite-dimensional. More specifically, the representations T_λ are unitary, irreducible, mutually inequivalent, and to within equivalence exhaust all the* (*non* 1*-dimensional*) *irreducible unitary representations of G.*

Note that there is a special emphasis on *unitary* representations. In general, a noncompact group G will have representations (on a Hilbert space) which are not unitarizable. For such representations, there is no fully adequate theory of harmonic analysis. Hence, one usually restricts attention to unitary representations. In particular, $\hat{G}$ is the *unitary dual*, composed of the equivalence classes of irreducible unitary representations.

At this point, the better part of discretion might lead one to end the exposition. For to go further with the harmonic analysis of G becomes quite technical. However, the Heisenberg group is in a sense "almost abelian," and as such it is a relatively elementary noncompact nonabelian group. Thus, we cannot resist the temptation to illustrate the infinite-dimensional, operator-theoretic nature of noncompact harmonic analysis. Our comments will be brief, and necessarily vague.

The Fourier transform $\hat{f}$ of a function f on G is given formally for all nonzero real numbers λ by the operator

$$\hat{f}(\lambda)=\int_G f(g)T_\lambda(g)dg, \tag{6.10}$$

where $dg=dr_1dr_2dr_3$ is (bi-invariant) Haar measure on G, and f is recaptured by the inversion formula

$$f(g)=(2\pi)^{-2}\int_{\hat{G}} tr\big(\hat{f}(\lambda)T_\lambda(g)^{-1}\big)|\lambda|d\lambda. \tag{6.11}$$

Using classical Fourier analysis, one can show that for sufficiently smooth functions f, the trace in (6.11) exists, and these formulas make rigorous sense. The measure $dm(\lambda)=(2\pi)^{-2}|\lambda|d\lambda$ on $\hat{G}$ is called *Plancherel measure* for G, and the formula (which follows from (6.11))

$$\int_G |f(g)|^2dg=\int_{\hat{G}}\|\hat{f}(\lambda)\|^2dm(\lambda) \tag{6.12}$$

is the *Plancherel formula* for G. Here, $\|\hat{f}(\lambda)\|$ denotes the so-called Hilbert–Schmidt norm of the operator $\hat{f}(\lambda)$. Finally, by means of the Plancherel formula, one obtains the decomposition

$$R \cong \int_{\hat{G}}^{\oplus} \infty\, T_\lambda\, dm(\lambda) \tag{6.13}$$

of the regular representation R of G as a direct integral in which each element T_λ of the infinite-dimensional unitary dual appears with infinite multiplicity.

7. Concluding remarks. The preceding exposition takes us to the brink of the postwar era in harmonic analysis. Indeed, the past three decades have witnessed a remarkable growth in our understanding of infinite-dimensional group representations and the associated noncompact harmonic analysis. Once again, in the interest of simplicity we shall be brief and mention but a few of the highpoints. Here, of course, my own prejudices will be revealed. I apologize for omissions of perhaps equally important contemporary developments that would be mentioned in a longer article.

The thrust of much of the recent work in harmonic analysis on locally compact topological groups has been directed toward solutions to the following two fundamental problems for a noncompact nonabelian group G.

1. Find all irreducible unitary representations of G, up to equivalence. That is, what is $\hat{G}$? This is the problem of existence of fundamental harmonics.

2. Decompose the regular representation of G on $L^2(G)$. Alternatively, find the Plancherel formula.

In stark deviation from the compact or abelian theory, there is essentially no hope for reasonable solutions in terms of the *generic* noncompact nonabelian group G.

Groups for which there exists a satisfactory decomposition theory are called *Type I*, or less commonly but more fittingly, *tame*. The definition is too technical to give here. The general abstract representation theory of such groups has been extensively studied and is well developed. For Type I unimodular groups, there is a general existence theorem for the Plancherel formula, due to I. E. Segal. However, for a specific group, the abstract theory gives no clue as to the construction of the Plancherel measure, nor to the structure of the dual.

Nonetheless, as one sees for the Heisenberg group, all is not so bleak if one considers more restrictive classes of such groups. Regrettably, the resultant harmonic analysis is most often extremely complicated. At the very least, one is forced to deal with such discomforts as interwoven combinations of discrete and continuous decompositions, infinite-dimensional irreducible representations, and the fact that not all elements of $\hat{G}$ appear in the Plancherel formula. We mention some of the high points.

Mackey theory. In the early 1950's, the algebraic and measure-theoretic foundations for infinite-dimensional representation theory were put in place by George Mackey. The central notion is that of an *induced representation*. Very roughly, Mackey's theory reduces the representation theory of a group to that of a normal subgroup, modulo questions of group extensions.

Harish-Chandra theory. Here one is concerned with harmonic analysis on noncompact real semi-simple Lie groups. The early work, especially for complex groups, was done by Gelfand and various collaborators. However, the subject is now dominated by the work of Harish-Chandra, whose numerous papers over a span of two decades finally culminated in 1968 in the Plancherel formula for real semi-simple groups.

Kirillov theory. In his doctoral thesis in 1962, A. A. Kirillov supplied a powerful geometric idea which completely described the harmonic analysis of nilpotent Lie groups, and which now dominates harmonic analysis on solvable groups. Roughly speaking, Kirillov's theory associates to each element of the dual of a Lie group, a so-called *orbit* which has the structure of a symplectic manifold. Extensions of Kirillov's method, by L. Auslander and B. Kostant, yielded the decomposition of the regular representation for all tame solvable groups, as well as the more general geometric theory, due to Kostant, of "quantization."

Typically, although not exclusively, the important applications of harmonic analysis (e.g., those in physics, number theory, and noncommutative real and complex analysis) arise from the decomposition of $L^2(X)$ under the action of a group of symmetries of the space X. For such applications, the reader is referred to the literature.

8. A Guide to Further Readings. *Section 1.* A more substantial introduction to Fourier series appears in the monograph by Weiss [1]. A treatment of the subject which brings out the diverse applications of Fourier series is given by Dym and McKean [4]. For a distribution-theoretic presentation of Fourier series, see the text by Beals [2]. Our historical citations are drawn from Grattan-Guiness [3] and the references therein. See also the exposition by Zygmund in [38]. The two volumes [6] by Zygmund form the definitive scholarly work in the field of Fourier series.

Section 2. Wussing's book [8] presents the historical development of group theory. (For a start, see the review by W. Waterhouse, Bulletin Am. Math. Soc., 78 (1972) 385-391.) Of course there are innumerable texts on the subject of finite groups. The book by Boerner [7] contains an elementary treatment of the representation theory.

Section 3. A mathematically substantive account of the development of Lie theory appears in the Historical Notes of [9].* Tidbits of philosophy and history can also be found in Weyl's important book [11]. See also the Historical Notes in [10] for information on the genesis of topological groups and Haar measure. For a short contemporaneous account of the Peter-Weyl Theorem from the pen of the master, see Weyl's lecture [12] to the Swiss Mathematical Society in 1927. For a current point of view, see the expository paper by G. Weiss in [38]. Weil's book [15] contains a more extensive treatment. The Cartan-Weyl theory is developed in Wallach [31].

Section 4. Vilenkin's book [16] provides extensive coverage of special functions from the harmonic analysis point of view. We refer to Stein and Weiss [17] for a classical treatment of spherical harmonics, and to Coifman and Weiss [32] for a group theoretic presentation. The elementary article [18] gives a somewhat more general point of view. The last chapter of Helgason's book [29] contains the general theory.

Section 5. Of course, there are many expositions of the classical theory, e.g., [1], [17], [20]. The encyclopedic treatise is Titchmarsh's book [19]. For the abstract theory, see [15], [21], and Graham's article in [38]. A development that centers around the Stone–von Neumann Theorem is given by Segal and Kunze [22].

Section 6. See the original article by von Neumann [33], the exposition by Pukanszky [20], or [22] for the Stone–von Neumann Theorem.

Section 7. There is no royal road through contemporary infinite-dimensional representation theory.‡ Most treatments or surveys are highly technical. Mackey's article [24] surveys the field through 1963,† and the A.M.S. symposium [27] brings matters up to 1972. The textbook [23] by Kirillov, which gives a broad introduction to current practice, contains a relatively elementary introduction to his orbit theory. The expository paper [38] by P. Sally describes the most easily accessible example in which Harish-Chandra's theory applies. One might also want to take a look at the survey [30] by Harish-Chandra, and the memoir [28] by Auslander and Moore. [34] treats

*After this paper was written, the preprint [41] by S. Helgason appeared. This is a written version of an invited address to the A.M.S. The introductory section is a highly attractive summary of the origins of Lie theory.

†It has been brought to my attention that the reissue in 1976 of Mackey's 1955 lectures at the University of Chicago contains a lengthy appendix that brings [24] up to date. In this addendum one finds a survey that touches not only upon "Mackey theory" itself, but nilpotent, solvable, and semi-simple representation theory, as well as applications in number theory, ergodic theory, and quantum mechanics.

‡Cf. the review by R. A. Kunze, Bull. Amer. Math. Soc., 84 (1978), 73–75.

number-theoretical applications, [35] concerns those in probability, and [36] is one among many treatments of applications to physics. Dyson's elementary article in [39] is the first place to look for group theory in physics. See also the expositions [25] and [26] by Mackey. Extensive bibliographies appear in [24], [27], [23], and Warner's treatise [37].

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Added in page proofs: G. W. Mackey has informed me that he has a lengthy article related to the theme of this paper to appear in 1979 in Rice University Studies, Proceedings of the Conference on History of Analysis.

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GEOMETRICAL OPTICS AND THE SINGING OF WHALES

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I must begin by confessing that the title of this talk is quite a come on. I am just as fascinated as anyone by the huge mammals that cavort in the sea and by their long mysterious and repetitive songs, but I am not at all an authority on how they make these sounds or what they are trying to communicate. What captured my fancy was the enormous distances sound travels in water and that whales appear to take advantage of this.

However, even more than that I felt that here is a simple example from nature which should arouse the applied mathematician in all of us to understanding, the analyst to rigorizing and, I hope, the teacher to explaining. It is, therefore, to the theoretical problems of mathematical physics that I will address myself.

That is the gist of my sermon and I am sorry that I cannot reproduce in print the beautiful "hymns" of the whales which have been so excellently recorded.* Certain features are characteristic. Early recordings sound like a lot of cocktail party noise and are as hard to analyze. Single whales in bays demonstrate that they repeat their sounds over and over. It is not hard to imagine that they are fascinated by the echoes from the walls of the bay. They also may be trying to send messages and we shall see how. The most intriguing sound of all is the low frequency note that they produce which can be transmitted for hundreds of miles, unlike the high frequency tones which are absorbed by the water. This note sounds more like a motor boat than a sound made by anything so human as a whale.

Turning to the mathematics, it is necessary to begin somewhere. We plunge in with the differential equation for the disturbance in the pressure:

$$\phi_{tt} = c^2 \Delta\phi,$$

here ϕ is a function of all the space variables and time, $\Delta = \operatorname{div}\operatorname{grad}$ and c is the speed of sound given in terms of pressure p as a function of density ρ by $\sqrt{dp/d\rho}$. Since water is almost incompressible, this speed is much greater in the ocean than in air. In the limit it seems $\Delta\phi = 0$. What in fact does this familiar limit mean? We shall come back to this later, but meanwhile we must justify the quantity c as the speed of propagation of something and also ask how much of the something actually goes with it.

To draw these conclusions from this equation is fairly complicated and the simplest way to go about it is to study high frequency disturbances. It is their behavior which is given by "geometrical optics." The helpful fact is that qualitatively they also describe not so high frequency behavior.

*The best example is "Songs of the Humpback Whale" by Roger S. Payne, Capital Records Stereo ST-620, and early recordings were made by W. E. Chaville and W. A. Watkins at the Woods Hole Oceanographic Institute. See also [7].

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Geometrical optics is usually understood as being given by the rule that light or sound takes the shortest path. But it can also be deduced as a high frequency phenomenon in the following way:

We simply set

$$\phi = e^{i\omega t}\Phi(x,y,z)$$

and limit ourselves to ω large. For Φ we get the equation

$$-\omega^2\Phi = c^2\Delta\Phi.$$

The trick of geometrical optics is to take Φ in the form

$$\Phi = e^{i\omega\chi}\psi$$

where ψ, the amplitude, and χ, the phase, do not change much with ω.

We write

$$\begin{aligned}\Delta e^{i\omega\chi} &= \operatorname{div}(i\omega\nabla\chi e^{i\omega\chi}) + \cdots \\ &= -\omega^2|\nabla\chi|^2 e^{i\omega\chi} + \cdots\end{aligned}$$

so we end up with

$$c^2|\nabla\chi|^2 = 1.$$

Combining the two exponentials to get ϕ, we see that if we move at a speed $|\nabla\chi|^{-1} = c$ we will move with the disturbance, thus justifying the name speed of sound.

To show how this agrees with the notions of least time let us suppose $c = c(y)$ and that we have two space variables (x,y). The curves orthogonal to the curves $\chi =$ constant satisfy $\chi_x dy - \chi_y dx = 0$ and hence we find by differentiating the differential equation for χ with respect to x that χ_x is constant along these trajectories and hence also that

$$\left(\left(\frac{dy}{dx}\right)^2 + 1\right)c^2 = \text{constant}.$$

On the other hand, the time of transit between two points is $\int dt = \int c^{-1}(y)\,ds$ and its variation with $x = X(y)$ is

$$\begin{aligned}\delta\int(\dot{X}^2+1)^{1/2}c^{-1}(y)\,dy &= \int\frac{1}{2}(\dot{X}^2+1)^{-1/2}c^{-1}(y)\dot{X}\delta\dot{X}\,dy \\ &= -\int\frac{d}{dy}\left((\dot{X}^2+1)^{-1/2}c^{-1}(y)\dot{X}\right)\delta\dot{X}\,dy.\end{aligned}$$

Thus a least time trajectory must satisfy the condition that $(\dot{X}^2+1)^{-1/2}c^{-1}(y)\dot{X}$ is constant, or as we have above

$$\left(\left(\frac{dy}{dx}\right)^2 + 1\right)c^2(y),$$

it is independent of y. So we have the equivalence in this case.

If we want to go a little further to study amplitudes, we must consistently expand in ω and we find to lowest order $\operatorname{div}(|\psi|^2\nabla\chi) = 0$ or

$$\int_{\text{surface}}|\psi|^2\frac{\partial\chi}{\partial n}\,d\sigma = \text{constant}.$$

If we pick a surface, a flux tube, on whose sides $\partial\chi/\partial n = 0$ and on whose ends $\partial\chi/\partial n = |\nabla\chi|$, we see that since $|\nabla\chi|^2 = c^{-2}$,

$$\int_{\text{end}}|\psi|^2c^{-1}\,d\sigma = \text{constant}.$$

This means that the amplitude or energy coming out at the end depends on the speed c and what goes in at the other end.

If we take the familiar source at a point and a constant speed of propagation, we get the amplitude $|\psi|$ behaving like $(\text{distance})^{-1/2}$ in two dimensions and like $(\text{distance})^{-1}$ in three. Thus if a whale were in an ocean of infinite depth and with a constant speed of sound, its sounds would die out too fast and it would not be audible. Fortunately, the ocean is not infinitely deep and its sound speed is not constant. In fact, c is a function of temperature of the water and of the hydrostatic pressure and is roughly indicated as a function of depth in Figure 1. The dotted line is for constant temperature.

From our least time example we have $c(1+(dy/dx)^2)$ is constant and thus from a point of minimum value for c there will issue a family of shortest paths or rays, to use their proper name, which refocus at the same depth some distance away. This is the so-called sonar channel. But a whale cannot dive three thousand feet and sing so loudly; at least the experts doubt it, although the whales seem to do very well at 1500 feet. How could they achieve an equivalent effect? One way is by reflecting sound off nearby vertical cliffs. Alternatively, a sound made at a depth of only 300 feet will refocus at thirty miles, distorted but distinct. If it is loud enough to begin with it will be audible.

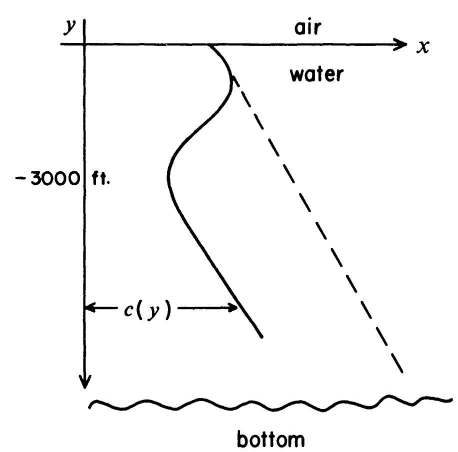

FIG. 1

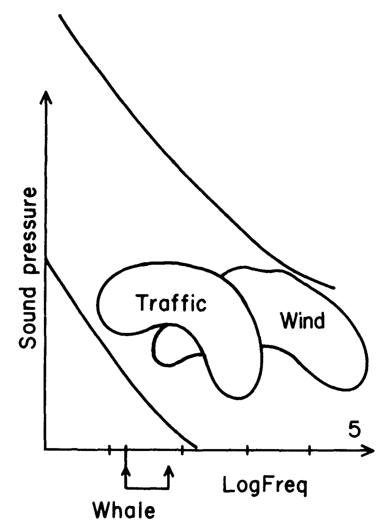

FIG. 2

At this point we have to examine what we mean by audibility. To be sure a sound can be heard, it must have an amplitude or pressure that is large enough to be heard above other sounds of the same frequency. The two main sounds in the ocean are traffic and wind. In Figure 2, we have a rough plot of pressure against frequency in the ocean. (This is taken from the work of Payne and Webb [8] on the subject, from which I have taken much information.) The low frequency whale sounds are at about 20 Hertz and we see that they are "killed" only by some storms and by traffic, i.e., propeller sound. Payne and Webb conclude that the whale is audible at distances given by the table of Figure 3. "Then" means when there was no propeller noise and under everyday wind conditions. Nowadays such conditions barely exist even in the most remote parts of the ocean.

	Spherical	Sonar	Cyl.
Now	45	525	2,025
Then	140	3,500	19,600
Then (quiet ocean)	450	11,500	202,500

FIG. 3. Distance in Miles to Listener for Bare Audibility

The table, of course, gives us only a qualitative picture. The true range of the ordinary whale sounds is certainly more than spherical and less than sonar. By comparison the two-dimensional world (cyl) would be incredibly noisy.

So much for the physics of long distance sound propagation. What about the verification of the mathematics?

In free space with variable speeds of propagation, our asymptotic expansion, including the phenomenon of focusing where the gradients become large and the remainder terms cannot be neglected, has been justified by Ludwig [5] and Duistermaat and Hörmander [1], using the methods of the Fourier Integral Operator. In developing all the analysis and algebra of the F.I.O. they laid the groundwork for other justifications. Of course, it should be added that special cases were known before. With boundary conditions the problem is still elusive, except for the simple case of constant sound speed and convex reflectors; see Ludwig and Morawetz [4]. Recently, Taylor [10] and Melrose [6] improved these results.

But to return to focusing, there is yet another way of looking at the problem. We write

$$\phi=\sum g_n(y,\omega)F_n(x,z,\omega)e^{i\omega t}$$

where g_n are bound states for

$$g_{nyy}=\left(\frac{-\omega^2}{c^2}+\mu_n^2\right)g_n, \qquad (\Delta_{xz}+\mu_n^2)F_n=0.$$

Here $\mu_n=\mu_n(\omega)$ is an eigenvalue (dispersive since it depends on ω). How does such a signal propagate? We write (a primitive example of a Fourier integral) a solution as a sum of such terms

$$\phi=\int\sum_n g_n(y,\omega)F_n(x,z,\omega)e^{i\omega t}\,d\omega=\int\sum_n g_n(y,\omega)e^{i(\omega t-\mu_n\chi_n)}\psi_n\,d\omega.$$

To estimate the behavior for large t we would like to integrate by parts. Let us do just one term of the sum in a case where μ_n is also large like ω say $\mu_n=\lambda_n\omega$. We write

$$g_n=\exp(i\omega f_n(y))(\tilde g_{0n}+\cdots)\cdot$$

We find

$$g_n=\exp\left(i\omega\int\sqrt{1-c^2\lambda_n^2}\;c^{-1}dy\right)(g_{01}+\cdots)+\exp\left(-i\omega\int\sqrt{1-c^2\lambda_n^2}\;c^{-1}dy\right)(g_{02}+\cdots)$$

Now we see that the behavior depends on the sign of $1-c^2\lambda_n^2$ which depends on y.

In fact, there are resonating modes, i.e., values of n where this asymptotic expression does represent a solution for $y\to\pm\infty$. This is a *turning point* problem and it is not easy to see that there is no exponential growth.

But let us incorporate this behavior into the integral taking just a typical term, i.e., no n's appear:

$$\phi=\int\exp i\omega\left\{\int\sqrt{1-c^2\lambda^2}\;c^{-1}dy+t-\lambda\chi(x,z)\right\}\{\tilde g_0(y)\psi(x,z,\omega)\}\,d\omega$$

Set $\omega\{\int\sqrt{1-c^2\lambda^2}\;c^{-1}dy+t-\lambda\chi(x,z)\}=h$. To integrate we set

$$\phi=\int e^{ih}\frac{U}{h_\omega}\,dh$$

and integrate by parts. This gives a smooth answer except where $h_\omega=0$. So the unsmooth signal moves so that

$$0=h_\omega=t-\mu_\omega\chi+\int(\omega^2-c^2\mu^2)^{-1/2}(\omega-c^2\mu\mu_\omega)c^{-1}dy$$

Its velocity (the group velocity) in the x,z plane is $(\mu_\omega)^{-1}$ and in the y direction

$$\frac{\sqrt{\omega^2-c^2\mu^2}}{\omega-c^2\mu\mu_\omega}c.$$

Note two points. First it is the use of the idea of looking where the signal is *not* that is so productive with Fourier integrals. Much of this is rigorizable. But in the general area of rigorizing the asymptotic expressions of eigenvalues and eigenfunctions there are many open problems. There is recent work by Ralston [9] on Gaussian beams; Keller and Rubinow [3] give elegant recipes for these eigenfunctions and values that check in special cases but much remains for the analyst.

Secondly comes the observation that there are various possibilities. If μ gets too big, there is no propagation. If μ is too small it's like the original problem. If μ is just right then the signal will spread out.

In fact, for various profiles of $c=c(y)$ it is possible for the group velocity $(\mu_\omega)^{-1}$ to behave quite differently depending on the frequency. Mostly, the higher frequency band has higher group velocity. However, some modes have high frequencies and lower group velocities.

That concludes the mathematical analysis if we start with the wave equation with variable sound speed. Let us examine where the equation comes from.

First be a little blind and look at the motion of an incompressible fluid. We have, using D_t as the time derivative along a particle path, from conservation of mass and momentum:

$$\operatorname{div} u = 0, \qquad D_t(\rho u) + \nabla p = 0.$$

Here u = velocity, p = pressure including hydrostatic, ρ = density = constant. For simplicity suppose we take one dimension. Then $u=u(t)$ and $p=p_1(t)x+p_2$. Nothing is propagated. The material seems to be rigid.

In fact, water is compressible, otherwise no sound could propagate. But sound speed becomes infinite in the limit of zero compressibility. What is the relation? This question has been studied closely by D. Ebin [2] who has developed the asymptotics or rather the convergence properties of the relationship quite rigorously for smooth flows. Shocks which can occur for any amount of compressibility play havoc with the asymptotics in the same way caustics and focusing foul up geometrical optics.

We could say: what about viscosity or entropy? They appear in the study of absorption but we will leave them aside. So again we have stepped back a little, but don't forget we have to put our foot on the bottom somewhere. That somewhere is: the laws controlling the conservation of mass and momentum:

$$\rho_t + \operatorname{div} \rho u = 0$$
$$D_t(\rho u_t) + \nabla \rho = 0$$

and now a gas law, $p=p(\rho)=A\rho^\gamma + B$.

So what should be the relation between this and the incompressible model? We can hardly expect these two sets of equations each to have a solution that are near each other unless they start out that way. So we assume that we are talking about compressible flows that satisfy initially

$$\operatorname{div} u = 0$$
$$\rho = \text{constant}.$$

It's best now to draw an analogy with a simple mechanical model. A rod on a pivot moves in a plane with constant angular velocity. With polar coordinates, θ_0, r_0, $\theta_{0t}=\text{const.}=\lambda$, $r_{0t}=0$. Now replace the rod by a spring with a very high modulus of elasticity k. There will be some explicit equations of motion which we just write as

$$\theta_{tt} = g(\theta_t, r_t, \theta, r, k),$$
$$r_{tt} = h(\theta_t, r_t \theta, r, k).$$

Suppose initially

$$r_t = 0$$
$$\theta_t = \lambda.$$

Then one can show fairly easily that as $k \to \infty$

$$\theta_{tt} \not\to \theta_{0tt}$$
$$r_{tt} \not\to r_{0tt}$$

but one can develop the solution as

$$\theta = \theta_0 + \frac{1}{k}\theta_1(\sqrt{k}\,t) + \cdots$$
$$r = r_0 + \frac{1}{k}r_1(\sqrt{k}\,t) + \cdots,$$

i.e., the gross motion is the same but the correction is on a different time scale.

Note: Two differentiations cancel out k in the expansion.

For compressible flow we get very similar behavior if we use the position ζ as the variable. Thus

$$\zeta = \zeta(\xi, t)$$

is the position at time t of a particle (of ink say) which was at ξ initially (Lagrangian variables). Then we find using the pressure law

$$\zeta_{tt} = H(\zeta, 1/k)$$

where $k = A\gamma$; H is a complicated functional involving nothing worse than second derivatives of ζ and those linearly. We have replaced four equations of first order by three of second order.

For incompressible flow we have correspondingly

$$\eta_{tt} = H(\eta, 0).$$

Ebin shows that we can write

$$\zeta = \eta + \frac{1}{k}\eta_1(\sqrt{k}\,t, \xi) + \cdots$$

where η is an incompressible fluid flow. Here everything in sight is assumed smooth. For our purposes what is important is that η_1 satisfies

$$\eta_{1tt} - \operatorname{div}\big(c^2(y)\nabla\eta_1\big) = G.$$

This looks a little different from our equation but by converting to pressure as variable we get $\phi_{tt} - c^2\Delta\phi = g$ as before where g involves second *derivatives* of η, the incompressible flow. No first derivatives occur.

If the fluid is in a constant state $G \equiv 0$. Thus if the incompressible leading term is smooth, the deviation from incompressibility propagates signals as we expect. On the other hand, if there is a region of rapidly varying incompressible flow it can generate noise that propagates like a source of light propagates. For example, whenever you have your blood pressure taken, the doctor listens to noises of this kind. And certainly when he tells you that you have a heart murmur that's what it is. It's an extra noisy vortex being shed off the valve backwards.

But this kind of analysis brings me back to the sound of the whale. How does it make those noises? It has no vocal chords. It does have a larynx. Some whales have teeth and use them in making sound. But no one really knows how it actually works. Could that mystery, especially the origins of that low frequency sound, be explained by studying what properties of the sound generating mechanism appear in the received sound pattern? Perhaps a study of such problems using the mathematics of these equations would reveal something about the creation of the sound. But so far only very little is known and that in very special problems, definitely not including whales.

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CREATING DIFFERENTIABILITY AND DESTROYING DERIVATIVES

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The purpose of this expository article is to discuss questions concerning the "creation" or "destruction" of certain properties related to differentiation. We shall emphasize recent work which can be presented without much technical preparation. Thus, most of what we describe should be accessible to anyone familiar with the basic concepts related to Lebesgue integration. Where we do use a fact which might not be entirely "basic," we make this clear. For purposes of completeness, we also discuss some work of a more technical nature. This work, which can be omitted by the casual reader without serious loss, appears mostly at the end of the article.

As we shall see very soon, the vehicle for creating or destroying desirable properties will be homeomorphism. We use the perspective of "creation" in Section 1 because it is more optimistic. We use the pessimistic perspective of destruction in Section 2 because that is how the subject of that section arose historically.

1. Creating Differentiability. Consider the function $F(x)=|x|$ on the interval $[-1,1]$. This function is not differentiable at the origin where the left and right derivatives do not agree. By a suitable homeomorphic change of variables, however, F can be transformed into a differentiable function. Thus, if $h(x)=x^3$, then h maps $[-1,1]$ onto itself homeomorphically and the function $(F\circ h)(x)=|x^3|$ is differentiable. Similarly the function $G(x)=x\sin 1/x$, $(G(0)=0)$, is not differentiable at the origin, while the function $G\circ h$ is everywhere differentiable.

Let us increase the complexity a bit. Let K be the Cantor function. This function is continuous, nondecreasing, constant on each interval complementary to the Cantor set, and it maps the Cantor set C onto $[0,1]$. Is it possible to transform K into a differentiable function by a suitable homeomorphic change of variables? We shall answer this question presently, but let us first look and see what is involved. The function K has the property that if I is any open interval containing points of the C, then the lower right Dini derivative of K, D_+K, takes on every non-negative extended real number on

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$C \cap I$. In fact, if $0 \leqslant \alpha \leqslant \infty$, then the set $\{x : D_+K(x) = \alpha\} \cap C \cap I$ has cardinality c. (This is not a trivial fact about K. A proof can be found in [**17**].) On the other hand, D^+K is infinite at every bilateral point of accumulation of C. Thus, the behavior of K with respect to questions of differentiability on C is rather complex. Furthermore, if $K \circ h$ were differentiable for some homeomorphism h, then the function $K \circ h$ would still be a Cantor-like function: that is, $K \circ h$ would still be constant on each interval complementary to the nowhere dense perfect set $P = h^{-1}(C)$, but not constant on any open interval containing points of P. And $K \circ h$ would be differentiable! These considerations (the complexity of K and the Cantor-like shape of any transform of K) make it unclear how to proceed. Perhaps an appropriate h can be constructed, but how? Or, perhaps no such h exists. Let us return to the functions F and G. Perhaps considerations of these simple functions can provide us with some insights.

The functions $F \circ h$ and $G \circ h$ are both differentiable, but $(F \circ h)'$ is bounded while $(G \circ h)'$ is not. We might ask whether the fact that $(G \circ h)'$ is not bounded is a result of a poor choice of h or whether there is some intrinsic property which G possesses which would make it impossible to transform G into a function with a bounded derivative. Let us consider that question. We have already seen that the property of being differentiable can sometimes be created by a change of variables. But some properties can never be created in this way. Continuity is such a property: if a function is discontinuous, no homeomorphic change of variables can transform it into a continuous function. The property of being of bounded variation is also such a property: one can easily verify that the total variation of a function remains the same under any homeomorphic change of variables. Now the function G is continuous—but it is not of bounded variation. Thus no transform of it can be of bounded variation. But every function having a bounded derivative is of bounded variation. Thus G cannot be transformed as required. This discussion actually shows that a *necessary* condition for a function to be transformable into one with a bounded derivative is that the function be continuous and of bounded variation.

Now, the function F meets this necessary condition and the function $F \circ h$ does have a bounded derivative. What are the essential features here? A bit of geometric reflection (or, an inspection of an appropriate composite difference quotient) reveals just what one needs. It is the combination of F satisfying a Lipschitz condition where it is not differentiable and h smoothing things out there without destroying things elsewhere. In order for h to smooth things out, we see that we need $(h^{-1})'$ to be infinite at the point of nondifferentiability of F, and differentiable elsewhere. One can create this situation for every function satisfying a Lipschitz condition. To see this, note first that such a function is differentiable a.e. Now, given any set $Z \subset [0, 1]$ of Lebesgue measure zero, $(\lambda(Z) = 0)$, there exists a homeomorphism h of $[0,1]$ onto itself and a constant $\alpha > 0$ such that $(h^{-1})'(x) = \infty$ for all $x \in Z$ and $(h^{-1})'(x) > \alpha$ everywhere. (A proof of this non-trivial fact can be found in [**21**: Th. 8].) An inspection of an appropriate difference quotient now shows that every Lipschitz function can be transformed into one with a bounded derivative via a suitable homeomorphic change of variables. The Cantor function K is not a Lipschitz function. But it *can* be transformed into one. Any continuous function of bounded variation can! To see this, we need only observe that if B is such a function, then its graph has finite arc length β. If we let $A(x)$ denote the arc length of B between 0 and x, and let $g(x) = (A(x)/\beta)^{-1}$, then we can easily verify directly that $B \circ g$ satisfies a Lipschitz condition. Thus $B \circ g$ can be transformed into a differentiable function with a bounded derivative via a suitable homeomorphism h: $(B \circ g \circ h)'$ is bounded. We summarize this discussion as a theorem.

THEOREM 1. *Let F be defined on* $[0,1]$. *A necessary and sufficient condition for there to exist a homeomorphism of* $[0,1]$ *onto itself such that $F \circ h$ has a bounded derivative is that F be continuous and of bounded variation.*

COROLLARY. *The Cantor function K can be transformed into a function with a bounded derivative.* (*Thus, there are differentiable Cantor-like functions.*)

We can apply some of our reasoning to plane curves. Suppose we are told that a curve in R_2 has parametric representation $x=x(t)$, $y=y(t)$, $(0\leqslant t\leqslant 1)$, and that both x and y are differentiable functions with bounded derivatives. Perhaps we conjure up an image of a curve having a tangent at each point, the tangent being vertical at those points where x' vanishes. If so, our image doesn't do justice to the possibilities. *Every* curve of finite length admits such a parametric representation! Such a curve has a parametric representation with coordinate functions continuous and of bounded variation. The argument leading to Theorem 1 can be modified without difficulty to give rise to a change of variables h which simultaneously transforms both coordinate functions into ones with bounded derivatives. We state this as a theorem about curves in R_2.

THEOREM 2. *Let γ be a rectifiable curve in R_2. Then there exists a parametric representation for γ each of whose coordinate functions has a bounded derivative.*

(The proof works equally well for curves in R_n.)

The details of the proof of Theorem 2 can be found in **[7]**. Theorem 1 was proved in **[5]**, the authors being unaware of Theorem 2 at the time.

Let us return to the function K, and this time ask somewhat more of it. Is it possible to transform K into a function with a *continuous* derivative? The fact that any transform of K is still a Cantor-like function turned out to be irrelevant to the earlier question. This time it isn't. A Cantor-like function does have a zero derivative on a dense set. Thus, if such a function is differentiable, the derivative must be discontinuous at each point where it does not vanish. And there must be such points, of course. What functions can be transformed into ones with continuous derivatives? The Cantor function provides an important clue. If $x\in C\cap I$ (C being the Cantor set and I an open interval), then K is not constant on I, nor is K strictly increasing on I. Let us describe this behavior by saying that x is *a point of varying monotonicity for K*. This means precisely that there is no neighborhood of x on which K is constant, and there is no neighborhood of x on which K is strictly monotonic. Thus, for K, the set of points of varying monotonicity is the Cantor set, and K maps this set onto the interval $[0,1]$, a set of positive measure. And that's the difficulty! If F is any function mapping its set of points of varying monotonicity onto a set of positive measure, then F cannot be transformed into a continuously differentiable function. (We omit the argument, which is very similar to the one given for the function K.) For continuous functions of bounded variation, the converse is also true, but much more difficult to prove. A proof can be found in **[5]**.

THEOREM 3. *Let F be defined on* $[0,1]$. *A necessary and sufficient condition for there to exist a homeomorphism h of* $[0,1]$ *onto itself such that $F\circ h$ is continuously differentiable, is that F be continuous and of bounded variation, and that F map its set of points of varying monotonicity onto a set of measure zero.*

We began this section by observing that certain functions which aren't differentiable can be transformed into ones which are. Yet, our theorems gave conditions under which functions can be transformed into ones having *bounded* derivatives, or *continuous* derivatives. We have not yet determined which functions can be transformed into differentiable functions with no additional requirements on the derivatives. Although this is not difficult to do, it does involve notions of too specialized a nature to describe here in detail. To complete our circle of ideas, however, we state the theorem and follow it with a brief remark.

THEOREM 4. *Let F be defined on* $[0,1]$. *A necessary and sufficient condition for there to exist a homeomorphism h of* $[0,1]$ *onto itself, such that $F\circ h$ is differentiable, is that F be continuous and of generalized bounded variation in the restricted sense* (VBG_*) **[11]**.

REMARK. The concept "VBG_*" is discussed in detail in **[18]**. Theorem 4 has a simple proof based on Theorem 1, above, and a theorem of Tolstoff's **[19]** which gives conditions under which a function

can be transformed into a differentiable one via a homeomorphic change of variables which is also differentiable.

There are, of course, other classes we could have considered. What happens if we wish to transform F into a function with a summable derivative? Or if we allow an infinite derivative? In the first case we get no new class. If F is differentiable with a summable derivative, then F is continuous and of bounded variation. Thus, any function which can be transformed into such a function can also be transformed into one with a bounded derivative. In the second case we also get no new class if we restrict ourselves to continuous functions. This is so because any function F for which F' exists, finite or infinite, for all x, must be VBG_*. Thus, if a continuous function can be transformed into such an F, it can also be transformed into one with a finite derivative. We do not know, however, which discontinuous functions can be transformed into ones which have derivatives (finite or infinite) at each point. A natural conjecture would be that every function which is of generalized bounded variation in the restricted sense can be transformed as required.

Our discussion so far has been concerned with creating differentiability through compositions with "inner" homeomorphism. We interpreted this as a change of variables or a "change of scale of the domain." We could equally ask about creating differentiability through a "change of scale of the range": more precisely to which functions F correspond homeomorphisms h of R onto itself such that $h \circ F$ is differentiable? We discuss this question briefly.

Before embarking on the general question, let us examine the functions F, G and K above from this standpoint. It is immediately clear that for the functions F and G, the same homeomorphism $h(x) = x^3$ which worked from the inside, works also from the outside: $h \circ F$ and $h \circ G$ are differentiable functions. But what about the function K? To answer this question, we need to know a certain fact about differentiable functions. Every differentiable function satisfies Luzin's condition N; that is, it maps sets of measure zero onto sets of measure zero. But K maps the zero measure Cantor set C onto $[0,1]$. Thus, if h is any homeomorphism of R onto itself, $h \circ K$ will map C onto the interval $h([0,1])$. Thus $h \circ K$ cannot be differentiable. Once again, the function K has given us a clue. No function which maps a zero measure set onto an interval can be transformed into a differentiable function via an "outer" homeomorphism. The condition that F not map zero measure sets onto intervals is thus a necessary condition for F to be transformable. This condition is not sufficient, but it is not far off target. The actual necessary and sufficient condition has recently been established in **[10]**. We state this condition as a theorem.

THEOREM 5. *Let F be defined on* $[0,1]$. *A necessary and sufficient condition for there to exist a homeomorphism h of R onto R such that $h \circ F$ be differentiable is that F be continuous and that to each interval J there corresponds a number $\varepsilon_J > 0$ such that if $E \subset [0,1]$ and $\lambda(E) < \varepsilon_J$, then the set $F(E)$ does not contain J.*

Simply stated, F doesn't map sets of arbitrarily small measure onto some fixed interval. The condition is, as we suggested, related to the condition that zero measure sets not be mapped onto intervals, but it is not equivalent. (See **[10]** for a discussion of a number of related conditions.) We mention that every absolutely continuous function meets this condition.

Theorem 5 is analogous to Theorem 4. This same condition works for the analogue to Theorem 1. It would be interesting to know the analogues to Theorems 1 and 3.

We have seen that differentiability can sometimes be created through compositions (either inner or outer) with homeomorphisms. It can, of course, also be destroyed. For example, if F is not differentiable but $F \circ h$ is, then h^{-1} destroys the differentiability of $F \circ h$; $(F \circ h) \circ h^{-1} \equiv F$ is not differentiable. We have taken the "creative" viewpoint in this section. In Section 2, below, we shall consider some analogous questions but shall take the "destructive" viewpoint instead.

2. **Destroying derivatives.** In 1921, Wilkosz **[20]** showed that the function $f(x) = \cos 1/x$, $(f(0) = 0)$ is a derivative, but the function f^2 isn't. (In saying that f "is" (not "has") a derivative, we mean that

there is a differentiable function F such that $F'(x)=f(x)$ *for all* x in the domain of f.) Wilkosz' example shows simultaneously that the class of derivatives is not closed under multiplication nor under outer composition with continuous functions. The fact that the class of derivatives is not closed under multiplication causes certain difficulties in dealing with derivatives. (See the recent survey article [9] for a development of all that is currently known about questions concerning products of derivatives.) The fact that the class of derivatives is not closed under outside composition with continuous functions is responsible for the fact that the class of derivatives cannot be characterized in terms of associated sets: i.e., sets of the form $\{x : f(x) < \alpha\}$ or $\{x : f(x) > \alpha\}$, $\alpha \in R$. (A discussion of this difficulty and others that one encounters when trying to characterize derivatives in various ways can be found in [3].)

Wilkosz was primarily concerned with bounded derivatives f for which f^2 is also a derivative, and he showed that the requirement that both f and f^2 be derivatives imposes considerable restrictions on f. In particular, such an f must be approximately continuous, a property not shared by all derivatives. (Every bounded approximately continuous function is a derivative, but a derivative need not be approximately continuous even if it is bounded.) He also proved that if f, g and g^2 are bounded derivatives, then so is fg. He attributed this result to Banach and regarded it as "the key to the whole importance of [the class of bounded derivatives whose squares are also derivatives]".

Thus the "squaring function" enjoys a special role with regard to bounded derivatives. If it does not destroy a certain bounded derivative f (i.e., if f^2 is also a derivative) then f must possess a number of desirable properties. What is it that singles out this squaring function for this special role? To get some insights to this question, we first investigate ways in which composition can destroy derivatives.

If h is any linear function ($h(x)=ax+b$), then h cannot destroy a derivative: if $f=F'$ then $h\circ f=(aF+b)'$. Suppose that h is any homeomorphism of R onto R which is not linear. This implies that there are numbers r and $s>0$ such that

$$h(r+s)+h(r-s)\neq 2h(r). \tag{1}$$

We shall construct a derivative f on $[0,1]$ which exploits the inequality (1).

Let $\{I_n\}$ be a disjoint sequence of open intervals in $(0,1)$ whose endpoints approach 0 as $h\to\infty$ such that $d(\bigcup I_{2K})=d(\bigcup I_{2k-1})=\frac{1}{2}$, where $d(A)$ denotes the right density of the set A at the origin: i.e.,

$$d(A)=\lim_{t\to 0+}\frac{\lambda(A\cap[0,t])}{t}.$$

(Intuitively this means that for small positive values of t, about half of the interval $[0,t]$ consists of points in $\bigcup I_{2k}$, about half of points in $\bigcup I_{2k-1}$, and a negligible part of $[0,t]$ consists of points in the remaining subintervals of $[0,t]$.) Let f be a function on $[0,1]$ meeting the following conditions:

(i) $f(0)=r$

(ii) $f(x)=r+s$ if $x\in\bigcup \bar{I}_{2k}$

(iii) $f(x)=r-s$ if $x\in\bigcup \bar{I}_{2k-1}$

and

(iv) f is linear and continuous on each remaining closed subinterval of $[0,1]$.

It is a routine matter to verify that f is the derivative of its integral on all of $[0,1]$. The only difficulty occurs at the origin, the point at which f is discontinuous. But near that point, f takes on the value $r+s$ "half the time," $r-s$ "half the time" and other values between $r-s$ and $r+s$ a "negligible part of the time." Thus, on the average (near $x=0$) f takes on the value r. And $f(0)=r$, so f is the derivative of its integral at $x=0$ as well. (All this is easy to verify analytically by expressing the integral of f over $[0,t]$ as a sum of three integrals, dividing by t, and letting $t\to 0$.)

Now consider the function $h\circ f$. The behavior of $h\circ f$ is very similar to the behavior of f: this function is still constant on $\bigcup \bar{I}_{2k}$ and on $\bigcup \bar{I}_{2k-1}$; these sets still have right density $1/2$ at the

origin, the function is still bounded on the negligible remaining set and is still continuous everywhere except at the origin. But there is one major difference: the derivative of its integral at the origin is $\frac{1}{2}[h(r+s)+h(r-s)]$, while its value at the origin is $h(r)$. Thus, because of (1), $h\circ f$ is *not* the derivative of its integral at the origin. But this implies that $h\circ f$ is not a derivative. (If a bounded function is the derivative of some function F, then F, having a bounded derivative, is absolutely continuous and is therefore the integral of its derivative.) We summarize as a theorem.

THEOREM 6. *If h is **any** nonlinear homeomorphism of R onto R, then there exists a bounded derivative f such that $h\circ f$ is not a derivative.*

Thus **every** nonlinear function destroys some derivative. (The requirement that h be a homeomorphism in Theorem 6 is unnecessary.)

A careful analysis of the discussion preceding Theorem 6 shows that two features of our argument were essential. These are the inequality (1) and the fact that f is not approximately continuous at $x=0$. (That is, there is no set E such that $d(E)=1$ and $f|E$ is continuous at $x=0$.) With somewhat more delicate reasoning of a similar nature, one can actually prove the following theorem.

THEOREM 7. *If f is a bounded derivative which is not approximately continuous, and h is a strictly convex function defined on the range of f, then $h\circ f$ is **not** a derivative.*

On the other hand, if f is approximately continuous, and h is continuous, then $h\circ f$ is approximately continuous. Since every bounded approximately continuous function f is a derivative, no continuous h can take such an f out of the class of derivatives. Combining this observation with Theorem 7, we see that if some strictly convex function destroys a derivative f, then **every** strictly convex function destroys f. Stated positively (or is it negatively?), if some strictly convex function does not destroy a derivative f, then none does.

We are now ready to answer the question we asked several paragraphs ago: "What is it that singles out the squaring function for this special role?" The answer is — NOTHING! **Any** strictly convex function can play the same role.

Theorem 6 was first stated, without proof, by Choquet [7]. Theorem 7 and a number of related results can be found in [**4**]. In particular, one finds there that the quantifiers in Theorem 6 can be reversed: there is a single derivative f such that $0\leqslant f(x)\leqslant 1$ for all x and if h is any nonlinear homeomorphism, of $[0,1]$ onto itself, then $h\circ f$ is not a derivative. Furthermore, if h is nowhere linear, then $h\circ f$ is not a derivative on any interval. Also, the "typical" (in the sense of Baire category) bounded derivative is destroyed by **every** strictly convex homeomorphism on **every** interval: i.e., if h is a strictly convex homeomorphism and I is a subinterval of $[0,1]$, then $h\circ f$ is not a derivative on I.

What happens if we do not assume f bounded? It is still true that the condition that both f and f^2 be derivatives on $[0,1]$ imposes considerable restrictions on f. One sees immediately, for example, that such an f must be summable (if $G'=f^2\geqslant 0$, then G is increasing so $G'=f^2$ is summable, because the derivative of a monotonic function must be summable). If, for example, we let $F(x)=x^2\sin 1/x^2$ ($F(0)=0$), then F is differentiable but F' is not summable so $(F')^2$ is not a derivative.

Iosifescu [**13**] showed that if f is a square summable derivative, then f^2 will be a derivative if and only if

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}|f(t)-f(x)|^2\,dt=0 \qquad \text{for all } x$$

(i.e., each point in $[0,1]$ is a Lebesgue point of the second kind). This, of course, imposes severe restrictions on f.

Once again we can ask what special role the squaring function plays. To what extent can this function be replaced by any strictly convex function h? Do we need to assume that h is bounded below (or above) in order to get comparable results? These questions do not seem to have been answered yet.

Let us now turn to the composition $f \circ h$, where f is a derivative on $[0,1]$ and h is a homeomorphism of $[0,1]$ onto itself. Must such a change of variables preserve the property of being a derivative? There are two properties shared by all derivatives: each derivative is in the first class of Baire (that is, it is the pointwise limit of a sequence of continuous functions) and each derivative has the Darboux property (sometimes called the intermediate value property). Each of these properties is preserved under compositions with continuous functions. Thus, at the very least, if f is a derivative and h a homeomorphism, $f \circ h$ must be a Darboux function in the first class of Baire. Unfortunately, this class $\mathfrak{D}\mathfrak{B}_1$ of functions is much larger than the class of derivatives. (See **[6]** for several comparisons of $\mathfrak{D}\mathfrak{B}_1$ with the class of derivatives.) So the information we have so far about the composition $f \circ h$ is not very helpful. Let us try to construct a specific example of a derivative f and a homeomorphism h such that $f \circ h$ is no longer a derivative. We first observe that if there is to be any hope for success, we must take f to be discontinuous. (If f is continuous, then so is $f \circ h$, and therefore $f \circ h$ is the derivative of its integral.) What happens if we take the function f which arose in the discussion preceding Theorem 6? To be specific, take $r=0$, $s=1$. This function is discontinuous at the origin, but is the derivative of its integral "because of cancellations." Let h be a homeomorphism which stretches each of the intervals I_{2k} and shrinks each of the intervals I_{2k-1} in such a way that $d(\bigcup h(I_{2k}))=\frac{2}{3}$ and $d(\bigcup h(I_{2k-1}))=\frac{1}{3}$. (Thus, $f \circ h^{-1}=1$ about two-thirds of the time and $f \circ h^{-1}=-1$ about one third of the time near $x=0$.)

A simple computation shows that $f \circ h^{-1}$ is not a derivative: the derivative of its integral equals $\frac{1}{3}$ at $x=0$ but $f \circ h^{-1}(0)=0$.

Thus inner homeomorphisms can destroy derivatives. Actually, the argument we gave can be modified to apply to any discontinuous derivative (although it is somewhat more delicate in general because derivatives need not be constant on conveniently placed intervals).

THEOREM 8. *Let f be a derivative on* $[0,1]$. *A necessary and sufficient condition that $f \circ h$ be a derivative for every homeomorphism h of* $[0,1]$ *onto itself is that f be continuous.*

Thus, every discontinuous derivative can be destroyed by some (inner) homeomorphism.

How bad can such a destruction be? That is, what desirable properties must $f \circ h$ have? We have already seen that $f \circ h$ must have the Darboux property, and must be in the first class of Baire. That's nice, but the class $\mathfrak{D}\mathfrak{B}_1$ is huge in comparison with the class of derivatives. (This vague statement can be justified in a number of ways **[6]**. For instance, two functions in $\mathfrak{D}\mathfrak{B}_1$ can agree everywhere except at one point, while two derivatives must differ on a set of positive measure if they differ at all.) So what other properties must $f \circ h$ have? The answer is none. Maximoff **[16]** showed that **every** function in $\mathfrak{D}\mathfrak{B}_1$ is of the form $f \circ h$, f a derivative, h a homeomorphism (see also **[7]** for a proof under the assumption that f is semicontinuous).

If we restrict the class of homeomorphisms, a number of positive results can be stated. The problem is actually related to questions concerning products of derivatives. Suppose, for example, that h is a differentiable homeomorphism such that f/h' is a derivative for each derivative f. Let F be differentiable with $F'=f$. Then $(F \circ h)'=(f \circ h)h'$. Thus $(f \circ h)h'$ is a derivative and hence so is the function $(f \circ h)=(f \circ h)\cdot h' \cdot 1/h'$.

Thus h cannot destroy a derivative. Fleissner **[8]** has recently shown that if h' is continuous and $1/h'$ is of bounded variation, then f/h' is a derivative for each derivative f. Thus, such an h also has the property that $f \circ h$ is a derivative whenever f is.

A necessary and sufficient condition h preserve derivatives has recently been advanced by Laczkovich and Petruska **[14]**. Their condition is too complicated to develop here. As the authors point out, however, it is not very far from the condition that $1/h'$ be continuous and of bounded variation (the condition we just considered).

If we also restrict the class of derivatives, additional results can be stated. For instance, if both f and f^2 are derivatives and both h and h^{-1} satisfy a Lipschitz condition, then both $f \circ h$ and $(f \circ h)^2$ are

derivatives [2]. Thus this class appears again, and it is closed under bilipschitzian changes of variables. (The functions f and h defined just before the statement of Theorem 8 show that we cannot drop the requirement that f^2 be a derivative. In the present setting, h has to be constructed with a bit of care to guarantee that it and its inverse satisfy a Lipschitz condition.)

3. Related problems. The problems we considered in Sections 1 and 2 are examples of a certain type of problem. Let $\mathcal{F}$ be a class of functions defined on, say, $[0,1]$, and let $\mathcal{H}$ denote the class of homeomorphisms of $[0,1]$ onto itself. We can ask whether $\mathcal{F}\circ\mathcal{H}=\mathcal{F}$. If the answer is "no," then we ask for a characterization of the class $\mathcal{G}=\mathcal{F}\circ\mathcal{H}$. (How badly can $\mathcal{H}$ "destroy" $\mathcal{F}$, or, equivalently, out of what functions can we "create" functions in $\mathcal{F}$?) We can also ask a number of related questions which arise by restricting $\mathcal{F}$, $\mathcal{H}$ or both. The general problem is to determine conditions on f and h under which $f\circ h\in\mathcal{F}$ ($f\in\mathcal{F}$, $h\in\mathcal{H}$). Similar questions can be asked for the composition $\mathcal{H}\circ\mathcal{F}$.

In Section 1 we were concerned exclusively with the class of differentiable functions (or certain subclasses) and in Section 2, the class of derivatives (or certain subclasses). We saw that there still were a number of questions that haven't been answered. (For example, what characterizes $\mathcal{H}\circ\mathcal{F}$ when $\mathcal{F}$ consists of the functions with continuous derivatives?) We also do not have characterizations for the classes $\mathcal{H}\circ\Delta$, $\mathcal{H}\circ b\Delta$ where $\Delta(b\Delta)$ is the class of derivatives (bounded derivatives). In [21], Zahorski constructed a hierarchy of classes, $\mathfrak{M}_1\supset\mathfrak{M}_2\supset\mathfrak{M}_3\supset\mathfrak{M}_4\supset\mathfrak{M}_5$. Each of these classes is defined in terms of associated sets. The class $\mathfrak{M}_1$ turns out to be $\mathfrak{D}\mathfrak{B}_1$ and $\mathfrak{M}_5$ turns out to be the class of approximately continuous functions. Because these classes are defined in terms of associated sets, $\mathcal{H}\circ\mathfrak{M}_i=\mathfrak{M}_i$ for each i. Zahorski showed that for bounded functions $\mathfrak{M}_5\subset\Delta\subset\mathfrak{M}_4$ and in any case $\Delta\subset\mathfrak{M}_3$. Thus $\mathcal{H}\circ b\Delta\subset\mathfrak{M}_4$ and $\mathcal{H}\circ\Delta\subset\mathfrak{M}_3$. Are these inclusions proper? It would be of interest to know. If so, what classes does one get?

The class $\mathfrak{M}_5$ of approximately continuous functions is important to the study of derivatives in many ways. That $\mathcal{H}\circ\mathfrak{M}_5=\mathfrak{M}_5$ is well known, and that $\mathfrak{M}_5\circ\mathcal{H}=\mathfrak{D}\mathfrak{B}_1$ is due to Maximoff [15]. Under what conditions is $f\circ h\in\mathfrak{M}_5$ for $f\in\mathfrak{M}_5$, $h\in\mathcal{H}$? A number of conditions have been given in [1] but the question has not yet been fully answered.

There are, of course, a number of classes of generalized derivatives that have been studied for various purposes (e.g., approximate derivatives, symmetric derivatives, Dini derivatives, Peano derivatives, etc.). The questions we asked could be asked equally well of each of these classes. We are not aware of any work that has been done in this direction. And, of course, one need not restrict oneself to classes $\mathcal{F}$ related to differentiation. For example, the class $\mathcal{F}\circ\mathcal{H}$ has recently been studied for $\mathcal{F}$, the class of continuous periodic functions whose Fourier series converge everywhere [12].

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A CIRCLE-OF-LIGHTS ALGORITHM FOR THE "MONEY-CHANGING PROBLEM"

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I. Introduction. The classical "money-changing problem" asks the following question: If we have coins of k different values, namely $a_1,\ldots,a_k$ units respectively, then what sums of money can we change, assuming that an infinite supply of each of the k denominations is available?

For example, if we have only 3-cent coins and 5-cent coins, then we can change 0, 3, 5, 6 cents and any amount $\geqslant 8$ cents.

In general, we are given integers $0<a_1<a_2<\cdots<a_k$ and we suppose that g.c.d.$(a_1,\ldots,a_k)=1$. We ask for a description of the set (semigroup) of integers n which can be written in the form

$$n=x_1a_1+x_2a_2+\cdots+x_ka_k \qquad (x_i\geqslant 0,\quad i=1,\ldots,k). \tag{1}$$

It is well known that there is a least positive integer χ, called the *conductor* of $a_1,\ldots,a_k$, such that $n=\chi-1$ is not representable as in (1), but *every* $n\geqslant\chi$ is representable. Sylvester showed that for $k=2$ the conductor is $\chi=(a_1-1)(a_2-1)$, and more general results may be found in [1], [5].

Our concern here is with algorithms for finding the conductor, for determining whether or not a given integer is representable, for finding a representation of a given integer, and for determining the number of n which are not representable ("omitted values").

Consider first the calculation of the conductor. A. Brauer [1] suggested the following procedure: For each $r=0,1,\ldots,a_1-1$ let n_r denote the least value of n such that $n\equiv r \pmod{a_1}$ and n is representable in the form (1). Then we have

$$\chi=1-a_1+\max_r(n_r). \tag{2}$$

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The question of precisely how the n_r are to be found was left open.

Heap and Lynn ([3], [4]) have proposed another algorithm for finding χ. One would define a matrix C by

$$C_{ij}=\begin{cases}1 \text{ if } i-j=a_p-1 & \text{for some } p\\ 1 \text{ if } i-j=-1 & \\ 0 \text{ otherwise} & \end{cases} \tag{3}$$

for $i,j=1,\ldots,N$, where $N=a_k$. They showed that if γ denotes the least power of C all of whose entries are positive then $\chi=\gamma-a_k+1$. The index γ is found by bisection of some interval which is known to contain it. This algorithm needs $O(a_k^3(\log\chi)^2)$ time and about $2a_k^3$ bits of array storage, and so, while quite elegant, it would not be preferred for large problems.

II. The circle of lights. Suppose $1<a_1<\cdots<a_k$ are given relatively prime integers. For each $n=0,1,2,\ldots,$ let x_n denote the proposition that the integer n is representable in the form (1). Then we have

$$\begin{array}{ll}\text{(a)} & x_m=\text{false } (m<0)\\ \text{(b)} & x_0=\text{ true}\\ \text{(c)} & \text{For } m>0:\\ & \quad x_m=x_{m-a_1} \text{ or } x_{m-a_2} \text{ or} \cdots \text{or } x_{m-a_k}.\end{array} \tag{4}$$

Now imagine that we are calculating the x_m recursively from (4). Observe that if at any time we encounter a_1 consecutive values of m for which x_m is true, then all succeeding x_m are true, and so the calculation can halt and the conductor will be 1 greater than the last m such that x_m was false.

Further, since each x_m is found from only its $N=a_k$ predecessors, we can over-write x_{m-N} with x_m as soon as the latter is computed. Thus the array x requires only a_k bits of storage as a circular list.

The whole procedure can be visualized as a circle of N lights, numbered $0,1,\ldots,N-1$. Initially light 0 is on, the others off. We sweep around the circle in a clockwise direction, and as we encounter each light, we will turn it on if any of the k lights which are situated $a_1,a_2,\ldots,a_k$ lights back (counterclockwise) from the present are on, we leave it on if it was already on, otherwise we leave it off. The process halts as soon as any a_1 consecutive bulbs are "on."

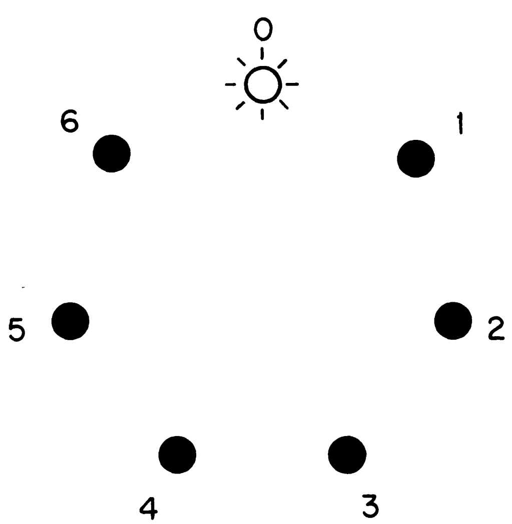

FIG. 1

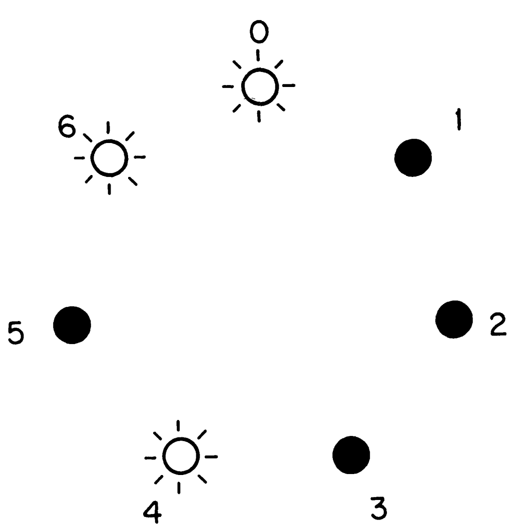

FIG. 2

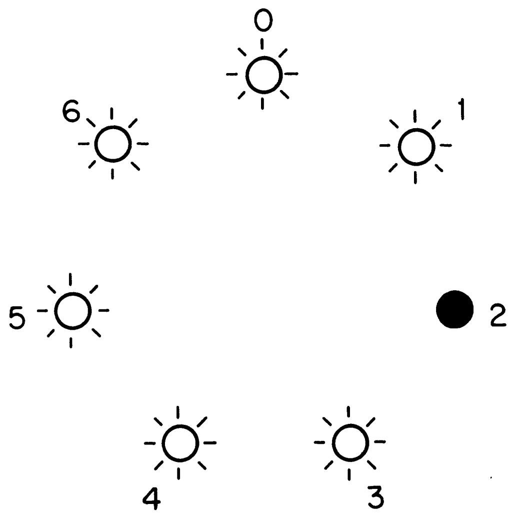

FIG. 3

As an example, consider the integers $(a_1,a_2,a_3)=(4,6,7)$. We have a ring of 7 lights which is initially as shown in Figure 1. After one full sweep around the circle the situation is as shown in Figure 2.

After a portion of one more sweep, going only as far as light number 5, the lights are as shown in Figure 3.

Now we have $\geqslant 4$ consecutive lights on, so the process can halt. The conductor is one more than the number represented by the most recent "off" bulb, i.e., it is $1+9=10$.

It is interesting that the algorithm provides a proof of its own complexity:

THEOREM. *Let $1<a_1<a_2<\cdots<a_k$ be relatively prime, and let x be their conductor. Then $\chi\leqslant a_k^2$.*

Proof. After each full sweep of the circle of lights, at least one more light must be "on," or else, since all subsequent sweeps would be identical, the conductor would be infinite. (Professor M. Koren has shown (p.c.) that actually every sweep interval of length a_1 produces a new light on.) After a_k sweeps they will all be on. ∎

We state a formal algorithm which is based on these ideas. Input are $1<a_1<\cdots<a_k$, relatively prime. Output are the conductor χ, and Ω, the number of omitted values. The counter q counts consecutive assumed values and is reset to zero at each omitted value.

(A) $q\leftarrow 0$; $N\leftarrow a_k$; $x_0\leftarrow$true; $x_\mu\leftarrow$false $(\mu=1,N-1)$; $m\leftarrow a_1$; $\Omega\leftarrow 0$
(B) [m is representable] $r\leftarrow m \pmod N$; $x_r\leftarrow$true
(C) $q\leftarrow q+1$; if $q\neq a_1$, go to (D); $\Omega\leftarrow m-a_1-\Omega$; $\chi\leftarrow m-a_1+1$; exit.
(D) [next m] $m\leftarrow m+1$; $i\leftarrow k$.
(E) [Is $m-a_i$ representable?] $r\leftarrow m-a_i \pmod N$; if x_r is true, go to (B); if $i=1$, go to (F); $i\leftarrow i-1$; go to (E).
(F) [m is omitted] $\Omega\leftarrow\Omega+q$; $q\leftarrow 0$; go to (D). ∎

The algorithm requires a_k bits of array storage and $O(ka_k^2)$ time. A Fortran program of 28 instructions found the conductor $x=13023$ of the set 271, 277, 281, 283 in 1,2 seconds of IBM 370/168 time, along with the number $\Omega=6533$ of values omitted by the form. Less than 75 words of array storage were used.

When the same algorithm was written for a little programmable calculator with 20 words of memory, the same problem was solved in about six hours!

To discover if a given n is representable, we would incorporate Brauer's suggestion: In addition to the array x we carry an array y of length a_1 words which stores the first occurrence of each residue class $\bmod a_1$. A given n is then representable if and only if $n \geqslant y_q$ where $q \equiv n \pmod{a_1}$.

To construct an explicit representation of a given n we would generate and store explicit representations of the y_j defined above.

We raise the following questions: (a) Is it true that for fixed k the fraction Ω/χ of omitted values is at most $1-(1/k)$ with equality only for the generators $k, k+1, \ldots, 2k-1$? (b) Let $f(n)$ be the number of semigroups whose conductor is n. What is the order of magnitude of $f(n)$ for $n \to \infty$?

I am pleased to thank Professor Nijenhuis for interesting discussions of this problem.

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DISCUSSION ON THE PROGRESS OF PURE ANALYSIS

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Editorial note. The story of Galois' life (1811–1832) is rather well known. (See, for example, E. T. Bell, *Men of Mathematics*, Simon and Schuster, New York, 1937; L. Infeld, *Whom the Gods Love: The Story of Evariste Galois*, Whittlesey House, New York, 1948.) The essay that follows (and has apparently not previously appeared in English) was written in 1832 as an introduction to a projected series of articles. You should keep in mind that what Galois meant by *Analysis* is not what we mean by the same word today: it included not only algebra but everything that involved calculation (rather than verbal or geometric reasoning).

We know that of all the bodies of human knowledge pure analysis is the most abstract, the most supremely logical, the only one which borrows nothing from the evidence of the senses. Many conclude from these facts that all in all it is the most methodical and the best coordinated subject. But this is an error. Take an algebra book, whether a textbook or some original work, and you will see nothing but a confused accumulation of propositions, whose individual logical structures contrast bizarrely with the disorder of the work as a whole. It seems that the ideas have already cost the author too much for him to take the trouble to tie them together and that the conception of the ideas that underlie his work has so exhausted him that his mind has lost the strength to give birth to an organizing idea.

If you do find an organization, a connection, a coordination, these are artificial and false. There are groundless subdivisions, arbitrary unifications, and entirely conventional arrangements. This defect, worse than the lack of any organization at all, is especially common in textbooks. Most of these are put together by men who have no clear understanding of the science that they teach.

All this will greatly astonish the man in the street, who generally takes the word *mathematics* to be a synonym for logical organization.

However, one will be still more astonished if one reflects that here as elsewhere science is the work of the human mind, which is destined rather to study than to know, to seek the truth rather than to

find it. One can conceive that a mind which had the power to perceive at one glance the entire body of mathematical truths—not just those known to us but all possible truths—could also deduce them methodically and mechanically, as it were, from a few principles by a uniform method. Then there would be no obstacles, nor any of those difficulties—so often imaginary—that the scholar finds in his explorations. But then there would also no longer be a role for the scholar. This is not the way it is. If the scholar's task is more difficult and therefore nobler, the progress of the science is also less methodical. The science progresses by a series of contrivances where chance plays more than a small role; its life is inorganic and resembles that of minerals that grow through juxtaposition. This applies not only to the science as it results from the work of a series of scholars but also to the individual research of each of them. The analysts would like to hide this from themselves; in vain. They do not deduce; they combine, they agglomerate. Completely abstract as it is, analysis is no more within our power than are other branches of knowledge. One must keep a close eye on it, probe it, plead with it. When the analysts do arrive at the truth, it is by virtue of groping around the sides of the hole into which they have fallen.

In common with creative works, textbooks necessarily share the defect that whenever the subject they treat is not thoroughly known, there cannot be a systematic presentation. Therefore a synthesized presentation is possible only for a very small number of topics. To give one for analysis would call for profound understanding. The futility of this undertaking repels those who might have the mental stamina for it.

It would be beneath the seriousness of this essay to introduce any personal sentiments towards scholars into such a discussion—be they sentiments of forbearance or of hostility. The author of these articles will avoid the one and the other of these two reefs. If a thorny past guarantees the former, a deep love for the science, which makes him respect those who cultivate it, will assure renunciation of the second.

In the sciences it is distressing to engage solely in the role of critic; we shall do so only when obliged or forced to. When our abilities permit, after noting a defect we shall indicate what in our eyes would be better. We shall thus often have occasion to call the reader's attention to the new ideas to which the study of analysis has led us. We shall allow ourselves to familiarize him with these ideas in our first articles in order not to have to return to them.

In less abstract subjects—in the case of works of art—it would be highly ridiculous to precede a critical discussion with one's own works. It would be too naive an admission of what at bottom is almost always true—that one makes one's self the standard of comparison. But here it is not a question of an objective, completed work, but of the most abstract ideas conceivable by man. Here criticism and discussion become synonymous, and to discuss is to throw one's own ideas into the arena with the ideas of others.

We shall therefore set forth in these articles whatever is most general and most philosophic in the researches that a thousand circumstances have prevented us from publishing sooner. We shall present them without the complications of examples and digressions, which usually submerge the generalized conceptions of the analysts. We shall, above all, explain them in good faith, indicating without evasiveness the path which led us to them as well as the obstacles which hindered us. For we want the reader to be as well informed as we are in the matters we shall treat. When this goal shall have been reached we shall be conscious of having done well, if not for the profit which the science will derive directly from it, then at least for the example of a heretofore unseen candor.

Translated by HELEN M. KLINE

The following paragraph was intended as part of the introduction to Galois' memoirs on the solution of algebraic equations by radicals; it shows the same mind at work in a vein similar to that of the preceding essay.

[These papers] contain at least as much French as algebra, so much so that when the printer was given the manuscripts, he really thought that they were only an introduction. Here I have no excuse;

it would have been so easy to reproduce an entire theory in detail on the pretext of presenting it as necessary for understanding the papers, or even better, with less work, to interlard a branch of science with two or three new theorems, without pointing out which ones! It would also have been so easy to substitute successively all the letters of the alphabet into each equation, numbering them in order to be able to see which combinations of letters the later equations belong to; this could have multiplied the number of equations indefinitely, if we reflect that after the Latin alphabet there is still the Greek alphabet, and after that is used up, there are still German letters, and nothing prevents us from using Syriac letters, and if necessary Chinese letters! It would have been so easy to transform each sentence ten times, carefully preceding each version with the pompous word "Theorem"; or again, to obtain by OUR ANALYSIS results known since good old Euclid; or finally to precede and follow each proposition with a formidable train of special cases! And from among so many possibilities I have not been able to select even one!

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PROGRESS REPORTS

EDITED BY P. R. HALMOS

Material for this Department should be sent to P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, Indiana 47401.

It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

HAUPTVERMUTUNG

H. SAMELSON

Sometimes long-standing and seemingly hopeless problems get solved. One that did is the "Hauptvermutung der kombinatorischen Topologie" (principal conjecture of combinatorial topology).

Put very briefly, the conjecture said that homeomorphism implies combinatorial equivalence. The answer as it is now known says that for manifolds the conjecture is nearly true; the deviation from the truth (the "nearly") is made quite explicit.

To explain what it all means, it is best to go back to the early days of topology. It is not surprising that in those days concepts such as curve, surface, polyhedron, manifold, and topological space were not very well defined. One used the engineering approach, where everybody is already supposed to know what is meant and where giving a name to an object is supposed to explain all its properties. Even Riemann's definition of a manifold in his famous lecture "Über die Hypothesen..." is of this type (although, of course, the necessary ideas are all present).

Poincaré's definitions were somewhat free and easy too. In fact, partly because of the very intuitive character of his definitions, he ran into trouble by overlooking "torsion" (that is, elements of finite order in the relevant groups). When this was pointed out to him (by Heegaard) he started over again with a more rigorous approach, considering his spaces as divided into "cells."

Even this needed sharpening; it led after a while to the concepts of "polyhedron" and "simplicial complex." [The latter is a finite set of simplices in $\mathbf{R}^k$ with the property that the intersection of any two of them is a face of both. A polyhedron is the union of the simplices in a simplicial complex, and is thus a (very special type of) subset of $\mathbf{R}^k$. A simplex of dimension r (or r-simplex) is the convex hull of $r+1$ independent points in $\mathbf{R}^k$, where "independent" means that the convex hull has dimension r, not less. Thus a 0-simplex is a point, a 1-simplex is a segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc.] There is an associated notion of piecewise linear (PL) map: a map that (after dividing the polyhedra, if needed, rectilinearly into smaller simplices) maps each simplex linearly into some simplex.

Combinatorial (or PL) manifolds of dimension n appeared on the scene. They are polyhedra that are not only locally like $\mathbf{R}^n$ (every point has a neighborhood homeomorphic to $\mathbf{R}^n$), but have the subtler property of satisfying this condition in the PL sense (every point has a neighborhood PL homeomorphic to $\mathbf{R}^n$—where it is understood that $\mathbf{R}^n$ is triangulated in the usual rectilinear way). The simplicial objects are, so to speak, finite, and therefore easier to handle; all definitions and operations become clear and crisp.

Brouwer managed to fit ordinary continuous maps into this scheme, through his ideas of simplicial approximation. This led to, e.g., the topological invariance of the combinatorially defined homology groups. What one really was after, however, were topological manifolds (locally homeomorphic to $\mathbf{R}^n$, with no PL or differentiability proviso), and the step from simplicial homeomorphism to plain homeomorphism seemed infinite. It seemed best to give up and simply to state the step as the Hauptvermutung (HV): *if two polyhedra are homeomorphic, then they are* PL *homeomorphic* (Steinitz, 1907). There is also the restriction of this to PL manifolds, and there is the related triangulation conjecture (Tr): *every topological manifold can be triangulated into a* PL *manifold.*

For the lowest dimension ($n=1$), there is no problem. For surfaces ($n=2$), HV and Tr were proved by Kerékjártó and by Radó (1923–24), and for 3-manifolds by Moise (1952). There were some early wrong proofs for the general case, and then the problem slumbered for a long time; homeomorphisms seemed just too complicated to handle.

In the meantime topology grew. The many new concepts that arose included vector bundles and fiber bundles, classifying spaces, characteristic classes, Thom's new decisive approach to classifying differentiable manifolds (up to the relation "bounding") by a systematic use of transversality (the implicit function theorem), and the corresponding notions for PL manifolds, surgery (the n-dimensional version of the old-fashioned scissors-and-glue topology), handle bodies, Whitehead torsion, s-bordism, Kervaire's example of a manifold that cannot be made differentiable, Milnor's exotic differentiable structures on the 7-sphere, etc., etc. A startling note was Milnor's example (1961) of two polyhedra that are homeomorphic but not PL homeomorphic; that disproved HV and it might have seemed like the end of the line. But eventually all this work, together with new ideas, led to positive results for manifolds. Sullivan (1967) and Lashof–Rothenberg (1969) gave an answer that says roughly

that HV holds for a manifold M if the cohomology group $H^3(M; \mathbf{Z}/2)$ is 0, and similarly Tr holds if $H^4(M; \mathbf{Z}/2)$ is 0. The conditions on H^3 and H^4 appearing here are surprising. The full story is now known; it is due to Kirby and Siebenmann (1969), and it explains the role of H^3 and H^4. A basic role is played by two (topological) groups called PL and TOP. Here PL_n means, roughly, the group of PL homeomorphisms, and TOP_n the group of ordinary homeomorphisms of $\mathbf{R}^n$, and PL and TOP are direct limits under the obvious maps $\mathrm{PL}_n \to \mathrm{PL}_{n+1}$ and $\mathrm{TOP}_n \to \mathrm{TOP}_{n+1}$. These groups yield TOP/PL, a suitably defined quotient group. That TOP and PL and TOP/PL enter is not surprising. The coordinate transformations in an atlas for a manifold come (at least locally) from TOP_n or PL_n, and one formulation of the whole problem is whether the coordinate transformations, which are in TOP_n to begin with, can be "improved" so as to become elements of PL_n.

The question would be easy to answer if the quotient TOP/PL were contractible; all "obstructions" would then be 0, and HV and Tr would be true without any restriction. It turns out that TOP/PL is not very far from being contractible. It fails by a small amount: its third homotopy group is $\mathbf{Z}/2$, and all others are 0. (Recall that the ith homotopy group of a space X consists of the homotopy classes of maps of the sphere S^i into X.) It is this $\mathbf{Z}/2$ that gives rise to the H^3 and H^4 above. There are non-trivial "obstructions" to Tr and HV in H^3 and H^4; if a manifold avoids them by having its H^3 or H^4 vanish, then Tr or HV hold. (In due course counterexamples to Tr and HV were found; they involved these groups, of course.)

It is difficult to give a brief description of the methods. They are complex, they involve most of the topics mentioned above, and they involve striking new constructions. They are much more sophisticated than the simple-minded early approaches, but they have the starting point in common with them: take the coordinate transformations that occur in the definition of a manifold and try to improve them.

It should be noted that in all this the dimension of M has to be $\geqslant 5$ (or $\geqslant 6$ if there is a boundary). There are still open questions in dimension 4, just as the Poincaré question in dimension 3 is still alive. (Is every simply connected, closed 3-manifold homeomorphic to S^3?) The most "difficult" dimensions are, in fact, 3 and 4; surely this is what the creator had in mind when he made the world 3 (or 4?) dimensional.

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MISCELLANEA

11. Complication is one of the manias of our era. If you are saying something really complex, you should do your best to spell it out for the reader in so far as is possible. ...In fine, it is no use showing off. You will only, when the chips are down, have produced the effect of having done so.

Pamela Hansford Johnson,
Important to Me, 1974, p. 244.
(Reprinted by permission of the author.)

MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

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SOME PRODUCT-SUM IDENTITIES

L. CARLITZ

1. Making use of the Borel-Cantelli lemma, Stern [**1**] has proved the following result. Let $\{a_n\}$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Then

$$\prod_{n=1}^{\infty} (1-a_n)^{-1} = 1 + \sum_{k=1}^{\infty} \sum_{S_k} \frac{a_{n_1}}{1-a_{n_1}} \frac{a_{n_2}}{1-a_{n_2}} \cdots \frac{a_{n_k}}{1-a_{n_k}}, \tag{1}$$

where $S_k = (n_1, n_2, \ldots, n_k)$ is any set of positive integers such that $n_1 < n_2 < \cdots < n_k$. Several special cases of (1) are also given.

It may be of interest to point out that (1) can be generalized in various ways and that such results can be proved easily without probabilistic arguments.

In the first place consider the finite product

$$P_N = \prod_{n=1}^{N} (1-a_n)^{-1}, \qquad a_n \neq 1$$

where the a_n are now arbitrary. Since $1/(1-a_n) = 1 + (a_n/(1-a_n))$, we have

$$P_N = \prod_{n=1}^{N} \left(1 + \frac{a_n}{1-a_n}\right).$$

Expanding the product on the right, we get

$$P_N = 1 + \sum_{k=1}^{N} \sum \frac{a_{n_1}}{1-a_{n_1}} \frac{a_{n_2}}{1-a_{n_2}} \cdots \frac{a_{n_k}}{1-a_{n_k}},$$

where the inner sum is over all sets of positive integers $n_1, n_2, \ldots, n_k$ such that $n_1 < n_2 < \cdots < n_k \leqslant N$.

If we now let $N \to \infty$ and assume that the a_n satisfy $\sum_{n=1}^{\infty} |a_n| < \infty$, it is clear that we get (1) without the restriction that the a_n be positive.

2. Consider next the finite product

$$Q_N = \prod_{n=1}^{N} \frac{1-a_n b_n}{1-a_n}, \qquad a_n \neq 1,$$

where the a_n, b_n are arbitrary. Since

$$\frac{1-a_n b_n}{1-a_n} = 1 + \frac{a_n(1-b_n)}{1-a_n},$$

we get, exactly as above,

$$Q_N = 1 + \sum_{k=1}^{N} \sum \frac{a_{n_1}(1-b_{n_1})}{1-a_{n_1}} \frac{a_{n_2}(1-b_{n_2})}{1-a_{n_2}} \cdots \frac{a_{n_k}(1-b_{n_k})}{1-a_{n_k}}, \tag{2}$$

where the inner sum is over all sets of positive integers $n_1, n_2, \ldots, n_k$ such that $n_1 < n_2 < \cdots < n_k \leqslant N$.

Now let $N \to \infty$ and assume that $\sum_{n=1}^{\infty} |a_n| < \infty$, $|b_n| < B$, $(n = 1,2,3,\ldots)$, where B is some positive constant. Then (2) gives

$$\prod_{n=1}^{\infty} \frac{1-a_n b_n}{1-a_n} = 1 + \sum_{k=1}^{\infty} \sum \frac{a_{n_1}(1-b_{n_1})}{1-a_{n_1}} \frac{a_{n_2}(1-b_{n_2})}{1-a_{n_2}} \cdots \frac{a_{n_k}(1-b_{n_k})}{1-a_{n_k}}, \tag{3}$$

where the inner sum is over all positive integers $n_1, n_2, \ldots, n_k$ such that $n_1 < n_2 < \cdots < n_k$.

Note that (2) can also be written in the form

$$\prod_{n=1}^{N} \frac{1-b_n}{1-a_n} = 1 + \sum_{k=1}^{N} \sum \frac{a_{n_1}-b_{n_1}}{1-a_{n_1}} \frac{a_{n_2}-b_{n_2}}{1-a_{n_2}} \cdots \frac{a_{n_k}-b_{n_k}}{1-a_{n_k}}. \tag{4}$$

A similar remark applies to (3).

The special case $b_n = -a_n$ of (4) may be noted:

$$\prod_{n=1}^{N} \frac{1+a_n}{1-a_n} = 1 + \sum_{k=1}^{N} 2^k \sum \frac{a_{n_1}}{1-a_{n_1}} \frac{a_{n_2}}{1-a_{n_2}} \cdots \frac{a_{n_k}}{1-a_{n_k}}.$$

Hence, for $N \to \infty$, we get

$$\prod_{n=1}^{\infty} \frac{1+a_n}{1-a_n} = 1 + \sum_{k=1}^{\infty} 2^k \sum \frac{a_{n_1}}{1-a_{n_1}} \frac{a_{n_2}}{1-a_{n_2}} \cdots \frac{a_{n_k}}{1-a_{n_k}},$$

where of course $\sum_{n=1}^{\infty} |a_n| < \infty$.

3. A further generalization is obtained by taking

$$R_N = \prod_{n=1}^{N} \frac{(1-a_n b_n)(1-a_n c_n)}{(1-a_n)(1-a_n b_n c_n)}, \qquad a_n \neq 1, \qquad a_n b_n c_n \neq 1.$$

Since

$$\frac{(1-a_n b_n)(1-a_n c_n)}{(1-a_n)(1-a_n b_n c_n)} = 1 + \frac{a_n(1-b_n)(1-c_n)}{(1-a_n)(1-a_n b_n c_n)},$$

we get

$$R_N = 1 + \sum_{k=1}^{N} \sum \frac{a_{n_1}(1-b_{n_1})(1-c_{n_1})}{(1-a_{n_1})(1-a_{n_1} b_{n_1} c_{n_1})} \cdots \frac{a_{n_k}(1-b_{n_k})(1-c_{n_k})}{(1-a_{n_k})(1-a_{n_k} b_{n_k} c_{n_k})},$$

where $n_1 < n_2 < \cdots < n_k \leqslant N$.

For $N \to \infty$, this yields the identity

$$\prod_{n=1}^{\infty} \frac{(1-a_n b_n)(1-a_n c_n)}{(1-a_n)(1-a_n b_n c_n)} = 1 + \sum_{k=1}^{\infty} \sum \frac{a_{n_1}(1-b_{n_1})(1-c_{n_1})}{(1-a_{n_1})(1-a_{n_1} b_{n_1} c_{n_1})} \cdots \frac{a_{n_k}(1-b_{n_k})(1-c_{n_k})}{(1-a_{n_k})(1-a_{n_k} b_{n_k} c_{n_k})}, \tag{5}$$

where $n_1 < n_2 < \cdots < n_k$ and

$$\sum_{n=1}^{\infty} |a_n| < \infty, \ |b_n| < B, \ |c_n| < C \ (n = 1,2,3,\ldots),$$

where B and C are constants.

There are various alternate forms of (5), in particular

$$\prod_{n=1}^{N}\frac{a_n(1-b_n)(1-c_n)}{(1-a_n)(a_n-b_nc_n)}=1+\sum_{k=1}^{N}\sum\frac{(a_{n_1}-b_{n_1})(a_{n_1}-c_{n_1})}{(1-a_{n_1})(a_{n_1}-b_{n_1}c_{n_1})}\cdots\frac{(a_{n_k}-b_{n_k})(a_{n_k}-c_{n_k})}{(1-a_{n_k})(a_{n_k}-b_{n_k}c_{n_k})},$$

where $n_1<n_2<\cdots<n_k\leqslant N$.

For $b_n=c_n=-1$, (5) reduces to

$$\prod_{n=1}^{N}\left(\frac{1+a_n}{1-a_n}\right)^2=1+\sum_{k=1}^{N}2^{2k}\sum\frac{a_{n_1}}{(1-a_{n_1})^2}\cdots\frac{a_{n_k}}{(1-a_{n_k})^2},$$

where $n_1<n_2<\cdots<n_k\leqslant N$. Finally, for $N\to\infty$, we get

$$\prod_{n=1}^{\infty}\left(\frac{1+a_n}{1-a_n}\right)^2=1+\sum_{k=1}^{\infty}2^{2k}\sum\frac{a_{n_1}}{(1-a_{n_1})^2}\cdots\frac{a_{n_k}}{(1-a_{n_k})^2},$$

where $n_1<n_2<\cdots<n_k$ and $\Sigma_{n=1}^{\infty}|a_n|<\infty$.

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A CHARACTERIZATION OF THE INTEGERS AMONG EUCLIDEAN DOMAINS

Steven Galovich

A **Euclidean domain** is an integral domain R together with a function $g:R^*\to N$, where $R^*=R-\{0\}$ and N is the set of nonnegative integers, such that:

(i) $g(ab)\geqslant g(a)$ for all $a,b\in R^*$,

(ii) if $a\in R$, $b\in R^*$, then there exist $q,r\in R$ such that $a=qb+r$, where either $r=0$ or $g(r)<g(b)$.

We will write (R,g) to indicate the domain R and its algorithm function g. We assume henceforth that R is not a field.

In certain cases the domain R is completely determined by properties of the function g. For example, if (R,g) is a Euclidean domain such that for each $a\in R$, $b\in R^*$, the pair q,r in (ii) is unique, then R is isomorphic to the ring of polynomials in one variable over a field. (For a proof, see **[2]** or **[3]**. A closely related result is given in **[1]**.) As another example, consider a Euclidean domain (R,g) whose algorithm satisfies: (a) $g(ab)=g(a)g(b)$ for all $a,b\in R^*$, (b) $g(a+b)\leqslant g(a)+g(b)$ for all $a,b\in R^*$ such that $a+b\neq 0$, (c) $g(a)=g(b)$ if and only if a and b are associates. Then Picavet **[3]** proves that $R=Z$, the ring of integers. In this note we give a characterization of Z as a Euclidean domain which is analogous to the characterization described above of the ring of polynomials in one variable over a field.

We say that (R,g) has the **double remainder property** (abbreviated **d.r.p.**) if, for each pair $a\in R$, $b\in R^*$ such that b does not divide a, there exist exactly two pairs $q_i,r_i,i=1,2$, such that $a=q_ib+r_i$ with $g(r_i)<g(b)$. Note that **d.r.p.** is equivalent to the condition that for $a\in R,b\in R^*$ with b not dividing a, there exist exactly two elements r_1,r_2 such that $g(r_i)<g(b)$ and $a\equiv r_i(\bmod b)$, $i=1,2$. It is quite easy to check that $(Z,|\;|)$ has **d.r.p.** Our main result asserts that as a Euclidean domain Z is characterized by this condition.

Theorem. *If* (Rg) *is a Euclidean domain with* **d.r.p.**, *then* $R=Z$.

Before beginning, let us indicate the key steps in the proof. First we show that $U(R)=\{\pm 1\}$, where $U(R)$ is the group of units of R. Then we show that for $x\in R^*, R/(x)$ is a finite ring with, say, $N(x)$ elements (Corollary to Lemma 4). Next we establish an important relationship between the algorithm g and the "norm" function N (Corollary to Lemma 6). This connection between g and N facilitates an inductive proof of the theorem.

We now fix some terminology. For a Euclidean domain (R,g), let $R_1=\{x\in R^* \mid g(x)\leqslant g(y)$ for all $y\in R^*\}$. By (i) and (ii), $R_1=U(R)$. For $n\geqslant 2$, let

$$R_n=\{x\in R^* \mid g(x)\leqslant g(y) \text{ for all } y\in R^*-R_{n-1}\}.$$

Clearly, $R_n\subseteq R_{n+1}$ for $n\geqslant 1$, and $\cup_{n=1}^{\infty}R_n=R^*$. For the remainder of the argument, we assume that (R,g) is a Euclidean domain with **d.r.p.**

LEMMA 1. *If* $u\in U(R)$ *and* $u\neq\pm 1$, *then* $1+u\in U(R)$.

Proof. Recall that if $v\in U(R)$, then $g(v)<g(r)$ for $r\in R^*-U(R)$. Observe that no two of $1, -u, u^2$ are equal to each other but that $1\equiv -u\equiv u^2 \pmod{(1+u)}$. Thus if $1+u\notin U(R)$, then **d.r.p.** is violated for the pair $1, 1+u$.

LEMMA 2. $U(R)\cup\{0\}$ *is not a field with respect to the operations of addition and multiplication in* R.

Proof. Assume that $U(R)\cup\{0\}$ is a field under the operations of R. Let $r\in R_2-R_1$. Since R has **d.r.p.**, there exist $u\in U(R)$, $u\neq 1$, and $q\in R$ such that $1=q\cdot r+u$. By assumption, $qr=1-u\in U(R)$, and hence $r\in U(R)$, a contradiction of the choice of r.

LEMMA 3. *As an element of* R, 2 *is a nonzero nonunit.*

Proof. If $2\in U(R)\cup\{0\}$, then Lemma 1 assures us that for all $u,v\in U(R)$, either $u+v=0$ or $u+v=u(1+u^{-1}v)\in U(R)$. As a result $U(R)\cup\{0\}$ is a field under the operations of R, a contradiction of Lemma 2.

Let char(R) denote the characteristic of the ring R. By Lemma 3, char$(R)\neq 2$ and $2\notin U(R)$. Thus we have the following result.

COROLLARY. *Char*$(R)=0$.

Proof of Theorem. We first show that $R_1=U(R)=\{\pm 1\}$. Assume that there exists $u\in U(R)$ with $u\neq\pm 1$. Then $u\equiv u-2\equiv u+2 \pmod 2$. However, by Lemma 1, $2+u=1+(1+u)\in U(R)$ and $u-2=-(1+(1-u))\in U(R)$. Thus we have contradicted **d.r.p.** Therefore, $R_1=\{\pm 1\}$.

LEMMA 4. *For each* n, R_n *is a finite set.*

Proof. We have just seen that R_1 is finite. Assume that R_{n-1} is finite and let $x\in R_n-R_{n-1}$. By **d.r.p.**, each nonzero coset of the ideal (x) contains exactly two elements of R_{n-1}. Thus, $R/(x)$ is a finite ring with $k=1+\frac{1}{2}(\# R_{n-1})$ elements. Since $R/(x)$ is a finite group (under addition) with k elements, it follows that in $R/(x)$, $k(1+(x))=k+(x)=(x)$. Therefore, $k=1+\frac{1}{2}(\# R_{n-1})$ is divisible by x. Because R is a unique factorization domain with a finite group of units, $1+\frac{1}{2}(\# R_{n-1})$ has only a finite number of divisors in R. The lemma is now proved.

Remark: Since R is an infinite set and since each R_n is a finite set, $R_n \subsetneq R_{n+1}$ for $n\geqslant 1$.

The proof of Lemma 4 yields the following important result.

COROLLARY. *If* $x\in R^*$, *then* $R/(x)$ *is a finite ring.*
We denote $\#(R/(x))$ by $N(x)$ and call this natural number the **norm** of x.

LEMMA 5. *For* $x,y\in R^*$, $N(xy)=N(x)N(y)$.

Proof. Since R is a principal ideal domain, the mapping $R/(x)\to(y)/(xy)$ which sends $a+(x)$ to $ay+(xy)$ is a bijection. Since

$$R/(y) \cong \frac{R/(xy)}{(y)/(xy)},$$

the statement follows.

LEMMA 6. *For* $n \geqslant 2$, $R_n - R_{n-1} = \{x \in R^* | N(x) = 1 + \frac{1}{2}(\# R_{n-1})\}$.

Proof. The proof of Lemma 4 shows that

$$R_n - R_{n-1} \subseteq \left\{ x \in R^* | N(x) = 1 + \tfrac{1}{2}(\# R_{n-1}) \right\}.$$

However, the sets $R_n - R_{n-1}$, for $n > 1$, are nonempty and their union is $R^* - UR$, while the sets $\{x | N(x) = 1 + \frac{1}{2}(\# R_{n-1})\}$ are disjoint. Thus, the lemma follows.

Lemma 6 provides an important connection between the algorithm g and the norm N.

COROLLARY. *For* $x, y \in R^*$, $g(x) < g(y)$ *if and only if* $N(x) < N(y)$. *Thus the Euclidean domain* (R, N) *also has* **d.r.p.** *Moreover, for* $y \in R^*$,

$$\# \{x \in R^* | N(x) < N(y)\} = 2(N(y) - 1). \qquad (*)$$

Next we show that $R_2 - R_1 = \{r | N(r) = 2\} = \{\pm 2\}$. Let $r \in R_2 - R_1$. Since $N(r) = 2$, $rs = 2$ for some $s \in R$. We show that $N(s) = 1$. This fact implies that s is a unit and that $r = \pm 2$. Observe that $N(r^2) = 4$ and that $1 \equiv r \equiv r^2 \pmod{(r-1)}$. By **d.r.p.** $N(r-1) \leqslant 4$. Similarly, $N(\pm 1 \pm r) \leqslant 4$. Now no two of $\pm 1 \pm r$ are equal, but they are all congruent mod 2. By **d.r.p.** $N(2) \leqslant 4$, and so $N(s) \leqslant 2$. If all of $\pm 1 \pm r$ have norm less than 4, then $N(2) < 4$ or else **d.r.p.** is violated; and if $N(2) < 4$, then $N(s) = 1$ as desired. Thus we can assume without loss of generality that $N(1-r) = 4$, and $N(s) = 2$. Since any element of norm 4 is a product of irreducibles dividing 2, and since r does not divide $1-r$, we must have $1 - r = \pm s^2$. Now

$$R_2 - R_1 = \{x | N(x) = 2\} = \{\pm r, \pm s\},$$

and

$$R_3 - R_2 = \{x | N(x) = 4\} = \{\pm r^2, \pm rs, \pm s^2\}.$$

Thus, since $N(1+r) \leqslant 4$, $1 + r = \pm s$. Therefore, $s^2 = (1+r)^2 = \pm(1-r)$. If $(1+r)^2 = 1 - r$, then $r = -3$ and $s = \pm 2$ which implies that $r = \pm 1$, a contradiction. Hence $(1+r)^2 = -1 + r$ or $r^2 + r + 2 = 0$; thus $r = (-1 \pm \sqrt{-7})/2$, and $s = -1 - r = -(1 \pm \sqrt{-7}\,/2$. Let $x \in R_4 - R_3 = \{y | N(y) = 7\}$. Clearly x divides $\sqrt{-7}$ and so x divides $r - s = 1 + 2r = \pm\sqrt{-7}$. Therefore, $r^2 \equiv rs \equiv s^2 \pmod{x}$ which contradicts **d.r.p.** Thus we conclude that $N(s) = \pm 1$, and hence $R_2 - R_1 = \{\pm 2\}$.

To conclude the proof, we argue by induction that $N(n) = n$ for all n. This result has been proved for $n \leqslant 2$. Suppose that $N(k) = k$ for $k < n$. If n is composite, then $N(n) = n$ by induction hypothesis. Let n be an odd prime. Then

$$N(n+1) = N(2(n+1)/2) = 2(n+1)/2 = n+1.$$

Since $1 \equiv 1 - n \equiv n + 1 \pmod{n}$, $N(n) \leqslant n + 1$. But 1 has additive order n in $R/(n)$ and so n divides $N(n)$ whence $N(n) = n$. Letting $y = n$ in (*), we learn that

$$\{x \in R^* | N(x) < n\} = \{\pm 1, \pm 2, \ldots, \pm(n-1)\}.$$

Taking the union over all n, we conclude that $R = Z$.

I am grateful to a number of mathematicians for their contributions to this paper. Sherman Stein first conjectured the theorem and encouraged me to consider the problem. In addition, Stein and several of his students meticulously read an earlier draft and volunteered many helpful comments. Finally, the referee both clarified and simplified the proof immensely. I thank them all for their assistance.

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NUM, A VARIANT OF NIM WITH NO FIRST-PLAYER WIN

J. G. MAULDON

0. The name of the game is Num. Num is a game between two opponents, called Left and Right, who play alternately just as in Nim [**1,2**], except that for each heap of matchsticks (or beans), certain predetermined sizes are forbidden to each player. These constraints are such that, if one player could in his turn leave (say) n matchsticks in a particular heap, then the other player could not. In particular, at most one of the players is entitled to clear any given heap.

Thus the *structure* of a particular game of Num is determined by an ordered k-tuple $\left((A_i^L,A_i^R):i=1,2,\dots,k\right)$ of pairs of sets A_i^L,A_i^R of non-negative integers, the two sets in each pair being disjoint. For example we might have $A_1^L=\{\text{evens}\}$, $A_1^R=\{\text{odds}\}$, $A_2^L=\{\text{primes}\}$, $A_2^R=\{\text{squares}\}$, $A_3^L=\{5,7,8\}$, $A_3^R=\varnothing$.

A *state* in Num, so structured, is an ordered k-tuple of non-negative integers, say $(n_1,n_2,\dots,n_k)$. The players move alternately from state to state by strictly reducing the value of any one n_i, subject to the condition that Left may only reduce n_i to a value occurring in A_i^L, whereas Right may only reduce n_i to a value occurring in A_i^R. The first player unable to make a legal move in his turn is the loser.

A *position* in Num, specifying both structure and state, is a k-tuple $P=\left((A_i^L,A_i^R,n_i):i=1,2,\dots,k\right)$ of triads (A_i^L,A_i^R,n_i).

The main object of the present Note is to prove Theorem 1, which shows that every Num position P is a real number in the sense of Conway [**1**]. Theorem 1 also shows how to compute the value of P as an ordinary real number which, in a precise sense, is the value (to Left) of the position P in the course of play. Specific applications of this interpretation are given in Section 5. It is not hard to see that the game of Num is isomorphic with the game of Red-blue Hackenbush [**2**, p.422] played on chains, and the result of Theorem 1 in this latter context is due to Elwyn Berlekamp [**3**, **1**, p.90].

Conway's theory [**1**, p.78; **2**, p.426] shows that the three exhaustive alternatives $P=0$, $P>0$, $P<0$ correspond respectively to a second-player win, a win for Left, and a win for Right. Consequently we have the following proposition, which explains the title of this Note.

THEOREM 0. *Given the initial position for a game of Num, then either the second player can surely force a win, or else (regardless of who starts) Left can force a win, or else (whoever starts) Right can force a win.*

These properties of Num contrast strongly with Nim, wherein no nonzero position is a number and for which it can reasonably be claimed that most positions are first-player wins, a case never occurring in Num at all.

1. The weight of a Num heap. A Num *heap* is a singleton Num position ($k=1$), so that a heap is a triad (A^L,A^R,n) with $A^L\cap A^R=\varnothing$ and $n\geqslant 0$. Writing $B^L=\{x\in A^L:x<n\}$ and $B^R=\{x\in A^R:x<n\}$, we see that the disjoint sets B^L,B^R specify the options available respectively to Left and to Right, so that the strategic value of a heap is completely determined by the ordered pair (B^L,B^R) of disjoint sets of nonnegative integers.

The *weight* of a heap H corresponding to the ordered pair (B^L,B^R) is a real number $w(H)=f(B^L,B^R)$, where f satisfies the condition $f(V,U)=-f(U,V)$, so that in particular $w(A^L,A^R,0)=f(\varnothing,\varnothing)=0$, and if B^R or B^L is empty we define $f(B,\varnothing)=-f(\varnothing,B)=$ the number of elements in the finite set B. In completing the definition of $w(H)=f(B^L,B^R)$ we may and shall assume that B^L and B^R are nonempty and $\min(B^L)<\min(B^R)$.

Arrange the set $B^L \cup B^R$ in increasing order and then substitute the figure 1 for each element of B^L and 0 for each element of B^R, thus obtaining a finite sequence S of ones and zeros starting with a one and including at least one zero. Let a denote the number of terms of S up to and including its first zero, let C denote the sequence S extended by a final 1 and with its first a terms $11\ldots10$ replaced by a binary decimal point, and let c denote the real number (strictly between 0 and 1) whose binary decimal representation is C. Finally define the weight $w(H)$ of the heap H by

$$w(H)=f(B^L,B^R)=(a-2)+c, \tag{1}$$

so that, in the present case $\min(B^L)<\min(B^R)$, we have $w(H)>0$.

For example, the weight of the heap $(\{\text{primes}\},\{\text{cubes}\},10)$ turns out to be $-35/32$, $w(\{0\},\{3\},8)=1/2$ and

$$w(\{\text{evens}\},\{\text{odds}\},n)=\tfrac{2}{3}(1-(-1/2)^n). \tag{2}$$

We shall see that, from Left's standpoint, these are precisely the values in play of the corresponding heaps regarded as Num positions.

Observe that $w(H)$ is a dyadic rational, and that for any dyadic rational w there exists a Num heap H with $w(H)=w$.

2. Numbers and games. According to **[1]** a *game* P (also called a *position*) is specified by a symbol of the form $\{P^L|P^R\}$, wherein P^L [resp. P^R] is representative of the (possibly empty) set of all the positions (i.e., games) which are options immediately available to Left [to Right] as the result of a single move from the position P. The Endgame, a position in which neither player has a move, is thus specified by the symbol $\{|\}$, and it is also denoted by 0. Defining the addition and ordering of games as in **[2**, pp.419-20, 425-6], we find that the Class of all games is an additive abelian Group which is partially ordered by $<$.

A *number* is a game $N=\{N^L|N^R\}$ in which (i) every N^L and every N^R is a number and (ii) for every N^L and every N^R we have $N^L<N<N^R$. [Condition (i) is in fact, but nontrivially, redundant.] Thus, for example, the Endgame 0 is a number.

Equality between games (and in particular between numbers) is a defined equivalence relation. The Class of all numbers (modulo equality) contains a subset which can be identified with the totally ordered additive group of the familiar real numbers. Under this identification the number $\{\{|\}|\}=\{0|\}$ is identified with the real number 1. In fact this position $1=\{0|\}$ gives Left an advantage of exactly one move, since Left can move to the Endgame 0, whereas Right cannot move at all. In general, if a game G is (or is equal to) a number N, then N is the value (to Left) of the position G in the course of play.

3. A preliminary lemma. If H is a Num heap whose weight $w(H)$ is not an integer, so that neither B^L nor B^R is empty, we define $w^L(H)$ to be the maximum weight of the positions to which Left can move from the position H, and $w^R(H)$ to be the minimum weight of those positions so attainable by Right.

LEMMA 1. *If H is a Num heap with $w(H)$ not an integer then, using the notation of* **[1]**, *we have*

$$w(H)=\{w^L(H)|w^R(H)\}. \tag{3}$$

Proof. Let d (>0) be the number of binary decimal places in the (terminating) binary decimal representation for $w(H)$. We readily verify that $w^L(H)=w(H)-2^{-d}$ and $w^R(H)=w(H)+2^{-d}$, so that Theorem 12 of **[1]** immediately yields (3).

4. The main theorem. We are now in a position to prove our main result.

THEOREM 1. (Berlekamp) (i) *If H is any Num heap, and if $w(H)$ is its weight as defined in Section* **1**,

then

$$H = w(H). \tag{4}$$

(ii) *If P is a Num position with component heaps $H_1, H_2, \ldots, H_k$, then*

$$P = \sum_1^k H_i = \sum_1^k w(H_i). \tag{5}$$

Furthermore, there exists a Num heap H such that $P = H$.

Proof. If $w(H)$ is an integer, (4) follows from the last sentence of Section 2, or from the remark five lines from the bottom of [**1**, p.74]. Otherwise our proof of (4) is by induction. We may either write $H = (A^L, A^R, n)$ and use induction on n initialized by $n = 0$ or, more elegantly [**1**, pp.79, 64], we may use induction over games. Either way, our inductive hypothesis implies that all the options of H are numbers, that the greatest Left option $= w^L(H)$, as defined in Section 3, and that the smallest Right option $= w^R(H)$. Then Theorem 68(ii) of [**1**] implies that $H = \{w^L(H) | w^R(H)\}$ and from (3) we deduce, as required, that $H = w(H)$. (5) follows immediately from [**1**, pp.74, 78(*Summary*)], and the last statement follows from the last sentence of Section 1.

In particular, (4) and (2) yield

$$(\{\text{evens}\}, \{\text{odds}\}, n) = \tfrac{2}{3}\left(1 - (-1/2)^n\right). \tag{6}$$

5. Generalization and examples. The game of Num may be generalized to allow numbers other than non-negative integers as admissible sizes for the heaps. For example, if ω is the first infinite ordinal, we have

$$(\{\text{evens}\}, \{\text{odds}\}, \omega) = \tfrac{2}{3}, \tag{7}$$

and we proceed to give an interpretation of (7) in terms of a particular game G.

The game G is three-heap Num in which Left must always leave an even number (e.g. zero) of matchsticks in any heap and Right must always leave an odd number, modified by the proviso that neither player may leave just a single nonzero heap. The initial sizes of the three heaps are originally unspecified and such specification (for one heap) is allowed as a move for either player.

THEOREM 2. *The game G is a fair game.*

Proof. The restrictive modification imposes no restriction on Right, and it is equivalent to allowing Right the privilege of clearing the last remaining nonzero heap, or of claiming two "passes," which in turn is equivalent to adding to the Num game a fourth Num heap $P = (\varnothing, \{0, 1\}, 2) = w(P) = -2$.

Writing $N = (\{\text{evens}\}, \{\text{odds}\}, \omega)$ we now have, by (7),

$$G = N + N + N + P = \tfrac{2}{3} + \tfrac{2}{3} + \tfrac{2}{3} + (-2) = 0. \tag{8}$$

This shows not only [**1**, p. 78] that G is a second-player win, consequently favouring neither Left nor Right when G is played in isolation, but also that G is neutral as a component in any disjunctive sum of games.

Using (6) and [**1**, p. 78], or simply as a corollary of Theorem 2, it is easy to see that *if the initial size of one of the heaps in the game G is arbitrarily pre-assigned, then either Left or Right can surely force a win (whoever starts) according as this preassignment is odd or even.*

More generally, and more precisely, the value to Right of any positive state $\{p, q, r\}$ in the game G is exactly a (positive, negative or zero) determinate fraction of a move, namely the fraction

$$f(p, q, r) = \tfrac{2}{3}\left((-2)^{-p} + (-2)^{-q} + (-2)^{-r}\right). \tag{9}$$

To quote from [**1**, p. 89]: "*It never ceases to amaze and amuse me that such statements have a precise meaning!*"

6. Concluding remarks. Although, when played in isolation, a game which is a win for the first player would seem to favour neither Left nor Right, yet such a game need not be a "fair game" in the sense of Theorem 2, since it may be strongly biased in favour of (say) Left. An interesting example is the game K, defined in terms of $\omega=\{0,1,2,3,\ldots|\}$ by

$$K=\{\omega|0\}, \tag{10}$$

which has the property that, if M is any ungeneralized Num position, then

$$\begin{array}{ll} K+M & \text{is a win for the first player if } M\leqslant 0,\\ K+M & \text{is a win for Left if } M>0,\\ K+K+M & \text{is a win for Left in every case.} \end{array} \tag{11}$$

Furthermore, if Q is any game whatever such that $K-Q$ is a win for Right (i.e. $Q>K$), then $K+Q$ is a win for Left whose value to Left is greater than any of the Num positions here considered, including the position $(\{\text{evens}\},\phi,\omega)=\omega$. The condition $Q>K$ is not, however, sufficient to imply that Q is a win for Left.

We observe that, if x and y are numbers with $x\geqslant y$, then

$$\{x|y\}+\{x|y\}=x+y \tag{12}$$

and in particular $K+K=\omega$ so that, although K is not equal to a number, the game $K+K$ is equal to a game of generalized Num, and to a positive number.

The proofs of these remarks are left to the reader.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

QUEEN SQUARES

CARL P. MCCARTY

Introduction. A famous problem consists in the placing of eight queens on a standard 8×8 chess board in such a manner that they cannot attack each other [1], [2], [4]. The problem has been generalized to the placing of n queens on an $n\times n$ board. For $n\geqslant 4$ it is well known that there is at least one solution. Since no more than n queens may be placed on an $n\times n$ board two other aspects of the problem can be considered:

(a) In how many ways can the n queens be placed?

(b) How many of these are unique after taking into account various symmetries of the board? Wells [5, p. 238] presents a chart which answers both of these questions for $n \leqslant 13$.

In this article a new type of generalization is proposed. Suppose the queen is endowed with added powers and allowed to roam over a cubical board with n^3 cells, call it an n-cube. Let the n-cube be constructed by stacking n levels or copies of an $n \times n$ board so that the left-hand corner of each level alternates color. Furthermore, let the queen be allowed to move diagonally up and down the cube as well as directly above and below the cell it occupies. Perhaps the best way to visualize these moves is to observe that a queen placed at the center cell of a 3-cube can attack all the other 26 cells.

Preliminary problem and a representation. Consider a rook with the extended power of being able to rise above or drop below itself to any level along the column it is in. Clearly, no more than n^2 such rooks can be placed into an n-cube so that they are mutually non-attacking; however, can exactly n^2 rooks always be placed into the n-cube?

The problem may be reformulated by considering the n-cube as an $n \times n$ grid which will either have placed into it an integer from 1 to n inclusive, according to which level the rook is on (letting 1 represent the bottommost level), or be left empty if no rook is in that column. For example:

1		
3	2	1
		3

FIG. 1

indicates the 3-cube with the rook arrangement

R		
		R

Bottom

	R	

Middle

R		
		R

Top

The placing of the n^2 non-attacking rooks into an n-cube is now reduced to the filling of an $n \times n$ grid with n copies of the set $\{1,2,\ldots,n\}$ such that no element is in the same row or column twice. This problem is equivalent to finding a latin square of order n [3, p. 15]. Such a square can be obtained, for example, by cyclically permuting the elements $\{1,2,\ldots,n\}$ and gives the desired maximal solution for any n.

Returning to the queen problem and realizing that two queens cannot occupy the same column, the same grid representation may be used but the conditions on placing the elements must be altered. As a simple example, it is easily seen that 3 non-attacking queens cannot be placed on a 3×3 board; however, it is possible to place 4 queens into a 3-cube as demonstrated in Figure 2.

1	3	
		1
2		

FIG. 2

Statement of the problems.

DEFINITION. A **queen square** of order n is a square arrangement of the elements from the set $S = \{1,2,\ldots,n\}$ and empty squares such that

(1) no element $i \in S$ may appear twice in the same row, column, or diagonal, and
(2) the elements $i, j \in S$ may not be placed $|i-j|$ squares apart from each other along any row, column, or diagonal.

Let $M(n)$ represent the maximum number of non-empty entries which a queen square of order n can contain, and let $R(n) = M(n)/n^2$ represent the ratio of $M(n)$ to the number of grid squares of an $n \times n$ board.

The following results were obtained by computer search. The chart provides lower bounds on $M(n)$ and $R(n)$ for $n=3$ to $n=18$. From these preliminary results it seems natural to separate odd and even values of n. The bounds obtained are not necessarily maximal.

	lower bound on			lower bound on	
n	$M(n)$	$R(n)$	n	$M(n)$	$R(n)$
3	4	.444	4	7	.438
5	13	.520	6	18	.500
7	27	.551	8	34	.531
9	43	.531	10	58	.580
11	68	.562	12	80	.555
13	96	.568	14	111	.566
15	132	.586	16	151	.590
17	171	.592	18	191	.589

The problems to consider are:
(1) Can a latin square be constructed which is also a queen square?
If yes, for which n is it possible?
If no, is there an upper bound on $R(n)$?
(2) Does there exist an algorithm for the maximal placement of the queens into the n-cube?

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CLASSROOM NOTES

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THE INTEGRAL DEFINITION OF THE LOGARITHM AND THE LOGARITHMIC SERIES

A. P. FRENCH

It may seem puzzling to many students (as it did to this writer) that the result of integrating $1/x$ is a function apparently so different from the integral of x^n for any $n \neq -1$. The purpose of this Note is

These results may be well known among professional mathematicians, but they do not seem to appear in introductory calculus texts, where their use might help to make the logarithmic function appear less mysterious.

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FUNCTIONS WITH ARBITRARILY SMALL PERIODS

R. CIGNOLI AND J. HOUNIE

The following result was proved by C. Burstin in 1915 [**3**]:

THEOREM 1. *A (Lebesgue) measurable function $f: R \to R$ having arbitrary small periods is a constant a.e.*

A simplified proof was given in 1937 by P. Hartman and R. Kershner ([**6**, footnote on p. 815]; see also [**7**, p. 32]), based on Lebesgue's theorem on the differentiation of the integral. Similar proofs were also given by R. P. Boas [**1**] and Z. Semanedi [**9**]. Boas' paper contains two more proofs of Burstin's theorem: one is based on the uniqueness of Fourier coefficients and the other on the existence of density points of sets of positive measure.

Our aim here is to present a very direct and simple proof of Burstin's theorem, based on elementary properties of Lebesgue integral. It is very well known that the only translation-invariant measures on the reals are the multiples of the Lebesgue measure. We show in Lemma 1 that this result is still valid if we replace "translation-invariant" by "invariant under a dense set of translations," and then we show that Burstin's theorem is an immediate consequence of this fact. We add two simple consequences of Burstin's theorem. The second one is a proof of the existence of non-measurable sets due to Sierpiński [**10**] that we think should be better known as an alternative to Vitali's example. The interest of Sierpiński's construction is that it is based on the prime ideal theorem for Boolean algebras, which is weaker than the axiom of choice [**5**] (see also [**8**, §10] and [**9**]).

By a Radon measure μ on R we understand a non-negative, σ-additive set-function defined on the Borel sets such that $\mu(B) < \infty$ for each bounded Borel set B.

LEMMA 1. *Let D be a dense subset of R and μ a Radon measure such that $\mu(d+[a,b)) = \mu([a,b))$ for every $d \in D$ and a,b in R. Then $\mu = k\lambda$, where λ is the Lebesgue measure and $k = \mu([0,1))$.*

Proof. If $c \in R$, $d_n \in D$ and $d_n \uparrow c$ we observe that

$$[a+d_n, b+c) \downarrow [a+c, b+c), [b+d_n, b+c) \downarrow \varnothing$$

so:

$$\mu([a,b)) = \mu([a+d_n, b+d_n)) = \mu([a+d_n, b+c)) - \mu([b+d_n, b+c)) \to \mu([a+c, b+c))$$

and we obtain $\mu([a,b)) = \mu(c+[a,b))$ for each a, b and c in R. It is well known that this implies that $\mu = k\lambda$. For the sake of completeness, we sketch a proof of this fact. If p,q are natural numbers, it follows that

$$\mu([0,p/q)) = \mu\left(\bigcup_{i=1}^{p}\left[\frac{i-1}{q}, \frac{i}{q}\right)\right) = \sum_{i=1}^{p} \mu\left(\frac{i}{q} + [0,1/q)\right) = p\mu([0,1/q)).$$

Analogously we have that

$$\mu([0,1)) = \mu([0,q/q)) = q\mu[0,1/q).$$

Then

$$\mu([0,p/q)) = (p/q)\mu([0,1)).$$

If a,b are rational numbers,

$$\mu([a,b))=\mu(a+[0,b-a))=(b-a)\,\mu([0,1))=\lambda([a,b))k.$$

So the Radon measures μ and $k\lambda$ coincide on the intervals $[a,b)$ with rational end points, and the result follows right away.

Recall that τ is said to be a period of $f:R\to R$ if $f(x+\tau)=f(x)$ for every x in R. It is easily seen that the set of periods of f is a subgroup $G(f)$ of the additive group of the reals, and therefore $G(f)$ is either discrete or dense. (See, for instance, [**2**, Prop. 1.1].)

Proof of Theorem 1. The hypothesis implies that $G(f)$ is dense in R. If f is non-negative and bounded, setting $\mu(A)=\int_A f(x)dx$ and applying the previous lemma we see that $\mu(A)=k\lambda(A)$, so

$$\int_A (f(x)-k)dx=0$$

for every measurable set A. Therefore, $f(x)=k$ a.e.

In the general case of a measurable function f, apply the previous result to the composite function $g(x)=\pi/2+\tan^{-1}f(x)$.

Remark. Since there are examples showing that the inverse image of a Lebesgue measurable set by a strictly increasing, continuous, positive function need not be measurable ([**4**, p. 83]), it is not obvious that the composition $g(x)=\pi/2+\tan^{-1}f(x)$ is measurable for an arbitrary measurable function f. That this is the case can be shown by using the following:

LEMMA 2. *Let $g:R\to R$ be continuously differentiable and such that $g'(x)>0$ for all x in R. Then the inverse image by g of a measurable set is measurable.*

Proof. Since g is continuous, the inverse image by g of a Borel set is a Borel set, and since each measurable set is the union of a Borel set and a set of measure zero, it is enough to show that the inverse image by g of a set of measure zero is a set of measure zero. Let h be the inverse function of g. Then h is strictly increasing and continuously differentiable. If E is a set of measure zero, write $E_n=E\cap[n,n+1]$, $n=0,\pm1,\pm2,\ldots$ and let M_n be an upper bound of h' in the closed interval $[h(n-1),h(n+2)]$. Given $\varepsilon>0$, we can find a sequence of open intervals (a_{nk},b_{nk}) so that

$$E_n\subseteq\bigcup_{k=1}^{\infty}(a_{nk},b_{nk})\subseteq[n-1,n+2]\quad\text{and}\quad\sum_{k=1}^{\infty}(b_{nk}-a_{nk})<\varepsilon/M_n.$$

Therefore

$$g^{-1}(E_n)\subseteq\bigcup_{k=1}^{\infty}(h(a_{nk}),h(b_{nk}))\subseteq[h(n-1),h(n+2)]$$

and

$$\sum_{k=1}^{\infty}(h(b_{nk})-h(a_{nk}))\leqslant M_n\sum_{k=1}^{\infty}(b_{nk}-a_{nk})<\varepsilon.$$

Thus $g^{-1}(E_n)$ is a set of measure zero and hence $g^{-1}(E)$ is of measure zero.

APPLICATIONS:

(i) *A proper measurable subgroup G of the additive group R has Lebesgue measure zero.*

Proof. If G is not discrete, it is dense. In that case, according to Burstin's theorem, the characteristic function f of G is a constant a.e., that is, either $f=0$ a.e. or $f=1$ a.e. Suppose $f=1$ a.e. If $G_1=G\cap[0,1]$, then $\lambda(G_1)=1$. Choose two numbers x and y such that $0<y-x<1$ and $y-x\notin G$. Then $(G_1+x)\cap(G_1+y)=\varnothing$ and $(G_1+x)\cup(G_1+y)\subseteq[x,y+1]$, which is a contradiction because $\lambda(G_1+x)+\lambda(G_1+y)=2\lambda(G_1)=2$, and $\lambda([x,1+y])=1+(y-x)<2$.

(ii) *Sierpiński's example of a non-measurable set* [**10**]:

Let N denote the set of natural numbers and let F be the ideal of the Boolean algebra 2^N formed by all finite subsets of N. Extend F to a prime ideal P and define $h:2^N \to \{0,1\}$ by $h(A)=0$ if $A \in P$ and $h(A)=1$ otherwise.

If $x \in [0,1]$, let $0 \cdot x_1 x_2 \cdots$ be the dyadic development of x (where we choose the finite development for the rationals with denominator 2^k) and set $N_x = \{n \in N : x_n = 1\}$. If $x \in R$ and $[x]$ denotes the integral part of x, set $f(x) = h(N_{x-[x]})$. Then it is not hard to see that: (1) f takes only the values 0 or 1; (2) f has all numbers of the form 2^{-n} as periods and (3) $f(1-x) = 1 - f(x)$.

If f were Lebesgue measurable, by (1), (2) and Burstin's theorem it would follow that either $f(x)=0$ a.e. or $f(x)=1$ a.e., and therefore that (4) $\lambda(f^{-1}\{0\}) \neq \lambda(f^{-1}\{1\})$. But from (3) it follows that $f^{-1}\{0\} = 1 - f^{-1}\{1\}$, and the invariance of Lebesgue measure under translations would imply that $\lambda(f^{-1}\{0\}) = \lambda(f^{-1}\{1\})$, in contradiction with (4).

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MATHEMATICAL EDUCATION

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CUPM ANNOUNCEMENT

The MAA's Committee on the Undergraduate Program in Mathematics (CUPM) has appointed a panel to study courses of the type called "Mathematics Appreciation," "Mathematics for Liberal Arts Students," etc. The goal of the panel is to produce a pamphlet which can be used as a source of information and guidance by those responsible for such courses. The panel wishes to have input from the mathematical community. The panel would like to hear about successful versions of these courses, suggestions concerning subject matter, teaching methods, helpful references, etc. Please communicate

your experiences, ideas, and suggestions to Professor Jerome A. Goldstein, Department of Mathematics, Tulane University, New Orleans, Louisiana 70118.

APPLIED MATHEMATICS IN A LIBERAL ARTS CONTEXT

JACK HACHIGIAN

Introduction. Despite the recent adverse publicity about New York City, it remains the center of corporate headquarters in the United States. Of the "Fortune 1000" there are 189 corporate headquarters in Manhattan, not including either major financial institutions, such as banks and Wall Street firms, or companies in the surrounding suburban areas. Moreover, there are many pharmaceutical firms in the surrounding industrial areas, and New York City has within its political boundaries many major medical research centers. Within this urban center lies Hunter College of the City University of New York, centrally located at East 68th Street and Park Avenue in Manhattan.

The introductory paragraph establishes the setting in which we at Hunter College formulated our solution to what J. Spanier [1] referred to as "realism within an academic environment," and with the belief that "mathematics applied to decision processes will, in time, change the emphasis in pure mathematics" [8].

Mathematics training has become increasingly abstract at both the graduate and undergraduate levels, with a general tendency to prepare students solely for academic careers. Little thought used to be given toward educating students in mathematics as a discipline within the liberal arts context that has broad applicability to many areas in a non-academic setting. Many changes are now being reported, however [3], [4], [5], [6]. The effort described below (begun in 1971) is an attempt to modify this tendency and direction in a constructive way with a view toward applications of mathematics to decision processes in the context of New York City and at a liberal arts college.

The Applied Mathematics Program. Long dormant, and deleted from the Hunter College catalog (1965), but never officially retired, was a Master's Degree in Applied Mathematics, which by its description seems to have been developed during the depression years of the 1930's. The orientation of this degree was classical and favored the physical sciences. Its existence was lucky, however, for two separate reasons: (a) Had we created an entirely new degree in 1971, in all likelihood it would never have received approval at the state level. Graduate degrees were being carefully reevaluated and many were eliminated, with the approval of new degree requests virtually impossible. (b) Most fortunately, the existence of the "old" Applied Mathematics Master's Degree forced upon us an "option" plan which helped in formulating our choices and gave us essential flexibility.

Our program has as its major premise that a student completing its requirements will be able to apply his/her acquired mathematical skills in various contexts. In some few cases, additional on-the-job training may be necessary to familiarize the student with the subject area, but in most cases there would be only a short period of acclimatization. We prefer to view our students as specialists in decision mathematics but generalists in the sense that they can apply their acquired skills in many different jobs: market research, capital investment, corporate planning, clinical trials, pharmaceutical testing, econometrics, etc. As will be noted below, we believe this premise has been vindicated by the success of our students in these various areas.

We view the modern application of mathematics to decision processes in the same relationship to data and experimental results as one views theoretical physics in relationship to experimental physics. Abstraction or generalization is not encouraged unless it specifically relates to, or clearly assists in, devising a model or method that explains the data. The point is that serious mathematics can be taught in this way, which isn't simply "cookbook" methodology or a group of techniques developed for use here and there.

Decision Theoretic Option. The Master's Degree requires 30 credits, where we require one year of probability theory (somewhere between Feller's volume 1 and volume 2), one year of statistical decision theory (somewhere between Ferguson's *Mathematical Statistics*, Hogg and Craig's *Introduction to Mathematical Statistics* and Brownlee's *Statistical Theory and Methodology in Science and Engineering*), and participation in the Case Seminar, which will be described below. The required credits amount to 15, leaving the equivalent of 15 credits (5 three-credit graduate courses) to be selected according to a student's own interests in the broad area of decision mathematics. These selections must be approved by the Graduate Advisor and may include a certain number of courses in other disciplines, e.g., economics, corporate finance, urban planning, sociology, physics, etc. In addition, computer programming skills are required, with at least two higher-order languages mastered without credit (e.g., APL, PL/I, FORTRAN, COBOL).

It is apparent that this flexibility provides students with many possible paths to their career goals, including the pursuit of a Ph.D. Optional paths are not narrowly defined, leaving students who cannot or should not proceed toward a higher academic degree the opportunity to pursue non-academic careers.

Advanced topics courses are designed to cover parts of various topics, or a single topic, during an entire semester. The subject matter taught in these courses is agreed upon by students and instructor during pre-registration. This additional flexibility helps us meet student needs without the required bureaucratic changes necessary to add and delete courses. Repetition of topics courses is permitted as long as the overlap with previous course work is small.

Case Seminar. A unique feature of the program has been what we have chosen to call Case Seminar. All students in the program are required to attend Case Seminar once each week. Academic or industrial applied mathematicians are invited to describe their efforts on a "case" basis, developing the application from its origin to the mathematical development necessary for a solution or partial solution. In some instances only a quasi-mathematical solution is possible, and as a result it becomes clear to the students during these seminars that formulating a problem in mathematical terms requires every bit as much ingenuity as the method of solution. The topics or "cases" have included analysis of weighted voting schemes, coding theory, econometric model development, analysis of fire engine response times in New York City, computational methods for sparse matrices, fast Fourier transforms, analysis of juvenile second-offenders, and graph-theoretic methods applied to garbage collection routes in New York City.

There is a second aspect to Case Seminar wherein students participate on mathematics projects, either as individuals or as members of a team. Individual projects were generally found to be related to a job the student concurrently held, e.g., in the Traffic Violations Bureau of New York City. Team projects, however, were preferred by the students and were far more exciting and educational. New students are presented the opportunity to participate in a team project within the first month of the fall semester, and these team projects require that a competitive proposal be written by the team and submitted to the National Science Foundation for funding by early November of the same semester.

Projects are limited by the liberal-arts training students receive at Hunter College as well as by the urban context of Hunter College. Technological and/or engineering problems do not arise. Potential projects are discussed within the group under faculty guidance to ensure that they are manageable with the time and manpower available. Discussions at this point frequently focus on two things: (1) What is of interest to the team members? (2) The various kinds of applied mathematical skills that will be necessary for attacking each problem with some degree of success.

Students' willingness and interest are crucial in the selection process, for very valid projects have been discarded by them on the grounds that the results may be used in ways not intended by the team. For example, during a recent hospital employees' strike in New York City, the question of whether the strike had caused an increase in the hospital death-rate was posed to one of our teams. Although anxious to reduce the death-rate (if indeed it had increased), the students suspected that

their efforts would not be used to deliver better hospital care but rather to undermine a union's right to strike. Rightly or wrongly they refused the project.

The process of selection is very intense, and upon selection of a problem a proposal must be written, a budget developed and reviewed, typed, signed by the Administration and sent to NSF before the deadline. Of course students are attending classes during this period and the impact on them is one of shock: What do we know? We can't possibly solve any of these! There isn't enough time. Why not wait 'til we've taken more courses?

With careful attention and guidance, faculty members encourage them to organize themselves, choose their leadership, and gather information about the possible projects. They then help the students choose a project which appears tractable for the ensuing summer.

The time constraint and the national character of the competition is frightening, but the experience is incomparable. After what appears to be an interminable waiting time the NSF announces its decision to award a grant to the team. (We have submitted two proposals, one in 1974 ($12,600) and one in 1975 ($16,700), and have been awarded both.) The joy of being awarded the grant gives way to the realization that this is just the beginning and that there is much to be accomplished before the actual project work commences during the summer: Other students in allied areas are to be hired, literature must be searched and read, understanding of organizational requirements must be mastered, e.g., payroll, purchase orders, hiring, and budgetary and time constraints, etc. This experience is very valuable in obtaining jobs later on.

A side effect now manifests itself in the classroom. Students' interest has been heightened by the prospect that they must solve a specific problem. Relevant questions now crop up, and requests are made for specific types of mathematics. Interest in the computing facilities is heightened and their accessibility becomes very important. A genuine camaraderie develops, since the team has a common goal and is anxious to succeed.

This period of intellectual ferment and peer-group interaction is of great educational value since individuals are forced to verbalize their questions and thoughts, and in a commuter school this group interaction is often impossible to achieve.

I will not describe all the projects (NSF or other) to date but will simply recount the various activities in one of them.

Like most urban centers, New York City has a large number of vacant buildings, primarily in ghetto areas. Fires in such buildings are likely to be more intense because of delays in reporting them. This intensity, together with structural deficiencies (missing stairs, etc.), makes such fires more hazardous for firemen, and the policy was to demolish such buildings. However, legal problems, as well as bureaucratic delays, prevented this policy from being widely implemented or effective. An alternative was needed.

One team of students undertook to determine whether there were differences in buildings and/or neighborhoods sufficient to distinguish those vacant buildings that were "prone" to fire and those that were not. Moreover, a solution, if it was found, was to be "operational," by which we meant that the variables used to identify fire-prone buildings were to be ones on which data could be obtained easily: data both timely and accessible to the Fire Department for rapid analysis. Census data would not do.

The scope of the project therefore included devising a sampling plan of the universe of vacant buildings, a subjective list of initial variables, the collection of data on those variables, devising computerized handling of data, on-site visits, dealing with various city agencies, search of records, etc., and, finally, an actual analysis of the data by first eliminating non-operational variables and then exploring the data for possible interrelationships.

Discriminant analysis was used on a sample of 250 vacant buildings whose history was known. Each building was classified as having had a fire or not according to the mean vector of the chosen variables. This classification was matched against the history of the buildings, and the number correctly classified was ascertained (73%). Although refinements are still needed to perfect the

methodology of classification, a report has been turned over to the N.Y.C. Fire Department, which has it under consideration for possible implementation. In addition, a reporting meeting was held, as required by NSF, with media coverage to make public the results obtained by the students. The presentations were made by the students themselves.

Success of the Program. We feel that we have achieved our primary goal, in devising a realistic applied mathematics program within a liberal-arts context, which was considerably more than preparing mathematics students for jobs. The measure of our success (for us, at least) was whether our students would be able to obtain employment in areas where their mathematical skills would be of continuing use. Our secondary long-term interest was to develop a program of mathematics at Hunter College related to decision and planning processes.

Our first graduates have now been employed for two years, and they report their success in terms of raises and promotions. We are happy to report that all our graduates found employment *either before or at most one month after* graduation, and employers with one student invariably request others. There is now a waiting list of employers. We have listed below the firms at which initial employment has been found simply to indicate their scope and range: Pfizer Pharmaceutical, U.S. Vitamin, N.Y.C. Traffic Bureau, New York Telephone, Pan American Airlines, Mt. Sinai Hospital, General Electric, and I.B.M. The median starting salary has been $16,750, with a maximum of $17,900 and a minimum of $14,000. Moreover, three students are pursuing Ph.D.'s, one each at Columbia University, Rutgers University, and the University of South Carolina.

Summary. We have developed an applied mathematics program in a liberal-arts context suitable to its economic and geographical location in New York City. Students are quite satisfied, and there are more employment opportunities than there are students, which will probably hold for some time. The only unfortunate aspect of the program is that it makes heavy demands on faculty time without any compensatory time. If the program is used as a model for a similar program elsewhere, this should be taken into consideration.

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GRADING ANSWER-UNTIL-CORRECT TESTS

JOE DAN AUSTIN

Musser and Thompson [1] have proposed an interesting test grading format which permits immediate feedback on the correctness of multiple-choice responses. This format is a multiple-choice test that is chemically treated so that a felt-tip marker indicates whether a response is correct. A student can answer until the correct response is found and then go to the next question. (This is called

the answer-until-correct format.) The grade on a question is a function of the number of responses needed to find the correct response. This note considers how to assign credit for the number of responses needed to find the correct response.

We assume that a student has a (unknown) probability α of knowing the correct response. If the correct response is not known, the student selects at random one of the n possible responses. If this response is incorrect, the student selects at random one of the remaining $n-1$ responses. This procedure is continued until the correct response is found. Each guess then has probability $1/n$ of giving the correct response.

Suppose one point is given for needing only one try to find the correct response. If two tries are needed c_2 points are given, if three tries c_3 points,..., and if n tries, c_n points. The c_i may be negative, in which case points are subtracted. The first question is what $c_2, c_3, \ldots, c_n$ give the same expected number of points α that results if no guesses are made and 0 points are given for an incorrect answer. This is the same as asking for an unbiased estimator of α. The expected number of points is

$$\alpha+\frac{1-\alpha}{n}(1+c_2+c_3+\cdots+c_n). \tag{1}$$

Therefore the expected number of points is α if and only if

$$c_2+c_3+\cdots+c_n=-1. \tag{2}$$

The next question is what $c_2, c_3, \ldots, c_n$ give the minimum variance unbiased estimate of α. The variance of the number of points is

$$\alpha(1-\alpha)+\frac{1-\alpha}{n}(1+c_2^2+c_3^2+\cdots+c_n^2). \tag{3}$$

Using Lagrange multipliers, we find that the minimum of (3) subject to the constraint (2) is obtained for

$$c_2=c_3=\cdots=c_n=-1/(n-1). \tag{4}$$

Note that this is the usual grading procedure for multiple-choice tests, i.e., the student receives one point for the correct answer in one try and loses $1/(n-1)$ points if the first answer is not correct.

Reference

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DIFFERENTIAL EQUATIONS BEFORE MULTIVARIABLE CALCULUS? TOWARDS A REALISTIC MATHEMATICS PROGRAM FOR UNDERGRADUATE ENGINEERS

T. M. CREESE

ABSTRACT. A course on ordinary differential equations preceding multivariable calculus can be made to suit present-day engineering curricula and resolve certain difficulties concerning the undergraduate engineering student's motivation, technical skill, ability to retain and use mathematical material, and potential for further growth.

The changing undergraduate engineer. As recently as the mid-1960's the population of undergraduate engineers contained comparatively large numbers of "analytically oriented" students. These were likely to take an interest in the ever more mathematically demanding basic engineering problems and in mathematical modeling, its uses and consequences. From this pool came today's engineering faculty, for many of whom detailed knowledge of substantial parts of applied mathematics is essential. To train such students the CUPM program [1] was reasonable.

By contrast, the "design-oriented" student predominates in the classes we see now. He is interested in specific designs and processes, management and sales. His interest may be strong but it is narrow, and he may have difficulty even recognizing the relevance of a basic engineering problem if it lies outside his active interests.

For the undergraduate engineer in general, and the more common design-oriented student in particular, the remembering and even the learning of any mathematics seems to be very much connected with the use of it. If a student sees a technique or fact put to use *immediately* on a problem which *he recognizes* as an application, then he can classify it as "practical" and hope to remember it. Anything not so classified he will call "theoretical," meaning, "Even if there is a use, I haven't seen it." Such a "theoretical" thing he does not remember, and when applications finally appear he may not recall ever having seen it. This may become the basis of complaints about a course's being "theoretical."

The changing mathematics curriculum for undergraduate engineers. In the past, mathematics departments have offered undergraduate engineers several analysis "service courses" in applied topics. (We will not consider the problem of the statistics courses.) These have included ordinary differential equations (O.D.E.), complex variables, and partial differential equations, or perhaps a year's course in "advanced calculus" or "engineering mathematics." In the past, the presence of graduate students and physics majors may have disguised the following phenomenon: only the most analytically oriented of the engineering undergraduates actually learn, remember, and use much mathematics from mathematics department courses beyond O.D.E. Thus the audience for such courses has declined over the past ten years, and engineering departments have responded in at least two ways. One is to change specific course requirements into mathematics elective requirements, allowing the student to choose courses in which he/she may find the motivation stronger. The other is to insert mathematical techniques into those places in engineering courses where they are used immediately.

Depending on the curriculum, the techniques so inserted may include certain numerical methods, finding the poles and residues of a rational function, solving boundary value problems for O.D.E.'s, and expansion in Fourier series or Fourier transform. Of the other mathematics department favorites, partial differential equations appear only infrequently and to only a few students, and power series methods seem not to appear at all.

Beyond O.D.E. the present-day engineering undergraduate usually does not take any more courses from the mathematics department, but collects mathematical techniques (and their uses) from his engineering courses. The engineering instructor may choose a technique for its efficiency in essentially one kind of problem, and the technique is shown and practiced at once. Typical prerequisites for its use are calculus or O.D.E., the idea of superposition (linearity), and a level of calculational proficiency and confidence. The technique is remembered because it is attached to a problem (application) whose importance is made clear at once.

By contrast, a course taught by a mathematician is ordinarily built on the supposition that in later professional life the student will face a wide variety of problems and will have to do the technique selecting on his own. Such courses usually give a more extensive treatment of each subject. (Short treatments seem to be either theoretical or cookbookish.) Very little reliance is placed on any given student's ability to handle the specific applications offered as illustration of the course material, and the instructor is not prepared to forecast how useful particular techniques will be to particular students. The material is organized around a few basic concepts considered to be the key to understanding and remembering or re-deriving. Unfortunately, with design-oriented students this memory device does not work of its own accord. The student regrets the lack of applications that really appeal to his own interests and fails to retain enough of the material to warrant the extended treatment given.

Of the undergraduate engineers who do enroll in upper-division analysis service courses, a majority appear to be suffering from misconceptions about the material to be covered, or to be resolving schedule conflicts.

Thus, the non-statistics portion of the present-day mathematics curriculum for undergraduate engineers is often effectively: (1) First year—calculus of one variable; (2) Second year—multivariable calculus or O.D.E., or both; (3) Thereafter—mathematical analysis techniques learned in engineering courses.

Except in schools where the analytical orientation of the students can be maintained by selective admissions, the day of the upper-division analysis service course would appear to have passed. It may be asked whether the first two years will long remain intact. (At the University of Kansas, some engineering departments require but one of the second-year courses. ECPD requirements **[2]** are even more flexible.) It may be that techniques selected from the second-year courses will be taken over into engineering courses. For mathematics departments, therefore, the critical point is the effectiveness and attractiveness of the second-year courses; for if the design-oriented students start dropping out after a year of calculus, then the engineering departments may be expected to take matters into their own hands.

Further, the pendulum may one day swing back in favor of the analytically oriented student. (The staffing of engineering faculties and research facilities will become a grave problem if it does not.) Confidence that mathematics departments are interested in both hearing and satisfying the needs of engineering students will be essential if those students are to be encouraged to seek undergraduate mathematics courses when the swing comes. For this reason too, the second-year courses need very careful attention.

An effective second year. Both multivariable calculus and O.D.E. are organized around linear ideas. They tend to combine with linear algebra, and also with each other. This has allowed us to develop the sort of extended-treatment course that we like, in a more or less unified second year, designed to motivate engineering students, especially the weaker ones.

O.D.E. is a natural course for applications, and so we begin with that. It serves to motivate the study of linear algebra, which we introduce after the study of Laplace transforms. The usefulness of linear operators is then confirmed in further analysis of the equations already solved, in additional methods, and in systems of equations (and in multivariable calculus the second semester). All the way through, motivation must be considered the key to the student's memory and understanding.

The apparent prerequisite problems may be turned into advantages. The student has no objection to using the formula for the Laplace transform of the convolution integral without having seen a derivation. The derivation may be given later, at the beginning of the discussion of iterated integrals. The derivation motivates the discussion of iterated integrals far more than does the calculation of area and volume.

Similarly, the exact first-order non-linear differential equations may be put at the beginning of the study of partial derivatives.

Descriptions of our courses are given in the Appendix. The principles on which they are organized are:

(1) Study only subjects for which the undergraduate engineering program has so much use that an extended treatment is really worthwhile.

(2) Begin the year with things of immediate appeal to second-year engineers.

(3) Anchor each new topic in ones that have been thoroughly treated previously, particularly in practical examples.

(4) Develop each topic or example to the point where it shows its practical usefulness, and require the student to use it.

(5) Reinforce relevant but troublesome parts of calculus, and be very explicit about the techniques for graphing the most commonly encountered functions.

(6) In the small and in the large, do applications first, progressing from there to techniques and only later to ideas. Leave the unifying concepts until it is clear that they do unify. From that point on, encourage the student to see the semester (or the year) as an integrated whole, compact in concept although diverse in technique and application.

Thus, we begin by making things easy to remember, by attaching them to applications in a way that appeals to all engineering students. We end by making things easy to remember by attaching them to unifying concepts.

The traditional second year is also offered. It is taken by a variety of non-engineers, by civil engineers, by a few other engineers who may prefer it, and by engineers with schedule conflicts. However, reports from engineering students and faculty indicate that the reorganized second year is generally considered much more effective than the traditional year.

Appendix. The following is an outline of the second year as we offer it.

First semester (5 hours)

Mixing problems, first-order linear equations
Setting up systems of equations modeling RLC circuits and spring-mass-dashpot systems
Eliminating variables from such systems, to obtain an nth order linear O.D.E. with one unknown
Linear O.D.E.'s with constant coefficients, using the Laplace transform as the primary tool
Systems analysis, transfer function, impulse response
Steady-state solutions, using complex arithmetic
Linear operator as a unifying concept
Linear differential operators with constant coefficients
Characteristic polynomial; undetermined coefficients
Equidimensional equations; reduction of order; variation of parameters
Matrices; systems of linear algebraic equations; determinants
Systems of first-order O.D.E.'s—further development

Second semester (3 hours)

Iterated integrals
Partial derivatives
Vector analysis in 3-space
Analytic geometry in 3-space
Matrices—further development
Non-linear transformations; Jacobian
Multiple integration

Teaching techniques, texts. The first semester was developed jointly by the mathematics and engineering departments. It is taught in a large lecture by a team consisting of one engineer and one mathematician. We have written the texts ourselves ([3], along with supplementary notes). The second semester is taught by the mathematics department in small sections. For a text we use appropriate parts of a standard calculus text, which may be supplemented by individual instructors.

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Answer to last month's problem: The howler quoted on p. 494 was perpetrated by Évariste Galois, *Annales de Mathématiques* 21 (1830–31), 182; reprinted in all editions of Galois's works. (Title: *Notes sur quelques points d'analyse*.)

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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Beginning in January, 1979, this Department will be edited by A. P. Hillman.

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

No solutions (except those accompanying proposals) should be sent to Professor Hillman.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, C A (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before December 31, 1978. Please enclose a self-addressed card or label (for acknowledgement).

E 2725. *Proposed by Solomon W. Golomb, University of Southern California*

Given positive integers a and b show that there exists a positive integer c such that infinitely many numbers of the form $an+b$ (n a positive integer) have all their prime factors $\leqslant c$.

E 2726*. *Proposed by Roy Streit, Naval Underwater Systems Center, New London, Connecticut*

For $(a,b)\in\mathbf{R}^2$ let $F(a,b)$ be the sequence $(c_0,c_1,c_2,\dots)$ where $c_n=[an+b]$. Which (a,b) have the property that

$$F(x,y)=F(a,b) \quad \text{implies} \quad (x,y)=(a,b)?$$

E 2727*. *Proposed by David P. Robbins, Hamilton College*

Two triangles $A_1A_2A_3$ and $B_1B_2B_3$ in $\mathbf{R}^3$ are equivalent if there exist three different parallel lines p_1,p_2,p_3 and rigid motions σ,τ such that $\sigma(A_i)$ and $\tau(B_i)$ lie on p_i $(i=1,2,3)$.

Find necessary and sufficient conditions for equivalence of two triangles.

E 2728. *Proposed by J. G. Mauldon, Amherst College*

Let a,b,c,d be radii of four mutually externally tangent right circular cylinders whose axes are parallel to the four principal diagonals of a cube. Characterize all quadruples a,b,c,d which arise in this way.

E 2729. *Proposed by John Goth, Austin, Texas*

Evaluate det (A) where $A=(a_{ij})$ is the $n\times n$ matrix given by

$$a_{ij}=\binom{im+j-1}{j} \qquad (i,j=1,\ldots,n),$$

m being a fixed positive integer.

E 2730. *Proposed by R. L. Graham, Bell Laboratories, Murray Hill, N.J.*

Describe all finite sets A of real numbers with the property that any two elements of A belong to some 3-term arithmetic progression in A.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Elementary Case of the Jordan Curve Theorem

E 2647 [1977, 294]. *Proposed by Daniel Gallin, University of San Francisco*

Let Γ_1,Γ_2, be two continuous maps of the unit segment $I=\{x|0\leqslant x\leqslant 1\}$ into the unit square I^2. Suppose that $\Gamma_1(0)=(0,0)$, $\Gamma_1(1)=(1,1)$, $\Gamma_2(0)=(0,1)$, $\Gamma_2(1)=(1,0)$.

Prove by elementary means (e.g., without using the Jordan Curve Theorem) that the two curves Γ_1 and Γ_2 meet.

Solution by Eli L. Isaacson, New York University (revised by the editor). Let $C_i=\{\Gamma_i(t)|t\in I\}$ and assume that $C_1\cap C_2=\varnothing$. Since C_i are compact, it follows that there exist disjoint open subsets U_i $(i=1,2)$ of I^2 such that $C_i\subset U_i$ $(i=1,2)$. Hence we may assume that Γ_i are simple polygonal curves, i.e., if we write $\Gamma_i(t)=(x_i(t),y_i(t))$ then each of the functions x_i,y_i $(i=1,2)$ is continuous and piecewise linear and moreover Γ_i $(i=1,2)$ is injective. By slightly perturbing the vertices of Γ_1 (except the endpoints) we may also assume that distinct vertices of Γ_1 have distinct abscissae.

For $0<s,t<1$ we shall say that s is *t-special* if $x_1(s)=x_2(t)$, $y_1(s)>y_2(t)$, and x_1 is monotonic in the interval $(s-\varepsilon,s+\varepsilon)$ for small $\varepsilon>0$. For a given t $(0<t<1)$ there exist only finitely many t-special values and we define $\varphi(t)$ to be the number of these values.

It is clear from the definition of φ that it is constant in every sufficiently small interval $(t-\varepsilon,t+\varepsilon)$ provided that $x_2(t)$ is not the abscissa of any vertex of Γ_1. Also it is clear that if $\varepsilon>0$ is sufficiently small then $\varphi(\varepsilon)=0$ and $\varphi(1-\varepsilon)=1$.

Now assume that $0<s,t<1$ are such that $x_1(s)=x_2(t)$ and $\Gamma_1(s)$ is a vertex of Γ_1. Recall that $\Gamma_1(s)$ is the unique vertex with abscissa equal to $x_2(t)$. If $y_1(s)<y_2(t)$ or if s is t-special then φ is constant in a small interval around t. Otherwise we have $y_1(s)>y_2(t)$ and x_1 is not monotonic in any small interval $(s-\varepsilon,s+\varepsilon)$. In this case we have either $\varphi(t+\varepsilon)=\varphi(t-\varepsilon)+2$ (Fig. 1) or $\varphi(t+\varepsilon)=\varphi(t-\varepsilon)-2$ (Fig. 2).

Thus φ is piecewise constant with even jumps $(=0,\pm 2)$. This contradicts the fact that $\varphi(\varepsilon)=0$ and $\varphi(1-\varepsilon)=1$ for small $\varepsilon>0$.

It follows from this contradiction that Γ_1 and Γ_2 intersect.

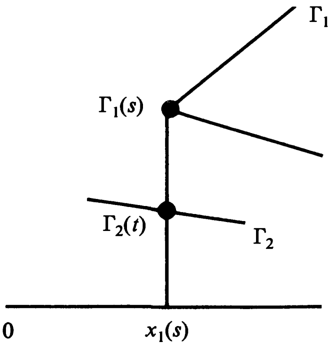

FIG. 1

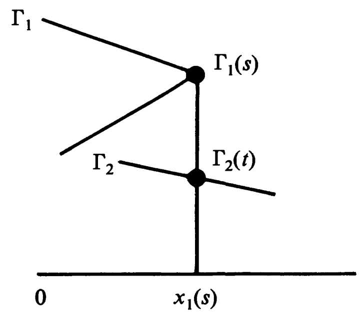

FIG. 2

Also solved by F. S. Cater, O. P. Lossers (Netherlands), and Arthur Solomon.

Editor's comments. Solomon reduces the problem to the polygonal Jordan Curve Theorem for which elementary proofs are known, e.g., B. H. Arnold, *Intuitive Concepts in Elementary Topology*, Prentice-Hall, 1962, pp. 91–92.

Cater proves that if C_i $(i=1,2)$ are two connected disjoint subsets of I^2, each of them containing two opposite vertices of I^2, then neither of them can be compact.

Nearly Doubled Primes

E 2648 [1977, 294]. *Proposed (part* (i)) *by R. P. Nederpelt, Eindhoven University of Technology, Netherlands, and (part* (ii)) *by R. B. Eggleton, Northern Illinois University, and John H. Loxton, University of New South Wales, Australia*

(i) Show that there is no infinite sequence of prime numbers $p_1,p_2,\dots$ such that $p_{k+1}=2p_k\pm 1$ for all k.

(ii)* Find a longest finite sequence of primes $p_1,p_2,\dots,p_n$ such that $p_{k+1}=2p_k+1$ for $1\leqslant k\leqslant n-1$.

I. *Solution of* (i) *by Stanley Wagon, Smith College.* Suppose $p_1,p_2,\dots$ is such a sequence. Clearly we may assume that $p_1>3$ and so $p_1\equiv\pm 1 \pmod 6$. Let $p_1\equiv -1 \pmod 6$. Since $2p_1-1$ is divisible by 3, we must have $p_2=2p_1+1$. Hence $p_2\equiv -1 \pmod 6$ and we can repeat the argument with p_2 instead of p_1. Thus $p_{k+1}=2p_k+1$ for all $k\geqslant 1$. Therefore

$$p_k=2^{k-1}p_1+2^{k-1}-1.$$

Thus $p_k\equiv 2^{k-1}-1 \pmod{p_1}$ and by Fermat's theorem we have $p_1|p_k$ for $k=p_1$. This is a contradiction.

The case $p_1\equiv 1 \pmod 6$ can be handled similarly.

II. *Discussion of* (ii)* *by Milton Eisner, Richmond, Virginia.* A sequence of length 6 is 89, 179, 359, 719, 1439, 2879.

Congruence arguments show that if a sequence is to have length greater than 6, it must satisfy all of the following:

$p_1\equiv 5 \pmod 6$
$p_1\equiv 9 \pmod{10}$
$p_1\equiv 5,\ 9,\ 11, \text{ or } 13 \pmod{14}$
$p_1\equiv 1,\ 3,\ 7, \text{ or } 21 \pmod{22}$
$p_1\equiv 1,\ 3,\ 5,\ 7,\ 15, \text{ or } 25 \pmod{26}$
$p_1\equiv 1,\ 5,\ 9,\ 11,\ 13,\ 19,\ 21,\ 23,\ 27, \text{ or } 33 \pmod{34}.$

Inspection of primes satisfying these congruences shows that there is no sequence of the required type of length greater than 6 with $p_1 < 2000$.

Also solved (part (i)) by Robert Breusch, James Davis & William Velez, Charles Delzell, Milton Eisner, Thomas Elsner, Enzo Gentile (Argentina), Marguerite Gerstell, Sidney Kravitz, O. P. Lossers (Netherlands), Jerry Metzger, John Riegsecker, James Tasky, David Wright, University of Wyoming Problem Group, and the proposer.

The sequence with $p_1 = 89$ was also found by Breusch, Elsner, Kravitz, and Wagon.

Comment. Kravitz proves that every sequence $a_1, a_2, \ldots$ with $a_{k+1} = 2a_k + 1$ $(k \geqslant 1)$ contains infinitely many composite numbers. This is stated as an exercise in D. Shanks, *Solved and Unsolved Problems in Number Theory*, p. 201.

Inequalities for Non-obtuse Triangles

E 2649 [1977, 294]. *Proposed by A. Oppenheim, University of Benin, Nigeria*

Let a,b,c and α,β,γ be the sides and the corresponding angles of a non-obtuse triangle. Show that

$$3(a+b+c) \leqslant \pi\left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}\right) \tag{1}$$

$$3(a^2+b^2+c^2) \geqslant \pi\left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}\right). \tag{2}$$

Solution by L. E. Mattics, University of South Alabama. We do (2) first. Let $f(x) = (1 - \pi/3x)\sin^2 x$ for $0 < x \leqslant \pi/2$ and $f(0) = 0$. Using the law of sines, the proof of (2) reduces to showing that $F(\alpha,\beta,\gamma) = f(\alpha) + f(\beta) + f(\gamma)$, subject to $\alpha + \beta + \gamma = \pi$, $0 \leqslant \alpha,\beta,\gamma \leqslant \pi/2$, is nonnegative. Using the Lagrange multiplier method, we see that if F has a local extremum in the interior of the region given by the constraints, then at that point $f'(\alpha) = f'(\beta) = f'(\gamma)$. We have

$$f(x) = \tfrac{1}{2}(1 - \cos 2x) + \frac{\pi}{3}\sum_{n \geqslant 1} (-1)^n \frac{(2x)^{2n-1}}{(2n)!},$$

$$f''(x) = 2\cos 2x + \frac{\pi}{3}\sum_{n \geqslant 2} (-1)^n \frac{2(2x)^{2n-3}}{n(2n-3)!},$$

$$f'''(x) = -4\sin 2x + \frac{\pi}{3}\sum_{n \geqslant 2} (-1)^n \frac{4(2x)^{2n-4}}{n(2n-4)!}.$$

Hence

$$f''(x) \geqslant 2\cos 2x + \frac{2\pi x}{3}\left(1 - \frac{4x^2}{9}\right),$$

$$f'''(x) \leqslant -4\sin 2x + \frac{2\pi}{9}(x^2-1)(x^2-3),$$

and so $f''(x) > 0$ for $0 \leqslant x \leqslant \pi/4$ and $f'''(x) < 0$ for $\pi/4 \leqslant x \leqslant \pi/2$. It is now easy to see that $f'(x)$ first increases and then decreases. Since $f''(\pi/3) > 0$, $f'(x)$ increases on $0 \leqslant x \leqslant \pi/3$.

The equations $f'(\alpha) = f'(\beta) = f'(\gamma)$ therefore imply that at least two of the angles α,β,γ are equal. If $\alpha = \beta = x$, $\gamma = \pi - 2x$, then

$$F(x,x,\pi-2x) = \frac{2(3x-\pi)}{3x(\pi-2x)} g(x)\sin^2 x,$$

where $g(x) = \pi - 2x - 4x\cos^2 x$. It is easy to check that $g(x) < 0$ for $\pi/4 < x < \pi/3$ and $g(x) > 0$ for $\pi/3 < x < \pi/2$. Consequently we have $F(x,x,\pi-2x) \geqslant 0$ for $\pi/4 \leqslant x \leqslant \pi/2$.

Hence if $F(\alpha,\beta,\gamma)$ has a negative global minimum, it must be on the boundary. Since $f'(\pi/7) < 0$ and $f(\pi/7) < -1/3$ we have $f(x) > -1/3$ for $0 \leqslant x \leqslant \pi/7$. For $\pi/7 \leqslant x \leqslant \pi/6$ we have $f(x) >$

$\frac{1}{4}(1-7/3)=-1/3$. Thus for $0\leqslant x\leqslant \pi/6$ we have $f(x)>-1/3$ and since then $\pi/3\leqslant \pi/2-x\leqslant \pi/2$ we also have $f(\pi/2-x)\geqslant 0$ and consequently $F(x,\pi/2-x,\pi/2)\geqslant 0$.

It remains to consider $F(x,\pi/2-x,\pi/2)$ for $\pi/6\leqslant x\leqslant \pi/3$. Since $f'(x)$ increases in $\pi/6\leqslant x\leqslant \pi/3$, the local minimum of $f(x)+f(\pi/2-x)$ must occur when $x=\pi/4$. But then $F(\pi/4,\pi/4,\pi/2)=0$. We finish (2) by noting that $F(\pi/2,\pi/2,0)=2/3$.

The proof of (1) is much easier. Let $u(x)=(1-\pi/3x)\sin x$ and $U(\alpha,\beta,\gamma)=u(\alpha)+u(\beta)+u(\gamma)$. This proof amounts to showing that $U(\alpha,\beta,\gamma)\leqslant 0$ when subjected to the same constraints as before. If $v(x)=x^{-1}\sin x$ then by the Mean Value Theorem

$$v(x)=v(\pi/3)+(x-\pi/3)v'(x_1),$$

where $x_1\in[x,\pi/3]$. Using this for $x=\alpha,\beta,\gamma$ we get

$$U(\alpha,\beta,\gamma)=\left(\alpha-\frac{\pi}{3}\right)^2 v'(\alpha_1)+\left(\beta-\frac{\pi}{3}\right)^2 v'(\beta_1)+\left(\gamma-\frac{\pi}{3}\right)^2 v'(\gamma_1).$$

Since $v'(x)=x^{-1}\cos x(x-\tan x)<0$ for $0<x<\pi/2$ we have $U(\alpha,\beta,\gamma)\leqslant 0$.

Also solved by the proposer.

Computing a Galois Group

E 2650 [1977, 294]. *Proposed by M. J. Pelling, Balliol College, Oxford, England*

Find the Galois group of the equation $x^9+x^3+1=0$ over the rationals.

Solution compiled from those of Lorraine L. Foster and J. Robert Henderson (jointly), California State University, Northridge, and Gerald J. Janusz, University of Illinois. Let $f(x)=x^3+x+1$ and $g(x)=f(x^3)$. Since f has no rational roots, it is irreducible over the rational field $\mathbf{Q}$. Since $f'(x)=3x^2+1$, $f(x)$ has precisely one real root, say a_1, and the remaining two roots a_2 and a_3 are complex conjugates of each other. Therefore the degree of $F=\mathbf{Q}(a_1,a_2,a_3)$ over $\mathbf{Q}$ is 6.

Let $\Delta=(a_1-a_2)(a_1-a_3)(a_2-a_3)$ and $D=\Delta^2$. It is well known that the discriminant D of a cubic polynomial x^3+px+q is given by $D=-4p^3-27q^2$. In our case we have $p=q=1$ and so $D=-31$.

Note that

$$a_1+a_2+a_3=0, \qquad a_1a_2a_3=-1. \tag{1}$$

Hence we can choose t_1,t_2,t_3 so that $t_i^3=a_i$ $(i=1,2,3)$, t_1 is real and $t_1t_2t_3=-1$. The roots of g are then $t_r\omega^s$ $(r=1,2,3;s=0,1,2)$ where ω is a fixed primitive cube root of unity.

Since $\sqrt{-3}\notin\mathbf{Q}(\sqrt{-31})$ we have $\omega\notin\mathbf{Q}(\Delta)$ and $\omega\notin F$. Thus $E=F(\omega)$ has degree 12 over $\mathbf{Q}$. It is clear that the splitting field K of g over $\mathbf{Q}$ contains E and that $K=E(t_1,t_2)$. Since $t_1^3=a_1\in E$, $t_2^3=a_2\in E$, and $\omega\in E$, we infer that $[K:E]$ is one of the numbers 1,3,9.

Let $u=t_1+t_2$. Then $u^3=a_1+a_2+3t_1t_2u$. Using (1) we obtain $u^3+a_3=3ut_1t_2$, whence we get

$$\left(u^3+a_3\right)^3=27u^3a_1a_2.$$

Using the fact that $a_1a_2a_3=-1$ and putting $v=u^3$ we have $a_3(v+a_3)^3+27v=0$ so that

$$24v+(v^3-3v-1)a_3+(3v^2-1)a_3^2=0. \tag{2}$$

By eliminating a_3 from (2) and $1+a_3+a_3^2=0$ we get

$$\begin{vmatrix} 24v & v^3-3v-1 & 3v^2-1 & 0 & 0\\ 0 & 24v & v^3-3v-1 & 3v^2-1 & 0\\ 0 & 0 & 24v & v^3-3v-1 & 3v^2-1\\ 1 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 1 \end{vmatrix}=0.$$

Thus v is a root of the polynomial

$$h(x)=x^9-24x^7-246x^6-78x^5+4260x^4-13685x^3-1542x^2-132x-1$$

and u is a root of $h(x^3)$.

One has the following factorization modulo 37:

$$h(x^3)\equiv(x^3+9)(x^6+20x^3+18)(x^6+24x^3+18)$$
$$(x^6+29x^3+3)(x^6+29x^3+30),$$

the factors being irreducible. If $h(x^3)$ is reducible over $\mathbf{Q}$ then it is reducible over the integers $\mathbf{Z}$ and the constant term of each irreducible factor must be ± 1. Since no monic proper factor of $h(x^3)$ (mod 37) has constant term ± 1, it follows that $h(x^3)$ is irreducible over $\mathbf{Q}$.

Thus $u\in K$ has degree 27 over $\mathbf{Q}$ and consequently we must have $[K:E]=9$. Since $M=\mathbf{Q}(\omega,\Delta)=\mathbf{Q}(\sqrt{-3},\sqrt{-31})$ is normal over $\mathbf{Q}$ it follows that $H=\mathrm{Gal}(K/M)$ is a normal subgroup of $G=\mathrm{Gal}(K/\mathbf{Q})$. Also $E=F(\omega)$ is normal over $\mathbf{Q}$, so $H_1=\mathrm{Gal}(K/E)$ is normal in G. Since $E(t_1)$ and $E(t_2)$ are two intermediate fields between E and K, the group H_1 is not cyclic but is elementary abelian of order 9.

H_1 is generated by σ and τ where these automorphisms are defined by

$$\sigma(t_1)=t_1,\quad \sigma(t_2)=t_2\omega,\quad \sigma(\omega)=\omega;$$
$$\tau(t_1)=t_1\omega,\quad \tau(t_2)=t_2,\quad \tau(\omega)=\omega.$$

Note that $\sigma(t_3)=\tau(t_3)=t_3\omega^2$ because $t_1t_2t_3=-1$.

There is an automorphism ρ of E such that $\rho(\omega)=\omega$, $\rho(a_1)=a_2$, $\rho(a_2)=a_3$, and $\rho(a_3)=a_1$. It can be extended to an automorphism of K and we denote this extension again by ρ. Then we must have

$$\rho(t_3)=t_1\omega^m,\quad \rho(t_1)=t_2\omega^n$$

for some $m,n=0,1,2$. Replacing ρ by $\pi\rho$ where π is a suitable element of $H_1=\langle\sigma,\tau\rangle$ we can arrange that $m=n=0$. Thus ρ satisfies

$$\rho(t_1)=t_2,\quad \rho(t_2)=t_3,\quad \rho(t_3)=t_1,\quad \rho(\omega)=\omega.$$

Hence $H=\langle\sigma,\tau,\rho\rangle$ is a nonabelian group of order 27 and exponent 3. It is easy to check that

$$\rho\sigma\rho^{-1}=\sigma^2,\quad \rho\tau\rho^{-1}=\sigma^2\tau$$

and so ρ commutes with $\sigma\tau$. Thus the center of H is $\langle\sigma\tau\rangle$.

Now let $\alpha=\mathrm{Aut}(K/\mathbf{Q})$ be the complex conjugation, i.e., $\alpha(t_1)=t_1$, $\alpha(t_2)=t_3$, $\alpha(\omega)=\omega^2$. Let β be the automorphism of $\mathbf{Q}(t_1,\omega,\Delta)$ which fixes t_1 and ω and sends Δ to $-\Delta$. There are three automorphisms β_1,β_2,β_3 of K which extend β because K has degree 3 over $\mathbf{Q}(t_1,\omega,\Delta)$.

Note that $\beta(a_2)=a_3$ and $\beta(a_3)=a_2$. Consequently $\beta_k(t_2)$ must be one of t_3, $t_3\omega$, $t_3\omega^2$. Since β_1,β_2,β_3 are distinct we must have that $\beta_1(t_2)$, $\beta_2(t_2)$ and $\beta_3(t_2)$ are distinct. Thus one of β_1,β_2,β_3 takes t_2 to t_3 and we call it again β. Thus $\beta(t_2)=t_3$ and since $\beta(t_1)=t_1$ and $t_1t_2t_3=-1$ we must have $\beta(t_3)=t_2$. Therefore $\beta^2=1$. Now one can check that $\alpha\beta=\beta\alpha$, and the group $V=\langle\alpha,\beta\rangle$ is a four-group.

Clearly G is a semidirect product of H and V and one can easily compute how V acts on H.

Partially solved by L. E. Mattics. The proposer shows that $[K:\mathbf{Q}]=108$, which was his original question. Foster and Henderson note that the same problem is stated in I. N. Herstein, *Topics in Algebra*, 1st ed., p. 186.

A Theorem of Grötsch

E 2651 [1977, 295]. *Proposed by Paul Erdös, Budapest, Hungary*

A finite number of pennies are placed flat on the plane so that no two overlap and no three touch each other. Prove that these pennies can be painted with at most three colors so that touching pennies bear different colors. (This is a variant of E 2527.)

Solution by Paul A. Catlin, Wayne State University. Regard the pennies as the vertices of a graph G, and consider two vertices to be adjacent in G if they represent touching pennies. By the conditions

of the problem, G is planar and triangle-free. Thus, by Grötsch's theorem (*Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin Luther Univ., Halle-Wittenberg, Math. Naturwiss. Reihe, 8 (1958), 109–119), the vertices of G (and hence the pennies) can be colored in 3 colors so that adjacent vertices (touching pennies) have different colors.

By a theorem of Grünbaum (*Grötsch's theorem on* 3-*colorings*, Michigan Math. J., 10 (1963) 303–310), even if there are as many as 3 triangles, one can still find a 3-coloring.

Also solved by David Hammer, L. E. Mattics, and Adam Riese.

Visible Lattice Points

E 2653 [1977, 386]. *Proposed by Albert A. Mullin, Huntsville, Alabama*

A lattice point $(x,y)\in\mathbf{Z}^2$ is visible if $\mathrm{GCD}(x,y)=1$. Prove or disprove: Given a positive integer n, there exists a lattice point (a,b) whose distance from every visible point is $\geqslant n$.

Solution by W. C. Waterhouse, Pennsylvania State University. Let p_{ij} be different primes for $-n\leqslant i,j\leqslant n$. By the Chinese remainder theorem choose a and b so that $a\equiv -i\ (\mathrm{mod}\, p_{ij})$ and $b\equiv -j\ (\mathrm{mod}\, p_{ij})$ for all i,j. Then for $-n\leqslant i,j\leqslant n$ the point $(a+i,b+j)$ is invisible, as the coordinates are both divisible by p_{ij}.

Hence (a,b) has the required property.

Also solved by forty-three other readers.

Editor's Comment. The same problem was proposed by Jan Mycielski and solved by M. Warmus in Colloquium Mathematicum 3 (1955) 203–205. Paul Erdös observes that the assertion follows from a result of Moser and himself in Canad Math. Bull 1 (1958). 5–8. Fritz. Herzog and B. M. Stewart observe that it also follows from their note in this MONTHLY, 78 (1971) 487–496. Blair Spearman notes that this problem appears as Theorem 5.29 in T. M. Apostol, *Introduction to Analytic Number Theory*.

Many solvers observe that the analogous result holds in $\mathbf{Z}^k$. Mycielski asks whether there exists a visible point in $\mathbf{Z}^2$ whose distance from every other visible point is $\geqslant n$. The answer is positive as shown by Herzog and Stewart (ibid.).

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to Professor R. C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before December 31, 1978.

6222. *Proposed by Emilie V. Haynsworth, Auburn University, Alabama*

Let A be an $n\times n$ matrix over the complex field. Let $\mathrm{Adj}\,A$ denote the standard adjoint matrix for A, that is $\mathrm{Adj}\,A=(C_{ji})$ where C_{ij} is the cofactor of a_{ij} in A. Prove that if $A+\mathrm{Adj}\,A=kI$, then

(1) A has at most two distinct eigenvalues, λ_1, and λ_2.

(2) The Jordan form, J, for A has blocks no larger than 2×2, and if $\lambda_1\neq\lambda_2$, A is diagonalizable.

(3) If $\lambda_1\lambda_2\neq 0$, and λ_1 has multiplicity m, then $\lambda_1^{m-1}\lambda_2^{n-m-1}=1$.

(4) If $\lambda_1=0$, $A\neq 0$, $n>2$, then λ_1 is a simple root and $\lambda_2^{n-2}=1$.

(5) Let $S=A+J-kI$. Then S^2 commutes with both A and J and, if S is nonsingular, $S^{-1}AS=J$.

(6) If A is nonnegative and λ_1 and λ_2 are both positive then A^{-1} is an M-matrix.

Conversely, if properties (1), (2) and (3) hold, then $A+\mathrm{Adj}\,A=kI$.

6223*. *Proposed by Harry D. Ruderman, Hunter College Campus School*

Let C be a convex curve. Let Q be a curve such that the two tangents to C from each point P of Q form an angle θ fixed in size. Assume that all points are in the same plane.

(1) If $\theta = 90°$ and Q is a circle, must C be a circle or an ellipse?

(2) If C is an ellipse and $\theta \neq 90°$ what is the nature of Q?

6224*. *Proposed by David P. Robbins, Hamilton College*

Suppose we are given N balls which are indistinguishable except that some are heavy and some are light (the heavy balls are alike in weight, as are the light balls). Using a pan balance what is the minimum number of weighings in which it is always possible

(a) to identify one heavy and one light ball?

(b) to determine the number of heavy and light balls?

6225. *Proposed by Edmund H. Anderson, Louisiana State University, Baton Rouge*

Construct a homotopically trivial mapping from the three-sphere onto the two-sphere such that the pre-images of points are simple closed curves.

6226. *Proposed by Marlow Sholander, Case Western Reserve University*

Domain D consists of the real numbers R from which a finite set is deleted. On domain D, the functions f, F, and G are continuous and satisfy the identity

$$f(r) - f(s) = (r-s)F(r)G(s).$$

Describe $f(x)$ on domain R.

6227*. *Proposed by D. M. Milosević, Pranjani, Yugoslavia*

Prove the following inequality in which $P_n(x)$ is a Legendre polynomial:

$$\int_{-1}^{+1} \frac{1-P_n(x)}{(1-x)^{5/4}}\,dx < 2^{5/4}\left(\sum_{k=1}^{n} \frac{n}{k}\right)^{1/2}.$$

SOLUTIONS OF ADVANCED PROBLEMS

A Fourier Transform in $\mathbf{R}^n$

6055 [1975, 942]. *Proposed by S. Zaidman, University of Montreal*

Let $u_\alpha(x, t)$ be the complex-valued function defined for $x \in \mathbf{R}^n$, $t \geqslant 0$, through the formula

$$u_\alpha(x,t) = (2\pi)^{-n/2} \int \cdots \int_{s_1^2 + \cdots s_n^2} \exp(-i(x_1 s_1 + \cdots + x_n s_n))\, g_\alpha(s_1, \ldots, s_n, t)\, ds_1 \cdots ds_n$$

where

$$g_\alpha(s_1, \ldots, s_n, t) = |s|^{-\alpha-2}(1 - e^{-|s|^2 t}),\ |s| \leqslant 1,\ t \geqslant 0,$$

$|s| = (s_1^2 + \cdots + s_n^2)^{1/2}$, α is a real number.

Find a number α such that

$$\lim_{t\to\infty} \int_{\mathbf{R}^n} |u_\alpha(x,t)|^2\, dx_1 \cdots dx_n = +\infty.$$

Solution by James A. Boa, State University of New York at Buffalo. Since $u_\alpha(x, t)$ is a Fourier transform, by Parseval's relation we are really asked to find α such that $\int |g|^2\, ds$ tends to infinity with

t. Since g is radially symmetric, $\|g\|_2$ is proportional to

$$I=\int_0^{\infty} g_\alpha^2(r,t)r^{n-1}\,dr=\int_0^1(1-e^{-r^2t})^2 r^{-2\alpha-4}r^{n-1}\,dr,$$

where $r=|s|$. We note that for convergence at zero, the integrand must be not as singular as r^{-1}. Since $1-e^{-r^2t}=O(r^2)$, this means that inequality

$$4-2\alpha-4+n-1>-1,$$

i.e., $\alpha<n/2$, is required.

If t is large, $1-e^{-r^2t}$ is very nearly 1 throughout most of the range $0\leqslant r\leqslant 1$. More precisely, given $\varepsilon>0$ it is possible to pick t sufficiently large so that the function $h_t(r)=1-(1-e^{-r^2t})^2$ is less than some prescribed number η, uniformly on the interval $\varepsilon\leqslant r\leqslant 1$. Now

$$I=\int_0^{\varepsilon} g_\alpha^2 r^{n-1}\,dr+\int_\varepsilon^1 r^{n-2\alpha-5}\,dr-\int_\varepsilon^1 h_t(r)r^{n-2\alpha-5}\,dr$$

$$>\frac{1}{4+2\alpha-n}\left[\varepsilon^{n-2\alpha-4}-1\right]-\int_\varepsilon^1 h_t(r)r^{n-2\alpha-5}\,dr.$$

Suppose $n-2\alpha-4$ is negative, i.e., $\alpha>\frac{1}{2}n-2$. Then given a large number M, pick ε so small that $(4+2\alpha-n)^{-1}\varepsilon^{n-2\alpha-4}$ is larger than M. Next pick t so large that $h_t(r)$ is uniformly less than $\varepsilon^{5+2\alpha-n}$ on the interval $\varepsilon\leqslant r\leqslant 1$, so that the integral $\int_\varepsilon^1 h_t(r)r^{n-2\alpha-5}\,dr$ is less than 1. Then

$$I+1+\frac{1}{4+2\alpha-n}>M,$$

so that

$$\lim_{t\to\infty} I=\infty \quad \text{if} \quad \tfrac{1}{2}n-2<\alpha<\tfrac{1}{2}n.$$

Also solved by P. G. Chauveheid (Belgium), Roger Cooke, Nathaniel Grossman, O. P. Lossers (Netherlands), and the proposer.

A Uniqueness Theorem in $\mathbf{R}^2$

6120 [1976, 817]. *Proposed by Jack Fishburn, University of Wisconsin at Madison*

Let f be a continuous function from the closed unit disk into the reals. If the line integral of f over every chord is zero, must f be identically zero?

What if the continuity of f is replaced by measurability? Must $f=0$ almost everywhere?

Solution by Kenneth J. Falconer, Corpus Christi College, Cambridge, England. We prove the more general result that if f is a plane integrable function whose integral along every line is zero then f is zero almost everywhere.

Because f is integrable, it has a two-dimensional Fourier transform, $\hat{f}(d)$ $(d\in\mathbf{R}^2)$, defined by

$$\hat{f}(d)=\int f(x)e^{ix\cdot d}\,dx.$$

In particular, we may express $\hat{f}(d)$ in terms of integrals along lines perpendicular to d, that is,

$$\hat{f}(d)=\int F(\mu,\theta)e^{i\mu|d|}\,d\mu \quad (d\neq 0)$$

where θ is a unit vector parallel to d, and $F(\mu,\theta)$ is the integral of f along the line perpendicular to θ and distance μ from the origin. (This follows by taking coordinates parallel to and perpendicular to d and applying Fubini's theorem.) $F(\mu,\theta)$ is zero for all μ and θ; therefore $\hat{f}(d)=0$ for all d (the Fourier transform of an integrable function is continuous) so that by the uniqueness of Fourier transforms f is zero almost everywhere.

We add that if f is a plane integrable function that vanishes outside some (unspecified) compact set it may be shown that if the integral of f along all lines in each of an infinite collection of directions is zero then f is zero almost everywhere.

Also solved by D. H. Armitage (N. Ireland), Ralph Henstock (N. Ireland), I. M. Isaacs, O. P. Lossers (Netherlands), W. Moran (N. Ireland), and Walter Rudin.

Editor's notes. Henstock and Armitage have offered references to the theorem that, if f is Lebesgue integrable over the plane and $\int f=0$ over every doubly infinite line, then $f\sim 0$:

1. A. S. Besicovitch, J. London Math. Soc., 33 (1958) 82–84.
2. J. W. Green, Proc. Amer. Math. Soc., 9 (1958) 758–762.
3. D. J. Newman, this MONTHLY, 64 (1957) 750–751.
4. A. Rényi, Acta Math. Acad. Sci. Hungar., 3 (1952) 131–141.

Isaacs proves in the case f is continuous that it is sufficient to assume that the line integrals vanish for chords whose inclinations lie in an arbitrarily small interval.

An Integer Expression

6121 [1976, 817]. *Proposed by Harry D. Ruderman, Hunter College Campus School*

For all positive integers $a_1, a_2, \ldots, a_n$ the following is always an integer:

$$\prod_{t=1}^{n} (na_i)!\Big/\left[n\prod_{i=1}^{n}(a_i!)\right]^{n-1}\left(\sum_{i=1}^{n} a_i\right)!.$$

Prove the conjecture for $n=3$. Is it true in general?

Solution by Jerrold R. Griggs, Massachusetts Institute of Technology. For primes p, let $N(p)$ (respectively, $D(p)$) represent the number of times p divides the numerator (denominator). The expression is an integer if $N(p)\geqslant D(p)$ for all p.

Suppose first that $p\nmid n$. Letting $S=\Sigma_{i=1}^{n} a_i$, we have

$$N(p)=\sum_{i=1}^{n}\sum_{j=1}^{\infty}\left[na_i/p^j\right]$$

and

$$D(p)=(n-1)\sum_i\sum_j\left[a_i/p^j\right]+\sum_j\left[S/p^j\right].$$

Fix j and let $a_i/p^j=k_i+l_i/n+\delta_i$ (k_i, $l_i\in N$, $l_i<n$ and $0\leqslant\delta_i<1/n$),

$$(n-1)\sum_i\left[a_i/p^j\right]+\left[S/p^j\right]=(n-1)\sum_i k_i+\left(\sum_i k_i+\left[\sum_i\left(\frac{l_i}{n}+\delta_i\right)\right]\right)$$

$$<n\sum_i k_i+\sum_i l_i+1.$$

Thus,

$$(n-1)\sum_i\left[a_i/p^j\right]+\left[S/p^j\right]\leqslant n\sum_i k_i+\sum_i l_i$$

$$=\sum_i\left[na_i/p^j\right].$$

Summing over j gives $D(p)\leqslant N(p)$.

Now suppose $p|n$, say $n=qp^w$, $p\nmid q$, and $w\geqslant 1$. We shall use the following fact:

$$(*)\qquad \sum_{j=1}^{\infty}\left[\alpha/p^j\right]\leqslant\frac{\alpha-1}{p-1}\qquad(\alpha\geqslant 1).$$

(If $p^m \leqslant \alpha < p^{m+1}$, $\sum_{j=1}^{\infty}[\alpha/p^j] = \sum_{j=1}^{m}[\alpha/p^j] \leqslant (\alpha-1)/(p-1.)$

$$N(n) = \sum_i \sum_j [na_i/p^j] = \sum_i \left(qa_i(1+p+\cdots+p^{w-1}) + \sum_j [qa_i/p^j] \right)$$

$$= qS\left(\frac{p^w-1}{p-1}\right) + \sum_i \sum_j [qa_i/p^j]$$

$$\geqslant S\left(\frac{n-q}{p-1}\right) + q\sum_i \sum_j [a_i/p^j].$$

$$D(n) = w(n-1) + (n-1)\sum_i \sum_j [a_i/p^j] + \sum_j [S/p^j].$$

Thus, $N(n) \geqslant D(n)$ if

$$S\left(\frac{n-q}{p-1}\right) \geqslant w(n-1) + (n-q-1)\sum_i \sum_j [a_i/p^j] + \sum_j [S/p^j].$$

By (*), this is true if

$$S\left(\frac{n-q}{p-1}\right) \geqslant w(n-1) + (n-q-1)\sum_i \frac{a_i-1}{p-1} + \frac{S-1}{p-1}$$

which reduces to

$$\binom{*}{*} \qquad (n-q-1)n+1 \geqslant w(n-1)(p-1).$$

This can be verified directly for $w = q = 1$. *For* $q > 1$ and/or $w > 1$, $q(1+p+\cdots+p^{w-1}) > w$, so that

$$n-q-1 = q(p^w-1)-1 \geqslant w(p-1),$$

which easily implies $\binom{*}{*}$.

It would be interesting to have a combinatorial argument.

Also solved by L. E. Mattics, José Luis de Miguel (Spain), Harold Shapiro, Peter Ungar, and J. van de Lune (Netherlands).

The Nearest Point in a Compact Set

6122 [1976, 817]. *Proposed by Albert A. Mullin, Redstone Arsenal, Alabama*

Does there exist a compact set $S \subset E^2$ such that for each $x \in E^2 \setminus S$ there exist precisely two nearest points of S? Clearly S cannot be convex.

Solution by William J. Gilbert, University of Waterloo, Canada. The answer is no. If S is any compact set in E^2, it cannot be the whole of E^2, so there exists a point $y \in E^2 \setminus S$. Let s be a point of S nearest to y. The circle, center y, radius ys, contains no points of S in its interior; since S is compact, its radius is nonzero. Consider any point x lying on the radius between y and s. The circle center x and radius xs contains only one point of S, namely s, and no points of S in its interior. Hence $x \in E^2 \setminus S$ and has only one nearest point of S.

Also solved by Marek Anczura, D. H. Armitage (Northern Ireland), Dennis Berkey, J. M. Borwein (Canada), Dietrich Braess (F. R. Germany), William Bynum & David Stanford, John Cantwell & Raymond Freese & Dana Kamerud, Kenneth Falconer (England), Thomas Gard, Jerrold Griggs, Gustaf Gripenberg (Finland), Donald Hayman, Edward Howorka, Eli Isaacson, Elgin Johnston, Lee Keener (Canada), E. H. Kronheimer (England), O. P. Lossers (Netherlands), Alvin Martin, Mark Meyerson, Lee Mohler, John Morgan II, James Munkres, Edward Ordman, Bruce Peterson, John Rainwater, Adam Riese, Ira Rosenholtz, Martin Schechter, Wolfe Snow,

University of South Carolina Problem Group, Walter Stromquist, M. F. Walker (South Africa), L. E. Ward, Jr., Gregory Wene, Edward Wilson, and G. Yuval (Israel).

Editor's note. Several generalizations were given by the solvers of this problem, but in each case the proof was close to that given above. For example, Bynum and Stanford prove: If S is a proper subset of a strictly convex normed linear space V over the real or complex numbers, then there is a point in $V\setminus S$ for which there exist no more than one nearest point of S.

Union of Sets of Zero Dimension

6126 [1977, 61]. *Proposed by Harold Reiter, University of North Carolina, Charlotte*

A well-known theorem of dimension theory states: The countable union of closed sets each of dimension zero has dimension zero.

If X is a metric space and $(2^X, D)$ is the associated space of compact subsets, with the Hausdorff metric, the following proposition, if true, would generalize the above theorem. Let S be a zero-dimensional collection of compact zero-dimensional sets. Then $\cup\{C: C\in S\}$ is zero-dimensional. Prove or disprove the proposition.

Solution by the proposer. The proposition is not true. Let X be the real line. Let C be the Cantor set and let $C+x=\{y+x: y\in C\}$. Then the set $\{C+x: x\in C\}$ is a zero dimensional collection of Cantor sets, but $\cup\{C+x: x\in C\}=C+C$ is the closed interval $[0,2]$.

$\Sigma\zeta(n)x^n$ for x rational

6127 [1977, 62]. *Proposed by M. J. Pelling, Balliol College, Oxford, England*

Sum the series $\sum_{n=2}^{\infty}\zeta(n)(a/b)^n$ where $0<a/b<1$ is rational. (The answer should not involve the gamma function.)

Solution by Robert B. Israel, University of British Columbia, Canada. We have

$$\sum_{n=2}^{\infty}\zeta(n)(a/b)^n=\sum_{n=2}^{\infty}\sum_{m=1}^{\infty}(a/bm)^n=\sum_{m=1}^{\infty}\frac{(a/bm)^2}{1-a/bm}=a\sum_{m=1}^{\infty}\left(\frac{1}{bm-a}-\frac{1}{bm}\right).$$

Let b be fixed, and let $S_N(a)=\sum_{m=1}^{n}(bm-a)^{-1}$. If $\omega\neq 1$ is a bth root of unity then

$$\sum_{a=0}^{b-1}\omega^{-a}S_N(a)=\omega+\omega^2/2+\cdots+\omega^{bN}/bN\to-\ln(1-\omega)$$

as $N\to\infty$, where we take the branch of ln with imaginary part from $-\pi$ to π. Now

$$S_N(a)-S_N(0)=b^{-1}\sum_{\omega}(\omega^a-1)\sum_{j=0}^{b-1}\omega^{-j}S_N(j)$$

so that

$$\sum_{n=2}^{\infty}\zeta(n)(a/b)^n=-\frac{a}{b}\sum_{\omega}(\omega^a-1)\ln(1-\omega)$$

where the sum Σ_ω is over all bth roots of unity except 1.

Also solved by Paul Chauveheid (Belgium), L. E. Clarke (England), Irving Gerst, John Hatcher, José Luis de Miguel (Spain), O. P. Lossers (Netherlands), and the proposer.

EDITED BY J. ARTHUR SEEBACK, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Use of Mathematical Literature. A. R. Dorling, Editor. Butterworths, London, 1977. xii + 260 pp. $24.95. (Telegraphic Review, February 1978.)

One winter afternoon during my senior year at the University of Illinois, Harry Levy took me to the mathematics library and spent an hour showing me around. He told me about the Jahrbuch and the Zentralblatt (the first issue of Mathematical Reviews was still seven years in the future), he sketched the Dewey decimal classification of mathematics, he showed me the most important journals (such as the Annals, the Transactions, and the American Journal, Fundamenta, Mathematische Annalen, and Sbornik), and he introduced me to the most informative survey books in several fields (such as van der Waerden, Saks, Titchmarsh, Veblen and Young, and Lefschetz). I have been grateful to him ever since (but why did I never tell him so?), and I have considered that afternoon my first and most valuable step toward mathematical scholarship.

The problem of information retrieval (isn't that the current phrase for scholarship?) is much greater now than it was 45 years ago, and it is part of the job of every conscientious teacher to help the next generation cope with it. The problem exists at all levels: where can a high school student learn about the nine-point circle, where can an undergraduate fill the gap between his physics lecture and the truth about partial differential equations, where can a teacher collect the list of basic books needed to prepare a new course, where can a graduate student find a connected topological group in which every element has order 2, where can a curious ring-theorist get straight on the undecidability of the continuum hypothesis, and, at the research frontier, where can an analyst who needs some polynomial therapy be vouchsafed a glance into the arcane mysteries of algebraic geometry?

There exist expository articles, survey books, review journals, bibliographies, and bibliographies of bibliographies. The book under review is offered as a guide to the primary sources of mathematics (articles and books), as well as to reviews of them and lists of them; it is somewhere between a bibliography and a bibliography of bibliographies. Any book of this kind is ephemeral (out of date in twenty years, for sure—perhaps in ten?), but it could be of great value while it lasts.

I don't know how many such books have been published (ever, or recently), but I know two others: one by Nathan Grier Parke III (Guide to the literature of mathematics and physics including related works on engineering science, McGraw-Hill, 1947), and one by John E. Pemberton (How to find out in mathematics, Pergamon, 1963).

Parke is heavy on physics; his headings include electroacoustics, fluorescence, optics, and pyrometry. About 80 pages (out of 200) are a textbook on how to read, how to study, and how to use a library. The rest is a sequence of articles, dictionary style, from adsorption to x-rays. The references under abstract algebra could still be useful to students; they include Artin's *Galois Theory*, Jacobson's *Theory of Rings*, and van der Waerden's *Moderne Algebra*. Algebraic geometry lists Coble's *Theta Functions*, Weil's *Foundations*, and Zariski's *Algebraic Surfaces*. The functional analysis entry refers (among others) to Banach's *Opérations Linéaires* and to Stone's *Hilbert Space*, and topology refers to Hurewicz and Wallman, the first edition of Lefschetz's colloquium volume, and Pontrjagin's *Topological Groups*. These are not lists to be ashamed of.

Pemberton is a librarian's book, more than a mathematician's; it mentions journals and bibliographies, more than subjects and books, but it steers a would-be student of probability, for instance, to Kolmogorov and to Feller and it tells about review journals (even *Revue Semestrielle*).

The present book does not have an author: it has 16 of them, including A. R. Dorling, the editor. There are 14 articles (of about 20 pages each), and most of them have to do with pure mathematics. The others include three introductory articles on generalities (such as organizations and journals), one on mathematical education, one on the history of mathematics, and one on mathematical programming.

The qualities of the articles vary, and their breadths depend, naturally, on the authors. Among the best are *Rings and Algebras* (by P. M. Cohn) and *Topology* (by J. F. Adams and A. R. Pears). A repeated source of inconvenience and annoyance is the ordering of the bibliographies. Cohn's piece, for example, is accompanied by a list of 182 numbered items, arranged in their order of mention in the text; I am pretty sure that MacLane's *Categories for the Working Mathematician* is in there somewhere, but I couldn't find it in the eight pages of the bibliography.

If I wanted to learn something about number theory, Dorling wouldn't help me. The phrase is in the index, but I could find no references to texts or expository articles. None of G. H. Hardy, J. E. Littlewood, and André Weil seems to be mentioned. (Parke's list contains Uspensky–Heaslet, as well as Hardy and Wright. That's not much from the nearly infinite literature, but it's a start.) Approximation theory is not in the index, and neither is operator theory; functional analysis is, just barely, but Banach and Stone are not. Harmonic analysis is mentioned in passing only (in S. J. Taylor's discussion of measure and probability), and I couldn't find integral equations at all. Unlike Pemberton (who refers to Parke), Dorling does not refer to either Parke or Pemberton.

I am glad I saw the book; it served to remind me that it is part of my job as a teacher to teach the seekers of the future how to stand on the shoulders of the giants of the past. There should be books like this, good ones; somebody ought to write one from time to time. This one, unfortunately, is not a good one.

P. R. HALMOS, Indiana University

MISCELLANEA

12. Doubtless we have known the excesses of a youthful romanticism in reaction against conservatism; it is well to retain one's critical sense when one is periodically invited to go into raptures over callow poets, kindergarten Picassos, or maestros in baby clothes. But, at least in mathematics, there is no possible doubt; without even mentioning the case of Galois, which is exceptional in many ways, dozens of famous examples show that in the continuing confrontation of the generations, the young almost always win, and their rear-guard opponents succeed only in making themselves ridiculous in the eyes of posterity. Let the work of Galois never cease to remind those aging mathematicians who tend to forget, that their duty to science compels them to make room for the young and their new ideas, and to aid the propagation and success of those ideas in every way.

J. Dieudonné, from the preface to
Écrits et mémoires mathématiques d'Évariste Galois, Paris, 1962
(translated and reproduced by permission of Professor Dieudonné).

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, T**(13-16: 1, 2), S, L. *Faces of Mathematics: An Introductory Course for College Students*. A. Wayne Roberts, Dale E. Varberg. T.Y. Crowell, 1978, xiii + 456 pp, $15.95. [ISBN: 0-7002-2507-2] An exciting and lively introduction to mathematics as a vehicle for learning how to think clearly, centered à la Pólya, on a vast array of intriguing puzzle-problems. Standard topics appear (e.g., computing, probability, matrices, number systems) but in refreshing contexts. The book is full of marginal nuggets (biographies, illustrations, humor, anecdotes, quotations, problems) that make browsing a delight. A superb option for a course in mathematics for liberal arts students or for prospective elementary school teachers. LAS

GENERAL, T(13: 1). *Introduction to Mathematics, Fourth Edition*. Bruce E. Meserve, Max A. Sobel. P-H, 1978, x + 499 pp, $13.95. [ISBN: 0-13-487553-2] This edition includes two new chapters, one on computers and calculators and one on the metric system. (*Second Edition*, TR, November 1969; *Third Edition*, TR, December 1973.) MU

GENERAL, T(13: 1). *Mathematical Ideas, Third Edition*. Charles D. Miller, Vern E. Heeren. Scott F, 1978, xi + 513 pp, $13.95. [ISBN: 0-673-15090-9] Completely new chapters on algebra and geometry. Eliminates chapter on transformational geometry, reduced coverage of functions and relations, increased coverage of the number systems. Much revision and updating elsewhere (*Second Edition*, TR, March 1974). Increase in abundance and quality of problems and of historical and biographical blurbs. TLS

GENERAL, S, L. *Mathematics and Science: An Adventure in Postage Stamps*. William L. Schaaf. NCTM, 1978, xiv + 152 pp, $7 (P). [ISBN: 0-87353-122-1] A chronology of science, technology, and mathematics based on the interesting but highly erratic data of postage stamps. Contains check lists of individuals and scientific topics featured on stamps. A good guide to the scientifically-minded philatelist, but a very misleading portrait of mathematics and science. LAS

GENERAL. *The Impartial Eye, A New Approach to Mathematics and Physics*. Ralph E. Bucknam. Sowers Printing, 1978, xv + 679 pp, $28.35. A mathematical, physical and philosophical *Ulysses* written by a technologically skilled patent lawyer. JAS

GENERAL, S(9-13), L. *The Tokyo Puzzles*. Kobon Fujimura. Transl: Fumie Adachi. Scribners, 1978, 184 pp, $8.95. [ISBN: 0-684-15536-2] 98 classic puzzles told by Japan's most popular writer of puzzle books. Many themes are well known in the West (false coins, moving matchsticks, logical liars) but cast here with novel angles; others are not known widely outside Japan. Try this: cut a cube of cheese into two pieces each with a cross section shaped like a regular hexagon. LAS

GENERAL, P, L. *The Phenomenon of Science*. V.F. Turchin. Columbia U Pr, 1977, xvii + 348 pp, $17.50. [ISBN: 0-231-03983-2] Offers a comprehensive cybernetic theory of biological and intellectual evolution, with emphasis on development of mathematical and scientific thought. Based on the central notion of "metasystem transition--the transition from a cybernetic system to a metasystem, which includes a set of systems of the initial type and controlled in a definite manner." This somewhat popular work offers interesting insights into many phenomena of intellectual history, including foundations of mathematics. Enjoyable reading. GHM

GENERAL, S(13-14). *Science Brain-Twisters, Paradoxes, and Fallacies*. Christopher P. Jargocki. Scribners, 1976, viii + 183 pp, $3.95 (P). [ISBN: 0-684-15585-0] 169 popular physical science questions, with excellent complete explanations. See hardcover edition, TR, December 1976. LAS

GENERAL, S(15-18), P, L. *Lexikon der Mathematik*. Walter Gellert, Herbert Kästner, Siegfried Neuber. VEB Bibliographisches Inst, 1977, 624 pp, 28M. A dictionary of mathematical terms, with brief biographies of important mathematicians and longer articles on important topics (e.g., groups, linear transformations). A bargain for those whose German is good. JD-B

GENERAL, T(14-16: 1, 2), S, P, L*. *Exploratory Data Analysis*. John W. Tukey. A-W, 1977, xvi + 688 pp, $18.75. [ISBN: 0-201-07616-0] An extraordinary, novel outline of rough and ready methods to tame mountains of data (e.g., rescaling, graphing, residuals, smoothing, two and three way fits) by hand using pencil and graph paper, rather than calculators or computers. It transforms numbers into pictures, especially pictures that "force us to notice what we never expected to see." It is, unfortunately, obscured by disheveled layout, stylistic anarchy and pervasive jargon (flog, froot, hanning). An unprecedented treatment of very useful ideas, this text will make an attractive elementary elective for students wishing to get their hands dirty with realistic problems that do not require sophisticated mathematical techniques. LAS

BASIC, T(13: 1). *Basic Business Mathematics*. John W. and Charlotte L. Ernest. Glencoe Pr, 1977, ix + 405 pp, $12.95 [ISBN: 0-02-472610-9]; *Answer and Test Manual*, 155 pp, (P). Basic arithmetic, simple and compound interest, financial statements, reading business graphs. Frequent exercises, review problems, answers at rear to odd-numbered problems. LAS

BASIC, T(13-14). *Mathematics for Business in a Consumer Age.* Stanley A. Salzman, Charles D. Miller. Scott F, 1978, 454 pp, $12.95. [ISBN: 0-673-15092-5] Part I reviews basic computational algorithms of arithmetic (whole numbers, decimals, percents, fractions) with no attention to underlying theory or conceptual understanding. Parts II-IV treat topics in mathematics (= arithmetic) of business (discounts, mark-up, profit, payroll, interest, depreciation, financial statements, annuities, installment buying, real estate, taxes, etc.). Emphasis on "how-to-do-it," with rote procedures outlined for each context. Enhanced by extensive testing materials and carefully graded exercises. Learning objectives clearly stated for each section. GHM

BASIC, T(13). *Modern Technical Mathematics.* Nathan O. Miles. Reston Pub, 1978, x + 853 pp, $17.95. [ISBN: 0-87909-509-1] High school mathematics plus a little on matrices, determinants and probability. Intended for technical students and written at a low intellectual level. JD-B

PRECALCULUS, T(13: 1). *College Algebra, Second Edition.* Margaret L. Lial, Charles D. Miller. Scott F, 1977, 369 pp, $12.95. [ISBN: 0-673-15039-9] Basically a revision of the 1973 first edition (TR, October 1973). JAS

PRECALCULUS, T(13: 1, 2). *Precalculus Mathematics with Elementary Functions.* Lawrence P. Runyan. Allyn, 1977, xiii + 601 pp, $15.95. [ISBN: 0-205-05573-7] A comprehensive survey of high school mathematics from elementary algebra through trigonometry, analytic geometry, matrices, and probability. Text consists largely of worked examples followed by exercises grouped under three headings: Fundamentals, Essentials, Applications. LAS

PRECALCULUS, T(13: 1). *College Algebra.* Bernard J. Rice, Jerry D. Strange. Prindle, 1977, x + 372 pp, $11.95. [ISBN: 0-87150-226-7] Contains, in addition to the usual core topics, linear programming, and vectors. Inexplicably, the layout in two colors frequently employs the PWS lion as a marginal dingbat without discernible pattern. JAS

PRECALCULUS T(13: 2-4). *Introductory Mathematical Analysis, Fifth Edition.* Edgar D. Eaves, J. Harvey Carruth. Allyn, 1978, xiii + 766 pp, $15.95. [ISBN: 0-205-05991-0] This new edition follows the same general outline as the *Fourth Edition* (TR, October 1974). However, the rigorous treatment of a number of topics has been replaced by an "example-motivated," "example-reinforced," less rigorous treatment. The number of illustrated examples has been increased. MU

EDUCATION, T(14: 1), S. *Mathematics for Elementary Teachers.* Eugene F. Krause. P-H, 1978, xvii + 471 pp, $12.95. [ISBN: 0-13-562702-8] A solid first mathematics course for prospective elementary school teachers. The standard topics are covered along with an introduction to probability. Number systems are studied from the concrete, algorithmic, and structural points of view. Lots of good problems. CEC

EDUCATION, S(16), L. *A Parents' Guide to School Mathematics.* Alan Tammadge, Phyllis Starr. Cambridge U Pr, 1977, vii + 182 pp, $13.50; $3.95 (P). [ISBN: 0-521-21108-5] A School Mathematics Project Handbook that describes the new mathematics program of the primary and secondary schools in the United Kingdom, with emphasis on secondary. Includes problems (with solutions) and some entertaining cartoons. PJ

EDUCATION, S(16), P, L. *Notes on Mathematics for Children.* Association of Teachers of Mathematics. Cambridge U Pr, 1977, xvi + 233 pp, $15.95; $5.95 (P). [ISBN: 0-521-20970-6; 0-521-29015-5] A thought-provoking collection of anecdotes, expositions, examples and arguments that focus on some significant problems and issues related to the way in which children learn mathematics and how it is taught. CEC

EDUCATION, T(13: 1), S. *Theory and Applications of Mathematics for Teachers, Second Edition.* Jason L. Frand, Evelyn B. Granville. Wadsworth, 1978, vii + 498 pp, $14.95. [ISBN: 0-534-00535-7] This book, for prospective teachers at the elementary level, focuses on the structure of the reals, numeration, divisibility, the metric system and measurement. The *Second Edition* includes more problems and an expanded discussion of the metric system. CEC

HISTORY, P, L**. *The Bicentennial Tribute to American Mathematics 1776-1976.* Ed: Dalton Tarwater. MAA, 1977, vii + 225 pp, $13. [ISBN: 0-88385-424-4] Lectures and panel presentations from the 1976 San Antonio M.A.A. meeting, ranging from history of American mathematics (authors: Struik, Grabiner, Birkhoff, Halmos, *et al.*, Rees, Hamming, Lax) to current trends in teaching and prospects for the near future. Contains many analyses and insights of value to teachers of college mathematics. LAS

HISTORY, P. *Briefwechsel zwischen Alexander von Humboldt und Carl Friedrich Gauss.* Ed: Kurth-R. Biermann. Akademei-Verlag, 1977, 202 pp, 36M. A scholarly edition of the extant letters between Gauss and Humboldt, all previously published. JD-B

HISTORY, P, L*. *Mathematical Demography.* David Smith, Nathan Keyfitz. Biomathematics, V. 6. Springer-Verlag, 1977, xi + 514 pp, $35.90. [ISBN: 0-387-07899-1; 3-540-07899-1] Excerpts from 56 key papers in the history of mathematical demography, ranging from a life table in Roman Law to recent papers in extinction probabilities and cohort methods. A unique resource for historians, and a good source for classroom enrichment. LAS

FOUNDATIONS, P. *Proof Theory.* Kurt Schütte. Transl: J.N. Crossley. Grund. math. Wissenschaften, B. 225. Springer-Verlag, 1977, xii + 302 pp, $23.10. [ISBN: 0-387-07911-4; 3-540-07911-4] Systematic presentation of the most important results to grow out of Hilbert's program for a constructive foundations of mathematics (i.e., "finitary" consistency proofs). Completely rewritten and updated translation of the 1960 edition. Includes Gödel's interpretation of pure number theory via functionals of finite type. Also consistency proof for a formal system of Π_1^1-analysis, following Takeuti but using a recently developed constructive system of ordinal notation. GHM

FOUNDATIONS, P. *Mathematical Logic, the Theory of Algorithms and the Theory of Sets.* Ed: S.I. Adjan. Proc. of Steklov Inst. of Math., No. 133. AMS, 1977, iv + 274 pp, $39.60 (P). [ISBN: 0-8218-3033-3] Twenty-three papers dedicated to the logician P.S. Novikov on his seventieth birthday. Translated from the Russian. Includes three survey articles on Novikov's scientific and pedagogical activity, on his work in descriptive set theory and in algorithmic problems of algebra. Some of the remaining articles also contain survey material. GHM

FOUNDATIONS, T(16-17: 1), S. *Einführung in die Mengenlehre.* Heinz-Dieter Ebbinghaus. Wissenschaftliche Buchgesellschaft, 1977, xiii + 177 pp, (P). [ISBN: 3-534-06709-6] A clearly and carefully written introduction to set theory. Makes a real effort to help the reader understand what is going on. Problems, historical remarks. JD-B.

FOUNDATIONS, T(16-18: 1), P, L. *Advanced Logic for Applications.* Richard E. Grandy. Reidel, 1977, xiii + 168 pp, $24. [ISBN: 90-277-0781-2] Excellent exposition of the central results of classical mathematical logic (completeness, effectiveness, incompleteness, undecidability, undefinability of truth, intuitionistic theories, etc.). However, the title is misleading: no "applications" are given or even suggested. Intended for wide audience of mathematicians, philosophers, linguists, psychologists, who may (!) themselves discover applications in their own fields. GHM

FOUNDATIONS, S(16-17), P. *Naive Mengen und abstrakte Zahlen I.* Walter Felscher. Bibliographisches Institut, 1978, 260 pp, (P). [ISBN: 3-411-01538-1] The first of three volumes on set theory and the real numbers. This one contains a discursive treatment of the former (but without transfinite methods) and a development of the natural numbers. JD-B

FOUNDATIONS, P. *A Theory of Possibility.* Nicholas Rescher. U of Pittsburgh Pr, 1975, xvi + 255 pp, $19.95. [ISBN: 0-8229-1122-I] Subtitle: a constructivistic and conceptualistic account of possible individuals and possible worlds. Propounds a systematic formal theory of possibilia, with a very wide range of applications to problems in modal logic, philosophical analysis and metaphysics. Lucid and appealing argument. Essential reading for those of an essentialistic bent. GHM

LINEAR ALGEBRA, T*(13-14: 1, 2). *Computational Linear Algebra with Models, Second Edition.* Gareth Williams. Allyn, 1978, xv + 480 pp, $15.95 [ISBN: 0-205-05998-8]; *Instructor's Manual,* 226 pp, (P). Earlier treatment of matrices (now preceding rather than following vector spaces) provides the student with more powerful tools early in the course. The extensive, realistic applications fully integrated with the text and the careful use of computer-based exercises continues to make this one of the best elementary applied linear algebra texts. (See TR of first edition, May 1975.) LAS

COMBINATORICS, S(18), P. *Lecture Notes in Mathematics-612: Infinitary Combinatorics and the Axiom of Determinateness.* Eugene M. Kleinberg. Springer-Verlag, 1977, 150 pp, $8.30 (P). [ISBN: 0-387-08440-1; 3-540-08440-1] Tidy exposition of infinite exponent partition relations implied by the axiom of determinateness. Axiom of determinateness is used only to derive two fundamental combinatorial relations on aleph one and aleph two. Remainder is almost exclusively combinatorial. Elegant proofs and very readable text. GHM

NUMBER THEORY, T*(17: 1), S, P, L. *p-Adic Numbers, p-Adic Analysis, and Zeta-Functions.* Neal Koblitz. Grad. Texts in Math., V. 58. Springer-Verlag, 1977, x + 122 pp, $12.80. [ISBN: 0-387-90274-0; 3-540-90274-0] A nicely written introduction to p-adic numbers which includes algebraic extensions and closures, power series, Mazur's construction of the p-adic zeta-function and Dwork's proof of the rationality of the zeta-function of a system over a finite field. Lots of good exercises make this book very usable. CEC

ALGEBRA, P. *Lecture Notes in Mathematics-625: Odd Order Group Actions and Witt Classification of Innerproducts.* J.P. Alexander, P.E. Conner, G.C. Hamrick. Springer-Verlag, 1977, 202 pp, $11.50 (P). [ISBN: 0-387-08528-9; 3-540-08528-9]

ALGEBRA, S(15-18). *On Freely Acting Groups.* Temple H. Fay. J. of Undergrad. Math. (Dept. of Math., Guilford Coll., Greensboro, NC 27410), 8 pp, $2 (P). Intended to introduce "mathematically mature" undergraduates to the notions of free semigroup, group and product, and the corresponding topological counterparts. Written almost telegraphically, with only sketches of proofs. Arrives quickly at interesting applications and even research problems suitable for undergraduates. Avoids category theory, but remains formalistic with little motivation. Lists many examples. GHM

ALGEBRA, P. *Lecture Notes in Mathematics-635: Serre's Conjecture.* T.Y. Lam. Springer-Verlag, 1978, xv + 227 pp, $12.40 (P). [ISBN: 0-387-08657-9; 3-540-08657-9] A relatively self-contained exposition of the solutions of Quillen, Suslin, and Vaserstein to Serre's problem [Question: Is every projective module over a polynomial ring free? Answer: Yes.] Includes a discussion of "classical" results on the problem, Horrocks' Theorem, and the quadratic analogue of Serre's problem. A very well-written book.SG

FINITE MATHEMATICS, T(13-14: 1). *Introduction to Mathematics, Second Edition.* Larry Goldstein, David Schneider. Kendall/Hunt, 1977, iv + 410 pp, $7.25 (P). [ISBN: 0-8403-1566-X] An entrance-level course for students in biological, social and management sciences intended to precede a "short calculus" course. The treatment is designed carefully for beginning students. The topics are optimization (linear programming and quadratic functions); systems of equations (including stochastic matrices and an introduction to non-linear problems); mathematical modeling (finite difference equations, logarithm and exponential functions), and trigonometry. JAS

FINITE MATHEMATICS, T(13). *Finite Mathematics.* Daniel P. Maki, Maynard Thompson. McGraw, 1978, x + 452 pp, $13.95. [ISBN: 0-07-039745-7] Three main parts: probability models (counting, stochastic processes, statistics), linear models (matrices, input-output economics, linear programming), and applications (Markov chains, two-person zero-sum games, digraphs and networks, investment options). Clear presentation of material with many examples. Independence of sections allows for flexible use. TRS

FINITE MATHEMATICS, T(13: 1), *Mathematics, A Practical Approach.* Kenneth Kalmanson, Patricia C. Kenschaft. Worth, 1978, xx + 737 pp, $15.95. [ISBN: 0-87901-085-1]; *Instructor's Manual,* 113 pp, (P). An index of applications stresses its practical approach. Includes most of the material from the second author's *Linear Mathematics* (TR, June 1978). Includes several chapters of calculus, extends the topics in probability and adds some statistics. LLK

CALCULUS, T(13-14: 3), *Calculus with Analytic Geometry.* Robert Ellis, Denny Gulick. HarBrace J, 1978, xiv + 1073 pp, $20.95. [ISBN: 0-15-505728-6] An appealing three-semester text for calculus through Green's, Stokes' and Divergence Theorems. Differential equations included in appendix. Key terms, formulas, and review exercises at end of each chapter. Good format, but wins the prize for heaviest book of the year. LLK

CALCULUS, S(13), *Calculus of Rational Functions.* G.E. Shilov. Transl: V.I. Kisin. MIR (Imported by: Imported Pub, 320 W. Ohio St., Chicago, IL 60610), 1976, 51 pp, $1.25 (P). In the foreword the author states, "If we restrict ourselves to a comparatively narrow class of rational functions and utilize the illustrative language of graphs, we can present the concepts of derivative and integral in a few pages... ." He has managed this in 50 pages. LLK

CALCULUS, T(13-14: 3), *Calculus with Analytic Geometry.* Harley Flanders, Justin J. Price. Acad Pr, 1978, xiv + 1041 pp, $20. [ISBN: 0-12-259672-2] A complete rewriting of *Calculus,* H. Flanders, R. Korfhage, and J.J. Price (1970). It is still a text that stresses practice rather than theory. Sections on multivariable calculus are especially good. No chapter on differential equations included.LLK

CALCULUS, T(13: 1-3), *College Mathematics for Management, Life, and Social Sciences.* Raymond A. Barnett. Dellen Pub, 1978, xii + 641 pp, $15.95. [ISBN: 0-89517-000-0]. This very substantial book begins with a review of basic algebra and progresses through linear systems, probability and statistics, differential and integral calculus. It is clearly written and nicely set in print, with abundant examples and numerous interesting, noncontrived exercises drawn from a variety of areas. The breadth of topics is nearly staggering, but by careful use of heuristic arguments and instructive examples the burdensome aspects of technical mathematics are minimized. A book worth considering. MU

CALCULUS, T(13), *Essentials of Calculus for Business and Economics.* Louis Leithold. Har-Row, 1978, xiv + 479 pp, $13.95. [ISBN: 0-06-043947-5] Calculus primarily from the social science point of view. Covers basic one-variable calculus plus differential calculus of functions of 2 to 3 variables (including Lagrange multipliers). Some proofs are omitted, but ideas are well-motivated, definitions and theorems are clearly stated. Many exercises. SG

COMPLEX ANALYSIS, P, *Complex Analysis: Proceedings of the SUNY Brockport Conference.* Ed: Sanford S. Miller. Lect. Notes in Pure and Appl. Math., V. 36. Dekker, 1978, xii + 177 pp, $18.75 (P). [ISBN: 0-8247-6725-X] Papers from a June 1976 conference featuring Malcolm Robertson as principal lecturer. Includes a list of 30 open problems compiled by the conference participants. LAS

COMPLEX ANALYSIS, P, *Cauchyjev Račun Ostataka sa Primenama* [*Cauchy's Calculus of Residues with Applications*]. Dragoslav S. Mitrinović, Jovan D. Kečkić. Matematički Problemi i Ekspozicije. Naučna Knjiga, Beograd, 1978, 271 pp (Serbo-Crotian; English summary). All about residues: how to find them, how to use them to prove theorems, evaluate integrals and sums, solve differential, integral, and difference equations, and discuss special functions; together with a biographical sketch of Cauchy. References as recent as 1977. RPB

COMPLEX ANALYSIS, T(15-16), L, *Komplexe Analysis für Ingenieure, Band I.* Peter Henrici, Rita Jeltsch. Birkhäuser, 1977, 160 pp, sFr. 17 (P). [ISBN: 3-7643-0861-3] A presentation of the basics for students with one to two years of calculus. The emphasis is on fundamental concepts with rigorous proofs presented only where they yield genuinely deeper insight. Exercises emphasize ideas more than computation. In Volume I a section on potential problems, and in the proposed Volume II (contents included in Volume I), a section on the Laplace transform emphasizes the applicability of the material presented. JAS

DIFFERENTIAL EQUATIONS, T(14-15), *Differential Equations for Engineers.* Thomas M. Creese, Robert M. Haralick. McGraw, 1978, xvi + 552 pp, $17.50. [ISBN: 0-07-013510-X] The four chapters cover first order equations, models of engineering systems, transform methods for linear equations with constant coefficients, and linear operator (with constant and non-constant coefficients). An appendix deals with power series methods. Easily the most attractive feature is the wide variety of applications to science and engineering. An interesting text. SG

DIFFERENTIAL EQUATIONS, P, *Elliptic Partial Differential Equations of Second Order.* David Gilbarg, Neil S. Trudinger. Grund. math. Wissenschaften, B. 224. Springer-Verlag, 1977, x + 401 pp, $35.90. [ISBN: 0-387-08007-4; 3-540-08007-4] An essentially self-contained exposition of portions of the theory of second order elliptic partial differential equations, with emphasis on the Dirichlet problem in bounded domains. Includes preparatory chapters on potential theory and functional analysis. Exercises TRS

DIFFERENTIAL EQUATIONS, P, *A Handbook of Methods of Approximate Fourier Transformation and Inversion of the Laplace Transformations.* V.I. Krylov, N.S. Skoblya. Transl: George Yankovsky. MIR (US Rep: Four Continent Book Corp, 156 Fifth Ave., NY 10010), 1977, 271 pp. A compendium of the present state of the problem of approximate computation of inverse Laplace transforms where existing tables of functions and their transforms fail. Text is designed for scientists and engineers who have basic knowledge of analysis and complex variables. TRS

DIFFERENTIAL EQUATIONS, P, *Estimates for the $\bar\partial$-Neumann Problem.* P.C. Greiner, E.M. Stein. Princeton U Pr, 1977, 194 pp, $6 (P). An exposition of recent work by the authors in which they have constructed parametrices and obtained sharp estimates for solutions of the $\bar\partial$-Neumann problem. JAS

COMPUTER PROGRAMMING, T. *Illustrating Basic (A Simple Programming Language).* Donald Alcock. Cambridge U Pr, 1977, ix + 134 pp, $10.95; $3.95 (P). [ISBN: 0-521-21703-2; 0-521-21704-0] Introductory text with informal, light-hearted style. The unusual graphic layout might be both enticing and confusing to a beginner. BNF variant used for combined syntax definition and index. TH

COMPUTER PROGRAMMING, T. *BASIC, A Computer Programming Language with Business and Management Applications.* C. Carl Pegels, Robert C. Verkler. Holden-Day, 1976, xi + 226 pp, $7.95 (P). [ISBN: 0-8162-6663-8] Introductory text, second edition. Chapters on business and economics problems, production-management problems, random numbers and simulation, corporate financial models, and statistics problems. Compares features of eight different Basics in appendix. Instructor's guide. TH

COMPUTER PROGRAMMING, T. *Programming in BASIC for Business.* Bruce Bosworth, Harry L. Nagel. SRA, 1977, viii + 229 pp, $7.95 (P). [ISBN: 0-574-21090-3] Introductory text. Chapter on business applications. Summary of statements inside front cover. TH

COMPUTER PROGRAMMING, S, P. *COBOL with Style, Programming Proverbs.* Louis J. Chmura, Henry F. Ledgard. Hayden, 1976, 148 pp, $5.45 (P). [ISBN: 0-8104-5781-4] Assumes familarity with Cobol. "...intended for Cobol programmers who want to write carefully constructed, readable programs." Chapters on programming proverbs, top-down programming, and program standards. Exercises. TH

COMPUTER PROGRAMMING, T. *Basic Programming for Business.* Irvine H. Forkner. P-H, 1977, xii + 237 pp, $10.50 (P). [ISBN: 0-13-066423-5] Introductory text for business students. TH

COMPUTER PROGRAMMING, T(13: 1), S*. *FORTRAN.* Samuel L. Marateck. Acad Pr, 1977, xvi + 671 pp, $9.95 (P). [ISBN: 0-12-470460-3] As in his book on *BASIC* (TR, December 1975), the author employs a double page format for his presentation of Fortran: the left-hand pages are thorough discussions of the programming techniques while those on the right capsulize and support the text with diagrams, programs and tables. No prior computer experience required. TRS

COMPUTER SCIENCE, S(15-17), P. *Betriebssysteme.* Lutz Richter. Teubner, Stuttgart, 1977, 152 pp, (P). [ISBN: 3-519-02335-0] Based on lectures given in the period 1972 to 1976, this volume gives an overview of operating systems with the purpose of preparing the reader to use the current literature. Discusses problems of parallel processors. JAS

COMPUTER SCIENCE, P. *Parallel Computers-Parallel Mathematics.* Ed: M. Feilmeier. North-Holland, 1977, xii + 354 pp, $36.75. [ISBN: 0-444-85042-2] Proceedings of the symposium held at the Technical University of Munich by the International Association for Mathematics and Computers in Simulation and Gesellschaft für Informatik. Contains 8 invited papers and 51 contributed papers. List of participants. RJA

COMPUTER SCIENCE, P. *On-Line Data Bases.* Infotech Inter, 1977. *Part 1: Analysis and Bibliography,* v + 324 pp; *Part 2: Invited Papers,* ii + 374 pp, (P). [ISBN: 8553-9370-X] Part 1 discusses philosophy, implementation factors, data base system designs, performance and reliability questions, and distributed data bases. References. Contributor and subject indices. Part 2 contains 21 invited papers. Subject index. RJA

COMPUTER SCIENCE, T(15-18: 1), S, L. *Programs and Machines: An Introduction to the Theory of Computation.* Richard Bird. Wiley, 1976, x + 214 pp, $16.95. [ISBN: 0-471-01650-0] Develops the subject from programming concepts, not mathematical theory. Begins with three basic types of program (flowcharts, "while" programs, and procedure definitions) and the distinction of program from machine. Proceeds to the equivalence of programs, the limitations of programming, the correctness of programs, and the theory and use of recursion. Chapter exercises. Chapter bibliographic remarks. References and index. RJA

COMPUTER SCIENCE, S(15). *Microcomputer Design and Applications.* Ed: Samuel C. Lee. Acad Pr, 1977, x + 346 pp, $14.50. [ISBN: 0-12-442350-7] Five papers on the design of microcomputers including arithmetic, multiprocessors, software, and files. Eleven papers on a variety of applications mostly in engineering. RWN

COMPUTER SCIENCE, S(13-14), L. *Modern Digital Communications.* E.J. Ross. TAB Books, 1977, 308 pp, $10.95; $6.95 (P). [ISBN: 0-8306-7955-3; 0-8306-6955-8] An introduction to digital communications including modems, equalization, multiplexing, A to D and D to A conversions, and binary codes. Glossary. For popular consumption. RWN

COMPUTER SCIENCE, S(13), L. *Beginner's Guide to Microprocessors.* Charles M. Gilmore. TAB Books, 1977, 181 pp, $8.95; $5.95 (P). [ISBN: 0-8306-7995-2; 0-8306-6995-7] Fundamentals of microcomputers for the hobbyist. Discusses binary arithmetic, logic, CPU's, memories, I/O, basic instruction sets, programming and the 6800 and 8080 in particular. At times unnecessarily imprecise. RWN

COMPUTER SCIENCE, T(13-14: 1), S. *Further Computer Appreciation.* T.F. Fry. Butterworths, 1977, 202 pp, $9.95 (P). [ISBN: 0-408-00239-5] A sequel to *Computer Appreciation* (TR, February 1974). Primarily concerned with an understanding of computer organization: I/O, CPU, memories and systems. Also considers files and some software. RWN

COMPUTER SCIENCE, T(15-18: 1, 2), S, L. *The Theory of Computer Science, A Programming Approach.* J.M. Brady. Chapman and Hall, 1977, xiii + 287 pp, $17.95. [ISBN: 0-470-99103-8] Part one--a metatheory of computer science: abstract machines, finite state machines, Turing machines, computability, recursive functions, Turing's thesis. Part two--theory of computer science: McCarthy's formalism, meaning of programming languages, software reliability. Material is consistently related to intuitions gained directly from actual computing practice. Appendices on the mathematical and programming prerequisites. References. Author and subject indexes. RJA

APPLICATIONS, T(15-17: 2), S, P*, L**. *Catastrophe Theory and its Applications.* Tim Poston, Ian Stewart. Fearon-Pitman, 1978, xviii + 491 pp, $49.75. [ISBN: 0-273-01029-8] The first complete exposition of catastrophe theory, integrating basic theory with extensive applications. The mathematics is expressed as a natural variation on multidimensional calculus rather than as an exotic fruit of differential topology, thereby rendering the volume accessible to most serious scientists. The applications are primarily in physical science (stability, fluid mechanics, optics, elasticity, phase transitions, laser physics), including many very recent ones. Concluding chapters examine (briefly) biological and behavioral applications, and speculate about the future. Contains an up-to-date bibliography of 500 items, plus over 200 references. LAS

APPLICATIONS, P. *Proceedings of the 1977 Army Numerical and Computers Analysis Conference.* US Army Research Office, P.O. Box 12211, Research Triangle Park, NC 27709, 1977, xiii + 601 pp, (P). Papers from the March 1977 conference held at the Mathematics Research Center in Madison. LAS

APPLICATIONS, T*(16-17: 1, 2), S*, P, L. *An Introduction to Mathematical Modeling.* Edward A. Bender. Wiley, 1978, x + 256 pp, $16.95. [ISBN: 0-471-02951-3] Each section deals with the application of a particular mathematical technique to a range of problems. First part requires only calculus and basic probability; second part uses ordinary differential equations. Rich source of interesting real-world, non-trivial, open-ended problems--especially suitable for teams of students since most require discussion and may have no single best answer: mechanical approaches are inappropriate. "Learn by doing" approach includes formulating, analyzing, and criticizing models. LCL

APPLICATIONS (ARTIFICIAL INTELLIGENCE), T(16-18: 1, 2), S, P, L. *A Structure for Plans and Behavior.* Earl D. Sacerdoti. Elsevier Sci Pub, 1977, xv + 126 pp, $8.95 (P); $15.95. [ISBN: 0-444-00209-X] A plan is a sequence of actions to be performed by a robot. This work contains techniques for generating a plan hierarchy; introduces the idea of representing a plan as a potentially ordered sequence that makes no more than the necessary commitments to the time-ordering of steps; develops mechanisms enabling planning systems to examine their own plans. Presents the automatic planning system NOAH. Three appendices. References. RJA

APPLICATIONS (BIOLOGY), P. *Computer Analysis of Neuronal Structures.* Ed: Robert D. Lindsay. Plenum Pr, 1977, xvi + 210 pp, $22.50. [ISBN: 0-306-30964-5] Eleven independent chapters written by different authors who have pioneered use of computer techniques in neuroanatomy. LAS

APPLICATIONS (BUSINESS), T(13-14: 2). *Mathematics: Fundamentals for Managerial Decision-Making, Third Edition.* Michael L. Kovacic, Gordon L. Shilling. Prindle, 1978, x + 628 pp, $15.95. [ISBN: 0-87150-246-1] Changes include: elimination of chapter on sets, expanded material on calculus, format uses boxes and color, virtually all the exercises are new; *Solutions Manual* is available. (*First Edition,* TR, February 1972; *Second Edition,* TR, December 1976.) LCL

APPLICATIONS (CONTROL THEORY), T(16-17: 1, 2), S, L. *An Introduction to Linear Control Systems.* Thomas E. Fortmann, Konrad L. Hitz. Dekker, 1977, xiv + 744 pp, $29.75. [ISBN: 0-8247-6512-5] Reviews the background needed from linear algebra and differential equations including Laplace transforms. Integrates modern and classical approaches. Stability analysis, frequency response, feedback, compensation, and multivariable systems. Problems. Bibliography. RWN

APPLICATIONS (ECONOMICS), P. *Price and Non-Price Competition, Dynamics of Marketing.* M.M. Metwally. Asia Pub, 1976, x + 144 pp, $11.95. [ISBN: 0-210-40568-6] A study of marketing where advertising, market share, and other facets of competition are built into the models. JAS

APPLICATIONS (ECONOMICS), P. *Arrow Impossibility Theorems.* Jerry S. Kelly. Acad Pr, 1978, xi + 194 pp, $17.50. [ISBN: 0-12-403350-4] Kenneth Arrow received the Nobel Prize in economics largely for his 1950 paper proving that certain desirable properties of social choice (i.e., voting systems) are logically inconsistent. Kelly here provides a systematic guide to the diversity of such impossibility theorems, especially those that have appeared in recent years. This field guide to axiomatic ethics concludes with unsolved problems and a comprehensive bibliography. LAS

APPLICATIONS (FLUID MECHANICS), P. *Annual Review of Fluid Mechanics, V. 10.* Ed: Milton van Dyke, J.V. Wehausen, John L. Lumley. Annual Reviews, 1978, 475 pp. [ISBN: 0-8234-0710-0] Tenth annual volume containing 20 invited survey articles, each with bibliography. This volume also includes a cumulative index to volumes 6-10. LAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Mathematics-570: Differential Geometrical Methods in Mathematical Physics.* Ed: K. Bleuler, A. Reetz. Springer-Verlag, 1977, viii + 576 pp, $19.40 (P). [ISBN: 0-387-08068-6; 3-540-08068-6] Proceedings of the Symposium held at the University of Bonn, July 1-4, 1975. This volume presents a relatively expository interchange between physicists and mathematicians concerning such recent interfaces as Riemannian geometry and relativity, connections and gauge theories, and graded Lie algebras and supersymmetry. JAS

APPLICATIONS (PHYSICS), P. *Inverse Problems in Quantum Scattering Theory.* K. Chadan, P.C. Sabatier. Springer-Verlag, 1977, xxii + 344 pp, $29.80. [ISBN: 0-387-08092-9; 3-540-08092-9]

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Ralph P. Boas, Northwestern University; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Timothy Hoel, St. Olaf; Paul Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D. C. 20036.

PERSONAL ITEMS

Duquesne University: Associate Professor Robert G. McDermot has been promoted to Professor. Professor Charles F. Sebesta will be Chairman of the Mathematics Department for one year, effective September 1978.

Seattle Pacific University: Professor Samuel L. Dunn has been appointed Director of the School of Natural and Mathematical Sciences. Assistant Professor Edward C. Beardslee has been promoted to Associate Professor. Associate Professor Dick A. Wood has been promoted to Professor.

College of the Holy Cross: Mr. James Stormes of Brown University has been appointed Visiting Lecturer and Dr. Thomas Cecil of Vassar College has been appointed Assistant Professor. Assistant Professor James Noonan has been promoted to Associate Professor. Associate Professor Melvin Tews has been appointed Chairman of the Mathematics Department.

North Carolina State University: Associate Professors Joe A. Marlin and Carl D. Meyer have been promoted to Professors. Assistant Professor Robert G. Savage has been promoted to Associate Professor. Professor and Associate Head Hubert V. Park has retired. Associate Professor James E. Huneycutt has resigned.

Broome Community College, Binghamton, New York: Associate Professors Thaddeus Czupryna and Mary Diegert have been promoted to Professors.

Ohio Northern University: Professor Jeffrey T. McLean has resigned as Chairman of Mathematical Sciences to become Chairman of Mathematical Sciences at the College of Saint Teresa. Assistant Professor Robert A. Hovis has been promoted to Associate Professor and has been named Acting Chairman.

Williams College: Associate Professor Victor E. Hill has been promoted to Professor. Professor Robert Kozelka will be on sabbatical leave 1978-79 at the University of North Carolina at Chapel Hill.

Associate Professor Mary Katherine Bell, Lamar University, is the recipient of a Minnie Stevens Piper Foundation Award, one of ten outstanding Professors selected annually from the State of Texas.

Dr. Jack Alanen has been appointed Director of the Andrew R. Jennings Computing Center, Case Western Reserve University.

Assistant Professor August W. Waltmann, Wartburg College, has been promoted to Assoc. Professor.

Professor David E. Flaspohler, Xavier University, has been named Dean of the Graduate School. During the past year he wrote an 18-part series on consumer mathematics for WCET-TV, educational station in Cincinnati.

Professor Richard F. McCoart, Loyola College, Maryland, is spending the academic year 1978-79 on sabbatical leave at the University of Hamburg, Germany.

Lecturer Richard Levaro, San Francisco State University, has been appointed Assistant Professor.

Assistant Professor John R. Hubbard, Lycoming College, Williamsport, Pennsylvania, has been promoted to Associate Professor with tenure.

Professor Floyd Bowling, Tennessee Wesleyan College, Athens, Tennessee, has retired with the title of Professor Emeritus after forty-three years of teaching.

Professor Laurence Sigler, Bucknell University, has been selected to receive one of three Lindback Awards for Distinguished Teaching in 1977-78.

Dr. James R. Overman, Dean Emeritus of Bowling Green State University, died on May 22, 1978, at the age of 90. He was a member of the Association since its charter in 1915.

Professor Sidney G. Roth, formerly Vice Chancellor for Federal Relations at New York University and temporarily with the Department of Energy in Washington, died on May 16, 1978, at the age of 65. He was a member of the Association for forty-five years.

Lecturer Michael G. Greening, University of New South Wales, Australia, died on April 16, 1978, at the age of 60. He was a member of the Association for eleven years.

Professor Joaquin B. Diaz, Rensselaer Polytechnic Institute, died on June 16, 1978.

ANNOUNCEMENT OF A NEW JOURNAL

The *INTERNATIONAL JOURNAL OF MATHEMATICS AND MATHEMATICAL SCIENCES* has begun publication as a quarterly from March, 1978, with Lokenath Debnath as Managing Editor. Published in the *JOURNAL* are original research papers, short research notes, book reviews, and broad expository and survey articles with emphasis on unsolved problems, controversial issues and open questions in all branches of Mathematics and Mathematical Sciences. Twenty-five highly recognized mathematicians are on the Editorial Board. Further information may be obtained from Dr. L. Debnath, Managing Editor, Mathematics Department, East Carolina University, Greenville, North Carolina 27834.

GREGORY LEE RECEIVES FIRST AMS-MAA-SIAM CONGRESSIONAL FELLOWSHIP

The first Congressional Fellowship to be sponsored by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics has been awarded to Dr. Edmund Gregory Lee, Assistant Professor of Mathematics at Fordham University, Bronx,NY.

Dr. Lee, who is 29 years old, is a graduate of Reed College and received his doctorate in mathematics from the Massachusetts Institute of Technology in 1975. He will serve, along with five Fellows selected by the American Associaton for the Advancement of Science and some sixteen other Fellows sponsored by professional societies in science and engineering, under an annual program, now in its fifth year, coordinated by the AAAS. Each Fellow will spend his fellowship year, which begins in September, working on the staff of an individual congressman or a congressional committee or in the congressional Office of Technology Assessment.

FIFTH INTERAMERICAN CONFERENCE ON MATHEMATICAL INSTRUCTION

The 5th INTERAMERICAN CONFERENCE ON MATHEMATICAL INSTRUCTION will take place from the 13th to the 16th of February, 1979 in the city of Campinas, São Paulo State, Brazil. The conference will consist of lectures, round table and panel discussions on mathematical instruction problems on all levels in all countries of the continent. Papers will be presented mainly in Portuguese and Spanish, though English and French will also be official languages of the meeting. Details about programming, registration and lodging will be included in the second announcement, which will be sent out in the middle of August, 1978 to those persons who request it at the address: 5° CIAEM, Cidade Universitária-UNICAMP, Caixa Postal, 6063, 13.100 Campinas - SP, BRASIL.

SECOND VOLUME OF HANDBOOK OF STATISTICS

The second volume in the series "Handbook of Statistics" will be devoted to the theory and applications of the techniques of "Classification, Pattern Recognition and Reduction of Dimensionality". The co-editors of this series are P. R. Krishnaiah, University of Pittsburgh, and L. Kanal, University of Maryland, and a partial list of the members of the editorial board include T. Cover, K. S. Fu, C. R. Rao and J. Van Ryzin. Any suggestions about the material that should be covered in this volume and possible contributors are welcome and should be sent very soon to one of the editors. The first volume is devoted to Univariate and Multivariate Analysis of Variance and it is expected to go to the press towards the end of 1978.

CUPM ANNOUNCEMENT

The MAA's Committee on the Undergraduate Program in Mathematics (CUPM) has appointed a panel to study courses of the type called "Mathematics Appreciation", "Mathematics for Liberal Arts Students", etc. The goal of the panel is to produce a pamphlet which can be used as a source of information and guidance by those responsible for such courses. The panel wishes to have input from the mathematical community. The panel would like to hear about successful versions of these courses, suggestions concerning subject matter, teaching methods, helpful references, etc. Please communicate your experiences, ideas, and suggestions to Professor Jerome A. Goldstein, Department of Mathematics, Tulane University, New Orleans, Louisiana 70118.

SEMINAR ON NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

The Mathematics Research Center at the University of Wisconsin-Madison will hold a two and one-half day Advanced Seminar on Numerical Methods for Partial Differential Equations, October 23-25, 1978. The program will include four mini-series (at least two connected lectures) by C. W. Hirt, C. K. Chu, H. O. Kreiss and K. J. Bathe. In addition there will be invited lectures by J. H. Bramble, J. Oliger, S. A. Orszag, and R. Temam. Further information may be obtained from S. V. Parter, Mathematics Research Center, University of Wisconsin-Madison, 610 Walnut Street, Madison, Wisconsin 53706.

EMPLOYMENT INFORMATION IN THE MATHEMATICAL SCIENCES

Beginning with the November 1978 issue, several important changes will take place in EIM (Employment Information for Mathematicians).

FIRST. In order to reflect more accurately the fact that this publication is (as it has been since its inception six years ago) devoted to employment for all those within the mathematical sciences, the name of the publication will be changed to "Employment Information in the Mathematical Sciences".

SECOND. Second class postal service will be discontinued. All subscribers in the U.S. will receive their copies by first class mail. Subscribers in Canada may also receive their copies by first class mail, but an option for air mail delivery will be available. Subscribers in other countries will receive their copies by air mail.

In addition, there will be modest increases in subscription rates to cover the improved postal service. A reclassification of the institutional subscribers has also been made to simplify the calculation of prices for institutional subscriptions; subscription rates for academic institutions vary with the type of program offered in the mathematical sciences.

THIRD. Because of continuing substantial deficits borne by the sponsoring organizations, a charge will be made for listings. The charge will be $25 per listing, with the option to repeat the listing in one or both of the following two issues.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE SOUTHEASTERN SECTION

The fifty-seventh annual meeting of the Southeastern Section was held on March 31-April 1, 1978, at Clemson University, South Carolina. A total of 338 persons attended the meeting, including 86 students and 204 members of the Association. The local arrangements were handled by Professor John Kenelly.

Three invited addresses were given: J. Harvey Carruth (Section Lecturer) of the University of Tennessee at Knoxville on *Topological Algebra: A Mathematical Push-Me Pull-You,* Harley Flanders of Tel Aviv University and Georgia Tech on *Some Mathematical and Pedagogical Aspects of Three Dimensional Drawings,* and Dorothy L. Bernstein (President-Elect of the Association) of Goucher College on *The Role of Applications in Pure Mathematics.*

There were eight sessions for contributed papers. The presiders were Ivey C. Gentry (Chairman of the Section), Joel V. Brawley and John Kenelly for the general sessions and K. P. Layton, W. Ruckle, J. Luedemann, J. Caveny, J. Reneke, J. Reay, W. Hare and R. Vandervelde for the special sessions.

Officers elected for 1978-79 were: Chairman, John Kenelly, Clemson University; Chairman-elect, John D. Neff, Georgia Tech; Vice-Chairman, Jack C. Wilson, University of North Carolina-Asheville; Section Lecturer, Trevor Evans, Emory University, all for one-year term. Ivey C. Gentry, Wake Forest University, was elected Secretary-Treasurer for a three-year term.

At the business meeting, it was announced that there were two winners of the Section prize for the best performance on the Putnam Examination. The Section voted to give the $25 prize to each, Ronald Ausbrooks of Vanderbilt University and Hon Tam, Georgia Tech. The Section voted to hold its 1980 meeting at Appalachian State University in Boone, North Carolina.

The following papers were presented:

Curves in the complex plane, Ann Kinsinger, Maryville College.

Characterizations of the Cantor set and the irrationals, K. Boone and P. Manz, Guilford College.

Metrics on continuous function spaces, Morris M. Galloway, Jr., Presbyterian College.

Computer simulation of a supermarket, E. W. Womble, Presbyterian College.

An application of the Laplace transform to pharmacokinetics, George E. Haborak, College of Charleston.

Using SAS for simulation - some industrial applications, R. L. Anderson, Milliken and Company.

An informal mathematics clinic, Michael Willet and David Herr, University of North Carolina at Greensboro.

Convergence of series and the strong law of large numbers, Benjamin Klein, Davidson College.

A two-test method of teaching precalculus, a preliminary report, John Nichols, Maryville College.

Advantages of plasticine models when teaching calculus in three dimensions, Andrew N. Harrington, Georgia Institute of Technology.

An elementary theorem on maximal and minimal areas, J. B. Stroud, Davidson College.

Systems of equivalent statements are basic mathematics, William G. Roughead, North Georgia College.

Good method for teaching quadratic trinomial factoring, John A. Bond, Jr., Macon Junior College.

Metric education: a few problems and partial solutions, and a comparison with India, Satish Chandra, Tuskegee Institute.

Köthe Space, Robert M. Scott, Clemson University.

The Hankel spectrum of a function, David H. Lubbers, Clemson University.

Best approximation in a Banach space with a Schauder decomposition, Hideaki Kaneko, Clemson U.

Generalized strong summability of Walsh-Fourier series, Lyndell Kerley, East Tennessee State U.

On the centennial of the "American Journal of Mathematics", Joe Albree, Auburn University of Montgomery.

On linear programming duality and Landau's characterization of tournament scores, Allan B. Cruse, University of San Francisco and Emory University.

Johnson's circle theorem - a new proof and two extensions, C. Ray Wylie, Furman University.

Some results on the structure of "free" distributively generated near-rings, David John, Emory U.

What are geometric games?, William H. Ruckle, Clemson University.

Further results on a generalization by Chung and Liu, of Ramsey theory, Michael Jacobson, Emory U.

Appropriate CAI programs for mathematics classes, James C. Kropa, Clayton Junior College.

A design for computer managed self-paced instruction in elementary algebra, Richard C. Detmer and Clinton W. Smullen III, University of Tennessee at Chattanooga.

Calculators in the mathematics classroom: some observations, G. Marvin Eargle, Appalachian State University.

The implementation of a calculator oriented freshman mathematics course, John Nichols, Presbyterian College.

A set of sequences used in the numerical search for pi, e, and gamma, Charles H. Frick, White Rock, South Carolina.

Higher order derivatives in multidimensional calculus, John V. Baxley, Wake Forest University and Sandria Kerr, Winston-Salem State University.

Applications of higher order trigonometric and hyperbolic functions to the solutions of a vibrating beam problem, Frederick C. Stevens, Tennessee Technological University.

Proof of correctness of a simple problem, Robert Beaudoin, Clemson University.

John D. Neff, *Secretary-Treasurer*

APRIL MEETING OF THE OKLAHOMA-ARKANSAS SECTION

This meeting was held at Henderson State University on March 31-April 1, 1978, in Arkadelphia, Arkansas. The attendance was approximately 100 with 22 papers given by faculty and students in the section. A unique feature of this meeting was an undergraduate session contributing ten papers. Leading the way in this session were 8 papers given by students from Hendrix College in Conway, Arkansas. The highlights of the meeting were the Court Lecture given by Dr. R. B. Deal, Professor of Biostatistics and Epidemiology, University of Oklahoma, Health Sciences Center and the Invited Address given by Dr. David Roselle, Professor of Mathematics, Virginia Polytechnic Institute and State University and MAA Secretary.

The following talks completed the program:

Hahn Products that are Almost-n_α Groups, R. E. Beasley, Central State University.

A Strong Maximum Principle for Vector-Valued Harmonic Functions, James L. Meek, University of Arkansas at Fayetteville.

On the Putnam Problem B-3 of 1976, Naoki Kimura, University of Arkansas at Fayetteville.

The Optimal Management of a Renewable Resource: The Fishery, Alan M. Johnson, University of Arkansas at Little Rock.

An Application of Matrix Theory in Aircraft Design, Jeanne Agnew, Oklahoma State University.

Basic Cryptology, Louis R. Beck, Hendrix College.

A General Algorithm for Finding Prime Divisors, Steven Brown and Stan Obenhaus, Oklahoma Christian College.

Generalized Libschitz Criteria for First Order Differential Equations, Mark L. Burton, Hendrix College.

Completeness Criteria on Ordered Fields, John W. Merrill, Hendrix College.

An Investigation of Optimal Control, Janet Dillahunty, Henrix College.

Rolling Cones, Alma E. Posey, Hendrix College.

On Additive Functions Being Linear, Joe VanDenHeuvel, Hendrix College.

Non-Linear Additive Functions, Julie Anderson, Hendrix College.

Applications of the Liouville Theorem to Pitch Angle Distributions of Cosmic Rays, Jeffrey Jackson, Oral Roberts University.

An Application of Some of Heaviside's Theorems to Solutions of Linear Differential Equations, Cassandra Schrimshire, Hendrix College.

Some Thoughts and Applications in Teaching Applied Mathematics, Marvin Keener, Oklahoma State U.

Are High School Students Prepared Mathematically to Enter College? Earl C. Rice, Central State U.

Formal Reasoning in the Mathematics Classroom, Verbal Snook, Oral Roberts University.

Stein Estimation in Football Pool, David H. Robinson, University of Iowa.

Blending Functions with Applications, Dale Doty, University of Tulsa.

Characterization of Minimal Regular Spaces, Larry L. Herrington, University of Arkansas at Pine Bluff.

Some Remarks on an Hyperbolic Plane, Tetsundo Sekiguchi, University of Arkansas at Fayetteville.

The following officers will function during the 1978-79 year:
Chairman: William Durand, Henderson State University
Past Chairman: Verbal Snook, Oral Roberts University
First Vice-Chairman: Jim Choike, Oklahoma State University
Second Vice-Chairman: Andrew D. Coe, Westark Community College
Secretary-Treasurer: John Jobe, Oklahoma State University
Arkansas Chairman, High School Contest: Claude Duplissey, University of Arkansas
Oklahoma Chairman, High School Contest: Tom Cairns, University of Tulsa
Governor: Paul Long, University of Arkansas

E. K. McLachlan, *Secretary-Treasurer*

MARCH MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the MAA was held at *New Mexico Institute of Mining and Technology*, Socorro, New Mexico on March 17-18, 1978. Twenty members and two guest registered their attendance.

The invited speaker was Professor Victor L. Klee, University of Washington, former president of the MAA and recipient of numerous awards from the mathematical community. Professor Klee's invited address was entitled *How Many Angels Can Dance on the Point of a Needle?*

The following papers were contributed:

A Conjecture on a Generalization of a Theorem Concerning Response Times in a Processor-Sharing Operating System, and its Investigation by Simulation, Robert Cooper, New Mexico Institute of Mining and Technology.

Approximations of Certain Quadratic Surds, Jeffrey L. Winter, Arizona State University.

The Mathematics Course for Engineers and Scientists at Sandia Laboratories, Darrell L. Hicks, Sandia Laboratories.

Matrix Derivatives-Some History and Formulae, Gerald S. Rogers, New Mexico State University.

Optimal Control of a Gas-Water Reservoir, Michael S. Waterman, Los Alamos Scientific Laboratory.

Conformal Mappings and Mirrors, David Thomas, New Mexico Institute of Mining and Technology.

Exponentials Reiterated, R. Arthur Knoebel, New Mexico State University

Projective Physics, A. Swimmer, Arizona State University

A Numerical Hydrodynamics Model, Ralph McGehee

Taylor Polynomials and Difference Quotients, Richard Bagby, New Mexico State University.

Clyde M. Dubbs, Chairman of the Southwestern Section acted as meeting coordinator and chairman of the session on contributed papers. Newly elected officers are: Carl E. Hall, University of Texas, El Paso, Chairman; Dennis Bonnett, Northern Arizona University, Vice Chairman.

A. Swimmer, *Secretary*

APRIL MEETING OF THE KANSAS SECTION

The sixty-third annual meeting of the Kansas Section of the MAA was held on April 1, 1978 at Wichita State University, Wichita, Kansas. Section President, Dr. Frank Brenneman, presided. Approximately 150 people attended.

Invited addresses were *Brouwer's Contribution to the Philosophy of Mathematics* and *Beyond Brouwer* by Errett Bishop, University of California at San Diego.

The following contributed papers were presented:

Group-theoretic Methods in Reactive Shock Hydrodynamics, J. David Logan and Jose Perez, Kansas State University.

An Identification Algorithm for Linear Tim Invariant Systems of Difference Equations, Gary L. McGrath, Pittsburg State University.

Kenneth O. May, Teacher, Mathematician, . . ., and Friend, Elaine L. Tatham, Johnson County Community College.

The Monte Carlo Method, Prem N. Bajaj, Wichita State University.

Some Old Results in the Theory of Minimal Surfaces Revisited, Alan Elcrat, Wichita State Univ.

External Tournament Theory, Saul Stabl, The University of Kansas.

Some Combinatorial Results on (0,1) Matrices, Dharam Chopra, Wichita State University.

A Class of Trigonometric Integrals Evaluated by Cauchy's Theorem, Joseph L. Strecker, Wichita State University.

GeneralizedHilbert Kernels, Lucio Arteaga, Wichita State University.

The five top performers from the section in the 1977 Putnam Examination were awarded one year memberships in the MAA. They are: Arthur S. Parker and Bruce K. Holmes, University of Kansas, Linden G. Willis and Steven Gregg, Kansas State University, and David Wiebe, Bethel College.

Newly elected officers are: Chairman, John Hutchinson, Wichita State U.; Vice Chairman, Elaine Tatham, Johnson County Community College, Overland Park, Kansas; Associate Chairman Two-Year Colleges, Francis (Duane) Forbes, Barton County Community College, Great Bend, Kansas.

Ellen Veed, *Secretary-Treasurer*

KENTUCKY SECTION ANNUAL MEETING

The Sixty-First Annual Meeting of the Kentucky Section of the MAA was held at Northern Kentucky University, Highland Heights, on April 7-8, 1978. There were 90 persons in attendance including 62 members of the MAA.

The invited addresses included: *The Reliability of Long Cables* and *The Lifetime of Cables under Random Loads* by Howard M. Taylor, Cornell University, and *The Placement Testing Program Developed by MAA* by Richard Prosl, College of William and Mary.

The following contributed papers were also presented:

Decision Tables and Flowcharting, Jeff Wiener, Northern Kentucky University.

Acceleration of Root Finding Algorithms Through Chebychev Interpolation, Richard Daugherty (Student), Western Kentucky University.

Factorization in GL(2,Z) Using Continued Fractions, Maurice Shrader-Frechette, Spalding College.

A Note on Rearranging and Associating Infinite Series, James Barksdale, Western Kentucky U.

Personal Computers in Education, Roger Geeslin, University of Louisville.

Personal Computers in a Differential Equations Laboratory. Thomas Jenkins, University of Louisville.

Alignment Properties in Interval Convexity Spaces, Jack Wilson, Murray State University.

Counting Pockets - A Report on the NCTM China Study Visit, Pauline Lowman, Western Kentucky U.

During the business session, reports were given on the high school mathematics contest, the MAA Building Fund, the possibility of publishing a newsletter, and distribution of the NCTM-MAA Joint Report on recommendations for preparation of high school students for college mathematcis courses. The section voted to contribute $500 to the MAA Building Fund and to encourage members to make individual contributions.

Jackqueline Moss, Paducah Community College, has been elected section governor for 1979-1981.

Joe K. Smith, *Secretary-Treasurer*

APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The 1978 annual meeting of the Allegheny Mountain Section of the MAA was held at the University of Pittsburgh, Pittsburgh, PA on April 14-15, 1978. The host was the Department of Mathematics and Statistics at the University of Pittsburgh, with Earle Myers, Governor of the Section, as chairman in charge of arrangements.

There were three highlights of the Friday evening portion of the program. The first was a talk by James Maynard, Pennsylvania State University, Capitol Campus, entitled *A Linear Programming Model for Scheduling Prison Guards*. The second was a panel on Industrial Opportunities for Mathematicians moderated by Christine Cancro, Gulf Research and Development, Harmarville, PA. The other members of the panel were Thomas Tomczyk, Blue Cross of Western Pennsylvania; Robert Kohm, Alcoa Laboratories; and Simon Kellman, Westinghouse. The third was a puzzle session by Barbara Faires, Westminster College.

There were three invited talks Saturday morning in addition to a business meeting:

Distributional Weight Functions for Orthogonal Polynomials, Allan Krall, Pennsylvania State University, University Park.

Coding Theory and Finite Geometries, Mario Benedicty, University of Pittsburgh.

Our Friend, the Harmonic Series, Ralph Boas, Northwestern University, and Editor of the American Mathematical Monthly.

Contrubuted talks were given by faculty members Friday evening. There were seven 15-minute talks as follows:

The Semigroup of all $n \times n$ Fuzzy Matrices Over the Set X with $|X| = k$, Yong M. Lee, Trenton State College, and Jin Bai Kim, West Virginia University.

Imaginary Intersections of Lines and Conics, Wray G. Brady, Slippery Rock State College.

Better Probability Through Gambling, Lawrence D. Ramboski, California State College.

Nature's Cryptogram: The Fibonacci Numbers, Nicholas S. Ford, Pennsylvania State University - Fayette Campus.

Semi-original Research Problems for School and College Students (and Teachers), Frederick H. Bell, University of Pittsburgh,

The Impact of Metric Conversion on Teaching Technical Mathematics, Kalman Mecs, Community College of Allegheny County - Allegheny Campus.

Mathematicians Beware, the Pygmies are Rising, Lawrence Miller, Butler County Community College.

Saturday morning, 15-minute contributed talks were given by students from institutions within the section:

Some Numerical Calculations on the High Gradient Magnetic Separation Process, Robert Caflisch, West Virginia University.

Symmetric Derivatives, Beth Morrison, Penn State Behrend.

An Interactive Computer Module which Solves Markov Chains, Philip Hirsch and Sunita Kulkarni, Indiana University of Pennsylvania.

Groups, Graphs, and Sequences, Dave Houseman, Allegheny College.

Affine Sub-geometries in the Euclidean Plane, Karen Walker, Allegheny College.

Double Error-Correcting Codes, Carol Walker, Allegheny College.

An Invariant Function for a Class of Convex Polygons, Jean Thomas, Allegheny College.

The business meeting of the Allegheny Mountain Section was called to order by Chairman Richard McDermot, Allegheny College. Francis Hall, Pennsylvania State University - Fayette and I. Dee Peters, West Virginia University reported on the high school contest in Western Pannsylvania and West Virginia, respectively. The list of the top ten students from the Allegheny Mountain Section in the Putnam Competition was read. Kevin Cline, Carnegie-Mellon University and Joshua Bernoff, Pennsylvania State University, will be given student memberships in the MAA. Michael Ireland, Pennsylvania State University; John Nestor, University of Pittsburgh; and Gary Storrich, University of Pittsburgh, will be given one-year subscriptions to the Mathematics Magazine.

The Allegheny Mountain Section will enter into a full co-sponsorship of the summer short course program begun by the Ohio Section. This year's short course will be on the Allegheny College campus June 13-17, 1978. Next year's meeting at Westminster College, New Wilmington, Pennsylvania, will be in concert with a regional meeting of the American Mathematical Society. A section newsletter will be established to be edited by Professor Kathleen Taylor, Duquesne University. Professor Earle Myers, University of Pittsburgh, gave a report as section Governor. Professor Ralph Boas, editor of the American Mathematical Monthly gave a short report on behalf of the Mathematical Association of America office. Officers elected were Frank Hiergeist, West Virginia University, Second Vice-Chairman; Richard Lundgren, Allegheny College, Coordinator of Student Program; and John Milsom, Butler County Community College, Secretary-Treasurer. The governor of the Allegheny Mountain Section from July, 1978, until June, 1981, will be Melvin Woodard, Indiana University of Pennsylvania. Continuing as Chairman of the section is Richard McDermot, Allegheny College. The First Vice-Chairman will be Ms. Carol Booth, West Virginia University.

John W. Milsom, *Secretary-Treasurer*

APRIL MEETING OF THE NORTH CENTRAL SECTION

The 1978 Spring meeting of the North Central Section of the MAA was held at the College of St. Thomas, St. Paul, Minnesota, on April 21-22, 1978. There were 137 persons in attendance including 107 MAA members.

The program featured three hour-long lectures in addition to eleven short papers. The principal speaker at the Saturday session was Warren Loud, University of Minnesota, whose talk was entitled *On Generalized Inverses of Matrices*. A special guest lecture, *From Prodigy to Dotigy*, was delivered by Paul Erdos, currently at the University of Colorado. At the Friday evening session, Joseph Gallian of the University of Minnesota, Duluth, spoke on *The Marvelous Mathieu Groups*.

At the business meeting the Section's By-Laws were amended to include the Newsletter Editor as a member of the Executive Committee, recognition was given to top finishers and their advisors for this year's Putnam examination (Paul Vojta of the University of Minnesota finished in the top five), and reports concerning the new National Headquarters, the Minnesota and North Dakota high school contests, and the next (1979) summer seminar of the North Central Section were delivered.

In addition, the following elections were completed. President-elect — Hubert Walczak, The College of St. Thomas, St. Paul, Minnesota, three-year term on the Executive Committee; Secretary-Treasurer — Charles Heuer, Concordia College, Moorhead, Minnesota; Member-at-large — Julie Guelich, Normandale Community College, Bloomington, Minnesota.

The sessions were presided over by Milton Legg, President of the NCS/MAA, Moorhead STate University (Saturday morning) and Hubert Walczak, The College of St. Thomas (Friday evening and Saturday afternoon).

Short papers (all contributed by members of the Section) included:

An Eigenvalue Problem—Numerical Annihilation, Gary Anderson (Hamline University).

Connectedness and Separation, George Brauer (University of Minnesota-Minneapolis).

Countable Compactifications of Completely Regular Spaces, James Hatzenbuhler (Moorhead State University).

Matrices and Continued Fractions, Roger Kirchner (Carleton College).

A Characterization of the Derivation-Bounded Languages, Mark Luker (University of Minnesota-Duluth).

Mathematics and Computer Graphics, Pierre Malraison (Control Data Corporation).

Grooming Spheres with the Calculus, William Perrizo (North Dakota State University).

Biased Estimators, Robert Raymond (University of Minnesota-Morris).

The Group of Homeomorphisms of the Real Line, Tom Savage (St. Olaf College).

Three Computer Approaches to Waring's Problem, William Serbyn (College of St. Thomas).

Sequences and Cycles, Virginia Vaske (South Dakota State University).

Charles Heuer, *Secretary-Treasurer*

APRIL MEETING OF THE IOWA SECTION

The 65th regular meeting of the Iowa Section, MAA, was held on the campus of the University of Northern Iowa, Cedar Falls, Iowa, on April 21 and 22, 1978. Chairperson Ellen Oliver presided. There were 34 present on Friday, 25 of them were members of the section. On Saturday, 50 people were in attendance, 32 of whom were members of the section.

The program consisted of the contributed papers, an invited address by Dr. Dorothy Bernstein, Governor's report by James Cornette, and the business session. Frank Kosier, State University, Iowa City, was elected as Chairperson-Elect, and James Peake, Iowa State University, Ames, was elected as Secretary-Treasurer. W. L. Waltmann moved that the Iowa Section adopt the following resolution, seconded by James Cornette: "In view of the 11 years of service given by Basil Gillam to the Iowa section of MAA as Secretary-Treasurer, that the members extend a vote of appreciation by a standing round of applause."

The program, arranged by Donald V. Meyer, consisted of the following:

Nonstandard models for arithmetic and analysis, Alexander Abian, Ames.

Choice and chance in the history of Elliptic integrals, Billie C. Carlson, Ames.

Two box-filling problems, William H. Cutler, Waverly.

Typesetting mathematics on a computer, Michael Folk, Des Moines.

Examples of elementary theory, Irvin R. Hentzel, Ames.

Counting rectangle on a lattice: an illustration of mathematical induction, Bonnie H. Litwiller and David Duncan, Cedar Falls.

Panel discussion: *Recommendations for the preparation of high school students for college mathematics courses.* Led by John C. Friedell, Dubuque.

Invited address: *The role of applications in pure mathematics.* Dr. Dorothy Bernstein, President Elect of the Mathematical Association of America, Professor of Mathematics, Goucher College, Baltimore, Maryland.

B. E. Gillam, *Secretary-Treasurer*

APRIL MEETING OF ROCKY MOUNTAIN SECTION

The 61st Annual Meeting of the Rocky Mountain Section of the MAA was held April 28-29, 1978 at the South Dakota School of Mines and Technology, Chairman Vern Nelson, Metropolitan State College, presiding. There were 81 in attendance, including Professor Henry Alder, President of the MAA, Professor Erdös, and Professor Forest Fisch, Section Governor.

Dr. Alder presented the banquet address *Prime Generating Functions and Congruences* and an address on *Recommendations for the Preparation of High School Students for College Mathematics Courses.* Professor Erdös presented *Unconventional Problems in Number Theory and Problems on Consecutive Integers.*

There were two panel discussions:

Ways to Encourage More Women to take Mathematics Courses, Professor Ruth Struik, moderator; Professor B. Gimmestad, University of Colorado at Colorado Springs; Professor J. Hodges, Colorado U; Ms. Ann Pape, Denver.

Employment Problems of the Undergraduate B.A. and B.S. in Mathematics, Professor Dale Rognlie, moderator; Professor V. Nelson, Metro State; Professor D. Elliott, UNC; Professor A. Magnus, CSU; Professor David Ballew, SDSM&T.

The following papers were contributed:

The Approximation of π, Professor Arne Magnus, CSU.

Some Matrix Models in Accounting and Finance and some Exponential and Logarithmic Models in Economics, Professor C. G. Mendez, Metro State.

A Lemma Useful in Multivariate Analysis, Professor Hung C. Li, USC.

When are $a^2 + b^2$ *and* $a^2 + (b-na)^2$ *Perfect Squares,* Professor Gerald E. Bergum, SDSU.

The Probability of a Number Being Prime, Professor David Ballew, SDSM&T.

A Problem Concerning Polygons in Networks, Professor B. Jones, CU.

Mathematics Competition for High School Students, Professor D. W. Hardy, CSU.

Avoiding Variation of Parameters, Professor Roger Opp, SDSM&T.

Geometrical Modification of Continued Fractions, Professor John Gill, USC.

Color Symmetry and Group Theory, Professor R. L. Roth, CU.

A Geometrical Problem in Graph Theory, Professor Laurel Rogers, CU at Colorado Springs.

Boundary Value Problem at Resonance, Professor M. Martelli, CU (University of Florence).

Hand Held Calculator Supplement for Calculus Courses, Professor Lewis Huff, CSU.

In addition to the above papers and addresses, exhibits were presented by the MAA, The Association for Women in Mathematics, Worth Publishers, Inc.

The business meeting was convened on Saturday, April 29 by Professor Nelson who presided. Thirty-five members attended. New officers for 1978-79: Professor John Hodges, University of Colorado, Chairman; Professor Laurel Rogers, University of Colorado at Colorado Springs, Chairman Elect; Professor Allan Skillman, Casper College, Vice Chairman; Professor David Ballew, Secretary-Treasurer; Professor David Hector, Denver University, Program Chairman. Professor Duan Porter has been elected Governer of the Section.

It was noted that Professor Clarence Swanson had passed away. Professor Swanson has been a long-time member of the Section and had served as Chairman. Professor George Donovan of Metropolitan State was formally recognized for his long service as Chairman of the High School Lectureship Program. Professor Ballew reported on prospective by-law changes to increase registration fees at the meetings. Professor Forest Fisch gave the Governor's report and discussed the proposed new building for MAA headquarters. Christopher Bretherton of the University of Colorado was given a membership in MAA for his high placement on the Putnam examination.

The 1979 meeting will be in April at the University of Denver; the 1980 meeting will be March 29-30 at the University of Colorado.

David Ballew, *Secretary-Treasurer*

NEW JERSEY SECTION MEETING

The New Jersey Section of Mathematical Association of America held its annual spring meeting in cooperation with AMTNJ (Association of Mathematics Teachers of New Jersey) on Saturday, April 29, 1978 at Steinert High School, Trenton, New Jersey.

During the two morning sessions of the conference, attendees could choose among six simultaneous presentations. Professor Anneli Lax of Courant Institute of Mathematical Sciences spoke on Linear Transformations and Function Theory. MAA-sponsored short student talks were given by Mark Kleiman, winner of the International Mathematics Olympiad (*Generating Functions*); Brian Farrell, St. Peter's College (*A Model for Pedestrian Crossing*); Jose Fernandez, St. Peter's College (*A Model for Music Theory*); Betsey Taylor, Douglass College (*Surfaces of Crystals*); and Erick Schweber, Monmouth College (*Maxwell Functions*).

The afternoon session was a panel discussion inspired by the MAA publication *Recommendations for the Preparation of High School Students for College Mathematics Courses.* The panel was moderated by Section Chairman, B. Melvin Kiernan, and included Daniel Flegler, Waldwick High School; Dr. John K. Reckzeh, Jersey City State College; and Dr. Francis T. Rush, St. Peter's College. A lively discussion among MAA and AMTNJ members followed. Slides of the new national headquarters for MAA in Washington were shown, and the advantages of the purchase were discussed.

The November meeting of the New Jersey Section will take place on November 4, 1978 at St. Peter's College, Jersey City, New Jersey.

Jean Lane, *Secretary*

MAY MEETING OF THE ILLINOIS SECTION

The Illinois Section of the MAA held its 57th annual meeting at Western Illinois University on May 5-6, 1978. Chairman John Bradburn presided. Dr. John Schumaker, past-governor of the Illinois Section and head of the department of mathematics at Rockford College, was the banquet keynote speaker. Professor Schumaker's address was entitled *Roving Mathematicians of the Renaissance.*

Dr. Henry Pollak of Bell laboratories gave the afternoon invited address *On the Relationship Between the Applications of Mathematics and the Teaching of Mathematics.* Saturday's invited address was *Applications in Undergraduate Mathematics* by Ross L. Finney, Project Director, Education Development Center in Newton, Massachusetts. Other sessions and their topics included:

What is Numerical Analysis? by Benny Neta, Northern Illinois University.

Complexity Theory: The Limitations of Computers, by Lowell Carmong, Southern Illinois University, Carbondale.

Number-Theory-finite, infinite, algebraic, geometric, by Larry Eggan, Illinois State University.

Two Views of Courses and Content in a Tech Math Sequence, by Robert Custu, Black Hawk College.

At the annual business meeting, committee reports were presented, and Governor Jon Laible spoke on the new MAA headquarters building in Washington, its funding, and showed several slides of the building. Professor Gordon Mock of Western Illinois University was named chairman for 78-79, Professor Robert Johnson of Augustana College, chairman-elect, and Dr. Patrick McCray of Searl Laboratory, 1st vice-chairman.

H. C. Saar, *Secretary*

MAY MEETING OF THE SEAWAY SECTION

The Spring meeting of the Seaway Section of the MAA was held at Brock University, St. Catharines, Ontario, on May 5-6, 1978, with a registered attendance of 58 people, 55 of them being members of the Association. Paul Schaefer, Chairman of the Section, presided.

At a dinner meeting held Friday evening at the Brock University Faculty Club, Edith Luchins, Rensselaer Polytechnic Institute, gave a talk entitled, *Sex Differences in Mathematics: What Not to Do About Them.*

On Saturday morning R. C. Buck, University of Wisconsin, Immediate Past First Vice-President of the MAA, gave a talk, *There were giants . . .*

Erik Hemmingsen, Syracuse University, member of the Board of Governors of the MAA, presented a report on the New MAA Headquarters in Washington, D.C.

Gail Young, University of Rochester, presented the eleventh annual Harry M. Gehman Lecture, entitled, *Our Future in Mathematics.*

At the business meeting the following officers were elected: Chairman, Violet Larney, State University of New York at Albany; First Vice-chairman, Howard Bell, Brock University; Second Vice-chairman, Peter Lindstrom, Genesee Community College; Secretary-Treasurer, Donald Trasher, State University College at Geneseo.

A proposed revision of Article 3 of the By-Laws of the Seaway Section was presented by Chairman Schaefer and adopted by the membership.

It was announced that Neal Madras, student at McGill University, had received the highest score of anyone in the Section in this year's Putnam Mathematical Contest, and that congratulations and a check for $10 would be sent to him from the Section.

A decision was made that $100 should be sent from the Section to the MAA Headquarters in Washington as a contribution to the Building Fund in honor of Harry M. Gehman, longtime stalwart of the MAA and of the Seaway Section.

Contribured papers presented during the morning and afternoon sessions were:

General Implicit and Explicit Solutions for Bernouilli Type Differential Equations, by J. P. Rivet, College Militaire Royal de Saint-Jean.

Discipline Oriented Problems and Individualized Sets of Data for Service Courses in Mathematics, by E. R. Muller, Brock University.

Harmonic Faber Polynomials, by E. T. Hoefer, Rochester Institute of Technology.

Visualizing the Derivative in Elementary Calculus, by R. C. Williams, Alfred University.

The Algebraic Structure of Ethical Language, by W. J. Rapaport, State University College at Fredonia.

Locating Objects from Satellite, by Constance Elson, Ithaca College.

The smallest n for which an n-square theorem is valid in certain fields, C. W. Kohls, Syracuse University.

Emmet Stopher, *Secretary-Treasurer*

APRIL MEETING OF THE NEBRASKA SECTION

The annual meeting of the Nebraska Section was held at the University of Nebraska at Omaha, April 14-15, 1978 with 65 persons in attendance including 50 members of the Mathematical Association of America. Section Chairman Paul A. Haeder presided. Invited lectures were given by Dr. A. B. Willcox and by Professor Alexander P. Mehaffey.

At the business meeting of the Section, Professor Stanley Luke gave a report on the 1978 High School Mathematics Contest in Nebraska and South Dakota. Members discussed the advisability of assessing sectional dues and passed a motion requesting that the MAA collect an additional one or two dollars and rebate same to the section. The report of Henry M. Cox, Secretary-Treasurer was read, in absentia, by the chairman; it showed a membership of 118 and a balance of $119 on January 1, 1978.

Officers for 1978-1979 were elected, as follows:

Chairman, Thomas Shores, University of Nebraska-Lincoln; Past-Chairman, Paul A. Haeder, University of Nebraska at Omaha; Chairman-Elect, Mildred Gross, Doane College; Secretary-Treasurer, Henry M. Cox, University of Nebraska-Lincoln; Chairman of Contest Committee, Stanley Luke, Nebraska Wesleyan University.

Dr. A. B. Willcox, Executive Director of the Mathematical Association announced results of the election of Sectional Governor: Gary H. Meisters of the University of Nebraska-Lincoln.

Papers were presented as follows:

Odd [M, 3] Group Codes for the Gaussian Channel, John Karlof, University of Nebraska at Omaha.

Gaussian Channel Codes for Regular Solids, Charles Downey, University of Nebraska at Omaha.

Karl Gauss, the Man and His Work, Alexander Mehaffey, Jr., University of South Dakota.

Can College Students Use Proportions?, Melvin C. Thornton, University of Nebraska-Lincoln.

The Computer Before it was God, Wayne W. Gutzman, University of South Dakota.

$Sin^2x + cos^2x = 1$, Allan Peterson, University of Nebraska-Lincoln.

Properties of Solutions of the Two Body Problem, John P. Maloney and John C. Kasher, University of Nebraska at Omaha.

Existence of Solutions to Certain Degenerate Quasilinear Elliptic Equations, Allan V. Lair, University of South Dakota.

Discussion: M.A.A. Headquarters Building, M.A.A. Programs and Policies, Mildred Gross, Regional Governor, and Alfred B. Willcox, Executive Director of the Mathematical Association of America.

Practical Aspects of Fuzzy Sets, Richard H. Warren, University of Nebraska at Omaha.

A Diophantine Equation and Some Finite Differences, Bernard J. Portz, Creighton University.

Gauss' Use of Infinitesimals in His Investigations of Curved Surfaces, Gary Meisters, University of Nebraska-Lincoln.

The Structure of Cyclic Convolutional Codes, P. A. von Kaenel, University of Nebraska at Omaha.

On the Distribution of Nonzero Elements in Certain Sparse Matrices, Dale M. Mesner, University of Nebraska-Lincoln.

Results and Statistical Summary of the 1978 Annual High School Mathematics Contest, Stanley D. Luke, Nebraska Wesleyan University.

An Existence Result for Second Order Boundary Value Problems, Dwight V. Sukup, University of South Dakota.

A Geometrical Construction of the Full Elliptic Integral of the First Kind Using a Compass and Straightedge, Gregory A. Kriegsmann, University of Nebraska-Lincoln.

Some Bridges To and From Mathematics, Alfred B. Willcox, Executive Director, The Mathematical Association of America.

Henry M. Cox, *Secretary*

MAY MEETING OF THE METROPOLITAN NEW YORK SECTION

The thirty-seventh annual meeting of the Metropolitan New York Section of the MAA was held at Queensborough Community College on Sunday, May 7, 1978, with approximately 150 persons in attendance. Professor Robert J. Bumcrot of Hosfstra University, Chairperson of the Section, presided at the meeting, which began with the business meeting at 9:45 a.m. Dean James Eastham, Professor Emeritus of Mathematics of Queensborough Community College delivered the Address of Welcome.

The business meeting included the following:

The first Charles Salkin Award to the highest regional scorer in the MAA High School Math Contest was presented to Mr. Fred Helenuis of Stuyvesant High School (in absentia), who received a perfect score on the test. The Section Awards to the highest scorer in the Putnam Mathematics Competition were presented to the two students who received the same score: Renato E. Mirollo, Columbia University and Peter P. Soni, New York University. It was decided that the next annual meeting will be May 5, 1979, at Adelphi University.

The principal speaker, Professor John Thorpe, SUNY at Stony Brook, gave his invited address: *Space, Time and Geometry*. The main part of the afternoon session was a panel discussion in the form of a skit on: *Bayesians vs. Non-Bayesians: Gambling Situation*. The afternoon session concluded with the following student and faculty papers given in four parallel sessions:

Research on a Classical Diophantine Problem of Format, Joseph Arking, New York Academy of Sciences.

Theory of Groups, Madeline Becker, North Shore High School.

On Minimum Points of Monotone Norms, Richard Bielak, Brooklyn College.

Formulas for Any Triangle's Centroid, Circumcenter, Incenter, and Orthocenter, Haig Bohigan, John Jay College of Criminal Justice.

An Interesting Class of Functions Induced by Permutations, Michael W. Ecker, Lehman College.

A Triangle Exists If and Only If Its Circumcircle is an Ellipse, Leon Gerber, St. John's U.

The Trapezoid Rule in Three Dimensions, Sheldon P. Gordon, Suffolk Community College.

The Case Against Changing to the Metric System, Morton J. Hellman, Long Island, University.
On Super Abundant and Deficient Numbers, Harvey J. Hindin, Polymathic Associates.
Specializations of the Mean Value Theorem, Hagop Ketchedjian, Brooklyn College.
Math and Genetics, Lisa Kirsch, Valley Stream North High School.
A Review of Some "Greedy" Algorithms, Deborah F. Kornblum, Western Electric.
It Takes More than Abortion to Prevent One Birth, Rochelle Wilson Meyer, Suffolk County Community College.
An Integral Transform Proof of the Law of Large Numbers, James V. Peters, St. Bonaventure U.
Zeroing in on Determinants, Janet Pomeranz, SUNY Maritime College.
Rings, Algebras, and Other Set Theoretic Fruits, Jay Schiffman, St. John's University.
Incompleteness in Quantified K1.1, Steven Schmidt.
Cartesian Products of Topological Spaces, Samuel Weinberg.

Lily E. Christ, *Secretary*

SPRING MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section was held at Stephen F. Austin State University in Nacogdoches, Texas on March 31, April 1, 1978. There were 198 registered persons in attendance.

Presenting invited addresses were: Professor S. Ulam who spoke on *Mathematics in Atomic Energy During the War*; and Professor Calvin A. Lathen, editor of *Two-Year College Mathematics Journal*, who spoke on *Peer Group Instruction: Alternative to Lecture-Discussion*. There was a panel discussion on *Preparing for College Mathematics: Recommendations of MAA and NCTM* presented by W. K. McNabb of Skyline Center, R. S. Pieters of Hochaday School, Bob Langston of Tarrant County Junior College, and Margaret Hutchinson of the University of St. Thomas. A distinguished service citation for unusual contributions to mathematics and to the Texas Section was presented to Professor H. E. Bray of Rice University.

Officers for 1978-79 are: Chairman: R. G. Dean, Stephen F. Austin State University; First Vice Chairman: Dalton Tarwater, Texas Tech University; Second Vice Chairman: Bill Anderson, East Texas Texas University; Level I Director: David Sanchez, San Antonio College; Level II Director: Margaret Hutchinson, University of St. Thomas; Director at Large: Roger M. Thrall, Rice University; Secretary-Treasurer: Glen Mattingly, Sam Houston State University; and High School Contest: J. R. Boone, Texas A & M University.

Contributed papers were:
Equivalence of Certain Summability Conditions, David F. Dawson, North Texas State University
Limit Preserving Summability of Subsequences, Thomas A. Keagy, Wayland Baptist College
On the Summation of Series by Using the Gauss Multiplication Theorem, Russell Cowan, Lamar U.
Minimizing Sums of Distribution of Integrals of Distributions, E. Green, Abilene Christian U.
Functions Representable as Integrals of Functions, Frank N. Huggins, Univ. of Texas at Arlington
Fixed point Theorems in Banach Spaces with Uniformly Normal Structure, A. A. Gillespie and B. B. Williams, University of Texas at Arlington
The 3-2 Intersection Property and Extreme Functional, Russell Bilyeu, North Texas State U.
The Minkowski-Farkas Lemma and the Namioka-Bauer Theorem, Ronald Teemley, North Texas State U.
A Unified Method for Some Geometry of the Triangle, J. M. Stark, Lamar University.
Enrichment Materials for Secondary Mathematics, William E. Beeman, University of Texas at Arlington
The Skyline Center Talented Mathematics Student Program, William K. McNabb, Dallas Independent School District
Statistics for High School Students, R. S. Pieters, Hockaday School, Dallas
Teaching the Reading of Mathematics, Shirley Tucker, Austin
An Evaluation of PSI in Introductory College Mathematics, J. C. Bolen and G. B. Turney, University of Texas at Arlington
Teaching Mathematics on Television, Larry F. Heath, University of Texas at Arlington
Quotients of Vector Subspace Lattices are Hardly Ever Complete, Don E. Edmondson, University of Texas at Austin
S^P Algebra, Gary Wiggins, Texas Tech University
Translates of (Unitary) Perfect Polynomials Over GF(q) are (Unitary) Perfect, Jacob T. B. Beard, Jr., University of Texas at Arlington
An Integral Domain with an "Almost" Division Algorithm, Nick Vaughn, North Texas State U.
Dedeking-like Conditions in Commutative Rings, H. S. Butts and Robert W. Yeagy, Stephen F. Austin State University
Large Collections of Group Preserving Block Disjoint Group Divisible Designs, William B. Poucher, Abilene Christian University
Roots of Polynomials, Ali R. Amir-Moez, Texas Tech University
Circular and Hyperbolic Functions—A Transformation Approach, John Huber, Pan American University
An Analytic Approach to the Nine Point Circle, John F. Lamb, Jr., East Texas State University
A Concise Introduction and Derivation of the Normal Density Function, Lawrence P. Maher, North Texas State University
Right Answer, Wrong Reasoning, But Why Did it Work 90% of the Time?, James Caristi, Texas Lutheran College
Relating Mathematics to the Real World: Industrial Needs for Mathematics, Fred S. Patterson, Execucom System Corporation
Regulated Functions: Bourbaki's Alternative to the Riemann Integral, S. K. Berbereau, University of Texas at Austin

Isomorphisms of Classical Groups, Morton L. Curtis, Rice University
A Model for Discrete Analysis in the Complex Plane, J. C. Harman, Lamar University (Visiting)

James C. Bradford

APRIL MEETING OF THE OHIO SECTION

The Ohio Section of MAA held its annual Spring meeting at the University of Akron, Akron, Ohio, April 28 and 29, 1978. Approximately one hundred and seventy people were in attendance. Section Chairman W. H. Beyer presided; J. H. Carney was the Program Chairman.

Invited addresses included: *Bidding Models: An Example of Applied Mathematics*, by P. Minton, Virginia Commonwealth University; and *Error-Correcting Codes: At the Intersection of Computer Science, Communications Theory, and Modern Algebra*, by N. J. A. Sloane, Bell Telephone Laboratories. Meeting highlights included a panel discussion on Statistics, with discussion leaders N. Bartlett, Standard Oil of Ohio; B. Chandrasekaran, Ohio State University; D. Leckie, Republic Steel Corporation; and P. Minton; and also an Open Forum on the joint publication of MAA and NCTM (Recommendations for the Preparation of High School Students for College Mathematics Courses).

The following contributed papers were also presented:

Sets of Factors of Sets of Consecutive Integers, S. F. Barger, Youngstown, State University

Distribution-Free Procedures for Inference on a Statistical Measure of Diversity, with Applications, T. N. Bhargava and N. S. Rajaram, Kent State University

A Statisticians' Tour of a Steel Mill, N. P. Bresky, Jones & Laughlin Steel Corporation

Optimizing Properties of Formulations, A. Church, Jr., General Tire & Rubber Company

Affine Geometry, K. Cummins, Kent State University

Finding Trees in A forest—A Computer Pattern Search, R. M. Dieffenbach, Miami University, Middletown Campus

Stochastic Model for Epidemic, P. J. Gingo, University of Akron

Probability Models in Computer Operating Systems, T. A. Hern, Bowling Green State University

A Use of Computers in Number Theory, J. C. Hintz, University of Akron

What Makes a Problem "Interesting"?, D. J. Horwath, John Carrol University

The Cost Accounting Problem, The Input-Output Model, and Markov Chains, D. O. Koehler, Miami U.

Applications of Secondary School Mathematics: An In-Service Program for Teachers, D. E. Kullman, Miami University

Non-Orthogonal Data: A Graphical Approach, S. G. Lindle, Bowling Green State University and D. M. Allen, University of Kentucky

Bezier Polynomials in Computer-Aided Geometric Design, C. A. Long and V. Norton, Bowling Green State University

A Computer Method for Fitting a Smooth Curve Through Points in Three Space, L. H. Miller, Ohio State University

Generating Pythagorean Triples, L. D. Rodabaugh, University of Akron

On Handling an Extreme-Point Theorem in a Linear Programming Course, F. D. Ryan, John Carrol U.

Mathematical Models for Epidemic, P. H. Schmidt, University of Akron

High School Problems Applicable to College Courses, 1978, L. J. Schneider, John Carrol U.

The Comprehensive School Mathematics Program, The Cambridge Report, and the Slow Learner, J. L. Smith, Muskingum College

Computers in Elementary Statistics, A. Sterrett and Z. Karian, Denison University

"Swap" or discussion sessions presented included: *Remediation from Pre-Calculus to Arithmetic*, led by W. W. Hokman, University Akron; *Support Courses—Liberal Arts, Mathematics of Finance, Mathematics for the Elementary Teacher, etc.*, led by T. L. Demen, Youngstown State University; and *Beginning Programming Courses*, led by D. J. Horwath, John Carrol University.

Section officers for the academic year 1978-79 are: M. D. Wetzel, Denison University, Chairperson; D. O. Koehler, Miami University, Chairman-Elect; W. H. Beyer, University Akron, Past Chair; G. Mavrigian, Youngstown State University, Secretary-Treasurer; C. A. Long, Bowling Green State U., Program Chairman; D. J. Horwath, John Carrol University and H. W. Vayo, University Toledo, Program Committee. R. L. Wilson, Ohio Wesleyan University, serves as Sectional Governor; S. W. Hahn, Wittenberg University, serves as MAA representative on the Committee-on-Sections; L. J. Schneider, John Carrol University, serves as Supervisor of the MAA High School Mathematics Competition; and C. P. Yang, Miami University, Middletown Campus, serves as representative to The Two Year College Mathematics Journal.

Gus Mavrigian, *Secretary-Treasurer*

APRIL MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held at Earlham College at Richmond on Saturday, April 22, 1978, with approximately 60 persons in attendance. The chairman of the Section, G. Sherman of Rose-Hulman Institute of Technology, presided. The meeting was held in conjunction with the Friendly Math Competition, a team-oriented contest that saw 13 of the small Colleges and Universities in Indiana furnish 3-man teams.

The following papers were presented:

A non-linear integral and a bang-bang theorem, A. deKorvin, Indiana State University

Some integers that are congruent to their powers, R. Hood, Franklin College

Division algorithms for prime factorization, J. Anderson (student), Rose-Hulman Institute of Technology

Periodic designs in the Euclidean and hyperbolic planes, H. Alexander, Earlham College

Surjective functions and combinatorial identities, R. Grimaldi, Rose-Hulman Institute of Technology

The geometry of binocular visual space, by W. Zage, Ball State University

The new MAA Headquarters, by M. Mansfield, IUPU-Fort Wayne

An isoperimetric problem: The farmer and his heirs, C. Cowen, University of Illinois

CONTACT: Mathematics and cryptology, B. Winkel, Albion College

Panel discussion on the MAA and NCTM recommendations for the preparation of high school students for college mathematics courses. Panelists: Moderator, D. Deal, Ball State University; R. Poland, Richmond Community Schools; W. Ritter, Rose-Hulman Institute of Technology; D. Jorge, Purdue University; A. Weinheimer, North Central High School, Indianapolis.

L. Cote, Purdue University, presented a book prize to James Keller, North Side High School, Fort Wayne, who for the second year was recognized for solving problems appearing in the Indiana School Mathematics Journal.

At the business meeting M. Mansfield, IUPU-Fort Wayne, as chairman of the Nominating Committee, presented the following slate of officers for 1978-79 (which was unanimously approved): Chairman, M. Jerison, Purdue University; Vice-Chairman, M. Nyman, Manchester College; Secretary-Treasurer, D. Wilson, Wabash College. G. Sherman, Rose-Hulman Institute of Technology, was announced as the new Governor of the Section.

Memberships in the MAA were awarded to Kendall S. Stanley, Purdue University and Richard M. Priem, Rose-Hulman Institute of Technology, in recognition of their performances on the Putnam examination.

D. Wilson, *Secretary-Treasurer*

APRIL MEETING OF MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The 1978 Spring Meeting (April meeting) of the Maryland-District of Columbia-Virginia Section of the MAA was held at the Virginia State College on Saturday, April 22, 1978. There were 72 persons in attendance including 63 members.

The principal speaker was Professor William M. Sanders of the James Madison University. The title of Professor Sanders' Invited Address was *Those Neglected Cubics*. The chairman of the Section, Professor Orville M. Thomas, presided over the business session. He announced that the Fall Meeting will be held at the U. S. Naval Academy on November 18, 1978.

The contributed papers included:

Periodic Solutions of the Equation $y'' = f(y)$, Nathaniel Withers, University of Richmond

More than you really wanted to know about Complete Regularity, David A. Schedler, Virginia Commonwealth University

The Metamathematics of Computer Programming, John Hays, Naval Research Laboratory

Cutting the d-Cube, Jim Lawrence, National Bureau of Standards

On the History of Topological Spaces, C. E. Aull, Virginia Polytechnic Institute and State U.

A Byway in Modern History of Recreational Mathematics, B. L. Schwartz, Private Consultant

The Isoperimetric Inequality for Curves with Self-Intersection, Andrew Vogt, Georgetown U.

Fisheries Stock Surplus Allocation—Stochastic Models, Ernest Mabrey, U.S. Bureau of the Census

E. E. Kummer's Ideal Numbers, Charles J. Parry, Virginia Polytechnic Institute and State U.

Nonspherical Gravitational Effects, R. N. Pal, M. Fan, and J. W. Kennedy, Computer Sciences Corporation

A Characterization of the Adjoint, William P. Wardlaw, U.S. Naval Academy

Space-fillers at Sixes and Sevens, Michael Goldberg, Washington, D. C.

The following officers were elected: Chairman-elect, John Smith, George Mason University, Richmond, Virginia; Vice-Chairman, Joseph Kent, University of Richmond.

Reuben Drake, *Secretary*

THE 1978 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The 39th Annual William Lowell Putnam Mathematical Competition will be held at participating institutions on Saturday, December 2, 1978. This competition is supported by the William Lowell Putname Prize Fund for the Promotion of Scholarship and is administered by The Mathematical Association of America. All colleges and universities in Canada and the United States may register eligible undergraduates. Registration forms will be mailed to institutions that participated in the 38th competition by September 22, 1978. Other institutions that wish to enter undergraduates should request registration forms from Professor L. F. Klosinski, Director; The William Lowell Putnam Mathematical Competition; University of Santa Clara; Santa Clara, CA 95053. Completed registrations must be received by the Director no later than October 23, 1978.

Further details are given in the Announcement Brochure that is mailed with the registration material. Reports of previous competitions, including examination questions and outlines of solutions are in past issues of this MONTHLY; the most recent of these reports were in the issues of January 1978, November 1976, November 1975, December 1974 and November 1973.

REPORT OF THE TREASURER FOR THE YEAR 1977

Herewith is a summary of the report of the Treasurer of the Association for the year 1977. The full report has been approved by the Finance Committee and accepted by a vote of the Board of Governors. Any member of the Association who wishes to have a copy of the full report may obtain one by writing to the Washington Office of the Association.

ASSETS	Dec. 31, 1977
Cash	$214 407
Short-term investments	17 333
Accounts receivable	70 633
Publication inventory	116 811
Prepaid expenses	4 530
Securities (at cost)	323 328
Furniture and equipment	32 477
Deferred publication costs	103 405
Due to or from other funds	(8 289)
Total Assets	$874 635

LIABILITIES AND FUND BALANCES	
Accounts payable	$ 57 789
Accrued liability	11 212
Unearned income	
Dues and subscriptions	367 463
Other	26 632
Unexpended grant receipts	27 786
Fund balances	383 753
Total Liabilities and Fund Balances	$874 635

OPERATING INCOME	
Dues	$387 403
Publications	367 024
Dividends and Interest	32 117
Contributions	17 866
Registration fees	21 333
Grant reimbursement	19 214
Miscellaneous	319
Total Income	$845 276

OPERATING EXPENDITURES	
Salaries	$324 695
Office expenses	172 320
Publications (excl. salary & office exp.)	241 789
Travel and meeting expenses	73 392
Taxes and Fees	14 180
Dues and Contributions	11 975
Awards and Grants	6 125
Allocated indirect costs	13 511
Miscellaneous	1 618
Total Expenditures	$832 583
Income over (under) Expenditures	$ 12 693

Leonard Gillman, *Treasurer*

ANNOUNCEMENT OF ALLENDOERFER, FORD AND POLYA AWARDS

At its meeting on January 28, 1978, in St. Louis, Missouri, the Board of Governors authorized a number of awards to authors of expository articles published in the MONTHLY, to be named after Lester R. Ford, Sr., MATHEMATICS MAGAZINE, to be named after Carl B. Allendoerfer, and the TWO-YEAR COLLEGE MATHEMATICS JOURNAL, to be named after Goerge Polya. A maximum of two Carl B. Allendoerfer Awards, five Lester R. Ford Awards, and two George Polya Awards will be made annually; each award is in the amount of $100. The articles are to be selected by committees appointed by the President of the Association for this purpose and the Chairman of the Committee on Publications is to be an *ex officio* member of each of these committees. The recipients for Allendoerfer Awards for articles published in 1977 were the following:

Human Population Growth: Stability or Explosion?, David A. Smith, MATHEMATICS MAGAZINE 50 (1977), 186-197.

Tilings by Regular Polygons, Branko Grunbaum and Geoffrey C. Shephard, MATHEMATICS MAGAZINE 50 (1977), 227-247.

The recipients for Ford Awards for articles published in 1977 were the following:

Immersions and mod-2 Quadratic Forms, T. F. Banchoff and L. H. Kauffman, MONTHLY 84 (1977), 168-85.

Partial Sums of Infinite Series, and How They Grow, Ralph P. Boas, MONTHLY, 84 (1977), 237-58.

Error-Correcting Codes and Invariant Theory: New Applications of a Nineteenth-Century Technique, Neil J. A. Sloane, MONTHLY, 84 (1977), 82-107.

The recipients for Polya Awards for articles published in 1977 were the following:

Statistical Inference for the General Education Student—It Can be Done, Allen H. Holmes, John LeDuc, Walter Sanders, TYCMJ, 8 (1977), 223-30.

Surface Area and the Cylinder Area Paradox, Frieda Zames, TYCMJ, 8 (1977), 207-11.

The recipients of the Allendoerfer, Ford and Polya Awards for articles published in 1977 were announced by President Henry L. Alder at the business meeting on August 9, 1978.

David P. Roselle, *Secretary*

NEWLY ELECTED MEMBERS OF THE BOARD OF GOVERNORS

Melvin Woodard Indiana University Indiana, Pennsylvania	Allegheny Mountain Section
G. J. Sherman Rose-Hulman Institute/Technology Terre Haute, Indiana	Indiana Section
Jacqueline C. Moss Paducah Community College Paducah, Kentucky	Kentucky Section
Harold N. Shapiro Courant Institute New York, New York	Metropolitan New York Section
Gary Meisters University of Nebraska Lincoln, Nebraska	Nebraska Section
Kenneth Rebman California State University Hayward, California	Northern California Section
Paul Long University of Arkansas Fayetteville, Arkansas	Oklahoma-Arkansas Section
A. Duane Porter University of Wyoming Laramie, Wyoming	Rocky Mountain Section
Gary B. Klatt University of Wisconsin Whitewater, Wisconsin	Wisconsin Section

Lucienne Stec, *Secretary to Dr. Willcox*

CALENDAR OF FUTURE MEETINGS

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday–Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, spring. Deadline for papers 2 weeks before meeting.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, St. Peter's College, Englewood Cliffs, November 4, 1978.

NORTH CENTRAL, University of Saskatchewan, Saskatoon, October 20–21, 1978.

NORTHEASTERN, Bunker Hill Community College, Charlestown, Massachusetts, November 18, 1978.

NORTHERN CALIFORNIA, first or second Saturday in February., Ohio Northern University, Ada, October 20–21, 1978.

OHIO, Northern University, Ada, October 20–21, 1978.

OKLAHOMA–ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 weeks before meeting.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers 6 weeks before meeting.

PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.

ROCKY MOUNTAIN, University of Denver, Denver, spring 1979.

SEAWAY, University of Rochester, New York, November 10–11, 1978

SOUTHEASTERN, University of Tennessee, Chattanooga, spring 1979.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

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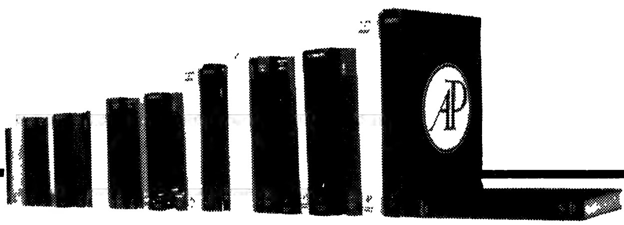

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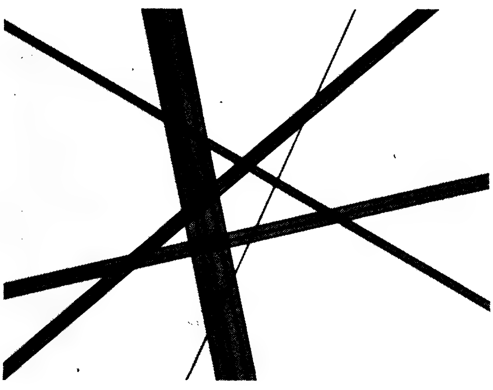
THE AMERICAN MATHEMATICAL MONTHLY

Volume 85, Number 8

E. G. Begle—In Memoriam

Approximation and Abstract Boundaries



Division of the Plane by Lines (pp. 647 and 660)

Strategy for Indefinite Integration

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Subseries of the harmonic series
Cauchy functional equation

Detailed contents facing cover 3

1978

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EDWARD GRIFFITH BEGLE

MARTHA ZELINKA

The mathematical community has lost one of its most distinguished leaders—Edward G. Begle died March 2, 1978.

In 1969, the Mathematical Association of America presented him with the Distinguished Service Award, a well-deserved honor for a man who has given so much to so many. Not even a decade has elapsed since then, and we are now honoring his memory. The debt we owe him is as incalculable as his impact on the school mathematics curriculum is timeless.

For fear that my heart would run away with me in my love and esteem for Ed and that only my thoughts and impressions would be recorded here, I invited comments from friends and colleagues in different professional positions, academic and non-academic, with different philosophies, who had played different roles in the big School Mathematics Study Group (SMSG) family. I was delighted with the numerous and voluminous responses. They are more eloquent than I could ever hope to be. From a group of very different personalities, such as only SMSG can gather, there resounds a message, clear and loud, a testimony to Ed's very triumph, so characteristic of his monumental achievement, moulding diversity into unity: Ed, we love you, respect you, and are forever indebted to you.

Edward G. Begle took his place as a mathematician early, and his deep concern for the student became apparent from the outset. Let me quote from Professor Charles E. Rickart's remarks: "Ed and I first taught Algebra to Army ASTP students, and then Calculus to Yale freshmen, spending hours discussing the problems of that teaching. Out of this grew Begle's elementary calculus text, which was unique at the time in that it contained serious mathematics written not for colleagues but for the students themselves.... Ed began his career as a topologist, with a Ph.D. from Princeton University under Professor Lefschetz, and was on his way to becoming a first-rate research mathematician. However, his destiny lay in another direction."

And Professor Edwin E. Moise states: "I should mention that while Ed's career as a research mathematician was brief, his work was solid and important. His papers looked good when they appeared, and after twenty-five or thirty years, they still do. (For example, he wrote the first valid proof of the Vietoris Theorem.)"

Ed's talent for administration was recognized. He was called upon to serve as Secretary of the American Mathematical Society, and it was quite natural that he was chosen to head up the School Mathematics Study Group when it was first organized. The SMSG experiment was one of magnitude and complexity, in scope, depth, and size. Professor Donald E. Richmond notes: "Ed guided without dominating. His exceptional integrity ensured that SMSG was administered with scrupulous honesty.... Throughout, Ed dictated no solutions but strove to harmonize the opinions of all...no curriculum was to be imposed on any school system."

Ed's leadership qualities were exercised in bringing together a wide variety of individuals, from elementary-school to college-level teachers, recognized educators and distinguished research mathematicians; they learned to communicate with each other, to respect each other, and to form efficient writing groups. It is difficult to imagine that anybody but Ed could have orchestrated such a performance. Let me illustrate, by quoting from the responses, what adding the right chords means to leading discord into harmony. The references go back to Chicago meetings in the early sixties:

Philip J. Davis: "...At that time I was a mathematical conservative (perhaps a revisionist) and I

Martha Zelinka received her M.A. degree from the University of Vienna and teaches mathematics at Weston High School, Weston, Massachusetts. She was extensively involved in SMSG and has also been active in connection with the New Mathematical Library, the Advanced Placement Mathematics Program, and the High School Mathematics Contests.—*Editors*

did not think well of attempts to reduce school mathematics to dry-as-dust axiomatics or abstract structure. Perhaps Begle wanted me on his Committee for balance. In any case, I certainly enjoyed the sense of vitality that issued from the 'New Math' in those days. It was infectious. As chairman, Begle was a firm, gentlemanly, and energetic man.... The meeting in Chicago broke into acrimonious polarization, and the dogmatism and self-righteousness of the abstractionist party and the fury with which they held to their views was matched only by the firmness and sweet reasonableness of my own henchmen. Ed Begle chaired this meeting, listening to all patiently, never once losing his cool, successfully controlling this scrapping zoo of mathematicians... ."

Mario L. Juncosa: "...We were meeting to consider devising tests which would distinguish the two classes of students (those that had 'New' math training and those who hadn't). Ed Begle was a great chairman in letting everyone speak out so that all were heard, no rancor developed, but work was accomplished. A balanced, comprehensive, and detailed outline of what should be tested evolved from a group of about sixteen people with fairly strong opinions—quite an achievement... ."

Jeremy Kilpatrick: "...To me, his most impressive quality, apart from his intelligence, was his absolute and unwavering integrity. He staked out the territory of mathematics education in a wilderness full of predators, some of whom would sell their mother for a state textbook adoption. Putting together teams of people who had never even met before, he inspired them to produce more than they might have thought possible. He had a knack for judging who could do what and for getting them to do it. Like some patriarch from the Old West, he commanded allegiance because of what he was and what he stood for... ."

One can only hope that Ed knew how many friends he had who felt so deeply and sincerely about him and his work.

The impact of SMSG on the school scene was immediate. Ed's ingenious plan included centers where teachers who used the experimental texts met with a consultant from a college or university. "He taught us to be concerned about mathematics in the classroom," says Professor Vincent Haag.

Dedication, enthusiasm, and hard work brought good results. Let me quote a remark of a great teacher, Martha Hildebrandt: "Dr. Begle wanted students to enjoy their work in mathematics so much that they would want to own and keep their textbooks, as you would a friend.... I myself have seen excellent thinking and learning as well as teaching done in the elementary and secondary classroom because of the SMSG texts. Some of it has been most beautiful and amazing, all of it the finest tribute which could be paid to Dr. Begle, mostly by people who did not know him personally but enjoyed their mathematics and were more capable of using it because of his efforts."

The above can be appreciated only if you know the excitement in a classroom when students take an active part in using new, interesting materials and become participants—how proud they were when their comments and criticisms were sent to headquarters at Yale and later at Stanford. There was a lot of learning going on. And, in the hands of the right person, the books are as good now as ever.

In 1961, from a newspaper in New Haven, Connecticut: "Yale University announced that it has revamped its freshman mathematics program to meet the needs of students with better high school training in the subject. The announcement was the first indication from a major American university of the value of the post-Sputnik surge to improve high school mathematics courses. Professor Charles E. Rickart, chairman of the Yale mathematics department, said that traditional freshmen math courses—analytical geometry and calculus—were replaced by a three-category program designed to meet the individual background of entering students... ."

The books produced by SMSG, and books that borrowed heavily from them, are among the best teaching materials used widely in the United States; they, that is Edward G. Begle, influenced the nation and, far beyond our boundaries, the globe.

Let me quote Bryan Thwaites (Westfield College, University of London): "Not being a pure mathematician, I had not heard of Professor Begle in his younger days as a creative researcher in his own subject. But as soon as I and a few others in England began to gather together in the late 1950's to talk about the deteriorating situation of mathematics and especially about the need to reform the

content of school mathematics, the name of Begle sprang out of nowhere, so to speak, apparently heading every list and every communication which came to us from other countries.... So by 1961 he and his work with the SMSG were already well known to us.... And that was the year of his visit to this country as a participant of—and (little did he suspect) a guide of incalculable value to—the Southampton Mathematical Conference 1961.... How, you may ask, have we in the U.K. viewed Professor Begle's achievements as head of that (to us) vast organization of the School Mathematics Study Group? First he led the way to team authorship, a startling innovation for conservative Europeans, but one which we adopted with fervour and moulded to our own fashion.... Second, the sheer drive and administrative skill with which he prosecuted the SMSG's activities were abundant causes for astonishment and admiration on this side of the Atlantic.... Third,...groups all over the world would constantly be referring back, or across, to what was going on at Stanford. Whether or not there was invariably agreement is beside the point: what mattered was the sense that something very definite was happening under Ed's leadership. Fourth,...One felt that there was no sense of competition in him; rather that he was carrying out some preordained strategy whose merits would ultimately be tested by the experience of millions of children and their teachers...."

It is impossible to portray the spirit and excitement of the old SMSG days, when so much was accomplished, so much was possible. In addition to the development of Sample Textbooks for School Mathematics, Ed initiated the National Longitudinal Study of Mathematical Abilities, an enormous undertaking, which he carried out with characteristic thoroughness, setting a standard for all subsequent studies. His fascination with research in mathematics education continued for the rest of his life.

Those of us who were privileged to work under Ed's guidance, who participated in summer writing sessions, have a vivid image of Ed. SMSG was housed on Stanford University campus in Cedar Hall —it can boast of no charm—a one-story-high, long-stretched-out building with one feature, seemingly built for Ed—a corridor that ran its full length. On either side were small rooms, where we did our writing, meeting, work. Ed Begle, looking stern, almost fear-inspiring, deep in thought, paced the hall from one end to the other. If you had to pass him, you practically stopped breathing, for fear that you might interrupt. The scene was the same during the regular school year; his taciturnity and pacing continued, as Jeremy Kilpatrick recalls from his days of graduate school: "I used to be so afraid of taking time from his crowded schedule that I would save up questions to ask of him for several days until I could find him alone in his office. I learned to phrase my questions in almost telegraphic style and to expect brief responses in return. In later years, I treasured letters of more than two sentences from Ed as veritable volumes."

Professor Anneli Lax has been the Technical Editor of The New Mathematical Library, an SMSG Monograph Project taken over by the MAA in 1975. Quoting from her response: "One of my strongest recollections is corresponding with Ed. I would write a rather verbose letter, reporting on what I was doing and asking a question now and then. His prompt reply usually consisted of a single sentence; a few had the form: 'Dear Anneli: Yes. Best regards, Ed.' The occasions when Ed wrote more than one paragraph were so memorable that I wanted to frame such letters."

Ed was completely dedicated to his profession; he was a hard worker. He was a man of broad interests and tastes, devoid of any pomposity.

But the account would be incomplete if I did not mention that in all the years of his great achievements there was by his side a unique person, his wife Elsie. Together in their love and concern for each other, they could meet joys and sorrows. They brought up a beautiful family, now rich with memories. Ed, with his gruff exterior, was a gentle man, with a deep love for children, who in turn responded with warmth. As Professor Lowell Paige expressed it so well at the Memorial Service for Ed: "He comforted friends in sorrow, he shared the pleasures of their successes, and admonished them of any shortcomings."

Our loss is great but his legacy is rich.

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APPROXIMATION AND ABSTRACT BOUNDARIES

HEINZ BAUER

Introduction: The approximation theorems of Weierstrass and Korovkin. Weierstrass' theorem on polynomial approximation is usually presented as the classical representative of the many approximation theorems. It states that for every continuous real-valued function f on a compact interval $[a,b]$ of the real line $\mathbf{R}$, there exists a sequence (p_n) of real polynomials (of one variable) which converges uniformly to f on $[a,b]$. Among the many proofs for this theorem, we select one in which the approximating sequence (p_n) is defined explicitly by means of the polynomials introduced by S. Bernstein.

It suffices to consider the unit interval $[0,1]$ in $\mathbf{R}$. For the given continuous function f and $n=1,2,\ldots$ the nth Bernstein polynomial is then defined as

$$p_n^f(x)=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}. \tag{1}$$

Deviating from the original proof of S. Bernstein [8], we test the statement of the Weierstrass approximation theorem for three simple functions: the constant function 1, the function $f(x)=x$ which will be denoted by the symbol id (=identity), and its square id^2, i.e., the function $f(x)=x^2$. From the binomial theorem and some simple calculations, we obtain for every $n=1,2,\ldots$

$$\begin{aligned} p_n^1&=1;\\ p_n^{\mathrm{id}}&=\mathrm{id};\\ p_n^{\mathrm{id}^2}&=\frac{1}{n}\mathrm{id}+\frac{n-1}{n}\mathrm{id}^2.\end{aligned} \tag{2}$$

These formulas prove that for each of the three functions $f=1$, id, id^2 the sequence (p_n^f) converges uniformly to f on $[0,1]$.

According to a theorem of Korovkin [15], [16]—the so-called three function theorem—this simple test immediately finishes the proof of the uniform convergence of (p_n^f) to f for every continuous real-valued function on $[0,1]$. To make this clear, we have to give an appropriate interpretation to the polynomial sequence (p_n^f) in the context of the linear space $\mathcal{C}([0,1])$ of all real-valued functions on $[0,1]$. For every natural number n we attach to f the polynomial p_n^f and thus arrive at a map

$$B_n:\mathcal{C}([0,1])\to\mathcal{C}([0,1]) \tag{3}$$

of the space $\mathcal{C}([0,1])$ into itself. That is, at $f\in\mathcal{C}([0,1])$ the map B_n takes on the value $B_nf=p_n^f$, the nth Bernstein polynomial for f. From the definition (1) of these polynomials, one can see immediately that B_n is in fact linear, hence a *linear operator* of $\mathcal{C}([0,1])$ into itself which, in addition, is also *positive*. The latter property means that every non-negative function $f\in\mathcal{C}([0,1])$ is transformed into a function B_nf which is again non-negative. After this interpretation of our original situation, we may apply Korovkin's three function theorem to turn our test into a proof of the Weierstrass approximation theorem. The theorem in question is, more precisely,

KOROVKIN'S FIRST THEOREM. *Let $[a,b]$ be a compact interval in $\mathbf{R}$ and let (L_n) be a sequence of positive linear operators L_n of $\mathcal{C}([a,b])$ into itself. Suppose that (L_nf) converges uniformly to f for the*

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three test functions $f=1$, id, id^2. *Then* (L_nf) *converges uniformly to* f *on* $[a,b]$ *for all functions* $f\in\mathcal{C}([a,b])$.*

A quick proof can be given by means of a trick. Every $f\in\mathcal{C}([a,b])$ is bounded:

$$|f(x)|\leqslant\gamma \qquad \text{for all} \quad x\in[a,b]$$

where γ is a certain positive number. Also, f is uniformly continuous on $[a,b]$: given $\varepsilon>0$ there exists a number $\delta>0$ such that, for all $x,y\in[a,b]$,

$$|x-y|\leqslant\sqrt{\delta} \quad\Rightarrow\quad |f(x)-f(y)|\leqslant\varepsilon,$$

or, equivalently,

$$(x-y)^2\leqslant\delta \quad\Rightarrow\quad |f(x)-f(y)|\leqslant\varepsilon.$$

The trick consists of putting $\sqrt{\delta}$ where one normally puts δ. It leads for arbitrary points $x,y\in[a,b]$ to the inequality

$$|f(x)-f(y)|\leqslant\varepsilon+\alpha(x-y)^2,$$

with $\alpha=2\gamma\delta^{-1}$, which one immediately derives by considering the two cases $(x-y)^2\leqslant\delta$ and $(x-y)^2>\delta$. Thus we have for all $y\in[a,b]$ the following inequality between functions:

$$|f-f(y)|\leqslant\varepsilon+\alpha(\mathrm{id}-y)^2.$$

Linearity and positivity of the operators L_n then imply†

$$|L_nf-f(y)L_n1|\leqslant\varepsilon L_n1+\alpha\big(L_n\mathrm{id}^2-2yL_n\mathrm{id}+y^2L_n1\big).$$

Evaluating the above expression at $x=y$, we obtain

$$|L_nf-fL_n1|\leqslant\varepsilon L_n1+\alpha\big(L_n\mathrm{id}^2-2\,\mathrm{id}\,L_n\mathrm{id}+\mathrm{id}^2L_n1\big).$$

From the assumption on (L_nf) for the three functions $f=1$, id, id^2 and from the triangle inequality it now follows that L_nf converges to f uniformly. We even obtain explicit bounds for the supremum norm

$$\|L_nf-f\| = \sup_{x\in[a,b]}|L_nf(x)-f(x)|$$

provided that such bounds are known for the three test functions as is the case for the original operator sequence (B_n) according to (2).

Quite naturally, Korovkin's theorem led to a quest for a deeper understanding of the role of the three test functions involved. The above proof, based on the trick involving uniform continuity, does not, for example, provide answers to a number of questions. Can one replace the three functions 1, id, id^2 by other test functions? Can one get along with two test functions? Is there an analogous theorem in higher dimensions?

In what follows, a generalized version of Korovkin's First Theorem will be derived and analyzed in order to answer such questions in a natural way.

1. Test space and corresponding affine functions. The above-mentioned questions will be treated within a framework broad enough for interesting applications. The compact interval $[a,b]$ will be

*We denote by L_nf the image of f under the operator L_n and by $L_nf(x)$ the value of L_nf at the point x. However, $L_n(f(x))$ is the image of the constant function $f(x)$ under L_n, hence $L_n(f(x))=f(x)L_n1$ by linearity.

†All positive linear operators $L:\mathcal{C}([a,b])\to\mathcal{C}([a,b])$ are *increasing* since $f\leqslant g$ implies $g-f\geqslant0$ and hence $Lg-Lf=L(g-f)\geqslant0$, i.e. $Lf\leqslant Lg$. In particular, the inequality $|Lf|\leqslant L|f|$ is derived from $-|f|\leqslant f\leqslant|f|$ where $|g|$ denotes the function $x\mapsto|g(x)|$ $(f,g\in\mathcal{C}([a,b]))$.

replaced by an arbitrary *compact metric space* X.† The set $\{1, \mathrm{id}, \mathrm{id}^2\}$ of the former three test functions will be replaced by an arbitrary (finite or infinite) subset $\mathfrak{T}$ of the linear space $\mathcal{C}(X)$ of all continuous real-valued functions on X. (All functions in $\mathcal{C}(X)$ are bounded since X is compact.) The only assumption on the set $\mathfrak{T}$ will be that the constant function 1 is in it:

$$1 \in \mathfrak{T}.^{\ddagger} \tag{4}$$

$\mathfrak{T}$ will be called the set of *test functions* or the *test set*.

A sequence $(L_n)_{n=1,2,\ldots}$ of positive linear operators

$$L_n : \mathcal{C}(X) \to \mathcal{C}(X)$$

will be called $\mathfrak{T}$*-admissible* if the sequence $(L_n t)$ converges uniformly to t on X for all test functions $t \in \mathfrak{T}$, i.e.,

$$\lim_{n\to\infty} \|L_n t - t\| = 0 \qquad \text{for all} \quad t \in \mathfrak{T}, \tag{5}$$

where

$$\|f\| = \sup_{x \in X} |f(x)|$$

denotes the so-called sup-norm of a bounded real-valued function on X. A function $f \in \mathcal{C}(X)$ will be called a *Korovkin function* (with respect to $\mathfrak{T}$) if the sequence $(L_n f)$ converges to f uniformly on X for *all* $\mathfrak{T}$-admissible sequences (L_n) of positive linear operators of $\mathcal{C}(X)$ into $\mathcal{C}(X)$.

With the help of these definitions, the main problems to be studied in this paper can be formulated as follows:

PROBLEM 1. *Which functions $f \in \mathcal{C}(X)$ are Korovkin functions with respect to a given test set $\mathfrak{T}$?*

PROBLEM 2. *What conditions on $\mathfrak{T}$ imply that all functions $f \in \mathcal{C}(X)$ are Korovkin functions?*

The linearity of the operators L_n immediately leads to the observation that each linear combination

$$h = \lambda_1 t_1 + \cdots + \lambda_k t_k \qquad (\lambda_1, \ldots, \lambda_k \in \mathbf{R})$$

of test functions $t_1, \ldots, t_k \in \mathfrak{T}$ is a Korovkin function for $\mathfrak{T}$. In fact, the triangle inequality yields

$$\|L_n h - h\| \leqslant |\lambda_1| \|L_n t_1 - t_1\| + \cdots + |\lambda_k| \|L_n t_k - t_k\| \tag{6}$$

for all n. Hence all functions from the linear hull $\mathrm{lin}\,\mathfrak{T}$ of $\mathfrak{T}$ in $\mathcal{C}(X)$ are Korovkin functions. In addition, we remark that "Korovkin function with respect to $\mathfrak{T}$" and "Korovkin function with respect to $\mathrm{lin}\,\mathfrak{T}$" are equivalent properties for a function $f \in \mathcal{C}(X)$. In what follows $\mathrm{lin}\,\mathfrak{T}$ will be called the *test space* corresponding to the test set $\mathfrak{T}$.

The type of answer one expects to Problem 2 is that $\mathfrak{T}$ or $\mathrm{lin}\,\mathfrak{T}$ should be "sufficiently large." Hence we should try to measure the "size" of $\mathfrak{T}$ or $\mathrm{lin}\,\mathfrak{T}$. In the case of Korovkin's theorem, $\mathfrak{T}$ equals $\{1, \mathrm{id}, \mathrm{id}^2\}$; consequently, $\mathrm{lin}\,\mathfrak{T}$ is the set of all real polynomials $x \mapsto \alpha + \beta x + \gamma x^2$ of degree $\leqslant 2$. As a linear space of dimension 3, $\mathrm{lin}\,\mathfrak{T}$ is a small subspace of $\mathcal{C}([a,b])$. However, $\mathrm{lin}\,\mathfrak{T}$ is large in the sense that an arbitrary function $f \in \mathcal{C}([a,b])$ is the infimum of all functions $h \in \mathrm{lin}\,\mathfrak{T}$ satisfying $h \geqslant f$ and, respectively, the supremum of all functions in $\mathrm{lin}\,\mathfrak{T}$ below f. This is geometrically plausible. One just has to observe that the graph of a polynomial $x \mapsto \alpha + \beta x + \gamma x^2$ of degree 2 (i.e., with $\gamma \neq 0$) is a parabola with an axis of symmetry parallel to the y-axis:

†An inexperienced reader should choose for X a closed bounded subset of $\mathbf{R}^p$, $p = 1, 2, \ldots$. A major part of the following considerations even holds for arbitrary compact topological spaces.

‡A formally weaker assumption would be to demand the existence of a strictly positive test function $t_0 \in \mathfrak{T}$. However, the following transformations reduce this case to the one considered above. Choose $\mathfrak{T}^* = \{(t/t_0) \mid t \in \mathfrak{T}\}$ as a new set of test functions and replace every linear operator $L : \mathcal{C}(X) \to \mathcal{C}(X)$ by the linear operator L^* defined by

$$L^* f = \frac{1}{t_0} L(t_0 f).$$

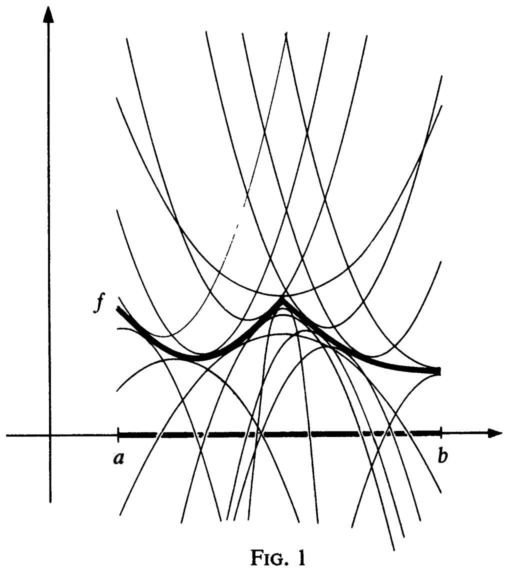

FIG. 1

As we will shortly see, it is exactly this behavior which hits the nerve of the problem and which leads to an adequate notion of $\mathfrak{T}$ being sufficiently large. In pursuing this claim, we define for each function $f \in \mathcal{C}(X)$ two, in general discontinuous, real-valued functions f^* and f_* on X by putting

$$\left.\begin{aligned} f^*(x) &= \inf\{h(x) \mid h \geqslant f, h \in \operatorname{lin}\mathfrak{T}\} \\ f_*(x) &= \sup\{h(x) \mid h \leqslant f, h \in \operatorname{lin}\mathfrak{T}\} \end{aligned}\right\} \quad (x \in X). \tag{7}$$

This definition makes sense since f is bounded on the compact metric space X and since, as a consequence of (4), $\operatorname{lin}\mathfrak{T}$ contains the constant real functions as elements. We have

$$f_* \leqslant f \leqslant f^*. \tag{8}$$

The sup-norm $\|f^* - f_*\|$ of the non-negative function $f^* - f_*$ in some sense measures how well or how badly f adjusts itself to the "shape" of the functions in the test space $\operatorname{lin}\mathfrak{T}$.

EXAMPLE 1. Consider the situation as in Korovkin's theorem: $X = [a,b]$, a compact interval in $\mathbf{R}$, and $\mathfrak{T} = \{1, \mathrm{id}, \mathrm{id}^2\}$. If one accepts the geometric argument preceding Figure 1, then one obtains $f_* = f = f^*$ for all $f \in \mathcal{C}([a,b])$. (A rigorous proof will be given later.)

EXAMPLE 2. Let again be $X = [a,b]$; but consider $\mathfrak{T} = \{1, \mathrm{id}\}$. The test space consists of all affine functions $x \mapsto \alpha + \beta x$. For all $f \in \mathcal{C}([a,b])$, f^* is the smallest concave (necessarily continuous) function $\geqslant f$ and, correspondingly, f_* the greatest convex (necessarily continuous) function $\leqslant f$ on $[a,b]$. Consequently, $f_* = f = f^*$ holds if and only if f is affine, hence is in $\operatorname{lin}\mathfrak{T}$.

Inspired by this example, we introduce the following notions, again in the framework of the general situation. A function $f \in \mathcal{C}(X)$ will be called *$\mathfrak{T}$-affine*, if $f_* = f^* (= f)$ holds. The set of all $\mathfrak{T}$-affine-functions will be denoted by $\mathfrak{T}^*$. All functions of the test space are trivially $\mathfrak{T}$-affine; therefore we have

$$\mathfrak{T} \subset \operatorname{lin}\mathfrak{T} \subset \mathfrak{T}^* \subset \mathcal{C}(X). \tag{9}$$

It is easily checked that $\mathfrak{T}^*$ is a linear subspace of $\mathcal{C}(X)$; one just has to observe that for all functions $f, g \in \mathcal{C}(X)$ and every real number $\lambda \geqslant 0$ we have

$$(f+g)^* \leqslant f^* + g^*, \quad (\lambda f)^* = \lambda f^*, \quad (-f)^* = -f_*. \tag{10}$$

We now have to decide whether the new notion of $\mathfrak{T}$-affine function really is relevant for Problem

1. We start with a preparatory consideration. Let f be a function in $\mathcal{C}(X)$ and let (L_n) be a $\mathfrak{T}$-admissible sequence of positive linear operators of $\mathcal{C}(X)$ into itself. For $g,h \in \operatorname{lin}\mathfrak{T}$ satisfying $g \leqslant f \leqslant h$, we have $L_n g \leqslant L_n f \leqslant L_n h$ for all n. From this, we obtain

$$g(x) \leqslant \varliminf_{n\to\infty} L_n f(x) \leqslant \varlimsup_{n\to\infty} L_n f(x) \leqslant h(x)$$

keeping in mind that $(L_n g)$ resp. $(L_n h)$ converge uniformly and hence pointwise to g resp. h. This being true for all such functions $g,h \in \lim \mathfrak{T}$, we deduce from (7) that

$$f_*(x) \leqslant \varliminf_{n\to\infty} L_n f(x) \leqslant \varlimsup_{n\to\infty} L_n f(x) \leqslant f^*(x). \tag{11}$$

Thus $(L_n f(x))$ converges to $f(x)$ for a given $x \in X$ if $f_*(x) = f^*(x)$. In particular we have proved that for a $\mathfrak{T}$-affine function f the sequence $(L_n f)$ converges at least pointwise to f.

From this observation there is only a small step to the following partial answer to our Problem 1:

PROPOSITION 1. *Every $\mathfrak{T}$-affine function is a Korovkin function.*

Proof. We try to modify the proof given for the case of pointwise convergence. First we remark that, given $f \in \mathfrak{T}^*$ and $\varepsilon > 0$, there exist finitely many functions $h_1', \ldots, h_k'$ and $h_1'', \ldots, h_k''$ in $\operatorname{lin}\mathfrak{T}$ such that we have

$$\underline{h} \leqslant f \leqslant \bar{h} \quad \text{and} \quad \bar{h} - \underline{h} < \varepsilon$$

for their corresponding envelopes

$$\underline{h} = \sup(h_1', \ldots, h_k') \quad \text{and} \quad \bar{h} = \inf(h_1'', \ldots, h_k'').$$

Indeed, it follows from $f_*(x) = f(x) = f^*(x)$ and (7) that for every $x \in X$ there exist functions h_x' and h_x'' in $\operatorname{lin}\mathfrak{T}$ such that

$$h_x' \leqslant f \leqslant h_x'' \quad \text{and} \quad h_x''(x) - h_x'(x) < \varepsilon.$$

Consequently, there is an open neighborhood U_x of x in x such that

$$h_x''(y) - h_x'(y) < \varepsilon$$

holds for all points $y \in U_x$. Obviously, $(U_x)_{x \in X}$ is an open covering of X. By the definition of compactness finitely many of the U_x, say $U_{x_1}, \ldots, U_{x_k}$, suffice to cover X. Then the functions

$$h_j' = h_{x_j}' \quad \text{and} \quad h_j'' = h_{x_j}'' \qquad (j = 1, \ldots, k)$$

have all desired properties.

Now let (L_n) be a $\mathfrak{T}$-admissible sequence of positive linear operators of $\mathcal{C}(X)$ into itself. Given $\varepsilon > 0$, there is a natural number n_0 such that

$$\|L_n h_j' - h_j'\| < \varepsilon \quad \text{and} \quad \|L_n h_j'' - h_j''\| < \varepsilon$$

for all $n \geqslant n_0$ and all $j = 1, \ldots, k$. Since all maps L_n are increasing, one has

$$L_n h_j' \leqslant L_n f \leqslant L_n h_j''$$

for all n and $j = 1, \ldots, k$. Consequently, for all $n \geqslant n_0$, we have

$$h_j' - \varepsilon \leqslant L_n f \leqslant h_j'' + \varepsilon \qquad (j = 1, \ldots, k)$$

and hence

$$\underline{h} - \varepsilon \leqslant L_n f \leqslant \bar{h} + \varepsilon.$$

By making use of the inequalities $\underline{h} \leqslant f \leqslant \bar{h}$, one finally obtains

$$|L_n f - f| \leqslant \bar{h} - \underline{h} + \varepsilon < 2\varepsilon \qquad \text{for all} \quad n \geqslant n_0$$

and from this uniform convergence of $(L_n f)$ to f. So each $\mathfrak{T}$-affine function f is a Korovkin function.

REMARK. Contrary to the proof of Korovkin's First Theorem where we have used the linearity of

the operators L_n in an essential way, the above proof does not use linearity explicitly. One only uses the fact that the operators L_n are increasing, which is a consequence of linearity and positivity. Therefore, if $\mathfrak{T}$ is a linear subspace of $\mathcal{C}(X)$, i.e., $\mathfrak{T}=\operatorname{lin}\mathfrak{T}$, Theorem 1 holds more generally for sequences (L_n) of increasing, not necessarily linear, operators of $\mathcal{C}(X)$ into itself with the convergence behavior expressed by (5).

To summarize, we can state that Problem 2 has found a first answer in Theorem 1. Indeed, if all functions in $\mathcal{C}(X)$ are $\mathfrak{T}$-affine, then each function in $\mathcal{C}(X)$ is a Korovkin function. According to Example 1, this is the situation in Korovkin's First Theorem.

A necessary condition for the validity of $\mathfrak{T}^*=\mathcal{C}(X)$ can now be formulated: $\mathfrak{T}$ must *separate points*. That is, given an arbitrary pair x_1,x_2 of distinct points in X, there exists a test function $t\in\mathfrak{T}$ satisfying $t(x_1)\neq t(x_2)$. Indeed, there certainly is a function $f\in C(X)$ satisfying $f(x_1)\neq f(x_2)$, e.g., $x\mapsto d(x,x_2)$, where d is the metric of X. As a consequence of $\mathfrak{T}^*=\mathcal{C}(X)$ and (7), then there also exists a function $h\in\operatorname{lin}\mathfrak{T}$ such that $h(x_1)\neq h(x_2)$. This finishes the proof since h is a linear combination of test functions. The condition of point separation is certainly not sufficient for the validity of $\mathfrak{T}^*=\mathcal{C}(X)$ as can be seen from Example 2.

2. Test set and Choquet boundary. Whether the above first answer to Problem 2 is of practical use depends greatly on the difficulty of proving the equality $\mathfrak{T}^*=\mathcal{C}(X)$ in a given concrete case. We will now develop a method for attacking this problem. This method is based on the observation that, according to (10),

$$\lim L_n f(x)=f(x)$$

holds for all $f\in\mathcal{C}(X)$ and all $\mathfrak{T}$-admissible sequences (L_n) at a given point $x\in X$ provided that

$$f_*(x)=f^*(x) \qquad \text{for all} \quad f\in\mathcal{C}(X).$$

This is the motivation for a deeper investigation of the set

$$\partial_{\mathfrak{T}}X=\left\{x\in X \mid f_*(x)=f^*(x) \qquad \text{for all} \quad f\in\mathcal{C}(X)\right\}. \tag{12}$$

In Example 1 this set equals $X=[a,b]$; in Example 2 it contains only the endpoints a and b of the interval as elements. In order to see this, one should observe that for the concave continuous function

$$f(x)=\begin{cases} x-a, & a\leqslant x\leqslant \dfrac{a+b}{2} \\ b-x, & \dfrac{a+b}{2}\leqslant x\leqslant b \end{cases}$$

we have $f^*=f$ and $f_*=0$. Consequently $f^*(x)=f_*(x)$ only holds for $x=a$ and $x=b$.

In these two examples, it is evident that each function of the test space $\operatorname{lin}\mathfrak{T}$ attains its global maximum and minimum on X at a point of $\partial_{\mathfrak{T}}X$. We mention without proof that this is true in general. (For a proof see **[2]**, p. 96.) Because of the formal analogy to the boundary maximum principle of function and potential theory, $\partial_{\mathfrak{T}}X$ is called an *abstract boundary*; more precisely it is called the *Choquet boundary* of X with respect to $\mathfrak{T}$.

By the very definition of this boundary the equality $\mathfrak{T}^*=\mathcal{C}(X)$ holds if and only if the Choquet boundary $\partial_{\mathfrak{T}}X$ equals X. This tautology will be transformed into a criterion for the validity of $\mathfrak{T}^*=\mathcal{C}(X)$ if we succeed in determining the Choquet boundary by using other tools. This can be done with the help of the notion of a measure and, in many important special cases, by means of the behavior of the zeros of functions in the test space.

As usual, a positive Radon measure μ on X is by definition a positive linear form on $\mathcal{C}(X)$, i.e. a positive linear map $\mu: C(X)\to\mathbf{R}$. (According to the Riesz representation theorem μ can be viewed as a regular Borel measure $\mu_0\geqslant 0$ on X. In fact there exists exactly one such measure μ_0 such that $\mu(f)=\int f\,d\mu_0$ for all $f\in\mathcal{C}(X)$.) For every point $x\in X$, $f\mapsto f(x)$ is such a measure; it will be denoted by ε_x and is called the *unit measure* (or *Dirac measure*) at x. In what follows, those measures which

operate on the test set $\mathfrak{T}$ like a unit measure will be of particular importance. A positive Radon measure μ on X will be called a *test measure* with respect to $\mathfrak{T}$ (or a $\mathfrak{T}$-*representing measure*) for a point $x \in X$ if

$$\mu(t) = t(x) \qquad \text{for all} \quad t \in \mathfrak{T}. \tag{13}$$

Obviously ε_x is always such a test measure. In general, there are many test measures for a given point x. In Example 2 with $X = [0, 1]$, the Lebesgue measure λ, which by definition is the linear form

$$\lambda(f) = \int_0^1 f(\xi)\, d\xi \qquad (f \in \mathcal{C}(X)),$$

as well as each of the measures $\frac{1}{2}\varepsilon_\alpha + \frac{1}{2}\varepsilon_{1-\alpha}, 0 \leqslant \alpha < \frac{1}{2}$, is a test measure for $x = \frac{1}{2}$ different from ε_x. However, it will soon turn out that the Choquet boundary $\partial_{\mathfrak{T}} X$ is exactly the set of those points $x \in X$ for which ε_x is the only test measure. The reason for this is the following close connection between the envelopes of (7) and the test measures.

LEMMA. *For every point $x \in X$ and every function $f \in \mathcal{C}(X)$, the following two sets coincide*:

$$\left[f_*(x), f^*(x) \right] = \{ \mu(f) \mid \mu \text{ test measure for } x \}. \tag{14}$$

Proof. Let μ be a test measure for x. Then $\mu(f)$ lies in the interval $[f_*(x), f^*(x)]$. In order to see this, consider functions $g, h \in \operatorname{lin} \mathfrak{T}$ satisfying $g \leqslant f \leqslant h$. Then according to (13)

$$g(x) = \mu(g) \leqslant \mu(f) \leqslant \mu(h) = h(x)$$

since g and h are linear combinations of test functions. Because of (7), this yields

$$f_*(x) \leqslant \mu(f) \leqslant f^*(x).$$

Conversely, let α be a real number in $[f_*(x), f^*(x)]$. On the linear subspace $\{\lambda f \mid \lambda \in \mathbf{R}\}$ of $\mathcal{C}(X)$ generated by f, the map

$$\lambda f \mapsto \lambda\alpha$$

defines a linear form μ_0 which is majorized by the function $p(g) = g^*(x)$ defined on $\mathcal{C}(X)$. Indeed, for $\lambda \geqslant 0$, we have

$$\mu_0(\lambda f) = \lambda\alpha \leqslant \lambda f^*(x) = (\lambda f)^*(x) = p(\lambda f),$$

and for $\lambda < 0$

$$\mu_0(\lambda f) = \lambda\alpha \leqslant \lambda f_*(x) = -\lambda(-f)^*(x) = (\lambda f)^*(x) = p(\lambda f).$$

According to (10), p is a sublinear form on $\mathcal{C}(X)$, that is, $p(g_1 + g_2) \leqslant p(g_1) + p(g_2)$ and $p(\lambda g) = \lambda p(g)$ for arbitrary functions $g_1, g_2, g \in \mathcal{C}(X)$ and arbitrary numbers $\lambda \geqslant 0$. Now the Hahn–Banach theorem ([**13**], p. 212) states that μ_0 can be extended to a linear form μ defined on $\mathcal{C}(X)$ which still is majorized by p. μ is a positive linear form since $f \leqslant 0, f \in \mathcal{C}(X)$, implies

$$\mu(f) \leqslant p(f) = f^*(x) \leqslant 0.$$

(Observe that the constant function 0 is in $\operatorname{lin} \mathfrak{T}$ and majorizes f.) Consequently, μ is a positive Radon measure on X. For functions $h \in \operatorname{lin} \mathfrak{T}$ we have $h_* = h = h^*$ which implies $\mu(h) \leqslant h^*(x) = h(x)$ as well as $-\mu(h) = \mu(-h) \leqslant (-h)^*(x) = -h_*(x) = -h(x)$. So a test measure μ for x has been found which by construction satisfies $\mu(f) = \mu_0(f) = \alpha$. This finishes the proof.

As already announced, we now obtain the crucial

PROPOSITION 2. *A point $x \in X$ lies in the Choquet boundary if and only if ε_x is the only test measure for x.*

Proof. According to definition (12), $x \in \partial_{\mathfrak{T}} X$ is equivalent to the validity of $f_*(x) = f^*(x)$ for all $f \in \mathcal{C}(X)$. By the preceding Lemma, this means that an arbitrary test measure μ for x satisfies $\mu(f) = f_*(x) = f^*(x) = f(x)$ for all $f \in \mathcal{C}(X)$. Equivalent to this is the statement that ε_x is the only test measure for x.

It is this Proposition by which the tautology

$$\mathfrak{T}^* = \mathcal{C}(X) \Leftrightarrow \partial_{\mathfrak{T}}X = X$$

is transformed into a non-trivial statement, namely:

COROLLARY. *All functions in $\mathcal{C}(X)$ are $\mathfrak{T}$-affine if and only if ε_x is the only test measure for an arbitrary point $x \in X$.*

Finally, we arrive at the decisive answer to our Problems 1 and 2.

THEOREM 1. *For a given test set $\mathfrak{T} \subset \mathcal{C}(X)$, the $\mathfrak{T}$-affine functions are the only Korovkin functions.*

THEOREM 2. *Let $\mathfrak{T} \subset \mathcal{C}(X)$ be a given test set. All functions in $\mathcal{C}(X)$ are Korovkin functions if and only if an arbitrary point $x \in X$ allows ε_x as the only test measure.*

In view of the preceding Corollary, the second theorem is an immediate consequence of the first one. So we only have to prove Theorem 1.

Proof. Because of Proposition 1, we need only show that a Korovkin function $f \in \mathcal{C}(X)$ is necessarily $\mathfrak{T}$-affine. According to our Lemma, this is equivalent to the statement that $\mu(f) = f(x_0)$ for all points $x_0 \in X$ and all test measures μ of x_0. So let $x_0 \in X$ and let a test measure μ for x_0 be given. Denote by d the metric of the space X. Let us consider for each natural number n the closed set A_n of all points $x \in X$ with distance $d(x, x_0) \geqslant (1/n)$ from x_0. It is well known that for non-empty A_n the function $x \mapsto d(x, A_n)$ (which measures the distance between x and A_n) is continuous and that $\alpha_n = d(x_0, A_n) > 0$. Putting

$$q_n(x) = \begin{cases} 1, & \text{if } A_n = \varnothing \\ \min\left(\dfrac{1}{\alpha_n} d(x, A_n), 1\right), & \text{if } A_n \neq \phi \end{cases}$$

we obtain a sequence (q_n) of continuous real-valued functions on X with the following properties.

$$0 \leqslant q_n \leqslant 1, \quad q_n(x_0) = 1, \quad q_n(x) = 0 \quad \text{on} \quad A_n \tag{15}$$

for all n. For $g \in \mathcal{C}(X)$

$$L_n g = \mu(g) q_n + (1 - q_n) g$$

defines a sequence (L_n) of positive linear operators of $\mathcal{C}(X)$ into itself. It is $\mathfrak{T}$-admissible since for $t \in \mathfrak{T}$ and $x \in X$ we have

$$|L_n t(x) - t(x)| = |t(x_0) - t(x)| q_n(x)$$

and hence because of (15)

$$\|L_n t - t\| \leqslant \sup\{|t(x_0) - t(x)| \,|\, x \in X \setminus A_n\}.$$

But the right-hand side becomes arbitrarily small for n sufficiently large since t is continuous and $X \setminus A_n$ is the open ball with center x_0 and radius $1/n$. For the Korovkin function f the sequence $(L_n f)$ converges uniformly to f, in particular

$$\lim L_n f(x_0) = f(x_0).$$

But according to our construction $L_n f(x_0) = \mu(f)$, and hence $\mu(f) = f(x_0)$. This proves that f is $\mathfrak{T}$-affine.

Whether a point x belongs to the Choquet boundary $\partial_{\mathfrak{T}}X$ can be decided in many examples by proving the existence of a function $h \geqslant 0$ in the test space $\operatorname{lin}\mathfrak{T}$, which has x as *the only zero*, i.e., $0 = h(x) < h(x')$ for all $x' \in X$, $x' \neq x$.

PROPOSITION 3. *Suppose that a function $h \geqslant 0$ of $\operatorname{lin}\mathfrak{T}$ has $x \in X$ as the only zero. Then ε_x is the only representing measure for x, i.e., x lies in the Choquet boundary $\partial_{\mathfrak{T}}X$.*

Proof. The proof is almost trivial if we use some measure theory. Indeed, for every test measure μ for x we derive from $\mu(h)=h(x)=0$ that the set of all $x'\in X$ satisfying $h(x')>0$, i.e., the complement of $\{x\}$ has measure zero. It then follows that $\mu=\varepsilon_x$ since the constant function 1 is a test function.

A direct proof can be given in the following way. Consider $f\in\mathcal{C}(X)$ such that $f(x)=0$. Then f is bounded; consequently $f\leqslant\gamma$ for some real number $\gamma>0$. Also, f is continuous at x; consequently, for a given number $\varepsilon>0$ there is an open neighborhood V of x such that $f(y)\leqslant\varepsilon$ holds for all $y\in V$. On the closed, hence compact, set $X\setminus V$ the function h is positive and thus minorized by some number $\eta>0$. Therefore,

$$f\leqslant\varepsilon+\frac{\gamma}{\eta}h$$

and hence

$$\mu(f)\leqslant\varepsilon\mu(1)+\frac{\gamma}{\eta}\,\mu(h)=\varepsilon$$

for an arbitrary test measure μ for the point x. One only has to observe that 1 and h are functions in the test space and that, accordingly, $\mu(1)=1$ and $\mu(h)=h(x)=0$. Since $\varepsilon>0$ was arbitrarily chosen, we conclude that $\mu(f)\leqslant 0$ for all $f\in\mathcal{C}(X)$ satisfying $f(x)=0$ and in fact $\mu(f)=0$, since these functions form a linear subspace. For an arbitrary $f\in\mathcal{C}(X)$ the function $f-f(x)$ is of the above type. So we have

$$0=\mu(f-f(x))=\mu(f)-f(x)\,\mu(1)$$

and thus $\mu(f)=f(x)$. This proves $\mu=\varepsilon_x$.

3. Applications. We now demonstrate the usefulness of our Theorems and of Proposition 3. First we return to

EXAMPLE 1. We have intentionally omitted, up to this point, a rigorous proof of the intuitive fact that all functions in $\mathcal{C}([a,b])$ are $\mathfrak{T}$-affine. This can now be easily shown. It suffices to observe that for each point $x_0\in[a,b]$ the function

$$x\mapsto(x-x_0)^2$$

lies in the test space $\operatorname{lin}\mathfrak{T}$. It is non-negative and its only zero is x_0.

Theorem 2, together with Proposition 3, therefore leads to a new proof of Korovkin's first theorem which is totally different from the one given in the Introduction. In this connection, the reader is again referred to the Remark in Section 1.

EXAMPLE 3. Let now X be the circle

$$\mathbf{T}=\{z\in\mathbf{C}\,|\,|z|=1\}$$

in the complex plane $\mathbf{C}$. By means of the map $x\mapsto e^{ix}$ we can identify $\mathbf{T}$ topologically with the interval $[0,2\pi]$ after having identified the endpoints 0 and 2π. As test set we choose

$$\mathfrak{T}=\{1,\mathrm{Re},\mathrm{Im}\},$$

where Re [resp., Im] denotes the real [resp., imaginary] part of the identity mapping, i.e., $z\mapsto\mathrm{Re}\,z$ [resp., $z\mapsto\mathrm{Im}\,z$]. After the above identification, this amounts to considering $\{1,\cos,\sin\}$ as test set.

The test space $\operatorname{lin}\mathfrak{T}$ then consists of all functions

$$x\mapsto\alpha+\beta\sin(x+\gamma)$$

with real coefficients α, β, γ. For $x_0\in\mathbf{T}$

$$h(x)=1+\sin\left(x+\frac{3}{2}\pi-x_0\right)$$

is a function $\geqslant 0$ in $\operatorname{lin}\mathfrak{T}$ with x_0 as its only zero in $\mathbf{T}$. So we arrive at another classical result of Korovkin as a consequence of Theorem 2:

Korovkin's Second Theorem. *For the circle* $\mathbf{T}$ *and the test set* $\mathfrak{T}=\{1,\mathrm{Re},\mathrm{Im}\}$ *every function in* $\mathcal{C}(\mathbf{T})$ *is a Korovkin function.*

This has the following nice application. Associate to an arbitrary function $f\in\mathcal{C}(\mathbf{T},\mathbf{C})$, i.e., to a complex-valued continuous function f on $\mathbf{T}$, its formal Fourier series

$$\sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

with the coefficients

$$c_n=\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt \qquad (n=0,\pm 1,\dots).$$

Define for every natural number n an operator $L_n:\mathcal{C}(\mathbf{T},\mathbf{C})\to\mathcal{C}(\mathbf{T},\mathbf{C})$ as follows:

$$L_nf=\frac{s_0+\cdots+s_{n-1}}{n} \qquad (f\in\mathcal{C}(\mathbf{T},\mathbf{C}))$$

where s_n is the nth partial sum

$$s_n(x)=\sum_{\nu=-n}^{+n} c_\nu e^{i\nu x} \qquad (x\in[0,2\pi])$$

of the Fourier series. Then L_n obviously is linear. According to the classical representation formula of Fejér (c.f. [**13**, p. 292]),

$$L_nf(x)=\frac{1}{2\pi n}\int_0^{2\pi} f(x+t)\left(\frac{\sin\frac{n}{2}t}{\sin\frac{t}{2}}\right)^2 dt$$

for $x\in[0,2\pi]$ and $f\in\mathcal{C}(\mathbf{T},\mathbf{C})$. Thus non-negative functions are mapped into non-negative functions, so L_n is a positive operator. After restricting L_n to $\mathcal{C}(\mathbf{T})$ we end up with a sequence of positive linear operators of $\mathcal{C}(\mathbf{T})$ into itself. For the function $f_0=1$ all Fourier coefficients are zero except for $c_0=1$. For $f_1(x)=e^{ix}$ the coefficient $c_1=1$ is the only one different from zero. We thus obtain

$$L_nf_0=f_0$$

$$L_nf_1=\frac{n-1}{n}f_1$$

and hence after decomposition of e^{ix} into real and imaginary parts

$$L_n\cos=\frac{n-1}{n}\cos,\qquad L_n\sin=\frac{n-1}{n}\sin$$

for all n. Consequently, the sequence (L_n) is $\mathfrak{T}$-admissible and Korovkin's Second Theorem yields uniform convergence of (L_nf) to f on $\mathbf{T}$ for all $f\in\mathcal{C}(\mathbf{T})$ and then even for all $f\in\mathcal{C}(\mathbf{T},\mathbf{C})$.

This is the classical theorem about the uniform Cesàro summability of Fourier series of continuous functions. From here one can easily go on to obtain Fejér's theorem about L^1-Cesàro summability of Fourier series of L^1-functions.

Example 4. Now let X be a compact subset of $\mathbf{R}^k$. We assume that for each point $x\in X$ there is at least one supporting hyperplane which intersects X just in x, that is, there exists a linear form l on $\mathbf{R}^k$ satisfying

$$l(x)<l(x') \qquad \text{for all}\quad x'\in X, x'\neq x. \tag{16}$$

As a test set we choose

$$\mathfrak{T}=\{1,p_1,\dots,p_k\}$$

where $p_j:X\to\mathbf{R}$ is the jth coordinate function (restricted to X), which assigns to each $x\in X$ its jth coordinate x_j $(j=1,\dots,k)$.

Every linear form on $\mathbf{R}^k$ is a linear combination of the k coordinate functions. Therefore,

condition (16) just states that every point $x \in X$ is the only zero of a non-negative function of the test space lin $\mathfrak{T}$. So by Proposition 3, we can again apply Theorem 2 in order to obtain: *All functions in $\mathcal{C}(X)$ are Korovkin functions.*

This result contains Korovkin's Second Theorem as a special case, since through every point of the circle $\mathbf{T}$ in $\mathbf{C}$ $(=\mathbf{R}^2)$ there is exactly one supporting line, the tangent.

EXAMPLE 5. Let X be an arbitrary metric space and let $\mathbf{F} \subset \mathbf{C}(X)$ be a set of real-valued continuous functions separating points (cf. the end of Section 1). If one takes as test set

$$\mathbf{T} = \{1\} \cup \mathbf{F} \cup \{f^2 | f \in \mathbf{F}\},$$

the set of powers f^j, $j = 0, 1, 2$, of functions f in $\mathbf{F}$, then *all functions in $\mathbf{C}(X)$ are Korovkin functions.*

Indeed, consider an arbitrary test measure μ for $x \in X$. The function $(f - f(x))^2$ is for each $f \in \mathbf{F}$ in the test space; furthermore

$$\mu((f - f(x))^2) = 0.$$

By using a well-known measure theoretic argument (cf. [**6**, p. 225 and Theorem 2.5.2]) we obtain for the support S_μ of μ

$$S_\mu \subset \bigcap_{f \in \mathbf{F}} \{x' \in X | f(x') = f(x)\}.$$

Point separation then yields

$$S_\mu \subset \{x\}$$

and finally $\mu = \varepsilon_x$ because of $\mu(1) = 1$. (A direct but longer proof can be given along the elementary ideas used in the proof of Proposition 3.)

In particular if X is a compact subset of $\mathbf{R}^k$, $k = 1, 2, \ldots$, we can choose $\mathbf{F}$ to be the set $\{p_1, \ldots, p_k\}$ where $p_1, \ldots, p_k$ are again the coordinate functions. So $\mathbf{T}$ equals

$$\{1, p_1, \ldots, p_k, p_1^2, \ldots, p_k^2\}.$$

We can even reduce the number of test functions by considering

$$\mathbf{T} = \{1, p_1, \ldots, p_k, p_1^2 + \cdots + p_k^2\}$$

as a test set. In both cases it suffices to apply Proposition 3:

$$h = \sum_{j=1}^{k} (p_j - p_j(x))^2$$

is a function $\geqslant 0$ from $\operatorname{lin} \mathbf{T}$ with x as the only zero.

Korovkin's First Theorem is a special case of this result: For $X = [a, b] \subset \mathbf{R}$ we have $p_1 = \mathrm{id}$ and are thus led to the test set $\{1, \mathrm{id}, \mathrm{id}^2\}$.

4. Geometric interpretation. We will now discuss the most important example of a Choquet boundary as well as its influence on the kind of problems we are treating.

Consider a *compact convex* set C in $\mathbf{R}^k$. Then C contains with each two of its points the segment joining them. As in Example 4, we introduce

$$\mathfrak{T}_C = \{1, p_1, \ldots, p_k\}$$

as test set. Then the test space lin $\mathfrak{T}_C$ is the space $\mathbf{A}(C)$ of all *affine* functions on $\mathbf{R}^k$ restricted to C, so all functions of the form

$$x = (x_1, \ldots, x_k) \mapsto \alpha_0 + \sum_{j=1}^{k} \alpha_j x_j$$

with real coefficients.

Let a and b be points of C and let x be a point of the segment joining a and b. Then x can be

written in the form

$$x=\lambda a+(1-\lambda)b$$

where λ is a real number in the unit interval. Thus

$$\mu=\lambda\varepsilon_a+(1-\lambda)\varepsilon_b$$

is a positive Radon measure on C satisfying $\mu(t)=t(x)$ for all test functions $t\in\mathfrak{T}_C$, so it is a test measure for x. It equals ε_x if and only if $x=a$ or $x=b$. Therefore, x is not a point of the Choquet boundary $\partial_{\mathfrak{T}_C}C$ if $a\neq b$ and $0<\lambda<1$; that is, if x is a point of a segment in C but different from the two end points. Points of C which can only be end points of segments in C on which they are situated are called *extreme points* of C. These are exactly those points $x\in C$ for which $C\setminus\{x\}$ is convex. If we denote by $\operatorname{ex} C$ the set of all extreme points of C, we have just proved that $\partial_{\mathfrak{T}_C}C$ is a subset of $\operatorname{ex} C$. However, one even has equality:

PROPOSITION 4. *For every compact convex set $C\subset\mathbf{R}^k$ and the test set $\mathfrak{T}_C=\{1,p_1,\ldots,p_k\}$ one has*

$$\partial_{\mathfrak{T}_C}C = \operatorname{ex} C. \tag{17}$$

We shall not give a proof of this result here, but rather refer the reader to Choquet [**9**, §29] or Phelps [**19**, §6] where Proposition 4 appears in much greater generality. An elementary introduction to the study of $\operatorname{ex} C$ can be found in Jacobs [**14**].

Let us now return to the general situation of a compact metric space X and a test set $\mathfrak{T}\subset\mathcal{C}(X)$. In addition to condition (4), according to which the constant function 1 belongs to $\mathfrak{T}$, we now demand that $\mathfrak{T}$ is a finite set:

$$\mathfrak{T} = \{1,t_1,\ldots,t_k\}, \tag{18}$$

and that $\mathfrak{T}$ separates points. (The case $\mathfrak{T}=\{1\}$ is uninteresting since $\partial_{\mathfrak{T}}X=X$ can hold only if $\mathfrak{T}$ is point separating, that is if X is a singleton.) The test space $\operatorname{lin}\mathfrak{T}$ then is of finite dimension. Finiteness of $\mathfrak{T}$ allows the definition of a continuous map $\Phi:X\to\mathbf{R}^k$ by means of

$$\Phi(x) = (t_1(x),\ldots,t_k(x)). \tag{19}$$

Consequently, $\Phi(X)$ is compact as a continuous image of X. Moreover, Φ is injective since $\mathfrak{T}$ separates the points of X. So, by a well-known theorem, Φ is a homeomorphism from X onto the image $\Phi(X)$. Hence we have embedded X homeomorphically in the euclidean k-space $\mathbf{R}^k$.

The relation between our new situation and the one considered in Proposition 4 becomes more evident by introducing the convex hull C of the compact set $\Phi(X)$ in $\mathbf{R}^k$. This convex hull is also compact (cf. [**24**, Theorem 3.10, p. 40]), so we arrive at a compact convex set in $\mathbf{R}^k$ which is associated with $(X,\mathfrak{T})$ in a very natural way. In terms of this set C, a new and more lucid interpretation can be given to the Choquet boundary $\partial_{\mathfrak{T}}X$.

PROPOSITION 5. *For every test set $\mathfrak{T}=\{1,t_1,\ldots,t_k\}$ separating the points of X, for the corresponding map $\Phi:X\to\mathbf{R}^k$ and the convex hull C of $\Phi(X)$ one has*

$$\Phi(\partial_{\mathfrak{T}}X) = \operatorname{ex} C. \tag{20}$$

In other words: A point $x\in X$ belongs to the Choquet boundary $\partial_{\mathfrak{T}}X$ if and only if its image $\Phi(x)$ is an extreme point of C.

We restrict ourselves to a sketch of the proof: By means of Φ we can shift the test functions $t_0=1,t_1,\ldots,t_k$ to $\Phi(X)$: the function $u_j=t_j\circ\Phi^{-1}$ on $\Phi(X)$ corresponds to t_j. Then u_0 is the constant function 1 and u_j is the jth coordinate function on $\Phi(X)$, $j=1,\ldots,k$, since for every point $y\in\Phi(X)$ and its pre-image $x=\Phi^{-1}(y)$ we have

$$u_j(y) = t_j(x) = p_j(\Phi(x))$$

because of (19). Thus the test set $\mathfrak{T}$ on X is transferred into the test set $\mathfrak{T}_C$ of Proposition 4 restricted to $\Phi(X)$. The homeomorphism $\Phi:X\to\Phi(X)$ also transforms, in a natural way, every test measure for

x (with respect to $\mathfrak{T}$) into a test measure for $\Phi(X)$ (with respect to $\mathfrak{T}_C$), which, however, is supported by $\Phi(X)$. Now Proposition 4 can be strengthened in such a way that $\operatorname{ex} C$ is the set of all points $y \in C$ for which ε_y is the only test measure for y (with respect to $\mathfrak{T}_C$) which is supported by $\Phi(X)$. Then Proposition 5 follows. Details can be found in [2] and [9].

The usefulness of Proposition 5 for our problems lies in the following consequence:

COROLLARY. *In the situation of Proposition 5 all functions in $\mathcal{C}(X)$ are Korovkin functions with respect to $\mathfrak{T}$ if and only if*

$$\Phi(X) = \operatorname{ex} C. \tag{21}$$

It suffices to observe that $\partial_{\mathfrak{T}} X = X$ is equivalent to $\Phi(\partial_{\mathfrak{T}} X) = \Phi(X)$, and hence to $\Phi(X) = \operatorname{ex} C$ because of (20). The result then follows from Theorem 2.

The equality (21) will enable us to decide in concrete cases by means of geometric arguments whether for a given test set all continuous functions are Korovkin functions.

EXAMPLE 6. Let X be a compact interval $[a,b]$ in $\mathbf{R}$ and let $\mathfrak{T} = \{1, t\}$ be a test set with 2 elements. If every function in $\mathcal{C}([a,b])$ is a Korovkin function $\mathfrak{T}$, the function t must separate points. The corresponding map $\Phi: X \to \mathbf{R}$ equals t. As a continuous image of a compact interval $\Phi(X) = t(X)$ is itself a compact interval $[c,d]$ in $\mathbf{R}$, hence convex and equal to its convex hull C. The extreme points of $[c,d]$ are the end points c and d. Consequently, $\Phi(X) = \operatorname{ex} C$ holds if and only if $\Phi(X)$ and hence X consists of just one point.

We have thus proved that on a compact interval $[a,b]$ which does not reduce to one point there is no test set $T = \{1, t\}$ of just 2 elements such that all functions in $\mathcal{C}(X)$ are Korovkin functions. This shows that the number of 3 test functions in Korovkin's First Theorem is *minimal*. It is this result which is called *Korovkin's Third Theorem*.

EXAMPLE 7. Again let X be a compact interval $[a,b]$ in $\mathbf{R}$ such that $a < b$. Consider a test set

$$T = \{1, \mathrm{id}, u\}$$

which is derived from the test set in Korovkin's First Theorem by replacing the function id^2 by a function $u \in \mathcal{C}([a,b])$ chosen arbitrarily for the moment. The set $\mathfrak{T}$ is point separating since the subset $\{\mathrm{id}\}$ already separates points, so we can use the map Φ from (19). We obtain

$$\Phi(X) = \{(x, u(x)) \mid x \in [a,b]\}$$

which is the graph of the function u. By passing from $\Phi(X)$ to its convex hull C, we arrive at a situation sketched in Figure 2.

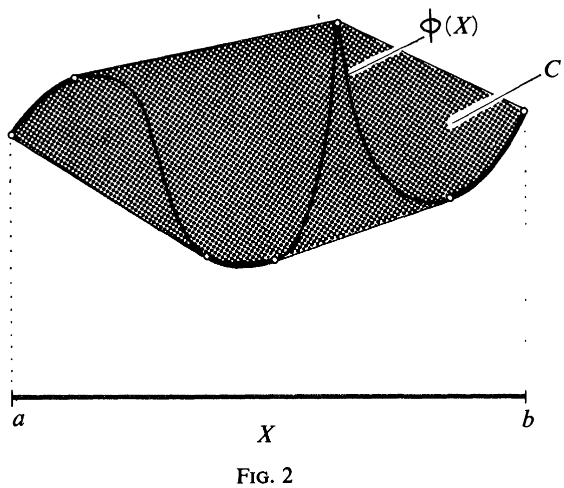

FIG. 2

Only the parts of the graph in boldfaced print consist of extreme points of C.

Now it is intuitively clear when the graph $\Phi(X)$ of u coincides with the set $\operatorname{ex} C$ of all extreme points of C. This is the case if and only if u is strictly concave or strictly convex. As usual $u\in\mathcal{C}([a,b])$ is called *strictly concave* if

$$\lambda u(x)+(1-\lambda)u(y)<u(\lambda x+(1-\lambda)y)$$

holds for all points x,y satisfying $a\leqslant x<y\leqslant b$ and all λ satisfying $0<\lambda<1$. We call u *strictly convex* if $-u$ is strictly concave. We thus are led to

PROPOSITION 6. *Let u be a continuous real-valued function on a compact interval $[a,b]$ in $\mathbf{R}$ such that $a<b$. All functions in $\mathcal{C}([a,b])$ are Korovkin functions with respect to the set $\mathfrak{T}=\{1,\mathrm{id},u\}$ if and only if u is strictly concave or strictly convex.*

We leave the (not very difficult) proof to the reader and content ourselves with the geometric illustration.

This result clarifies the role of the function id^2 appearing in Korovkin's First Theorem. It is strictly convex. By Proposition 6 it could be replaced by any other strictly convex function, for example, by the exponential function $x\mapsto e^x$. The proof for Korovkin's First Theorem given in the Introduction of course breaks down if we make this replacement.

EXAMPLE 8. Consider a compact convex set C_0 in $\mathbf{R}^k$ with a closed and hence compact set $X=\operatorname{ex} C_0$ of extreme points. According to (19) the map $\Phi: X\to\mathbf{R}^k$ is the identity map if we take

$$\mathfrak{T}=\{1,p_1,\ldots,p_k\}$$

as test set. Here p_j again denotes the jth coordinate function restricted to X. The convex hull of $X=\Phi(X)$ (up to now denoted by C) coincides with C_0. This follows from a classical theorem of Minkowski or from the Krein-Milman theorem (cf. **[9]**, **[13]**, **[19]**, **[24]**) according to which C_0 is the convex hull of the set of its extreme points. So we have $\Phi(X)=X=\operatorname{ex} C$. The Corollary following Proposition 4 then yields: *All functions in $\mathcal{C}(X)$ are Korovkin functions.*

This example contains Example 4 as a special case. If there exists a supporting hyperplane through a point $x\in X$, that is, if there exists a linear form l on $\mathbf{R}^k$ such that

$$l(x)\leqslant l(x') \qquad \text{for all} \quad x'\in X,$$

then X and consequently the convex hull C of X lie in the convex half space of all $x'\in X$ satisfying $l(x')\geqslant\alpha$ where $\alpha=l(x)$. The hyperplane

$$H=\{x'\in\mathbf{R}^k\,|\,l(x')=\alpha\}$$

therefore is a supporting hyperplane for C. Such a hyperplane always contains at least one extreme point since—as we have mentioned in Section 2 in connection with the introduction of the Choquet boundary—l attains its minimum α on C at a point of $\partial_{\mathfrak{T}_C}C=\operatorname{ex} C$. But in Example 4 H hits X only in the point x; the same is true if we replace X by C. Therefore, x must be an extreme point of C, and we have $X\subset\operatorname{ex} C$. We even have equality, since every point $c\in C$ is of the form $c=\sum_{j=1}^r\lambda_j x_j$ for distinct points $x_1,\ldots,x_r\in X$ and coefficients $\lambda_1\geqslant 0,\ldots,\lambda_r\geqslant 0$ satisfying $\sum_{j=1}^r\lambda_j=1$. A simple calculation shows that c can be an extreme point of C only if $\lambda_j=0$ for all $j\in\{1,\ldots,r\}$ except one.

EXAMPLE 9. Our last example intends to show that Proposition 5 also leads in more complicated situations to a simple determination of the Choquet boundary. Consider the interval $X=[-1,+1]$ in $\mathbf{R}$ and let $T=\{1,\mathrm{id},\mathrm{id}^3\}$ be the test set. From Example 7 we know that $\Phi(X)$ is the graph of the function $x\mapsto x^3$, the well-known cubic parabola. Figure 2 has to be replaced in this case by Figure 3. The extreme points of $C=\operatorname{conv}\Phi(X)$ are all points $(x,x^3)\in\mathbf{R}^2$ where $x\in[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]$. According to Proposition 5, $[-1,-\frac{1}{2}]\cup[\frac{1}{2},1]$ then is the Choquet boundary of $[-1,+1]$ with respect to $\mathfrak{T}$. A purely analytic proof based on Proposition 2 or even on formula (12) is much more complicated.

5. Bibliographical comments. Korovkin's famous three theorems first appeared in **[15]** and some-

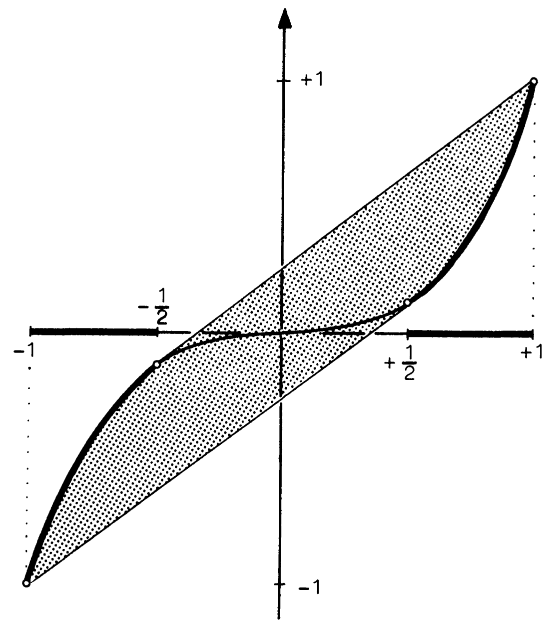

FIG. 3

what later in the book **[16]**. The tools of the present paper—in particular, the introduction of the space $\mathfrak{T}^*$ of $\mathfrak{T}$-affine functions (originally called $\mathfrak{T}$-harmonic functions) as well as properties of the Choquet boundary, including the geometric interpretation—appeared in the author's paper **[2]** in connection with potential-theoretic problems. Example 1 is already discussed there, however without reference to approximation theory. Šaškin **[20]** was the first to study conditions on a finite set of test functions on a compact metric space X which imply that all functions in $\mathcal{C}(X)$ are Korovkin functions. The connection with the Choquet boundary arises implicitly there. Explicitly, this connection appears in Wulbert **[27]** where—similar to Šaškin **[21]**—sequences (L_n) of (not necessarily positive) contraction maps $L_n : \mathcal{C}(X) \to \mathcal{C}(X)$ are also considered. The relevance of the space $\mathfrak{T}^*$ of $\mathfrak{T}$-affine functions was recognized by Šaškin **[22]** and Baskakov **[1]** though only in the form of an analogy for pointwise convergence. The methods of the present article have been developed in the author's paper **[3]** and independently at almost the same time by Berens–Lorentz **[7]**. In particular, Theorem 1 can be found there. Another proof of it appears in Šaškin **[23]**. Most of the examples of this article are part of the folklore in connection with Korovkin's theorems.

Further investigations on this subject can be found in **[3]**, **[4]**, **[7]**, as well as in the work of Donner **[10]**, **[11]**, Grossman **[12]**, Kutateladze **[17]**, Leha **[18]** and Wolff **[25]**, **[26]**.

The present article is essentially a translation of the author's paper **[5]**. The author is grateful to R. Jamison for assistance with the translation.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT ERLANGEN-NÜRNBERG, D-8520 ERLANGEN, WEST GERMANY.

ON THE DIVISION OF THE PLANE BY LINES

JOHN E. WETZEL

1. It occasionally happens in mathematics that the "real" reasons for the correctness of a result are quite unrelated to the arguments that comprise its proof. The reasoning used in proofs is severely restricted: the arguments must be carefully organized, logically correct, and complete. Every possibility has to be examined. The intuition is freer: intuitive insights may be fragmented, incomplete; rigor and logic are irrelevant; and even incorrect insights may sometimes be useful. While a good understanding of the proof of a result usually contributes to the full comprehension of the result, good intuitive insights may contribute even more. Indeed, appropriate intuitions can make an otherwise mysterious result seem virtually obvious.

One proof of a result may be preferred to another for many different reasons, some of them quite

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subjective. The preferred proof may be shorter, more surprising, more elegant; more (or less) elementary, more (or less) computational, more (or less) abstract. It may require less (or more) prior knowledge. It may be special, in the sense that it works only for the problem at hand, or it might be an instance of a more general method. Another, less obvious reason for preferring one proof over another lies in its accessibility to the intuition. The nearer the proof is, in some sense, to the intuition, the easier it may be to grasp.

In this note we illustrate this situation by giving three quite different elementary proofs of a little-known formula for the number of regions formed by an arbitrary arrangement of lines in the plane, a formula that was given in 1889 by Samuel Roberts **[15]** and rediscovered in 1963 by Albert Blank **[5]**. In section 2 we use some pretty heuristic arguments originally advanced by Roberts to find the formula. In the sections that follow we give first an unenlightening proof by mathematical induction, then an easy but *ad hoc* argument for counting the regions directly, and finally an intuitively appealing, elegant argument that is based on the notion of a sweep-line.

2. Suppose first of all that we have n lines in "general position," that is to say, n lines so arranged that each two meet in a point and no three pass through the same point. J. Steiner **[16]** proved in 1826 that n lines in general position divide the plane into

$$R=1+n+\binom{n}{2} \tag{1}$$

regions, of which

$$R'=1-n+\binom{n}{2}=\binom{n-1}{2} \tag{2}$$

are bounded. These formulas are easily proved by recursion, and we take them as known. They are discussed from the heuristics point of view by G. Pólya in **[14**, pp. 43–52 and problems 11, 15, 16, p. 54 (solutions, pp. 223, 224)]. A proof without frills appears in Golovina and Yaglom **[8**, p. 83].

Lines in the plane can fail to be in general position in two different ways: there may be more than two lines through a point, and there may be parallels. Both kinds of degeneracies reduce the number of points of intersection, because the intersection points coincide at multiple points, and parallel lines do not intersect at all. It is plain that regions and points of intersection are somehow closely related to each other—a point is produced when lines come together to pinch off a region—but the precise relationship is elusive.

Let us look more closely at each of these two kinds of degeneracies. First consider a multiple point M of multiplicity λ. We imagine the λ lines through M displaced a little (Fig. 1) to make an arrangement of λ lines in general position. According to (2), these λ lines in general position form

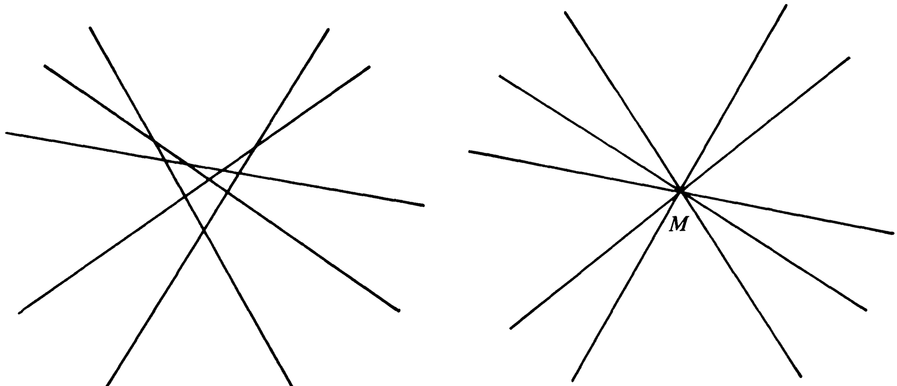

FIG. 1. The generation of a multiple point

$\binom{\lambda-1}{2}$ bounded regions, and all are lost when the lines are brought again to concurrency. So the concurrency of λ lines at M clearly causes the loss of $\binom{\lambda-1}{2}$ regions.

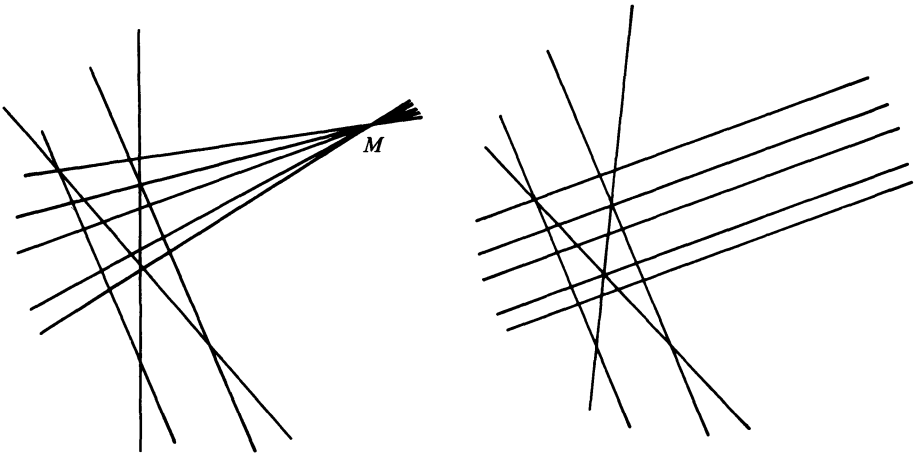

FIG. 2. The generation of a family of parallels

Now consider a family of μ parallel lines in a direction d. We imagine the lines displaced a little to pass through a common point M far away (Fig. 2), and we rebuild the parallels by letting M tend to infinity in the direction d. Then $\binom{\mu-1}{2}$ regions are lost at the point M, and $\mu-1$ further regions are lost beyond M. So the total loss due to the μ parallels is $\binom{\mu-1}{2}+(\mu-1)=\binom{\mu}{2}$.

If there are m multiple points $M_1, M_2, \ldots, M_m$ in the given arrangement of n lines with, say, $\lambda_i \geqslant 3$ lines passing through M_i, then the total loss of regions ought to be

$$\binom{\lambda_1-1}{2}+\binom{\lambda_2-1}{2}+\cdots+\binom{\lambda_m-1}{2}$$

regions; and if there are p parallel families with, say, $\mu_j \geqslant 2$ lines in the jth family, the total loss of regions due to the parallels ought to be

$$\binom{\mu_1}{2}+\binom{\mu_2}{2}+\cdots+\binom{\mu_p}{2}$$

regions. Hence one must have

$$R=1+n+\binom{n}{2}-\sum_{i=1}^{m}\binom{\lambda_i-1}{2}-\sum_{j=1}^{p}\binom{\mu_j}{2}. \tag{3}$$

This is Roberts' formula. It gives R as "the number of regions formed by n lines in general position" minus "the number of regions lost because of the multiple points" minus "the number of regions lost because of the parallels."

Roberts' heuristic arguments for (3) are convincing, not to say compelling. They make the formula seem almost obvious. Figure 3 pictures an arrangement of $n=12$ lines that meet to form four multiple points of multiplicities 3, 3, 3, and 4 and lie in parallel families containing 2, 3, and 5 lines. Both the formula and a visual count give $R=59$, so the formula is correct for this fairly complicated arrangement. Surely it must be correct!

And yet, these arguments do not immediately yield a correct proof. The difficulty is that moving one line to destroy a multiple point or a parallel family may significantly alter the arrangement in

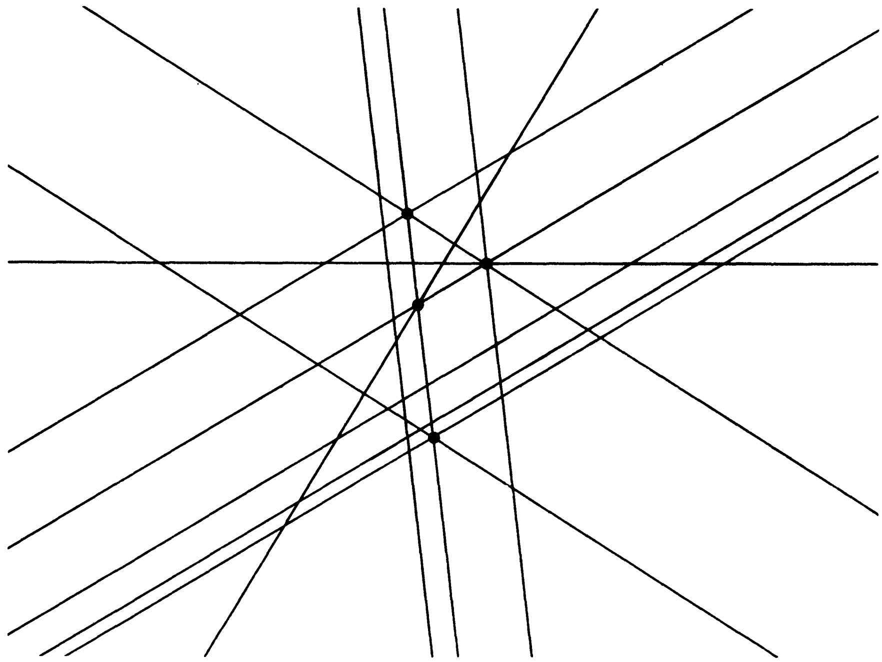

FIG. 3. An arrangement of 12 lines that divides the plane into 59 regions

some other way. Consider, for example, a line that passes through three collinear multiple points and belongs to a parallel family. Moving that line so as to destroy one of the multiple points necessarily changes at least one other multiple point, and new regions are produced if the line is turned. You cannot wiggle a part of a line without moving the whole line, and the prospect of keeping track of all the changes that might occur appears to be an overwhelming notational chore.

Nevertheless, the heuristic arguments are very persuasive. The formula surely cannot be false, and indeed it is not.

3. As our first proof of the formula we give a concise and correct argument by mathematical induction. To carry this proof out we need to introduce some notation.

Let α be an arrangement of n lines. For each point P of the plane, write $\lambda(P)$ for the number of lines of α that pass through P. We call P a *simple* point of α if $\lambda(P)=2$ and a *multiple* point of α if $\lambda(P)\geqslant 3$. For each direction d, write $\mu(d)$ for the number of lines of α that lie in the direction d. We call d a *multiple* direction if $\mu(d)\geqslant 2$.

If we adopt the usual combinatorial convention that $\binom{a}{b}=0$ when $b>a$, we can write

$$\sum_{i=1}^{m}\binom{\lambda_i-1}{2}=\sum_{P}\binom{\lambda(P)-1}{2}$$

and

$$\sum_{j=1}^{p}\binom{\mu_j}{2}=\sum_{d}\binom{\mu(d)}{2},$$

because $\binom{\lambda(P)-1}{2}=0$ unless P is a multiple point, and $\binom{\mu(d)}{2}=0$ unless d is a multiple direction. In this notation, Roberts' formula (3) becomes

$$R=1+n+\binom{n}{2}-\sum_{P}\binom{\lambda(P)-1}{2}-\sum_{d}\binom{\mu(d)}{2}. \tag{4}$$

Formula (4) can easily be verified for $n=0$ and 1 and also for the case in which all n lines are parallel. Suppose that (4) holds for every arrangement of n lines. Take any arrangement α' of $n+1$ lines not all of which are parallel, and denote its counters by λ' and μ'. Select a line L in α' and delete it, leaving an arrangement α of n lines for which, by the induction hypothesis, formula (4) holds. We restore L to rebuild α' from α and watch closely what happens.

If the chosen line L lies in the direction d_0, then the counters for α' are related to those of α by the formulas

$$\lambda'(P)=\begin{cases}\lambda(P) & \text{if } P\notin L\\ \lambda(P)+1 & \text{if } P\in L,\end{cases}$$

$$\mu'(d)=\begin{cases}\mu(d) & \text{if } d\neq d_0\\ \mu(d_0)+1 & \text{if } d=d_0.\end{cases}$$

A moment's thought shows that when L is restored, it meets the lines of α in exactly

$$N=n-\mu(d_0)-\sum_{P\in L}\binom{\lambda(P)-1}{1}\geqslant 1$$

points, for L does not cross $\mu(d_0)$ of the n lines of α, and a point of concurrency of $\lambda(P)$ lines adds only one point of intersection. These N points partition L into two rays and $N-1$ line segments, and each of these $N+1$ pieces divides a region of α into two parts, producing exactly $N+1$ new regions. Consequently there must be exactly

$$\Delta R=n+1-\sum_{P\in L}\binom{\lambda(P)-1}{1}-\mu(d_0) \tag{5}$$

new regions created when L is restored.

The two sums over points in (4) and (5) combine to give

$$\begin{aligned}\sum_{P\in L}\left[\binom{\lambda(P)-1}{2}+\binom{\lambda(P)-1}{1}\right]+\sum_{P\notin L}\binom{\lambda(P)-1}{2}\\ =\sum_{P\in L}\binom{\lambda(P)}{2}+\sum_{P\notin L}\binom{\lambda(P)-1}{2}\\ =\sum_{P}\binom{\lambda'(P)-1}{2}.\end{aligned}$$

Similarly, the two sums over directions give

$$\begin{aligned}\left[\binom{\mu(d_0)}{2}+\binom{\mu(d_0)}{1}\right]+\sum_{d\neq d_0}\binom{\mu(d)}{2}\\ =\binom{\mu(d_0)+1}{2}+\sum_{d\neq d_0}\binom{\mu(d)}{2}\\ =\sum_{d}\binom{\mu'(d)}{2}.\end{aligned}$$

It follows that α' defines exactly

$$\begin{aligned}R'&=R+\Delta R\\ &=1+(n+1)+\binom{n+1}{2}-\sum_{P}\binom{\lambda'(P)-1}{2}-\sum_{d}\binom{\mu'(d)}{2},\end{aligned}$$

which is Roberts' formula for α'. This completes the proof by induction.

This argument is correct, complete, logical, and not very enlightening. The key observation, that

the *new* regions are in one-one correspondence with the segments and rays formed on the new line by the points in which it meets the lines already in place, is buried in the middle. The mechanics of the transition are difficult to see through; the formula appears to come from a fortuitous combination of binomial coefficients that could hardly have been foreseen. Indeed, this proof contributes little more to our understanding of the formula than the fact, important though it is, that it is correct.

4. The intuitive content of Roberts' formula is rich and varied, and different insights can be found that lead to correct and elegant proofs. We develop one such argument next.

Our first observation is that translating a line of the arrangement does not alter the structure of the parallel families. What does it do to the region count?

If the line L to be translated initially does not pass through any multiple points, and if in its final resting position it does not pass through any multiple points, we claim that the region count is unaltered by the translation. Indeed, it is clear that the region count could change during the transition only when L slides through a point of intersection; and at each such point P precisely as many new regions are created when L leaves P as were lost when L came to P, viz., $\lambda(P)-1$.

Now we ask what happens when we tear down a multiple point by translating its lines away one by one. If λ lines pass through the multiple point M, the first line to go creates $\lambda-2$ new regions having M as a boundary point; then the next line to leave produces $\lambda-3$ more new regions; and so forth, until the point is reduced to a simple point. In the process, exactly

$$(\lambda-2)+(\lambda-3)+\cdots+2+1=\binom{\lambda-1}{2}$$

new regions are produced from the multiple point M. Note that if all of the multiple points of the arrangement are destroyed in this way, the fact that some of them may be collinear is immaterial.

Given an arbitrary arrangement α of n lines having m multiple points $M_1, M_2, \ldots, M_m$ of multiplicity $\lambda_1, \lambda_2, \ldots, \lambda_m$ and forming R regions, tear down all of its multiple points by making small translations of its lines. This procedure evidently produces a new arrangement α' having no multiple points and having

$$R'=R+\sum_{i=1}^{m}\binom{\lambda_i-1}{2} \tag{6}$$

regions.

Suppose the lines of the original arrangement α (and therefore those of the new arrangement α') fall into s parallel families of $n_1, n_2, \ldots, n_s$ lines, respectively, where to accommodate lines having no parallel partners we allow n_i to be 1. If there are p multiple directions (as in the notation of section 2), then the n_k sum to n, each μ_j is an n_k, and there are $s-p \geqslant 0$ lines that have no parallel partners. Unless the lines are all parallel, in which case Roberts' formula is trivially true, we select one line from each parallel family to obtain an arrangement of s lines in general position. By translating the other lines to new positions near their chosen partners (being careful not to introduce multiple points), we can arrange the lines in s "narrow" parallel families without changing the region count; and indeed we can make the families so narrow (Fig. 4) that no points of intersection come between any two parallels. For lines so arranged, we can count the regions by inspection.

First of all, there are clearly $(n_i-1)(n_j-1)$ little parallelograms formed where the n_i parallel lines in the ith family cross the n_j parallel lines in the jth family. (These regions are heavily shaded in Figure 4.) Summing, we find there are

$$\sum_{i<j}(n_i-1)(n_j-1)=\sigma_2-(s-1)n+\binom{s}{2}$$

regions of this kind, where σ_2 is the sum of the $\binom{s}{2}$ products $n_i n_j$ with $1 \leqslant i<j \leqslant s$.

There are evidently $s(n_i-1)$ further regions formed between the n_i lines in the ith family, because there are $s-1$ other intersecting families. So in all there are

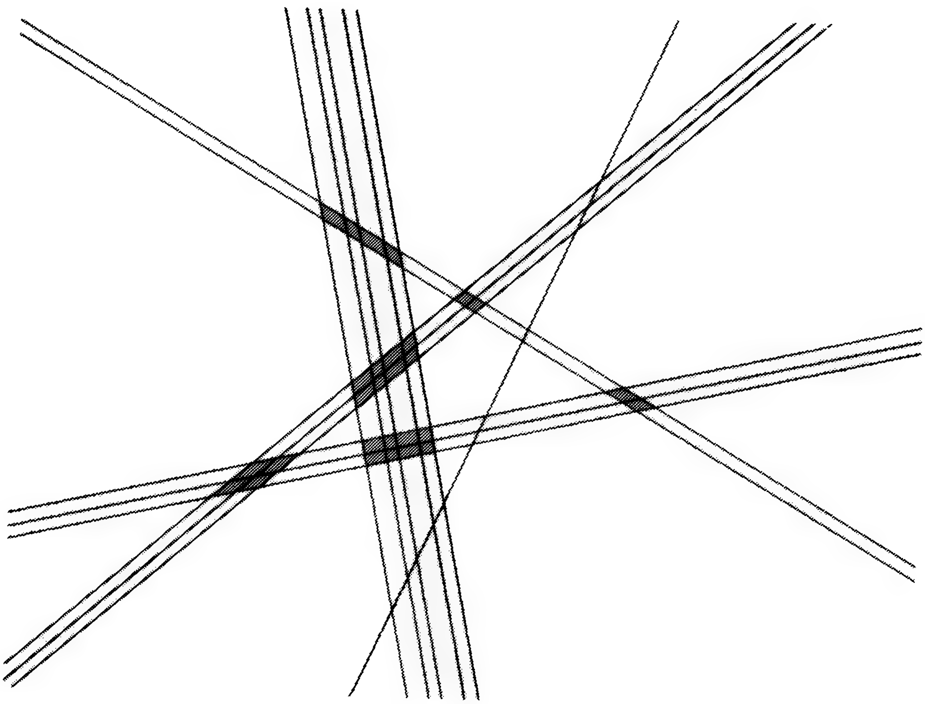

FIG. 4. An arrangement of 14 lines in 5 narrow parallel families

$$\sum_{i=1}^{s} s(n_i-1)=sn-s^2$$

regions of this sort (lightly shaded in Figure 4).

The same number of regions remain as are formed by s lines in general position, viz., $1+s+\binom{s}{2}$. So in all there are

$$\begin{aligned} R' &= 1+s+\binom{s}{2}+sn-s^2+\sigma_2-(s-1)n+\binom{s}{2} \\ &= 1+n+\sigma_2 \end{aligned} \tag{7}$$

regions in the arrangement α'. This pretty formula was proved (by recursion) by J. Steiner in 1826.

Using the easy identity

$$\sigma_2=\binom{n_1+n_2+\cdots+n_s}{2}-\sum_{t=1}^{s}\binom{n_t}{2},$$

which can be verified by simply expanding out, we can rewrite (7) in Roberts' form:

$$R'=1+n+\binom{n}{2}-\sum_{t=1}^{s}\binom{n_t}{2}=1+n+\binom{n}{2}-\sum_{j=1}^{p}\binom{\mu_j}{2}.$$

Roberts' formula for the original arrangement α now follows immediately from (6):

$$\begin{aligned} R &= R'-\sum_{i=1}^{m}\binom{\lambda_i-1}{2} \\ &= 1+n+\binom{n}{2}-\sum_{i=1}^{m}\binom{\lambda_i-1}{2}-\sum_{j=1}^{p}\binom{\mu_j}{2}. \end{aligned}$$

Although this argument is fairly natural, direct, and easy to follow, it seems somehow quite special.

There is little hint of a general method. If Steiner's formula (7) is a known prior result, then the deduction of Roberts' formula from Steiner's is short and elegant. Unfortunately, Steiner's formula is itself not very well known.

5. Finally we present a proof that is based on the ingenious notion of a sweep-line introduced in 1955 by H. Hadwiger **[10]** and developed by A. Brousseau **[6]** in a pretty paper concerned with the heuristics of partition problems. This argument is elegant, transparent, and intuitive. It throws considerable light on Roberts' formula and provides intuitive insights quite different from those given by Roberts' heuristic arguments and from those of the previous section. It has the further advantage that no prior knowledge of special cases need be assumed.

Take a line b, which with Brousseau we call a sweep-line, not parallel to any of the lines of the given arrangement and initially located so far away that all of the points of intersection of the arrangement are on the same side. In this initial position, b is divided into two rays and $n-1$ segments by the n points of intersection with the n given lines, and each of these segments and rays lies in (and so counts) a well-defined region. So the sweep-line initially identifies $1+n$ regions.

Now sweep the line b across the arrangement, moving it always parallel to its initial position. New regions are encountered precisely at the points in which the given lines intersect—one new region at each simple point, two at each point of multiplicity three, and in general, $\lambda-1$ new regions at each point of multiplicity λ. Suppose there are S simple points. Then the sweep-line encounters exactly

$$S+\sum_{i=1}^{m}(\lambda_i-1)$$

new regions during its transit across the plane, and since every region is eventually counted, exactly

$$R=1+n+S+\sum_{i=1}^{m}(\lambda_i-1) \tag{8}$$

regions are formed by the given lines.

A formula for the number S of simple points may be deduced from the obvious fact that each two lines either intersect or are parallel. If $\lambda_i \geqslant 3$ lines pass through M_i, there evidently are $\binom{\lambda_i}{2}$ pairs of lines that intersect at M_i. Consequently there are, in all,

$$k_1=S+\sum_{i=1}^{m}\binom{\lambda_i}{2}$$

intersecting pairs of lines in the arrangement.

If there are $\mu_j \geqslant 2$ parallel lines in the jth parallel family, then there plainly are $\binom{\mu_j}{2}$ pairs of lines in this family that do not meet. Consequently there are, in all,

$$k_2=\sum_{j=1}^{p}\binom{\mu_j}{2}$$

pairs of non-intersecting lines in the arrangement.

Since each of the $\binom{n}{2}$ pairs of lines in the arrangement is either intersecting or parallel, we must have $\binom{n}{2}=k_1+k_2$. The desired formula for S follows:

$$S=\binom{n}{2}-\sum_{i=1}^{m}\binom{\lambda_i}{2}-\sum_{j=1}^{p}\binom{\mu_j}{2}.$$

Substituting this formula for S into (8), we find that

$$R=1+n+\binom{n}{2}-\sum_{i=1}^{m}\left[\binom{\lambda_i}{2}-\lambda_i+1\right]-\sum_{j=1}^{p}\binom{\mu_j}{2},$$

which, in view of the identity

$$\binom{\lambda_i}{2}-\lambda_i+1=\binom{\lambda_i-1}{2},$$

is Roberts' formula.

The sweep-line plays a mathematical role in the argument, to be sure, but it plays an interesting psychological role as well. With its aid we focus our attention on the plane one cross-section strip at a time and obtain the number R of regions as the sum of two parts: the first an initial contribution that is entirely independent of the details of the arrangement, the second the sum of "local" changes that occur precisely at the points of intersection. Watching the sweep-line as it passes through a point of intersection, we see exactly what the change in the region count really is at that point, and why. This makes some precise sense of the observation that regions are somehow closely related to points of intersection.

An important aspect of the sweep-line method is that one need not know the result in advance, because the argument allows the result to be discovered. This makes the sweep method an important heuristic technique for plane partition problems. It generalizes in a natural way to ovals, to higher dimensional Euclidean spaces, and to projective spaces. The sweep idea is surely one of the most ingenious recent ideas of elementary mathematics.

6. Brousseau's pretty formula (8) can also be deduced from Euler's formula $f=2-v+e$ for connected plane graphs together with some simple incidence relations. Similar considerations permit yet another proof of Roberts' formula (see Alexanderson and Wetzel [**2**]). A variant of this argument for arrangements of pseudolines (where one can, in fact, wiggle just a part of a line) appears in Alexanderson and Wetzel [**4**].

There is an extensive literature on partition problems in Euclidean and projective spaces. Grünbaum [**9**] summarizes much of what is known and includes a large bibliography. Other partition problems are discussed in Alexanderson and Wetzel [**2**] and [**3**], Freeman [**7**], and Kerr and Wetzel [**11**], [**12**], and [**13**]. A survey of partition formulas for three- and four-dimensional Euclidean and projective spaces is presented in Alexanderson and Wetzel [**1**].

An entirely different approach to problems of this kind is developed with great success by Zaslavsky in [**17**] and [**18**].

It is a pleasure to thank Marianne Jankowski for preparing the figures and to acknowledge many fruitful conversations on the ideas of this note with G. L. Alexanderson.

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CORRECTIONS TO
"Extended Mean Values"

(THIS MONTHLY, 85 (1978) 84–90)

E. B. LEACH AND M. C. SHOLANDER

Due to mail malhandling on the southern shore of the Mediterranean, final corrections on galley sheets were not made in time for paper publication.

1. Table (3.1). Row 3, column 2, entry should be "$\sin w$". Row 4, column 3, entry should be "$(\tanh w)/w$".

2. To each of (3.8), (3.9), and (3.11) should be added the restriction $x \neq y$. In the proof of (3.11), parentheses around a fraction numerator are missing.

3. In the line sixth from the last, the sector should extend from $-\pi/4$ to $\pi/2$ radians.

4. In paragraph 1 of Section 1, it was intended that N be described as the "root-mean square." The authors are aware that statisticians use "root mean square" when referring to the root mean-square M_2.

5. In the introduction, it is mentioned that Tobey (in 1967) described means which had extended means as special cases. We are informed that similar credit should be given to H. Brøns' 1967 lecture notes at the Institute for Mathematical Statistics in Copenhagen. Following these notes, we have in Ann. Math. Stat. 40 (1969) 339–355 a paper "Generalized Means and Associated Families of Distributions" by H. Brøns, H. D. Brunk, W. Franck, and D. L. Hanson where, after such means are defined, various statistical applications are made.

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MISCELLANEA

13. Analysis takes back with one hand what it gives us with the other. I recoil with fear and loathing from that deplorable evil, continuous functions with no derivatives.

Hermite to Stieltjes, 20 May 1893.

(Note: Weierstrass's example of a continuous nowhere differentiable function had been published (by du Bois-Reymond) 18 years earlier.)

PROGRESS REPORTS

EDITED BY P. R. HALMOS

Material for this Department should be sent to P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, Indiana 47401.

It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

APPROXIMATELY FINITE VON NEUMANN ALGEBRAS

CALVIN C. MOORE

A big part of the early development of functional analysis was the attempt to find an appropriate infinite dimensional version of the classical theorem that a real symmetric matrix can be diagonalized by an orthogonal change of coordinates. As one knows now, the proper context for such a generalization is that of a self-adjoint operator on a Hilbert space; and the work of Stone and von Neumann in the late 1920's and early 1930's, which culminated a long series of developments, strongly suggested that in the process of studying a single self-adjoint operator one should also study the entire algebra generated by that operator, including not only polynomials but also more general functions of it. The stunning realization, which was made at approximately the same time, that ordinary quantum mechanics can be given a precise formulation using self-adjoint operators, also strongly suggested that one consider entire algebras of operators; the reason is that in this modeling of quantum mechanics one is naturally led to consider sums $A+B$ and "products" $AB+BA$ or $AB-BA$ of operators A and B. In this case it is natural to consider even noncommutative algebras.

Consequently, von Neumann and Murray were led to introduce and study in a series of papers in the 1930's and 1940's what they called rings of operators (which were subsequently rechristened "von Neumann algebras"). The definition is very simple to state: A von Neumann algebra is an algebra of operators on a Hilbert space H, containing the unit operator, closed under the formation of adjoints, and closed in a suitable topology (called "weak"). The emphasis is on the study of the intrinsic algebraic and topological properties of such algebras, and, as with any class of objects, a primary question is that of classification.

A prototypical example of a von Neumann algebra is $B(H_n)$, the algebra of all bounded operators on a Hilbert space of dimension n, where n is a positive integer or $\aleph_0$. (It is best to stick to Hilbert spaces with countable orthonormal bases.) Another example is L^∞ of a measure space, realized as

multiplication operators on L^2 of that measure space. These latter algebras are abelian, and one of the first theorems in the subject is that they constitute all of the abelian von Neumann algebras.

A first attempt to classify von Neumann algebras would be to mimic as far as possible the Wedderburn classification of finite-dimensional semi-simple algebras. It is in fact not difficult to isolate by intrinsic structural properties a class of von Neumann algebras where this technique will work. These were termed algebras of type I, and the Wedderburn technique leads quickly to a complete classification. It turns out that they can all be built up in a canonical way from $B(H)$'s and L^∞'s, and there are no algebras that one would not have foreseen at the outset of the study. Moreover, the models of classical quantum mechanics involve only algebras of type I. But one cannot help imagining that von Neumann had hopes of finding, and perhaps even classifying, some entirely new and exotic classes of algebras that might be used as models for more complicated quantum mechanical systems, such as quantum field theory, where the standard models seemed inadequate. Recently there has been some very substantial progress on such a program of classification and that is the subject of this report.

It was realized very early on that any von Neumann algebra can be expressed uniquely as a continuous direct sum of factor algebras—those whose center consists of scalar operators, or equivalently those which are "simple" in a certain well-defined sense. This decomposition theorem in effect reduces virtually all problems about von Neumann algebras to problems about factors. As one might have predicted, the only factors of type I are the algebras $B(H_n)$ for some n. The factors not of type I can be further divided into subtypes denoted II_1, II_∞, and III. This classification is based on the existence of an essentially unique "dimension function" defined on the set of projections in the algebra which measures the relative sizes of projections. In the type I case the range of this dimension function for $B(H_n)$ is the set of integral multiples up to and including n of a fixed number, say 1. The algebras of type II_1 are characterized by the property that the dimension function assumes all values in a finite interval, say $[0,1]$; the II_∞ factors are the ones for which the range of the dimension function is the entire positive line together with $+\infty$, and the factors of type III are those for which the range consists of the two points, zero and infinity. A II_∞ factor can always be realized as the algebra of infinite matrices $M_\infty(A)$ over a II_1 factor A, or equivalently as the tensor product, $A \otimes B(H_\infty)$, suitably defined, of A with all bounded operators on an infinite-dimensional Hilbert space.

In a first attempt to classify factors not of type I, von Neumann and Murray were led to introduce the notion of approximately finite algebras (they called them hyperfinite). The definition is simple and natural: it is just that there should exist in A an ascending chain A_n of finite-dimensional subalgebras whose union is weakly dense in the algebra A. In other words, the algebra A is approximately finite when it can be approximated (so to speak) by finite-dimensional subalgebras. It is easy to construct examples; if A_n is the algebra $B(H_{2^n})$ of $2^n \times 2^n$ matrices, then A_n can be naturally embedded in A_{n+1} as $2^n \times 2^n$ block matrices whose blocks are 2×2 scalar matrices. It is natural to form the "union" A_∞ of this ascending chain, and this union may then be completed appropriately to produce a von Neumann algebra which by its very construction is approximately finite. It is readily shown that this algebra is a factor and is in fact a factor of type II_1. (The generalized dimensions of projections in the subalgebra A_n are of the form $k/2^n$ with $0 \leqslant k \leqslant 2^n$; the completion contains projections of arbitrary dimension between 0 and 1.) This is the simplest example of an algebra not of type I, and von Neumann and Murray in one of their original papers went on to show that, up to isomorphism, it is the only approximately finite II_1 factor.

On the other hand, it is quite clear that if A is an approximately finite II_1 factor, then $M_\infty(A)$, which is a II_∞ factor, is also approximately finite. In view of the uniqueness theorem in the II_1 case, it is natural to ask if this is the only approximately finite II_∞ factor. Equivalently: does the assumption that $M_\infty(B)$ is approximately finite imply that B itself is approximately finite? This simple question turns out to be enormously difficult and subtle. Moreover, it was gradually realized in the late 1960's

and 1970's that the question contained the key to classifying all approximately finite von Neumann algebras. This classification program was based on work of Takesaki and Connes in which it is shown how to classify the seemingly intractable algebras of type III in terms of II_∞ and II_1 algebras and their automorphisms. At the same time, Connes laid further groundwork by analyzing in detail the automorphism group of the approximately finite II_1 factor. At that point the only missing link in the program was the question of the unicity of the approximately finite II_∞ factor. If unicity failed here, the classification program for approximately finite algebras would have been in a shambles. However, the answer turned out to be affirmative, and nearly 40 years after the problem was posed A. Connes proved (1976) that there is only one approximately finite II_∞ factor.

The classification of approximately finite factors is virtually complete; there is one remaining "very small" subclass of factors of type III where it is not yet nailed down. The classification scheme which would result is indeed extraordinarily elegant and can be put in several ways. One such formulation, which emerges from the combined work of Connes, Krieger, and Takesaki, is that there is a bijective correspondence between all approximately finite factors other than the ones of type I and type II_1 (which we regard as known) and all possible ergodic flows (on a measure space) which leave a measure quasi-invariant, but not necessarily invariant. The map from algebras to flows is the map that attaches to the algebra its so-called "flow of weights." It is known to be surjective, and is proved to be injective, except that it is not yet known whether there is more than one algebra whose corresponding flow consists of a measure space with one point that is fixed for all time. This is the "very small" subclass mentioned above. While one cannot classify these flows any more than one could classify the algebras at the beginning, the existence of such a bijective map is of considerable importance and interest and sheds light both on the structure of the algebras and on the structure of ergodic flows.

The class of approximately finite algebras is of importance above and beyond its intrinsic interest. There are some convincing plausibility arguments (supported in some cases by precise theorems) that the von Neumann algebras that should be used as models for more complicated quantum mechanical systems are the approximately finite ones.

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MATHEMATICAL NOTES

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Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

DISSECTIONS OF A SIMPLY-CONNECTED PLANE DOMAIN

JOHN E. WETZEL

1. Introduction. Let D be a plane domain (an open, connected set in the plane) having boundary ∂D. A line segment AB in $D \cup \partial D$ is a chord of D provided $AB \cap \partial D = \{A,B\}$. Suppose that D is partitioned into subdomains by a finite set of c chords that meet to form p points of intersection in D, and suppose further that each of these points is simple, i.e., that no three of the chords are concurrent inside D. Bauman [2] found that the number R of subdomains formed by the dissecting chords is given by the very pretty formula

$$R = 1 + c + p, \tag{1}$$

and he suggested a proof by mathematical induction.

For lines in the plane this formula was given in 1966 by Alfred Brousseau [3]; and it appears with many similar formulas in [1], under the assumption that D is convex. Bauman asserts that the formula holds for any region, an assertion that is evidently incorrect, as a glance at an annulus shows. But the formula does hold for any simply-connected domain, and the dissecting elements need not be straight nor must their intersections be simple. The purpose of this short note is to give an accurate statement and proof.

2. The result. Let D be a simply-connected domain in the plane. A simple arc γ in $D \cup \partial D$ with endpoints A and B is a *chord* (also called a *cross-cut*) of D provided $\gamma \cap \partial D = \{A,B\}$. There is no reason to insist that the endpoints be different. A finite set Γ of chords is *admissible* if each two chords intersect in only finitely many points (or are disjoint). Suppose an admissible set Γ of c chords $\gamma_1, \gamma_2, \ldots, \gamma_c$ is given in D. For each point P of D, let $\lambda(P)$ be the number of chords of Γ that pass through P, and write $\mathcal{P} = \{P \in D \mid \lambda(P) \geqslant 2\}$. Finally, suppose that $D - \cup(\Gamma)$ falls into R connected components $D_1, D_2, \ldots, D_R$. Under these circumstances,

$$R = 1 + c + \sum_{P \in \mathcal{P}} (\lambda(P) - 1). \tag{2}$$

Bauman's formula (1) arises when each point P of $\mathcal{P}$ has $\lambda(P) = 2$.

3. Proof. The fundamental separation fact involved in formula (2) is that a chord divides a simply-connected domain into two subdomains, both of which are also simply-connected. A proof of this basic assertion can be found, for example, in Newman [4] (Theorem 5.1 on page 145).

The proof of formula (2) is by induction on c. We need the fact that the subdomains formed in D by an admissible set of chords are simply-connected, and we prove this by induction at the same time.

This assertion of simple-connectedness and formula (2) are both correct for $c = 0$ and $c = 1$. Suppose both are correct for all admissible sets of c chords in D, and let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_c, \gamma\}$ be an admissible set of $c+1$ chords in D. Then the admissible set $\Gamma_0 = \Gamma - \gamma$ partitions D into R simply-connected subdomains, and R is given by formula (2). We will show that both sides of (2) increase by the same amount when γ is put into place and that all the new subdomains formed are simply-connected.

Suppose that γ is oriented and has initial point A and terminal point B, and suppose that γ meets the chords γ_i of Γ_0 in $k \geqslant 0$ points $P_1, P_2, \ldots, P_k$ in D, named in order along γ. Complete the notational pattern by writing $P_0 = A$ and $P_{k+1} = B$. The k points of intersection partition γ into $k+1$ simple subarcs $\gamma^1, \gamma^2, \ldots, \gamma^{k+1}$, named so that for each i, γ^i joins P_{i-1} to P_i.

The first simple arc γ^1 joins P_0 to P_1 and, except for its endpoints, lies in a well-defined simply-connected subdomain of $D - \cup(\Gamma_0)$, which it divides into two simply-connected subdomains. So

$$\Delta_1 = D - \left(\cup(\Gamma_0) \cup \gamma^1\right)$$

has $R+1$ components, all of which are simply-connected. Suppose that

$$\Delta_j = D - \left(\cup(\Gamma_0)\cup(\gamma^1\cup\gamma^2\cup\ldots\cup\gamma^j)\right)$$

has $R+j$ components, all of which are simply-connected. The simple arc γ^{j+1} joins P_j and P_{j+1} and, except for its endpoints, lies in a well-defined, simply-connected subdomain of Δ_j, which it divides into two simply-connected subdomains. So $\Delta_{j+1}=\Delta_j-\gamma^{j+1}$ has exactly $R+(j+1)$ components, one more than Δ_j.

Consequently $D-\cup(\Gamma)=\Delta_{k+1}$ has exactly $R+(k+1)$ components, all of which are simply-connected. The right side of (2) plainly increases by exactly $k+1$ when the new chord γ is put into place; and since R also increases by exactly $k+1$, formula (2) holds for Γ. This completes the proof.

4. Remarks. The proof works as presented for simply-connected domains on the sphere, and it follows that for unbounded plane domains the notion of "chord" can be extended to include simple curves with one or both endpoints at infinity.

A different proof of formula (2) can be given using Euler's formula for the dissected domain, together with some easily obtained incidence relations (see **[1]**).

Many of the results presented in **[1]** for convex domains dissected by (straight) chords can similarly be extended to simply-connected domains dissected by crosscuts. In particular, if each two crosscuts in a simply-connected domain have at most one point in common, then Roberts' "subtractive" formulas (formulas (7), (8), and (9) of **[1]**) correctly count the regions, arcs, and points of intersection determined by the crosscuts.

Using much more sophisticated algebraic machinery, Zaslavsky **[5]** gives a more general formula that includes (2) as a special case.

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SOME CONVERGENT SUBSERIES OF THE HARMONIC SERIES

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The series $\Sigma 1/n$ diverges, but it is known that if we omit the terms corresponding to the integers whose decimal representations contain a specified digit at least once (the digit 0, for example), the resulting series converges **[3]**. (See the references in **[3]** for the history of the topic.)

Let Z_i denote the set of integers having exactly i zeros in their decimal representation. Consider the subseries

$$\sum_{n_i\in Z_i} 1/n_i = s_i, \qquad \text{say},$$

of the harmonic series $\Sigma\ 1/n$.

It is known that $s_0=23.10345$ to 5 decimal places **[1]**. The author is informed **[2]** that Baillie has gone to 20 places. [An account of this work will appear soon in this MONTHLY.]

In this note we find some interesting properties of the sequence $\{s_n\}$.

THEOREM 1. *The sequence $\{s_n\}$ is strictly decreasing.*

Proof. We first show that $s_1 < s_0$. We write

$$s_1 = \left(\frac{1}{10} + \frac{1}{20} + \cdots + \frac{1}{90} + \frac{1}{110} + \cdots + \frac{1}{190} + \frac{1}{210} + \cdots\right)$$
$$+ \left(\frac{1}{101} + \cdots + \frac{1}{109} + \frac{1}{201} + \cdots + \frac{1}{209} + \cdots\right)$$
$$+ \left(\frac{1}{1011} + \frac{1}{1012} + \cdots + \frac{1}{1019} + \frac{1}{1021} + \cdots\right) + \cdots,$$

where the terms have been so arranged that the terms corresponding to the integers having zero in the first (unit's) place have been put in the first bracket, terms corresponding to the integers having 0 in the second place have been put in the second bracket and so on. We note that

$$s_1 < \frac{1}{10}s_0 + \frac{9}{100}s_0 + \frac{81}{1000}s_0 + \cdots = (.1)s_0\left[1 + .9 + (.9)^2 + \cdots\right] = s_0.$$

Now we come to the general case that $s_{n+1} < s_n$.

The terms of Z_{n+1} can be arranged in the following manner:

(i) *Terms having a zero at the first place.* These contribute to the sum for s_{n+1} a number $=(.1)s_n$.

(ii) *Terms having a zero at the second place, first place being occupied by a non-zero digit.* These contribute to the sum for s_{n+1} a number $<(.09)s_n$.

(iii) *Terms having a zero at the third place, first and second places being occupied by non-zero digits.* These contribute to the sum for s_{n+1} a number $<(.081)s_n$ and so on. Therefore

$$s_{n+1} < (.1)s_n\left[1 + .9 + (.9)^2 + \cdots\right] = s_n.$$

Now s_0 is known to be less than 23.104. Also $\{s_n\}$ is a strictly decreasing sequence of positive numbers. Therefore $\{s_n\}$ converges. We now prove

THEOREM 2. *For every n, $s_n > 19.28$.*

Proof. Consider an element $1/x_n$ of s_n, where x_n is an integer with exactly n zeros in its decimal representation. The number of such elements having $n+j+1$ ($j>0$) digits is $\binom{n+j}{j}9^{j+1}$. We notice that of these numbers, one ninth have leading digit 1, one ninth have leading digit 2, and so on; so of these $\binom{n+j}{j}9^n$ are between 10^{n+j} and $2\cdot 10^{n+j}$, and so on. Therefore the contribution of these numbers to the sum for s_n is at least

$$\binom{n+j}{j}(9^j/10^{j+n})\left(\tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{10}\right).$$

Therefore

$$s_n > \sum_{j=0}^{\infty}\binom{n+j}{j}(.9)^j(.1)^n\left(\tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{10}\right)$$
$$= (1-.9)^{-n-1}(.1)^n\left(\tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{10}\right) > 10(1.928) = 19.28.$$

It would be interesting to determine $\lim s_n$ more accurately.

From Theorem 2 it is clear that $\{s_n\}$ is bounded away from zero. This latter fact is *equivalent* to the following theorem due to R. P. Boas.

THEOREM 3. [1] *Let $\{\varepsilon_n\}$ be a sequence of positive numbers; then $\sum_{n=0}^{\infty}\varepsilon_n s_n$ converges or diverges according as $\sum_{n=0}^{\infty}\varepsilon_n$ converges or diverges.*

In fact Theorem 3 and the fact that $\{s_n\}$ is bounded away from zero are just two ways of looking

at the same phenomenon. It is trivial that Theorem 3 follows from $s_n \geqslant s > 0$. Conversely, suppose we know Theorem 3; we must show that $s_n \geqslant s > 0$ for some s. Suppose that $\underline{\lim}\, s_n = 0$. We can find a sequence $\{n_k\}$ such that $s_{n_k} < 1/k^3$. Taking $\varepsilon_n = 0$ except $\varepsilon_{n_k} = k$ $(k = 1, 2, \ldots)$, we get a contradiction.

Use of Abel's test gives the following:

THEOREM 4. *If Σa_n is a convergent series of real terms, then $\Sigma a_n s_n$ is convergent.*

By pursuing the same line of reasoning as used in the proof of Theorem 1, we have

THEOREM 5. *Let $a_0, a_1, \ldots, a_k$ be positive numbers and*

$$S_n^\alpha = \sum_{x_n \epsilon Z_n} 1/x_n^\alpha,$$

$$P_n^k = \sum_{x_n \epsilon Z_n} \left(a_0 x_n^k + a_1 x_n^{k-1} + \cdots + a_k\right)^{-1},$$

k, a positive integer. Then the sequence $\{S_n^\alpha\}$ converges for $\alpha > \log_{10} 9 = 0.95+$ and decreases strictly for $\alpha \geqslant 1$. The sequence $\{P_n^k\}$ also decreases strictly.

By using more elaborate reasoning and making use of Theorem 5, we can show that $\Delta s_n = s_n - s_{n+1}$ is strictly decreasing. This result is equivalent to

THEOREM 6. *The sequence $\{s_n\}$ is convex.*

COROLLARY 1. $n\Delta s_n \to 0$ *as* $n \to \infty$.

COROLLARY 2. $\Sigma(n+1)\Delta^2 s_n < \infty$.

If we write integers in base k instead of base 10, the corresponding results of this note still hold. For example, we see that each of the corresponding subseries s_n has a sum between

$$k\left(\tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{k}\right) \quad \text{and} \quad k\left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{k-1}\right)$$

and the corresponding $\{S_n^\alpha\}$ are convergent if $\alpha > [\log(k-1)]/\log k$ and the sum of each such subseries is between

$$k(k^\alpha - k + 1)^{-1}[2^{-\alpha} + \cdots + k^{-\alpha}] \quad \text{and} \quad k(k^\alpha - k + 1)^{-1}\left[1 + 2^{-\alpha} + \cdots + (k-1)^{-\alpha}\right].$$

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CAUCHY FUNCTIONAL EQUATION AGAIN

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THEOREM. *Let $f: (0, +\infty) \to R$ be a measurable function satisfying the Cauchy functional equation $f(x+y) = f(x) + f(y)$ $(x, y > 0)$. Then $f(x) = xf(1)$ $(x > 0)$.*

This theorem has numerous proofs (see [2]). Most of them use an intermediate step by showing that f must be continuous, or bounded on some interval. My favorite proof is a direct one and relies

on the following

RESULT. *If A and B are measurable subsets of R with positive measure, then $A+B$ (vectorial sum) contains an open interval* (see [**1**, p. 272] for a proof).

Now call *semi-group* any subset S of $(0, +\infty)$ such that $S+S\subset S$.

LEMMA. *A semi-group with positive measure contains a half line $(a, +\infty)$.*

Proof of the lemma. The result above implies that there exists an interval (α, β) contained in S. Hence

$$S\supset\bigcup_{n=1}^{\infty}(n\alpha, n\beta)\supset\left(\frac{\alpha\beta}{\beta-\alpha}, +\infty\right).$$

Proof of the theorem. Suppose that $f(x)/x$ is not a constant. Then there exists $a\in R$ such that

$$S_1=\left\{x;\, x>0, \quad \frac{f(x)}{x}\geqslant a\right\} \quad S_{-1}=\left\{x;\, x>0, \quad \frac{f(x)}{x}<a\right\}$$

are both non-empty.

The important observation is that S_1 and S_{-1} are semi-groups: if x and y are in S_ε ($\varepsilon=\pm1$) then it follows from

$$\frac{f(x+y)}{x+y}=\frac{x}{x+y}\cdot\frac{f(x)}{x}+\frac{y}{x+y}\,\frac{f(y)}{y}$$

that $x+y$ belongs to S_ε.

Now f measurable implies that S_1 and S_{-1} are measurable. Since $S_1\cup S_{-1}=(0, +\infty)$, one of them, say S_ε, has a positive measure and, from the lemma, contains some half-line $(a, +\infty)$. The other one, $S_{-\varepsilon}$, being non-empty, is unbounded. This contradicts $S_1\cap S_{-1}=\varnothing$.

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RESEARCH PROBLEMS

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In this Department the Monthly presents easily stated research problems deasling with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The Univeristy of Calgary, Calgary, Alberta, Canada, T2N 1N4.

DISSECTIONS AND INTERTWININGS OF GRAPHS

PETER UNGAR

We say a cubic graph is **cyclically *n*-connected** if it has at least one pair of disjoint circuits and it requires the removal of at least n edges to obtain two components, each of which has a circuit.

Problem: Is every cyclically 4-connected cubic (c.4-c.c.) graph the union of two edge-disjoint trees? An easy argument, given below, shows that the question is equivalent to the following:

Does every c.4-c.c. graph G have two disjoint subtrees T_1, T_2 such that every edge of G either is in T_1 or in T_2 or connects a node of T_1 and a node of T_2?

The problem arose in a conversation with Ian C. Percival, circa 1951, about Whitney's celebrated theorem that in a plane map with a c.4-c.c. boundary graph G one can draw a closed curve W (a **Whitney circuit**) wihich passes through every region once. (A notable generalization of Whitney's theorem was given by Tutte [**5**], but the reader who wishes to understand Whitney's basic idea without undue effort is advised to read [**6**].)

For a plane graph G a Whitney circuit W is equivalent to a two-tree decomposition. Indeed, given W, T_1 will consist of the nodes and edges of G inside W and T_2 will consist of the nodes and edges outside W. Conversely, if T_1 and T_2 are given, we get a curve passing once through every region by going around either one of the trees. (See Fig. 1.)

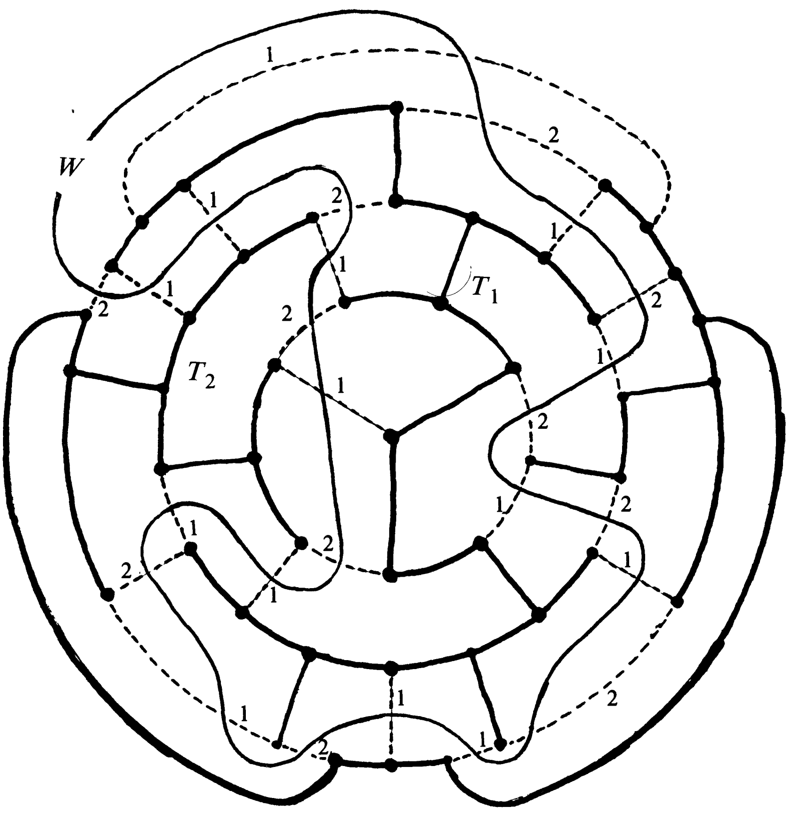

FIG. 1

We establish that the two forms of the problem are equivalent as follows. Suppose G is the union of two edge-disjoint trees G_1, G_2. Then T_i will consist of the nodes of G where 2 or 3 edges of G_i meet and the edges which connect these nodes. Conversely, suppose the two disjoint trees T_1, T_2 are given. Let B be the subgraph of G consisting of the edges of G which are neither in T_1 nor in T_2, and their endnodes. The nodes of B have degree 1 or 2 since each node of B is also a node of T_1 or T_2. Hence the components of B are paths and circuits. The circuits are even because B is bipartite. Hence the edges of B can be colored 1 and 2 so that no two edges of the same color meet at a node. We obtain the decomposition of G into two edge-disjoint trees by adjoining the edges colored i to T_i.

We see that Whitney's theorem is equivalent to the statement that every planar c.4-c.c. graph G has a 2-tree decomposition, and the problem we are posing is whether this remains true if the assumption of planarity is dropped.

The proof of Whitney's theorem relies on the Jordan curve theorem at every step, and this made it seem very unlikely at first that planarity could be dispensed with. However, we found that more or less randomly drawn c.4-c.c. graphs with up to 80 nodes could always easily be decomposed into two edge-disjoint trees. One can start anywhere and add edges to construct one of the trees, taking care not to disconnect the complement and attempting to reach all parts of the graph. When one cannot add more edges, either one has the desired decomposition or one can obtain it with a few easy modifications.

Since the desired decomposition is known to exist for planar graphs, it was of special interest to look at highly non-planar ones. We found that Balaban's 70-node graph of girth 10 could be decomposed without difficulty, and so could the 12-cage (the smallest graph of girth 12) which has 126 nodes and genus $\geqslant 17$. (Both graphs can be found in [**2**].)

Of course, such evidence is far from conclusive. It was more than half a century before Tutte found a counterexample to Tait's conjecture that every 3-connected planar cubic graph is Hamiltonian, even though some of these counterexamples are not very complicated. (The boundary graph in Figure 1 is a c.5-c.c. non-Hamiltonian graph due to Grinberg [**4**].) We therefore compared our empirical evidence for the 2-tree decomposition property with empirical evidence for Tait's conjecture. We took some planar 3-connected cubic graphs at random, including the boundary graph of Moore's map [**1**] and constructed a Hamilton line by a T_1, T_2 decomposition of the dual graph which is of course not cubic. We succeeded in all cases but it did seem to involve considerably less freedom and more backtracking than the 2-tree decompositions of c.4-c.c. graphs.

The reader who wishes to pursue this matter may find ideas on how to prove such theorems in the paper of S. Kundu [**3**], who proved the beautiful theorem that a 4-edge connected graph has two edge-disjoint spanning trees.

The second problem we wish to state was formulated in conversations with Martin Milgram about the missing link in attempted proofs that for any given surface S there are only finitely many graphs which are not representable on S and are minimal with respect to this property. Milgram defines a graph G to be an **intertwining** of two graphs H and K if G has no nodes of degree 2 and has subgraphs homeomorphic to H and to K and if no proper subgraph of G has these properties.

We conjecture that two graphs have only finitely many intertwinings.

L. Lovász told us that he had also made this conjecture.

One can define an intertwining of n graphs similarly and state the analogous conjecture, but it would clearly suffice to prove the conjecture for $n=2$.

Proving the conjecture for arbitrary graphs H and K seems difficult, so we tried particular pairs just for orientation. Milgram worked out a proof when H is the complete 5-graph and K is $K_{3,3}$. Ungar proved that if G has eight nodes A_i, B_i, C_i, $D_i (i=1,2)$, a pair of disjoint paths A_1A_2, B_1B_2 and also a pair of disjoint paths C_1C_2, D_1D_2 but no proper subgraph of G has such paths between these four pairs of nodes then G has at most 60 nodes of degree $\geqslant 3$.

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SEQUENCE TOPOLOGIES ON THE REAL LINE

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The usual Euclidean topology on the real line **R** determines the convergent sequences; but the converse is false. This note describes different topologies on **R** which determine the same convergent sequences as the usual topology. We also prove that there is no weakest topology on **R** possessing the above property.

Let $\mathfrak{U}$ be the usual topology on **R**. A topology $\mathfrak{T}$ is called a *sequence topology* provided every sequence in **R** converging to a point p relative to $\mathfrak{U}$ also converges to p relative to $\mathfrak{T}$, and conversely.

Every sequence topology $\mathfrak{T}$ is weaker than $\mathfrak{U}$. Indeed, if F is $\mathfrak{T}$-closed and $\bar{F}$ is its $\mathfrak{U}$-closure, then each point p of $\bar{F}$ is the $\mathfrak{U}$-limit of a sequence from F. Thus p is also the $\mathfrak{T}$-limit of the same sequence. It follows that $p \in F$, since F is $\mathfrak{T}$-closed, and hence $F=\bar{F}$.

Also, any sequence topology $\mathfrak{T}$ is T_1 and thus stronger than the cofinite topology. To see this let p, q in **R** be distinct points. Since the sequence $(p, q, p, q, \ldots)$ does not $\mathfrak{U}$-converge to p, it also fails to converge to p relative to $\mathfrak{T}$. Therefore some $\mathfrak{T}$-open set contains p and excludes q.

EXAMPLE 1: Let $\mathfrak{T}$ be the topology whose closed sets other than **R** are the countable, $\mathfrak{U}$-closed subsets of **R**. Since this topology is weaker than $\mathfrak{U}$, every $\mathfrak{T}$-convergent sequence is $\mathfrak{U}$-convergent. Thus, to prove that $\mathfrak{U}$ is a sequence topology, it suffices to show that $(x_n) \to p$ relative to $\mathfrak{U}$, whenever (x_n) is a sequence tending to a point p relative to $\mathfrak{T}$.

Suppose $(x_n) \not\to p$ relative to $\mathfrak{U}$; then some subsequence (x_{n_k}) tends to $+\infty$ (or $-\infty$) or some subsequence (x_{n_k}) tends to a point $q \neq p$ relative to $\mathfrak{U}$. Without loss of generality we can assume the subsequence in each case never takes on the value p. In the first case let $V=\mathbf{R}\setminus\{x_{n_k} | k=1, 2, 3, \ldots\}$. Clearly $p \in V$, V is $\mathfrak{T}$-open and no x_{n_k} is in V. Thus $x_n \not\to p$ relative to $\mathfrak{T}$. In the second case let $V=\mathbf{R}\setminus\{q, x_{n_1}, x_{n_2}, \ldots\}$. Again $p \in V$, V is $\mathfrak{T}$-open and no x_{n_k} is in V, so that $x_n \not\to p$ relative to $\mathfrak{T}$.

This sequence topology $\mathfrak{T}$ has some noteworthy properties which are listed below. Their proofs are left to the reader.

(a) A function $f: \mathbf{R} \to \mathbf{R}$ is $\mathfrak{T}$-continuous if and only if f is $\mathfrak{U}$-continuous and the inverse image of a countable set is either countable or all of **R**.

(b) For any a in **R** the maps $x \to a+x$, $x \to ax$ are $\mathfrak{T}$-continuous. The former is a $\mathfrak{T}$-homeomorphism; the latter is if $a \neq 0$.

(c) If U, V are $\mathfrak{T}$-open non-empty sets, then $U+V=U \cdot V=\mathbf{R}$. Thus addition is separately, but not jointly, $\mathfrak{T}$-continuous; and so **R** is not a $\mathfrak{T}$ topological group under addition.

EXAMPLE 2: Having seen that sequence topologies on **R** other than $\mathfrak{U}$ do exist, it is natural to ask if there is a weakest such topology. In this example we show that for any infinite $\mathfrak{U}$-closed set $F \neq \mathbf{R}$, there is a sequence topology $\mathfrak{T}$ in which F is not closed. It follows that the intersection of all sequence topologies on **R** is the cofinite topology, which is not itself a sequence topology.

We say that a subset S of the natural numbers N is of density 1 if $(1/n)|\{x \in S : x \leqslant n\}| \to 1$ as $n \to \infty$, and of density 0 if the above limit is 0. It is easy to see that a subset S of N is of density 0 if and only if its complement $N \setminus S$ is of density 1. Sets of density 0 are closed under finite unions, while those of density 1 are closed under finite intersections. In addition, any sequence $(i_1, i_2, i_3, \ldots)$ of natural numbers has a subsequence $(i_{n_1}, i_{n_2}, i_{n_3}, \ldots)$ such that the set of its elements is of density 0.

For any infinite $\mathfrak{U}$-closed set F different from **R**, fix a point $p \notin F$ and a strictly increasing (or, if necessary, decreasing) sequence of points $(t_1, t_2, t_3, \ldots)$ in F. To specify a sequence topology $\mathfrak{T}$ in

which F is not closed, we give a base $\mathfrak{B}$ of open sets for $\mathfrak{T}$ as follows. A subset V of $\mathbf{R}$ is to be in $\mathfrak{B}$ if and only if V is $\mathfrak{U}$-open and either

(1) $p \notin V$, or

(2) $p \in V$ and for some set $S \subseteq N$ of density 1, V contains $\{t_i : i \in S\}$.

That is, if V contains p it must contain all t_i's of F except possibly a set of them which is indexed by a set of density 0.

It is clear that $\mathfrak{B}$ is a base for a topology $\mathfrak{T}$ and that F is not $\mathfrak{T}$-closed, because p is in its $\mathfrak{T}$-closure. In addition, $\mathfrak{T}$ is weaker than $\mathfrak{U}$, which means that every $\mathfrak{U}$-convergent sequence is $\mathfrak{T}$-convergent. All that is needed for $\mathfrak{T}$ to be a sequence topology is the converse of this last remark.

PROPOSITION. *If a sequence (x_n) converges to a point x relative to $\mathfrak{T}$, then (x_n) tends to x relative to $\mathfrak{U}$. Hence $\mathfrak{T}$ is a sequence topology.*

Proof. Since the topologies $\mathfrak{U}$ and $\mathfrak{T}$ are identical in the neighborhoods of all points x different from p, the only case to check is that $(x_n) \to p$ relative to $\mathfrak{U}$ whenever $(x_n) \to p$ relative to $\mathfrak{T}$.

Suppose, to the contrary, some sequence (x_n) tends to p relative to $\mathfrak{T}$ but not relative to $\mathfrak{U}$. Then there is an $\varepsilon > 0$ and a subsequence (y_n) of (x_n) such that $(y_n) \to p$ relative to $\mathfrak{T}$ but $y_n \notin (p-\varepsilon, p+\varepsilon)$ for each $n \in N$.

For each t_i in F choose $\varepsilon_i > 0$ so that the open intervals $V_i = (t_i - \varepsilon_i,\ t_i + \varepsilon_i)$ are mutually disjoint. Since the set $\{t_i\}$ is discrete, this is easily done. Since $(y_n) \to p$ relative to $\mathfrak{T}$, (y_n) is eventually in the $\mathfrak{T}$-open set $V = (p-\varepsilon, p+\varepsilon) \cup (\bigcup_{i=1}^{\infty} V_i)$. Thus there is a subsequence (z_n) of (y_n) such that $z_n \in V \backslash (p-\varepsilon, p+\varepsilon)$ for each $n \in N$ and, moreover, $(z_n) \to p$ relative to $\mathfrak{T}$.

Because the V_i's are mutually disjoint, each z_n belongs to one and only one V_{i_n}. Take a subsequence $(i_{n_1}, i_{n_2}, i_{n_3}, \ldots)$ of the sequence of natural numbers $(i_1, i_2, i_3, i_4, \ldots)$ such that the set S of its elements $i_{n_1}, i_{n_2}, i_{n_3}, \ldots$ is of density 0. Let $W = (p-\varepsilon, p+\varepsilon) \cup (\bigcup_{j \notin S} V_j)$. This set is still a $\mathfrak{T}$-open neighborhood of p. However it fails to contain infinitely many elements of (z_n), contradicting the fact $(z_n) \to p$ relative to $\mathfrak{T}$. This proves the proposition.

Thus we have proved the following.

THEOREM. *The infimum of all sequence topologies on the real line is the cofinite topology.*

As a final comment we note that when F, in the construction of example 2, is an unbounded closed set, the t_i's can be chosen to converge to ∞ (or $-\infty$). The resulting topology $\mathfrak{T}$ will then be Hausdorff. On the other hand, when F is bounded, the sequence $(t_1, t_2, t_3, \ldots)$ tends to a point q different from p. It is then impossible to separate p and q using our topology $\mathfrak{T}$. Therefore, $\mathfrak{T}$ is not Hausdorff. In fact, whenever F is a $\mathfrak{U}$-compact set, it stays compact in any sequence topology on $\mathfrak{T}$. In this case, no Hausdorff sequence topology $\mathfrak{T}$ can be constructed such that F is not $\mathfrak{T}$-closed.

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INTEGRATION BY PARTS AND INVERSE FUNCTIONS

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Let f be a strictly increasing function with continuous derivative on a compact interval $[a,b]$. Integration by parts gives

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b xf'(x)\,dx. \tag{1}$$

Let $y = f(x)$, $x = f^{-1}(y)$; then (1) can be written

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy. \tag{2}$$

In [1] it is observed that equation (2) remains valid in case f is only assumed to be strictly increasing and continuous on $[a,b]$; in support of this claim reference is made to the reduction of a Lebesgue–Stieltjes integral to that of a Lebesgue integral.

The purpose of this note is to point out that the desired extension also follows from simple calculus. Indeed, if f is strictly increasing and continuous on $[a,b]$, then it admits an inverse function f^{-1} of the same type. Moreover, we may assume without loss of generality that the graph of f is situated in the first quadrant of the x,y-plane. But equation (2) permits a simple geometrical interpretation in terms of areas of regions represented by the integrals and the quantities $bf(b)$ and $af(a)$ viewed as areas of rectangles. The desired extension is therefore immediate.

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HILBERT AT VASSAR: AN UNDERGRADUATE SEMINAR

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A common feature of an undergraduate's mathematical education is a seminar in the senior year. At any school without a graduate program, and particularly at a liberal arts institution, such a course offers the opportunity to probe issues outside (if not beyond) the usual offerings of the curriculum. The seminar at Vassar is also designed to satisfy a campus-wide requirement that every senior participate in an event which acts as a capstone for that student's major course of study, a requirement met in some other departments by a thesis or research project. In addition, the department realizes some of its own goals for the seminar by asking that the students present the lectures. In this way the students are encouraged to work independently on a body of mathematics, incorporating such skills as the location and evaluation of resource material, the integration of information from diverse sources, and the organization and presentation of a coherent lecture.

Inherent problems in selecting an appropriate topic stem from diversities among the senior mathematics majors, all of whom are required to participate. The wide range of abilities makes it impossible to treat the graduate-level topics to which some students should be exposed, but academically unsound to reduce the choice to the level of the weaker students. The different preparations of the students require that prerequisites for the seminar subject matter come from core courses. Potential actuaries, high school teachers, research mathematicians, and computer programmers all should see the topic as of use in their planning. Finally, the material must have the staying power necessary to hold variable interest levels through an occasionally dull or inadequate exposition.

The purpose of this note is to report on a particularly successful seminar theme and concept. The germ of the idea was the observation that it is common practice for other academic departments to offer courses on individual practitioners in their fields. There seemed no immediate reason not to organize a mathematics course focusing on an individual mathematician. David Hilbert emerged as

PRESENTING A STRATEGY FOR INDEFINITE INTEGRATION

ALAN H. SCHOENFELD

Introduction and theoretical overview. Students in a first-year calculus class should, with a reasonable amount of practice, be able to evaluate indefinite integrals with much the same facility as their instructor does. At least, practice is apparently the telling advantage teachers have over their students. Unlike subjects such as "related rates," integration is sufficiently straightforward that the instructor's "cleverness," "insight," or "mathematical ability" should play a minor role except in the most complex problems. Equally important, integration is a technical skill in which one is little helped by theoretical knowledge. In many areas—the mean value theorem and its implications, for example—the instructor's background provides a perspective completely inaccessible to the student, so one could not expect the student to deal with the subject matter the way the teacher does. In integration, however, we would expect an instructor's perspective to be of no more help in evaluating an integral than would a knowledge of abstract group theory be of assistance in determining the product of two three-digit numbers.

Thus we could reasonably expect to set very high performance standards for our students. Yet they consistently have more difficulty evaluating integrals than they "should," in spite of many hours of working on practice problems. In this article, we argue (1) that these difficulties occur largely because students lack a coherent and efficient means of approaching problems in integration, and not because they have difficulty learning the important techniques; and (2) that we can enhance students' abilities to evaluate integrals, with no extra study time, by providing them with a coherent approach.

We discuss our methodology briefly, in the hope that our readers will consider applying it in other areas. It has been used successfully in areas ranging from artificial intelligence simulations (for example, Slagle's SAINT program for indefinite integrals) to instruction in the art of juggling. In broad outline, the major phases of experimentation are as follows.

1. Determine, as accurately as possible, what your experimental audience can and cannot do. (The next section of this paper assesses student facility with indefinite integrals, both before and after the "standard" presentation of the material.)

2. Examine, in minute detail, the action of "experts" as they perform the desired task. The purposes of this examination are to: (a) discover regularities, strategies, or implicit rules the "experts" may be using—often without being conscious of them—as they solve problems; and (b) decompose the problem-solving process into smaller, more manageable "chunks" which can be taught individually and then carefully combined. (The third section describes the process of isolating an effective strategy for indefinite integration.)

3. Create a model of the problem-solving process based on the "chunks" discovered in (b), making explicit use of the strategies discovered in (a). Design instructional materials which make *both* accessible to the students. (The integration materials are described in the fourth section.)

4. Test and revise both the model and the instructional materials. Preliminary test results for the integration strategy are described in the fifth section.

What CAN students do, and why? The first two of our findings will hardly surprise the experienced teacher. (1) Most students can learn to perform each of the individual techniques of integration (substitutions, parts, partial fractions, etc.) reliably, as long as they know which technique they should be using—for example, when working problems at the end of a particular section of text. (2) Students often find it very difficult to select appropriate approaches to problems encountered out of context (in sets of miscellaneous exercises or on tests). For example, a colleague who placed $\int (x/(x^2-9))dx$ at the beginning of an exam, hoping to bolster his students' confidence with a rapid solution via the substitution $u=x^2-9$, discovered that 44 of his 178 students had factored x^2-9 and used partial fractions and that 17 of them tried to use the substitution $x=3\sin\theta$, which they felt was suggested by

the term u^2-a^2 in the denominator! (3) Even worse, we find that when left to their own devices students often develop and employ inappropriate and counterproductive problem-solving strategies.

The roots of these difficulties lie in the standard presentations of text materials on integration. The student usually proceeds linearly through the sections of the text, obtaining good practice on each of the techniques in isolation. The prevailing assumption is that, while working the miscellaneous exercises at the end of integration chapters, students will develop their own coherent approaches to the subject. George B. Thomas, for example (Calculus and Analytic Geometry, 4th ed., Addison-Wesley, Reading, Mass., p. 281), says: "Perhaps a good way to develop the skill we are aiming for is for each student to build his own table of integrals. He may, for example, make a notebook in which various sections are headed by standard forms.... *Making* such a notebook probably has educational value. But once it is made, it should rarely be necessary to refer to it!"

The role of the teacher in sharpening students' skills should not be minimized, but we should be wary of thinking we have provided students with general strategies if we merely allude to global approaches while discussing particular problems in the classroom. Interviews with students who are asked to solve problems "out loud" provide evidence of the strategies they devise when left on their own. A week before her final exam, a student (who earned an A in the course) made the following comments as she examined $\int (1/(e^x+1))\,dx$: "Well, right now I'm just thinking of the various methods I know. There is substitution, which I like more than any of the others, and integration by parts, and partial fractions. Maybe if I use substitution...I'll play with that..." After a first unsuccessful attempt at the substitution $u=(e^x+1)$, she looked for ways to use, first, integration by parts, and then partial fractions.

Not at all coincidentally, her text (Sherman K. Stein's Calculus and Analytic Geometry) presents, in order, these techniques of indefinite integration: "some basic facts"; *the substitution technique*, using a table of integrals; *integration by parts*; *rational functions*; and *partial fractions*. The student had acquired a strategy by default! And although her fidelity to the order of text presentation may be somewhat unusual, one major aspect of her approach is not: given any problem, she tries a series of techniques on it in some prescribed order, rather than trying to determine what particular technique is appropriate for that problem. This has been called the "shopping list" approach to integration and is a poor strategy to employ for at least two reasons:

1. If the appropriate technique to employ lies near the bottom of the "shopping list," the student may waste much time considering others.
2. An awkward but ultimately successful approach near the top of the list may absorb much time and energy, while a more appropriately chosen technique further down the list might solve the problem with dispatch.

Unfortunately, experience indicates that this kind of "shopping list" approach to integration is rather common among students left on their own to develop integration strategies.

Isolating an effective strategy. A proficient problem-solver may well develop and employ a consistent problem-solving strategy without being aware of it. And even if he recognizes that there is a consistency to his behavior, he may not be able to elucidate it. (As a trivial example, close your eyes and try to create a sequence of instructions describing how to tie a pair of shoelaces. Your difficulty in delineating such a strategy in no way contradicts the fact that you've tied your shoelaces the same way, day in and day out, for years.)

When trying to develop any sort of problem-solving strategy, it is often profitable to examine the behavior of proficient problem-solvers ("experts") working on typical problems. In doing so, however, one should be wary of focusing only on the *solutions* which are produced. (By a "solution" I mean the application of the ultimately chosen technique to produce an answer.) Much more important than the product is the process, that is, the means by which experts arrive at the means of approach to the problem. As noted above, they may be unaware that they are employing a strategy, or be unable to delineate it in any case. A pattern may only become apparent after a large number of solution processes have been examined and compared.

I began the search for an explicit integration strategy by keeping track of all my thoughts and actions as I evaluated the more than one hundred miscellaneous integrals at the back of a standard text. While it was by no means apparent what kind of strategy I might have been using, there was no doubt that I had been using one: *in more than 75% of the problems I had chosen my method of approach and begun implementing it within 20 seconds!* The consistent speed with which I chose my approach rendered untenable any hypothesis of an unsystematic approach. And since more than a year had elapsed since I had last taught integration or solved many integrals, I was clearly not relying on memory alone. Thus there was a strategy, be it pattern recognition or something more elaborate. The task was to find it. More precisely, the task was to construct an implementable "program" that would replicate my decision-making pattern. The major facet of the strategy was clear from my observations: the *form* of the integrand should suggest the appropriate technique of integration. For example, the strategy should call for noting that the integrand in $\int x \sec^2 x\, dx$ is a product of dissimilar functions and should therefore be done by parts. Having made this observation, one notes that the terms in the integrand are a polynomial and a circular function. Under these conditions the problem-solver sets the polynomial to be "U" in the formula for parts. Thus a person trained in the strategy should decide "$\int x \sec^2 x\, dx$ should be approached by parts, with $U=(x)$ and $dV=(\sec^2 x\, dx)$" within perhaps twenty seconds of examining the problem—a performance comparable to mine.

We should stress here that the "program" developed for the novice problem-solver need not be the actual strategy that the "expert" uses—if, indeed, one can even determine what that strategy is. After much practice, for example, the expert may have developed an approach that relies heavily on pattern recognition. The strategy given to students may instead rely on a series of decision criteria. "Success" does not mean that students using the strategy *think* the way the "experts" do, but rather that they make the same decisions in comparable amounts of time. (R. P. Boas has written me of a comparable "success": "The NSF used to have a computer program that predicted fellowship awards extremely well, although it was known *not* to use the strategy that panelists used.") The first test of the strategy, then, is a "dry run": someone (not necessarily a student) is trained in the strategy and restricted to performing only those actions that the "program" dictates. The strategy is a success when the results, over a broad spectrum of problems, are comparable to an expert's unrestricted performance. One now faces the task of creating instruction which makes the strategy accessible to students.

An outline of the strategy, and the instructional materials. The strategy is divided into three major stages: simplifying, classifying, and modifying the integrand.

Simplifying the integrand. Start by looking for a preliminary simplification to reduce the complexity of the integrand. Among other things, consider various algebraic decompositions, the use of algebraic or trigonometric identities, and certain "obvious" substitutions: for the "inside" function $g(x)$ if the integrand contains a term of the form $f(g(x))$, for particularly complicated terms, and for the denominators of certain rational functions. The idea is to make tentative explorations hoping to discover a significant simplification of the problem, and to avoid lines of attack that begin to be complicated. (At this point one wants rapid returns. We can return to complex operations at another time.)

For example, the problem-solver should almost automatically consider the substitution $u=(x^2-9)$ in the problem $\int (x/(x^2-9))\, dx$ that we discussed in the second section and never have to consider the more complicated alternatives. As a second example, consider $\int (\cos x+\sin x)^2\, dx$. A rather sophisticated computer integration program had some trouble with this one because it had to integrate $\int \cos^2 x\, dx$, $\int 2 \cos x \sin x\, dx$, and $\int \sin^2 x\, dx$ separately. In our strategy, the problem-solver should be on the lookout for the identity $\sin^2 x+\cos^2 x=1$. If one applies this identity after expanding the integrand, the problem becomes trivial.

Classifying the integrand. Herein lies the meat of the strategy. After performing all plausible preliminary simplifications, the problem-solver uses the *form* of the integrand to indicate the

appropriate method of solution. If the integrand is:

1. a rational function of x, a five-step algorithm for decomposing the integrand and integrating the resulting terms is prescribed;

2. a product of certain types, integration by parts is suggested (a sub-strategy is provided for determining the choice of U and dV);

3. a combination of trigonometric functions, the particular combination will suggest either a substitution, the use of a reduction formula, or (in rare cases) a special substitution;

4. a function of $\pm(u^2 \pm a^2)^{n/2}$, or of e^x, or of $(ax+b)^{m/n}$, a particular type of substitution is recommended.

As a matter of fact, the techniques described in the first two stages above are sufficient to solve more than three-fourths of the miscellaneous exercises one sees in textbooks—enough to earn most students a solid B on a test. The third stage, when necessary, depends more on insight and pattern recognition. The purpose of the instruction is to lay out some of the options students may want to consider when faced with difficult problems.

Modifying the integrand. Given that the first two stages have failed to solve a problem, several options are available. In brief, the student can (1) try to exploit similarities between the current problem and previously solved problems, (2) resort to some special manipulations which are appropriate for small subclasses of problems, or (3) use a needs analysis to see what *would* make it possible to perform the integration successfully and find a way of introducing it into the problem.

The materials that provide students with the strategy serve as a replacement for miscellaneous exercises and are designed for use after students have seen the basic techniques of integration. They consist of a workbook text, *Integration: Getting It All Together*, and a solutions manual. The text is divided into three parts, which correspond to the three stages of the strategy described above. Each part of the text concludes with exercises designed to check both the students' use of the strategy and their ability to employ the particular techniques of integration. All of the exercises are solved, with extensive comments, in the solutions manual. Most solutions are presented in two stages. First the students select the techniques they think appropriate to solve the problem, and then they justify their choice. They compare their rationale with the one given in the solutions manual before actually working through a solution. After employing the suggested technique, they can compare solutions (for efficiency) with the one given.

Working through the materials is "slow going." It should be stressed, however, that the materials replace most or all of the time students spend working on the miscellaneous exercises. A comparison of total study times is given in the next section.

Results of preliminary testing. The real test of the strategy is, of course, whether students can use it. There are two major factors to be considered in evaluating the materials. First, do they enable students to evaluate integrals more consistently and effectively than the standard means of instruction? And second, what is the cost (in study time) to students? Since using this strategy burdens the student with learning the strategy in addition to the individual techniques of integration, whatever benefits might be obtained by using it should be carefully contrasted to possible costs in study time.

In order to minimize the possibility of both the Hawthorne effect and our own self-deception, we first tested the materials in an environment which was far less than optimal for their success. The instructor of a second-quarter calculus class agreed, four days before an examination, to distribute the materials to a randomly chosen half of his class. (Two students were absent the day he handed the materials out, so eleven of twenty-six students actually received them.) These eleven students were asked to prepare for the test using the materials, rather than by solving the numerous miscellaneous exercises in their text. No promises regarding the effectiveness of the materials were made. It was stressed that they had never been tested and that we had no way of knowing a priori whether they would help or hinder the students. The class was promised that the final grades for the experimental and control groups would be determined separately, should either group outperform the other.

Most important, we kept a "hands off" policy once the students were given the materials. No attempt was made to coordinate the materials with, or adapt them to, the text used for the class (Sherman K. Stein's Calculus and Analytic Geometry). No one was available to elucidate or clarify the strategy for the students in the experimental group, so whatever they learned would be from the materials alone. We could certainly expect better results if the instructor used the strategy as part of his classroom presentation. The test results were as follows.

Seven of the questions on the examination were indefinite integrals. The other two questions dealt with theoretical issues related to integration and serve as an additional check of the randomizing process. The tables below give the average percent scores for the two test groups on the two sets of questions.

	Q.1	Q.2	Q.3	Q.4	Q.5	Q.6	Q.7	Average
Test Group	82.7	56.4	80.0	74.6	83.6	31.8	33.5	63.2
Control Group	70.0	62.0	61.3	60.0	80.0	19.4	20.5	53.3
Difference	+12.7	−5.6	+18.7	+14.6	+3.6	+12.4	+13.0	+9.9

TABLE 1: Scores on Indefinite Integrals

	Q.1	Q.2	Average
Test Group	48.2	74.2	61.2
Control Group	59.1	64.5	61.8
Difference	−10.9	+9.7	−0.6

TABLE 2: Scores on Other Questions

Note that the experimental and control groups performed quite comparably on the two questions not calling for indefinite integration. The advantage accruing to the control group is negligible, and the closeness of the scores lends credence to the assumption that the random selection did indeed split the class into two groups of comparable ability. The two groups' performance on the indefinite integrals is clearly not comparable, however. The degree of statistical significance one obtains from these data depends, of course, on the test one chooses to apply. Of the standard statistical approaches, the worst results were given by comparing the means of the two small-size samples. Because of the large variance of scores in each of the two groups, the means were only significantly different at the $p<.15$ level. The best results were obtained by comparing the experimental and control groups on each of the seven questions, as paired dependent samples. Under the assumption of comparability, each group should outscore the other on approximately the same number of questions, and (in sum) the differences should even out. The fact that the experimental group outscored the control group on six of seven questions, and did so by more than ten percent on five of them, was statistically significant at $p<.01$. Thus, preliminary testing under far less than optimal circumstances suggests strongly that the materials enhance students' ability to evaluate indefinite integrals. We must then ask: At what cost in study time are these benefits obtained?

It was in answer to this question that the research produced its biggest surprise. The students who used the materials indicated that they averaged (during the four-day period) a total of 7.1 hours each preparing for the exam, and the students in the control group indicated that they averaged (during the same four days) a total study time of 8.8 hours each. In other words, *the students who used the materials performed better on the exam, after spending less total study time preparing for it.*

In retrospect, this should not be terribly surprising. The student working from miscellaneous exercises may feel compelled to keep practicing by the fear that something new may "crop up," while the student with the more clearly defined task of learning to apply the strategy can decide more easily when they have mastered it. The results of that mastery speak for themselves.

Conclusions. There are two components to effective problem-solving in any subject area. Problem-solvers must have (1) a mastery of the techniques that are commonly used in that area; and (2) a strategy that enables them to determine quickly and accurately which techniques they will bring

to bear on any particular problem.

In most subject domains, the standard teaching practice is to provide students with instruction on the techniques appropriate to the subject area being studied. It is generally left for each student to develop, through "experience" and "practice," an individual systematic approach to solving problems in that domain. The work on integration discussed here illustrates some claims we believe hold in many subject areas—and perhaps in all of mathematics. We assert the following and hope these points will be subjected to rigorous and extensive examination:

1. We cannot expect most students to develop coherent strategies for approaching problems even in narrowly prescribed subject domains, if they are left on their own to do so.

2. "Experts" in a subject area may consistently apply coherent strategies without being aware of it. Even if they are aware of a regularity in their behavior, they may not be able to elaborate the strategy which produces it.

3. With careful observation of "expert" behavior, it may be possible to delineate a strategy which, under ideal conditions, produces problem-solving behavior comparable to that of the experts.

4. Such a strategy, if properly presented to students, can have them solving problems more effectively after less total study time.

We hope this article will stimulate some further examination of the role of problem-solving strategies in mathematics.

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MATHEMATICS COURSES FOR ELEMENTARY TEACHERS

Marc Swadener

A persistent problem facing the mathematics education community is that of the mathematical competence of elementary school teachers. Although literature concerning the level of mathematical competence is almost nonexistent, there is widespread feeling that, in general, elementary school teachers have very limited backgrounds in mathematics. Since no specific information on this is available, one can only judge on the basis of what is specifically required for certification as an elementary teacher by each of the states. According to *Requirements for Certification, 1975–76*, by Elizabeth H. Woellner [1], twenty-seven states don't specifically mention mathematics as a requirement for elementary teacher certification, thirty-nine require *at most* one college-level mathematics course, ten require five, six, or seven semester hours, and only one requires nine semester hours. On this basis alone, it is easily seen that elementary teachers have minimal competence in mathematics, not approaching what is recommended by the Committee on the Undergraduate Program in Mathematics (CUPM) [2].

For what reasons would elementary teachers seek additional work in mathematics? There are three basic reasons. First would be to increase their competence as teachers, which could be considered their professional responsibility.

Second, they would seek additional work for increased pay, since most school systems are on incremental pay increases based on additional college work. In many instances, this additional work must be for graduate credit at accredited institutions. Third is for recertification as a teacher. Many states issue teaching certificates that are valid for a limited amount of time. Teachers then must complete additional college work in order to renew the teaching certificate. Once again, many times this work is required to be graduate work. The second and third reasons for seeking additional work are more mercenary in nature, but in today's society, this is a reality that must be faced.

Assuming, within this context, that to improve their competence the only realistic alternative is to take college or university courses and recognizing that education courses are appropriate for courses

that have a pedagogical orientation; those that have a mathematical content orientation would most appropriately be placed under the category of mathematics courses. Thus, the only remaining question is whether the courses should be for undergraduate or graduate credit. The rationale for graduate credit is the following:

(a) Frequently, undergraduate credit does not apply for pay increments or recertification.
(b) Many elementary teachers seek master's degrees in elementary education and many institutions will apply only graduate credit for such a degree.
(c) Graduate credit for a mathematics course for elementary teachers does *not* imply that it should be applicable for a graduate degree in mathematics. It merely implies that the course is valuable, applicable as advanced work in the chosen field of study (elementary education), and is intellectually honest in that context (it need not be watered down, but it also need not be advanced theoretical mathematics).

It is unlikely that the mathematical competence of elementary school teachers will increase unless they have some form of additional work in mathematics. This work should have the involvement of mathematicians and, if the large universities are in fact the leaders in education in this country, then large university departments should be providing leadership in this field. Thus, a logical question is, "To what degree are departments of mathematics in major universities involved in teacher education and, more specifically, in graduate courses for elementary teachers?"

To answer this question, the sixty-one highest-rated institutions on the scale "Effectiveness of Doctoral Programs" in mathematics [3] were surveyed.

Each of the departments of mathematics in the selected sixty-one institutions was asked (a) to indicate if they did or did not offer a mathematics course designed for elementary teachers, (b) to list the titles of such courses, (c) to list the texts used, and (d) to indicate whether each course was offered for graduate credit or not.

In addition to this information, it was determined which of the institutions were involved in teacher education in general and whether or not each was accredited by the National Council for Accreditation of Teacher Education (NCATE). This information was obtained from *Education Directory, 1971–72: Higher Education* [4]. The results of the survey are presented below. Data are presented in the form of a table. The list of textbooks used in the courses revealed no preponderance of use of any one of the textbooks mentioned, and thus is not included in this report.

Table 1. Number of departments of mathematics contacted concerning the offering of mathematics courses specifically designed for elementary teachers and summary of results of the survey.

Number of departments (institutions) contacted	61
(a) Number offering teacher preparation programs	51
(b) Number accredited by NCATE	48
Number of departments responding	50
(a) Number offering teacher preparation programs	41
(b) Number accredited by NCATE	37
(c) Number offering mathematics course designed for elementary teachers	24
(1) for undergraduate credit	21
(2) for graduate credit	8

Some discussion of Table 1 seems desirable. First a large portion of the institutions surveyed did offer teacher education programs (84%), and were accredited by NCATE (70%). Of the responding departments, forty-one (82%) were institutions which offered teacher education programs, and thirty-seven of these institutions were accredited by NCATE (90%). These facts show two things. First that teacher education is in fact an emphasis at the institutions and not something unrelated to departmental responsibilities. Second that the responding departments were representative of the sixty institutions in the sample.

Almost half (24 in number, or 48%) of the responding departments offer at least one mathematics course designed for elementary school teachers. Only 41 of the 50 institutions, however, have teacher education programs. Thus, more than half (59%) of the responding institutions that have teacher education programs offered at least one mathematics course for elementary teachers. This leads one to believe that many (perhaps 41%) of these institutions either (a) do not provide a mathematics course for elementary teachers, (b) require mathematics teachers to select their mathematics course(s) from regular mathematics offerings, or (c) provide the mathematics courses through the department of education.

Only one-fifth (20%) of the responding institutions with teacher preparation programs offer a graduate-credit mathematics course for elementary teachers. In terms of those institutions offering a mathematics course for elementary teachers, only one-third (33%) of these institutions offer at least one course for graduate credits. It is heartening to see these few, but disheartening to think that most institutions do not offer in-service elementary teachers any mathematics course that would count as graduate credit. Should an in-service elementary teacher wish to take a mathematics course for credit in one of these institutions (leaders in the nation), he or she has one of three alternatives: (a) take a non-graduate credit course (not a reasonable alternative for reasons cited previously), (b) take no course at all (not a good alternative since it does not improve the teacher's mathematical competence), or (c) take a course in the department of education. Even though the last of these alternatives is the best under the circumstances, it is not the best situation. The department of mathematics should be involved, but the data clearly show that in the nation's leading universities this is not true. In fact, several responses indicated that the question of mathematics for the elementary teacher was the responsibility of the department of education rather than the department of mathematics. In one case, an institution responded that it offered a master's degree in mathematics for elementary teachers.

A summary of the survey indicates that departments of mathematics in sixty-one of the nation's leading universities evidently are divided in their concern over the responsibility for undergraduate mathematics courses for prospective elementary teachers. Further, in a clear majority of cases (two-thirds or four-fifths, depending on how one counts) the mathematics departments offer no alternatives for the education of the in-service elementary teacher in the area of mathematics content.

This is not to say that departments of mathematics are solely at fault for the situation. Some institutions don't grant graduate credit for non-degree students, others are dropping master's degree programs, and the pressures within departments of mathematics are great to produce research mathematics products.

The writer suggests that these major institutions should exercise some leadership in addressing the problem of the mathematical competence of elementary school teachers. Certainly nothing can be gained by disregarding the problem. It is a fact that, with thousands of elementary school teachers, there is the potential for a large market for a program to improve their mathematical competence. One step in this direction would be to offer graduate credit for a well-designed course in mathematics for the elementary teacher.

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1. Elizabeth H. Woellner, Requirements for Certification, University of Chicago Press, Chicago, 1975.

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3. K. D. Roose and C. J. Anderson, A Rating of Graduate Programs, American Council on Education, Washington, D.C., 1970, 97.

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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Beginning in January, 1979, this Department will be edited by A. P. Hillman.

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

No solutions (except those accompanying proposals) should be sent to Professor Hillman.

*An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether or not you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.*

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

ELEMENTARY PROBLEMS

Solution of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary problems in this issue should be typed (with double spacing) and (if possible) submitted in duplicate, and should be mailed before January 31, 1979. Please enclose a self-addressed label or card (for acknowledgement)

E 2731. *Proposed by Bruce Reznick, Duke University*

Characterize all polynomials which satisfy $P(x,y)=P(y,x)$ and $P(x,y)=P(x,x-y)$.

E 2732. *Proposed by Peter Sjögren, University of Uppsala, Sweden*

It is easy to see that one can label the squares of an $n\times n$ chessboard by integers from 1 to n^2 so that the difference between labels of neighboring squares does not exceed n. Is this best possible? (Two squares are neighbors if they share a common side.)

E 2733. *Proposed by Jim Fickett, University of Colorado*

Let S_i, $i=1,2,\ldots,m$ be subsets of $[0,1]$; each S_i is a finite union of disjoint intervals. Let $l(S_i)$ be the sum of the lengths of these intervals. Assume that $l(S_i)=\epsilon$, $l(S_i\cap S_j)\leqslant \epsilon^2$ $(i\neq j)$, where $\epsilon>0$ is fixed. How large can m be?

E 2734. *Proposed by Melvin Hausner, Courant Institute, New York University*

Let $A=(a_{ij})$ be a real square matrix such that $a_{ij}>0$ for $i\neq j$. Show that all entries of e^A are positive.

E 2735. *Proposed by I. P. Goulden and D. M. Jackson, University of Waterloo, Ontario*

Let n be a fixed integer and define

$$f_k(x)=\sum_{r\geqslant 0} x^{nr+k}/(nr+k)! \qquad (0\leqslant k\leqslant n-1).$$

For $P\subset S=\{0,1,\ldots,n-1\}$, let $F(P;\,x)=F(P)$ be the square matrix whose entries are indexed by elements of P and the (i,j)th entry is $f_{i-j}(x)$, $i,j\in P$. (We put $f_r(x)=f_k(x)$ if $r\equiv k \bmod n$.)

If n is even, show that $\det F(P)=\det F(S\setminus P)$ for all $P\subset S$. Generalize.

E 2736. *Proposed by E. Ehrhart, University of Strasbourg, France*

Let Δ be a closed triangle and $P_0,A_0,P_1,A_1,\ldots$ an infinite sequence of points in a plane. Assume that $P_i\neq P_{i+1},A_i\neq A_{i+1}$, each A_i is a vertex of Δ and the midpoint of the segment $[P_i,P_{i+1}]$, and that $[P_i,P_{i+1}]\cap\Delta=\{A_i\}$.

Prove that $P_n=P_0$ for some positive n.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Polynomial Inequality

E 2655 [1977, 386]. *Proposed by Michael W. Chamberlain, University of Santa Clara*

Prove that for integral $n\geqslant 2$ and $0<x<n/(n+1)$ one has $(1-2x^n+x^{n+1})^n<(1-x^n)^{n+1}$.

Solution by Allen Stenger, American Express Company, Phoenix, Arizona. Let $0<x<1$, Then $1-2x^n+x^{n+1}$ and $1-x^n$ are positive and we can rewrite the given inequality as follows:

$$1-2x^n+x^{n+1}<(1-x^n)\sqrt[n]{1-x^n}\,,$$

$$(1-x^n)(1-\sqrt[n]{1-x^n}\,)<(1-x)x^n,$$

$$\frac{1-x^n}{1-x}<\frac{1-(1-x^n)}{1-\sqrt[n]{1-x^n}}\,.$$

Since $(1-t^n)/(1-t)$ strictly increases in $0<t<1$, the last inequality is equivalent to

$$x<\sqrt[n]{t-x^n}\,,\quad \text{i.e., to}\quad x<1/\sqrt[n]{2}\,.$$

It remains to notice that $1/\sqrt[n]{2}>n/(n+1)$ i.e. $(1+1/n)^n>2$ $(n\geqslant 2)$.

Also solved by D. M. Bloom, Clark Givens, Robert Breusch, L. E. Mattics, Otto Ruehr, Robert Shafer, Michael Skalsky, St. Olaf Problem Group, and the proposer.

Similarity Ratio of Some Simplices

E 2657 [1977, 386]. *Proposed by G. Tsintsifas, Thessaloniki, Greece*

Let $A=A_0A_1\cdots A_n$ and $B=B_0B_1\cdots B_n$ be regular simplices in $\mathbf{R}^n$. Assume that the ith vertex of B

lies on the ith face of A $(0 \leqslant i \leqslant n)$. What is the minimal value of their similarity ratio λ (λA congruent to B, $\lambda > 0$)?

Solution by Dimitri Nakassis, Washington D.C. Let $a_k = \overrightarrow{OA_k}$, $b_k = \overrightarrow{OB_k}$ $(0 \leqslant k \leqslant n)$ where O is the centroid of A. The dot products $a_k \cdot a_k = \alpha$ and $a_r \cdot a_k = \beta$ $(r \neq k)$ are independent of the indices and $\alpha > \beta$.

It follows from our hypotheses that there exists a row-stochastic matrix $P = (p_{ij})$ with $p_{ii} = 0$ $(0 \leqslant i \leqslant n)$ such that

$$b_i = \sum_{j=0}^{n} p_{ij} a_j \qquad (0 \leqslant i \leqslant n).$$

Using this we obtain for $i \neq j$

$$\begin{aligned} 2\lambda^2(\alpha - \beta) &= \lambda^2 |a_i - a_j|^2 = |b_i - b_j|^2 \\ &= \left| \sum_{k=0}^{n} (p_{ik} - p_{jk}) a_k \right|^2 \\ &= \alpha \sum_{k=0}^{n} (p_{ik} - p_{jk})^2 + \beta \sum{}' (p_{ik} - p_{jk})(p_{ir} - p_{jr}) \\ &= (\alpha - \beta) \sum_{k=0}^{n} (p_{ik} - p_{jk})^2, \end{aligned} \tag{1}$$

where Σ' is summation over all ordered pairs (k,r) with $k \neq r$.

The last equality in (1) follows after squaring

$$\sum_{k=0}^{n} (p_{ik} - p_{jk}) = 0.$$

It follows from (1) that $2\lambda^2 \geqslant p_{ij}^2 + p_{ji}^2$ and consequently $2\lambda \geqslant p_{ij} + p_{ji}$ $(i \neq j)$. Summing over all pairs (i,j) with $i < j$, we obtain

$$n(n+1)\lambda \geqslant \sum_{i<j} (p_{ij} + p_{ji}) = n + 1.$$

Hence $\lambda \geqslant 1/n$ and it follows from the above proof that $\lambda = 1/n$ if and only if $p_{ij} = 1/n$ for all $i \neq j$.

Also solved by E. G. Straus, and the proposer.

Integral Cyclic Quadrilaterals of Given Perimeter

E 2660 [1977, 487]. *Proposed by E. Erhart, University of Strasbourg, France*

A quadrilateral is *cyclic* if its vertices lie on a circle. Find the number of congruence classes of cyclic quadrilaterals having integral sides and a given perimeter n.

Solution composed from those of L. E. Mattics (University of South Alabama) and the proposer. Given any ordered 4-tuple of positive numbers (a,b,c,d) with $n = a+b+c+d$ and $a,b,c,d < n/2$, one can show that there is precisely one cyclic quadrilateral with consecutive sides a,b,c,d. In the following, a,b,c,d are positive integers $< n/2$ and n is fixed throughout.

Let S_i $(i = 0,1,2,3,4)$ be the sets consisting of all 4-tuples of the form

$$\begin{array}{ll} (a,b,c,d) & \text{for } S_0 \\ (a,a,b,c) & \text{for } S_1 \\ (a,a,b,b) & \text{for } S_2 \\ (a,a,a,b) & \text{for } S_3 \\ (a,a,a,a) & \text{for } S_4 \end{array}$$

where in each case the co-ordinates sum up to n. Thus, for instance S_4 is empty unless n is divisible by 4. Let T_i $(i=0,1,2,3,4)$ be the subset of S_i consisting of 4-tuples of the form listed above with the extra condition that a,b,c,d are distinct. Thus $T_4=S_4$.

If $s_i=|S_i|$, $t_i=|T_i|$, we have

$$\begin{aligned} &t_4=s_4, t_3=s_3-s_4, t_2=s_2-s_4,\\ &t_1=s_1-2(s_3-s_4)-s_2,\\ &t_0=s_0-6t_1-4t_3-3t_2-t_4=s_0-6s_1+3s_2+8s_3-6s_4. \end{aligned}$$

The number A_n which we should compute is given by

$$A_n=3\cdot\frac{t_0}{4!}+2\cdot\frac{t_1}{2}+2\cdot\frac{t_2}{2}+t_3+t_4.$$

Replacing t_i's in terms of s_i's we obtain

$$A_n=\frac{1}{8}(s_0+2s_1+3s_2+2s_4).$$

A direct counting gives

$$s_0=\binom{n-1}{3}-4\binom{n-k-1}{3}\qquad\text{where } k=\left[\frac{n-1}{2}\right].$$

$$s_1=\begin{cases}\frac{1}{8}(n^2-4n+8) & \text{if } n\equiv 0(\operatorname{mod} 4),\\ \frac{1}{8}(n-1)^2 & \text{if } n\equiv 1(\operatorname{mod} 4),\\ \frac{1}{8}(n-2)^2 & \text{if } n\equiv 2(\operatorname{mod} 4),\\ \frac{1}{8}(n-3)(n+1) & \text{if } n\equiv 3(\operatorname{mod} 4),\end{cases}$$

$$s_2=\frac{1+(-1)^n}{4}(n-2),$$

$$s_4=1\quad\text{if } n\equiv 0(\operatorname{mod} 4),\quad =0\text{ otherwise.}$$

Replacing these into the formula for A_n we obtain

$$A_n=\begin{cases}\frac{1}{96}(n^3-3n^2+20n) & \text{if } n\equiv 0(\operatorname{mod} 4),\\ \frac{1}{96}(n^3-7n+6) & \text{if } n\equiv 1(\operatorname{mod} 4),\\ \frac{1}{96}(n^3-3n^2+20n-36) & \text{if } n\equiv 2(\operatorname{mod} 4),\\ \frac{1}{96}(n^3-7n-6) & \text{if } n\equiv 3(\operatorname{mod} 4).\end{cases}$$

Note that if n is even then A_n is the integer closest to $n(n^2-3n+20)/96$ and if n is odd then A_n is the integer closest to $n(n^2-7)/96$.

Functional Characterization of Least Common Multiples

E 2661 [1977, 487]. *Proposed by Steve Galovich, Carleton College, Northfield, Minnesota*

Find all functions f which satisfy

(i) $f(x,x)=x$;

(ii) $f(x,y)=f(y,x)$;

(iii) $(x+y)f(x,y)=yf(x,x+y)$,

assuming that the variables and the values of f are positive integers.

Solution by Alan H. Stein, University of Connecticut. Since (ii) and (iii) imply that $f(x,y)$ for $x \neq y$ can be determined if the values of $f(u,v)$ are known for all (u,v) such that $u+v<x+y$, the solution is unique.

We claim that $f(x,y) = \mathrm{LCM}(x,y)$ satisfies all three conditions. This is obvious for (i) and (ii). The property (iii) follows from

$$(x+y)\{x,y\} = (x+y)xy/(x,y) = (x,y)xy/(x,x+y)$$
$$= y\{x,x+y\}$$

where $\{x,y\} = \mathrm{LCM}(x,y)$, $(x,y) = \mathrm{GCD}(x,y)$.

Also solved by the proposer and 95 other readers.

A Maximization Problem for (0, 1)–Matrices

E 2662 [1977, 487]. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada*

For an $n \times n$ (0, 1)–matrix A, let A' denote the complementary matrix, i.e., $A' = J - A$ where J is the matrix with all entries equal to one. Define $\sigma_n = \max \Sigma(AA')$ where $\Sigma(X)$ denotes the sum of all entries of a matrix X and the maximum is taken over all $n \times n$ (0, 1)–matrices A. Show that $\sigma_n \geqslant (n^3 - n)/3$. Does the equality hold for all n?

Solution by Ellen Hertz, Belford, New Jersey. Let $A = (a_{ij})$, $A' = (a'_{ij})$ and so $a'_{ij} = 1 - a_{ij}$. Note that the expression

$$a(i,j,k) = a_{ij}a'_{jk} + a_{jk}a'_{ki} + a_{ki}a'_{ij}$$

is equal to 0 or 1. Indeed, at most one of three summands in $a(i,j,k)$ can be equal to 1. Since $a(i,i,i) = 0$, for $1 \leqslant i \leqslant n$ we obtain

$$3\sum(AA') = \sum_{i,j,k} a(i,j,k) \leqslant n^3 - n.$$

The equality holds if we set $a_{ij} = 1$ for $i \leqslant j$ and $a_{ij} = 0$ for $i > j$. Hence $\sigma_n = (n^3 - n)/3$ for all n.

Also solved by Floyd Barger, Robert Bishop, James Boyce, Robert Breusch, Fred Buckley, Peter de Buda, Thomas Foregger, Patrick Gardner, Curtis Greene, Yasuhiko Ikeda, Daniel Johnson, Longwood College Problem Group, L. E. Mattics, Mark Merriman, Albert Nijenhuis, James Reeds, Martin Schaefer (Germany), and Pavol Tomasta (Czechoslovakia). Partial solutions by Kenneth Bernstein, J. C. Binz (Switzerland), Eli Isaacson, Jordan Levy, Gerald Thomson, and the proposer.

Editor's Comment. Greene has characterized those matrices A for which $\Sigma(AA') = \sigma_n$. This is the case iff there exists a permutation matrix P such that $B = P^{-1}AP$ can be partitioned so that

(i) its diagonal blocks are of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (1), \text{ or } (0),$$

(ii) all the entries below (resp. above) the diagonal blocks are ones (resp. zeros).

An Old Exercise

E 2663 [1977, 487]. *Proposed by Marius Solomon, student, University of Pennsylvania*

Let $f: (0, \infty) \to \mathbf{R}$ be differentiable and assume that $f(x) + f'(x) \to 0$ when $x \to \infty$. Show that $f(x) \to 0$ as $x \to \infty$.

Solution by H. Kestelman, University College, London, England. Let $\epsilon>0$ and choose $a>0$ so that $|f(x)+f'(x)|<\epsilon$ for $x\geqslant a$. If $x>a$, then by Cauchy's mean value theorem there exists $b\in(a,x)$ such that

$$\frac{f(x)e^x-f(a)e^a}{e^x-e^a}=f(b)+f'(b).$$

It follows that

$$|f(x)-f(a)e^{a-x}|<\varepsilon|1-e^{a-x}|,\ |f(x)<|f(a)|e^{a-x}+\varepsilon|1-e^{a-x}|.$$

Hence $|f(x)|<2\epsilon$ for sufficiently large x.

Also solved by the proposer and 61 other readers.

Editor's Comment. Many solvers note that the problem is old and appears, for instance, in G. H. Hardy, *A Course in Pure Mathematics*, 10th ed., 1967, p. 281, problem 50. For a more general result, see the solution of Advanced Problem 5875 [1974, 92].

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to Professor R. C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. Solutions of Advanced Problems in this issue should be type (with double spacing) on separate signed sheets and should be mailed before January 31, 1979.

6228. *Proposed by Ivan Vidav, University of Ljubljana, Yugoslavia*

Let A be a C^*-algebra with unit 1, and let e and f be two projections of A such that $e+f$ is invertible in A. Show that $e\cap f=2e(e+f)^{-1}f$. ($e\cap f$ is the supremum of the set of all projections $h\in A$ such that $h\leqslant e$ and $h\leqslant f$.)

6229*. *Proposed by David W. Erbach, Cambridge University, England*

Suppose that the plane is tiled with regular hexagons in the customary manner. Color each black or white independently with probability $\frac{1}{2}$. What is the expected size of a connected monochromatic component? What is the probability that there is an infinite component? (Similar questions have arisen in the attempt of geneticists to understand the coloration of mammals.)

6230. *Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France*

$X(t)$ is the perimeter length of the convex hull of $b(s)_{0\leqslant s\leqslant t}$, where b is the standard brownian motion in the euclidean plane. Compute $\mathbf{E}(X(t))$.

6231. *Proposed by Terry R. McConnell, Stanford University*

Let A be a subset of $\mathbf{R}^2$ with Lebesgue measure >0. Prove that A contains the vertices of a square.

6232*. *Proposed by Allan Wm. Johnson, Jr., Defense Communications Agency, Washington, D.C.*

Prove or disprove: Given any integer $G>13$, there exist distinct integers $x_i>0$ such that $G^3=\Sigma_{i=1}^5 x_i^3$.

6233. *Proposed by James Lynch and Jan Mycielski, University of Colorado*

Prove that $\Pi_{n=1}^{\infty}(1-a^{-n})$ is irrational for every integer a with $|a|>1$.

SOLUTIONS OF ADVANCED PROBLEMS

The Maximum in Random Samples

6050 [1975, 856]. *Proposed by Donald E. Knuth, Stanford University*

Let $X_1,X_2,Y_1,\dots,Y_{m+n}$ be independent random variables, where X_1 and X_2 have common

distribution F and $Y_1,\ldots,Y_{m+n}$ have common distribution G. Prove that

$$\frac{1}{2} \leqslant \Pr(X_1+\max(Y_1,\ldots,Y_m) \leqslant X_2+\max(Y_{m+1},\ldots,Y_{m+n})) \leqslant \frac{n}{m+n}$$

when $m \leqslant n$ and G is differentiable.

Solution by Harry Lass, Jet Propulsion Laboratory, California Institute of Technology. Let $Y_1, Y_2,\ldots, Y_m, Z_1, Z_2,\ldots, Z_n$ be independent random variables with a common p.d.f., say $f(y)$, whose c.d.f. is $F(y)=\int_{-\infty}^{y} f(\alpha)d\alpha$. Let $U=\max(Y_1, Y_2,\ldots, Y_n)$, $V=\max(Z_1, Z_2,\ldots, Z_n)$.

LEMMA 1. $\Pr(V \geqslant U)=n/(n+m)$. Clearly, if $n+m$ independent samples are chosen, the probability that the largest value belongs to the set $(Z_1, Z_2,\ldots, Z_n)$ is $n/(n+m)$.

LEMMA 2. If $g(w)$ is the p.d.f. for $W=V-U$, then $g(w) \geqslant g(-w)$ for $w \geqslant 0$, provided $n \geqslant m$.

From elementary probability theory the p.d.f. for V is $nf(v)[F(v)]^{n-1}$, the p.d.f. for U is $p_2(u)=mf(u)[F(u)]^{m-1}$, so that the p.d.f. for $W=V-U$ is

$$g(w)=mn\int_{-\infty}^{\infty} f(u)[F(u)]^{m-1}f(u+w)[F(u+w)]^{n-1}du,$$

and

$$g(-w)=mn\int_{-\infty}^{\infty} f(u)[F(u)]^{m-1}f(u-w)[F(u-w)]^{n-1}du$$

$$=mn\int_{-\infty}^{\infty} f(u+w)[F(u+w)]^{m-1}f(u)[F(u)]^{n-1}du.$$

From $F(u+w) \geqslant F(u)$ for $w \geqslant 0$, it follows that for $n \geqslant m$, $[F(u+w)]^{n-m} \geqslant [F(u)]^{n-m}$. It then follows that $g(w) \geqslant g(-w)$.

Let X_1 and X_2 be independent random samples from a continuous random variable X, and let $h(\xi)$ be the p.d.f. for X_1-X_2. From Lemma 2 it follows that $h(\xi)=h(-\xi)$, albeit this result is trivial. We are interested in

$$\Pr(X_1-X_2 \leqslant V-U)=\int_{-\infty}^{\infty}\int_{\xi}^{\infty} g(w)h(\xi)dwd\xi$$

$$=\int_{-\infty}^{0}\int_{\xi}^{\infty} g(w)h(\xi)dwd\xi+\int_{0}^{\infty}\int_{\xi}^{\infty} g(w)h(\xi)dwd\xi$$

$$=\int_{0}^{\infty}\int_{-\xi}^{\infty} g(w)h(\xi)dwd\xi+\int_{0}^{\infty}\int_{\xi}^{\infty} g(w)h(\xi)dwd\xi$$

$$=2\int_{0}^{\infty}\int_{0}^{\infty} g(w)h(\xi)dwd\xi-\int_{0}^{\infty}\left[\int_{0}^{\xi} g(w)dw-\int_{-\xi}^{0} g(w)dw\right]h(\xi)d\xi.$$

From

$$\int_{0}^{\infty} g(w)dw=\frac{n}{n+m}, \qquad \int_{0}^{\infty} h(\xi)d\xi=\frac{1}{2}, \qquad \text{and} \int_{0}^{\xi} g(w)dw \geqslant \int_{-\xi}^{0} g(w)dw,$$

it follows that

$$\Pr(X_1+U \leqslant X_2+V) \leqslant \frac{n}{n+m}, \qquad m \leqslant n.$$

Moreover, from

$$\Pr(X_1-X_2 \leqslant V-U)=\int_{-\infty}^{0}\int_{\xi}^{\infty} g(w)h(\xi)dwd\xi+\int_{0}^{\infty}\int_{\xi}^{\infty} g(w)h(\xi)dwd\xi$$

$$\geqslant \int_0^\infty \int_{-\xi}^\infty g(w)h(-\xi)\,dw\,d\xi + \int_0^\infty \int_\xi^\infty g(-w)h(\xi)\,dw\,d\xi$$

$$\geqslant \int_0^\infty \int_{-\xi}^\infty g(w)h(\xi)\,dw\,d\xi + \int_0^\infty \int_{-\infty}^{-\xi} g(w)h(\xi)\,dw\,d\xi$$

$$\geqslant \int_0^\infty \int_{-\infty}^\infty g(w)h(\xi)\,dw\,d\xi = \frac{1}{2},$$

we get

$$\frac{1}{2} \leqslant \Pr(X_1 + U \leqslant X_2 + V) \leqslant \frac{n}{n+m}.$$

Also solved by David Gootkind, Peter Hooper (Canada), Marcel Neuts, S. M. Samuels, Ronald Thisted, and the proposer.

Hooper notes that the differentiability of G is not needed to establish the result. The left-hand inequality is true for all G, whereas the second is true for all continuous G.

Bijection $2^\omega \leftrightarrow N^\omega$

6128 [1977, 62]. *Proposed by Martin Schechter and Peter Borwein, University of British Columbia*

Let 2^ω be the set of all sequences with entries 0 or 1 and let N^ω be the set of all sequences with entries from the nonnegative integers. Can one construct a bijection f from 2^ω onto N^ω with the property that for any sequence X in 2^ω one can compute the first n entries of $f(X)$ given only the first m entries of X (where m may depend on X and n)?

Solution by Marvin Solomon, University of Wisconsin, and Arthur Solomon, Western Illinois University. A first attempt is to let each digit of $f(X)$ represent the length of a sequence of consecutive zeros. For example, $f(001010111001\ldots) = 211002\ldots$. However, this method fails for any X having finitely many ones. In fact, every attempt to construct the required f must fail for essentially the same reason.

The proof uses König's infinity lemma for trees. This lemma has the curious property that it seems trivial until one sees an application. (See D. E. Knuth, *The Art of Computer Programming*, vol. I: Fundamental Algorithms, Addison Wesley, 1968, pp. 381–383.)

LEMMA. *Let T be a finitely branching tree. (Each node has a finite, although not necessarily bounded, number of sons.) Then either T is finite, or there is an infinite path from the root of T.*

Suppose that a function f satisfying the hypotheses of the problem exists. Say that a tuple x of zeros and ones *determines* the integer n, if the first entry of $f(X)$ is n whenever X begins with x. By the assumptions on f, every integer is determined by some x, and every element of 2^ω has a finite prefix that determines some integer. Say x is *minimal* if x determines some n, but no proper prefix of x determines any integer. A machine computing f cannot compute any entries of its output as long as the input "so far" is a prefix of a minimal tuple. Let T be the tree whose nodes are labeled with prefixes of minimal tuples, such that the root is labeled with the zero-tuple and the father of the node labeled x is labeled by the tuple obtained by deleting the last component of x. T is finitely branching; indeed, each node has at most two sons. T is also infinite, since each integer n is determined by a tuple labeling a distinct node of T. Thus T is an infinite path $n_0 n_1 \ldots$, where each n_k is labeled by a proper prefix of n_{k+1}. This path determines a sequence X such that no finite prefix of X determines any integer.

Also solved by José Luis de Miguel (Spain), Richard Enison, R. Israel (Canada), Jordan Levy, O. P. Lossers (Netherlands), Joel Spencer, William Myers, and the proposers.

Distance from a Simple Closed Curve

6129 [1977, 62]. *Proposed by E. H. Kronheimer, Birkbeck College, University of London, England*

Let S be a simple closed curve in the plane. Prove that, unless S is a circle, it is always possible to find four points, p, q, u, v on S and a point x inside S, such that u and v belong to distinct components of $S\setminus\{p,q\}$, and x is nearer to both p and q than it is to either u or v.

Solution by the proposer. Let A be the topological disk bounded by S. There is no difficulty provided we can assume that A contains a circle C, center k, such that the set $H=S\cap C$ has at least two points. For then, if H is not connected, we can take $x=k$, take p and q to be points in distinct components of H, and take u and v to be points—not in H—belonging to distinct components of $S\setminus\{p,q\}$. If, on the other hand, H is connected, then (as long as S is not a circle) H must be an arc whose end-points we can take as p and q. For u we take the midpoint (on H) of H, for v the first point at which the extension of the line-segment uk beyond k intersects S, and for x the midpoint of the line-segment uv.

Suppose therefore that A contains no such circle C. Then every point x of A has a unique nearest point $f(x)$ in S. The map f so defined is necessarily discontinuous since it would otherwise retract the disk A onto its boundary S. Since S is compact, it follows that A contains a sequence $\{a_n\}$, converging to some point a, whose image $\{f(a_n)\}$ converges to some point b in S, where $b\neq f(a)$. The continuity of $d=$"distance" as a function on $A\times A$ implies that the sequence $\{d(a_n,f(a_n))\}$ converges to $d(a,b)$; whereas the continuity of "distance from S" as a function on A implies that the same sequence converges to $d(a,f(a))$. Thus b and $f(a)$ are equidistant from the point a: a contradiction.

Also solved by Gustaf Gripenberg (Finland), and J. G. Mauldon.

A Partition of the Rational Points of the Plane

6130 [1977, 62]. *Proposed by Erwin Just, Bronx Community College, and Eugene Levine, Adelphi University*

Prove that there exists a partition of the rational points of the plane into an infinite number of everywhere dense subsets such that each straight line containing two rational points will have a nonempty intersection with each of the subsets.

Solution by Alvin F. Martin, Hamden, Connecticut. Let $\{A_n\}_{n=1}^{\infty}$ be a partition of the rationals into dense subsets. For example, let A_n contain all quotients a/b, where a and b are relatively prime integers, $b\neq 0$, and the nth prime is the first prime which divides b, and let A_1 also contain the integers. The desired partition is $\{B_n\}_{n=1}^{\infty}$, where B_n consists of all ordered pairs (x,y) of rationals such that $\max\{x,y\}\in A_n$. In fact, if L is a line in the plane containing two rational points, then $L\cap B_n$ is dense in L for all n, unless L is horizontal or vertical, and does not intersect the origin. In this case, $L\cap B_n$ is dense in $L-L_0$, where L_0 is an open finite segment of L.

Also solved by Roy Davies (England), Harold Enison, Marguerite Gerstell, William Gorman, George Graham, Jr., Burrell Helton, R. Israel (Canada), William Lambert, Jr. (Costa Rica), Jordan Levy, O. P. Lossers (Netherlands), J. G. Mauldon, John Morgan II, Gene Ortner, Richard Poppen, Martin Schechter, Arthur Solomon, and the proposers.

A Dense Set in $L^1(-\infty,\infty)$

6131 [1977, 62]. *Proposed by Lee A. Rubel, University of Illinois, Urbana*

Suppose $\phi\geqslant 0$ is in $L^1(-\infty,\infty)$, ϕ vanishes outside of $[a,b]$, and ϕ is strictly decreasing on $[a,b]$. Prove that the span of the translates of ϕ is dense in $L^1(-\infty,\infty)$.

Solution by Arnold S. Goldstein, student, Armstrong State College and Savannah State College. By its strict monotonicity, $\phi(x)\neq 0$ on $[a,b)$, a set of nonzero Lebesgue measure. Let $\hat{\phi}(x)=\int_{-\infty}^{\infty}e^{ixt}\phi(t)\,dt$, the Fourier transform of ϕ. By corollary 6E, p. 17 in R. R. Goldberg, *Fourier Transforms*, New York, 1961, we have $\hat{\phi}(x)\neq 0$ for $x\in(-\infty,\infty)$. Let

$$T_\phi=\left\{h\in L^1(-\infty,\infty):h(x)=\sum a_k\phi(x+c_k)\text{ for some finite set of real } a_k \text{ and } c_k\right\}.$$

Now, by Wiener's theorem (loc. cit., p. 33) we have $\overline{T}_\phi=L^1$.

Also solved by Paul Chauveheid (Belgium), Roger Cooke, Gustaf Gripenberg (Finland), R. Israel (Canada), A. A. Jagers (Netherlands), Harald Krogstad (Norway), J. van de Lune (Netherlands), and the proposer.

Editor's notes. Van de Lune shows that it is enough to assume that ϕ is decreasing on $[a,b]$ and strictly decreasing on some subinterval $[\alpha,\beta]$. The proposer notes: "It would be instructive to find a proof not using Wiener's theorem, since it could presumably also deal with the case of $L^p(-\infty,\infty)$ for other p."

A Function with the "Darboux Property"

6132 [1977, 140]. *Proposed by Mihai Eşanu, Liceul "Gheorghe Lazăr," Bucharest, Romania.*

Find all the functions $f:R\to R$ with the Darboux property such that for some $n\geq 1$, $f^n(x)=-x$ for all x. (Here $f^2=f\circ f$, etc.)

Solution by the Purdue–Calumet Coffee Club, Purdue University Calumet Campus. We assume that "Darboux property" means "intermediate value property."

Let f satisfy the iterative property (I) in the statement of the problem. Then f is clearly a bijection. This, combined with the Darboux property, says that f is monotone. Further

$$f(-x)=f[f^n(x)]=f^n[f(x)]=-f(x),$$

so f is odd. In particular, then $f(0)=0$. Then (I) implies f is decreasing and $xf(x)<0$ for $x\neq 0$.

Now pick any $x_0>0$ and let $x_k=f(x_{k-1})$. Then

$$(-1)^k x_k>0,\quad x_n=-x_0$$

hence $(-1)^n x_n=(-1)^{n+1}x_0>0$, so n is an odd integer.

Assume $x_1>-x_0$. Since f is decreasing and odd,

$$x_2=f(x_1)<f(-x_0)=-f(x_0)=-x_1,$$

so we obtain inductively,

$$(-1)^k x_k>(-1)^{k+1}x_{k+1}$$

which leads to the contradiction $x_0>-x_n$.

Similarly, the assumption $x_1<-x_0$ leads to the contradiction $x_n<-x_0$. Thus the only solution of (I) with the Darboux property is $f(x)=-x$.

Also solved by Thomas Foregger, William Gorman III, F. B. Strauss, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed

two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

FILMS

Similar Triangles. Produced by Bruce and Katharine Cornwell. Seven and one-half minutes, 16 mm., sound and color. Purchase or rent from International Film Bureau, 332 South Michigan Avenue, Chicago, IL 60604. (Film Guide, free of charge.)

This film, which is advertised as a "natural sequel to *Trio for Three Angles*," is a colorful, fast-moving collage of similar triangles. There is a musical sound track, but there is no narration and there are only two captions; hence the film would need appropriate commentary by the instructor both before and during the film viewing if it is to be used in a junior or senior high school class. However, this lack of verbal accompaniment does allow this seven-and-a-half-minute color film to be used as a meaningful mathematical enrichment exercise for grade school children.

In an opening segment, which seems unrelated to the title, three angles and their interiors are exhibited via color-coded concentric arcs which are intersected to form a triangle. The film then uses color-coding of congruent angles in the display of similar triangles as nested sets. This is well done; but unfortunately no other relative positions for similar triangles are shown.

In the next section, the sides of a triangle are bisected and the three bisection points are joined to form four triangles interior to the given triangle. The angles are again color-coded and the middle triangle is successively superimposed on the other three. In this case, the emphasis seems to be more on the demonstrated congruence relation among the four interior triangles than on the similarity relation between the interior triangles and the original. The remainder of the film extends these notions by partitioning the sides of a triangle first into three, and then into four, congruent segments and joining these parts to form smaller similar triangles. The graphic display of these triangles moving to coincide with each other, and the patterns produced by the coloring used to distinguish directly and oppositely congruent triangles, make this second section more attractive, both pedagogically and visually, than the first.

Richard J. Allen and Judith N. Cederberg
St. Olaf College, Northfield, Minnesota

MISCELLANEA

14. [In mathematics] as in all sciences, every era has its pressing questions: some questions come alive and simultaneously attract the most enlightened intellects, as if in spite of themselves and without any mutual agreement. It often seems that the same ideas occur to many people at once, as if by revelation; if we wonder how this happens, it is easy to find sources in the works of our predecessors, where the ideas were generated without their creators' being aware of them.

Up to now, science has not made much of these frequent coincidences in research. Distasteful competition, undignified rivalry have been their principal product. Nevertheless it is not hard to recognize that the facts show that scientists are no more made for isolation than anyone else, that they too belong to their era, and that sooner or later they will increase their abilities tenfold by association. When that happens, think how much time will be saved for science!

E. Galois, 1832.

(*Écrits et Mémoires mathématiques d'Évariste Galois*, edited by R. Bourgne and J.-P. Azra, Paris, 1962, p. 19. Suggested by A. A. Mullin, free translation by the Editor.)

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL. *Mathematics, Statistics, and Systems for Health.* Norman T.J. Bailey. Wiley, 1977, x + 222 pp, $18.50. [ISBN: 0-471-99500-2] Cursory, often bewildering discussions for the lay public concerning various quantitative methods used in health related fields--including statistics, modelling, computers, operations research, systems analysis, and system dynamics. LCL

BASIC, T. *Geometry in Modules, An Informal Course.* Muriel Lange. A-W, 1975. *Teachers' Manual,* x + 222 pp, $8.36 (P) [ISBN: 0-201-04129-4]; *Module A,* 128 pp; *Module B,* 124 pp; *Module C,* 128 pp; *Module D,* 128 pp, $2.48 each. A self-study, discovery approach for those who find a formal high school geometry course too difficult. New terms are presented pictorially. Modules A & B cover measurement; Module C--congruence and parallelism; Module D--similarity, circles and spheres. JNC

BASIC, T(9-13), S. *The Mathematics of the Energy Crisis.* R. Gagliardi. Intergalactic Pub, 1978, viii + 72 pp, (P); *Solutions to Problems,* 10 pp, (P). A small collection of practical arithmetical word problems pertaining to energy production and consumption. LCL

PRECALCULUS, T(13: 1). *College Algebra, Third Edition.* Abraham Spitzbart. A-W, 1978, viii + 390 pp, $12.95. [ISBN: 0-201-07482-6] A straightforward, unexciting presentation of the standard topics beginning with properties of the real numbers. An appendix offers a concentrated review of elementary algebra. JNC

PRECALCULUS. *College Algebra and Trigonometry, Second Edition.* William L. Hart, Bert K. Waits. Heath, 1978, xiv + 482 pp, $13.95. [ISBN: 0-669-01460-5] One more edition of the many permutations of topics put out by William L. Hart over the years. LLK

PRECALCULUS, T(13). *Algebra and Trigonometry for College Students.* Richard S. Paul, Ernest F. Haeussler, Jr. Reston Pub, 1978, xii + 560 pp, $15.95. [ISBN: 0-87909-031-6] Begins with arithmetic of real numbers and algebraic expressions. Covers polynomial functions, linear equations, exponential, logarithmic, and trigonometric functions, inequalities, conic sections. SG

PRECALCULUS, T(13: 1). *Lessons in College Algebra.* Robert A. Nowlan. Har-Row, 1978, xv + 671 pp, $13.95. [ISBN: 0-06-044862-8] This book is written to serve both as a precalculus text and as a text for a terminal course in mathematics. It has a clear monochrome printing with an abundance of examples and exercises. MU

PRECALCULUS, T(13: 1). *Algebra and Trigonometry.* Margaret L. Lial, Charles D. Miller. Scott F, 1978, 560 pp, $13.95. [ISBN: 0-673-15133-6]; *Solutions Guide: Complete Answers and Selected Solutions,* 110 pp, free (P) [ISBN: 0-673-17609-6]; *Instructor's Guide and MathLab with Quizzes and Tests,* free (P); *Study Guide,* Laura Cameron, 222 pp, $4.95 (P). [ISBN: 0-673-15139-5] Assumes basic algebra but includes three chapters of review preceded by a diagnostic pretest. Supplementary materials make this a complete package for a self-paced program. LLK

PRECALCULUS, T(13: 1). *Algebra and Trigonometry: A Functions Approach, Second Edition.* Mervin L. Keedy, Marvin L. Bittinger. A-W, 1978, xvi + 715 pp, $13.95 (P) [ISBN: 0-201-03870-6]; *College Algebra: A Functions Approach, Second Edition,* xiii + 528 pp, $12.95 (P). [ISBN: 0-201-03866-8] Retains the format of the first edition (TR, October 1974) with the addition of the changes made in the 1977 textbook, *Fundamental Algebra and Trigonometry,* by these same authors. *College Algebra* volume contains eleven of the fifteen chapters in the combined text. LLK

PRECALCULUS, T*(13: 1). *Plane Trigonometry, Second Edition.* Bernard J. Rice, Jerry D. Strange. Prindle, 1978, x + 291 pp, $12.50. [ISBN: 0-87150-250-X] This edition incorporates chapter tests and many more examples and exercises, while maintaining the basic approach of the first edition (TR, August-September 1975). A very nice text. JNC

PRECALCULUS, T(13: 1-2). *College Algebra.* Richard Thompson. Prindle, 1978, x + 646 pp, $12.95 (P). [ISBN: 0-87150-299-1] First covers linear and quadratic equations to their graphs then considers functions (universe, rational, exponential, log), systems of linear equations, matrices, determinants, inequalities, complex numbers, polynomials, sequences, elementary combinatorics. SG

EDUCATION, P. *Capes mathématique, Préparation à l'oral: Nouvelles leçons développées et commentées.* Denis Richard, Ibrahim Rihaoui, Jean-Marc Braemer. Hermann (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978. *Première Partie: Lecons 19-27,* 174 pp, 36F (P) [ISBN: 2-7056-1389-7]; *Deuxième Partie: Lecons 28-35,* 165 pp, 36F (P). [ISBN: 2-7056-1390-0] Continuation of 1977 work of the same title (TR, February 1978): lesson guides for teachers on topics such as equivalence relations, indefinite integrals, and greatest common divisor. LAS

HISTORY, L. *Early Editions of Euclid's Elements.* Charles Thomas-Stanford. Alan Wofsy Fine Arts, 1977, ix + 67 pp, $65. [ISBN: 0-915346-29-X] A limited re-edition of a 1926 publication. Consists of a short commentary upon dates, type and illustrations of early editions; an annotated bibliography of these editions with some sample reproductions as well as twelve beautiful plates. A nice coffee table book for rich mathematicians. JNC

HISTORY, S(13-18), L. *Géometrie Grecque*. Paul Tannery. Arno Pr, 1976, vii + 187 pp, $11. [ISBN: 0-405-07340-2] Reprint of an essay, first published in 1887, of the same name and by the same author. This text is a history of the elementary geometry of some of the ancient Greek geometers. RJA

HISTORY, S, L**. *Whom the Gods Love: The Story of Evariste Galois*. Leopold Infeld. NCTM, 1978, xvii + 323 pp, $9.78. [ISBN: 0-87353-125-6] Reprint of the 1948 Whittlesey House edition, the seventh in NCTM's "Classics in Mathematics Education" series. Infeld's tale is entirely non-scientific, concentrating instead on the powerful, romantic themes of politics and revolution in nineteenth century France. LAS

FOUNDATIONS, P, L. *A Preface to Logic*. Morris R. Cohen. Dover, 1972, xi + 209 pp, $3.50 (P). [ISBN: 0-486-23517-3] Unaltered republication of 1944 edition. Essays dating from 1916 to 1944 which explore the periphery of logic, the relations of logic to the rest of the universe, the philosophical presuppositions which give logic its meaning, and the applications which give it importance. GHM

FOUNDATIONS, S(15-18), P, L. *Studies in Inductive Probability and Rational Expectation*. Theo A. F. Kuipers. Reidel, 1978, xii + 145 pp, $22.50. [ISBN: 90-277-0882-7] An attempt to explicate the concepts of confirmation and rational belief in set-theoretic terms. FLW

FOUNDATIONS, P, L. *Fuzzy Automata and Decision Processes*. Ed: Madan M. Gupta, George N. Saridis, Brian R. Gaines. North-Holland, 1977, xiv + 496 pp, $37.50. [ISBN: 0-444-00231-6] Seventeen papers on the basic ideas and theory of fuzziness, along with seven papers on applications. Many of the papers are from a 1975 conference held at MIT. Extensive bibliography. FLW

FOUNDATIONS, P. *Lecture Notes in Mathematics-611: First Order Categorical Logic*. Michael Makkai, Gonzalo E. Reyes. Springer-Verlag, 1977, xviii + 301 pp, $14.30 (P). [ISBN: 0-387-08439-8; 3-540-08439-8] Self-contained monograph on results and methods in the theory of topoi and related categories motivated by model theory and first order logic. GHM

ALGEBRA, T(17-18: 1, 2), P. *Semisimple Lie Algebras*. Morikuni Goto, Frank D. Grosshans. Lect. Notes in Pure and Appl. Math., V. 38. Dekker, 1978, vii + 480 pp, $37.50 (P). [ISBN: 0-8247-6744-6] Systematic, self-contained (presumes only an understanding of linear algebra) exposition of semisimple Lie algebras, which stresses their connection with Lie groups and algebraic groups. Concludes with discussion of representation theory and the classification of real simple Lie algebras. Includes exercises. LCL

ALGEBRA, T(14-15: 1). *An Introduction to Abstract Algebra*. Thomas A. Whitelaw. Blackie, 1978, ix + 166 pp, £4.95 (P). [ISBN: 0-216-90488-9] Careful introduction to the most basic topics in group theory and ring theory. Emphasis on leading the student to an understanding (and appreciation) of the structure of proofs and abstract thinking. For example, includes very helpful discussions of quantifiers, logical implication and proof by contradiction, fundamental topics which most texts leave for students to sort out for themselves. Author succeeds in presenting ideas very clearly, judged from the undergraduate student's point of view. An excellent primer. GHM

ALGEBRA, S(16), P, L. *Lectures on Group Theory and Particle Theory*. H. Bacry. Gordon, 1977, xvii + 580 pp, $70. Translation from French of lectures on applications of group representations to particle physics. Extensive preliminary information, many exercises, and a bibliography arranged by subject. The price seems exorbitant. JEG

FINITE MATHEMATICS, T(13: 1, 2). *Introduction to College Mathematics with A Programming Language*. E.J. LeCuyer. Springer-Verlag, 1978, xii + 420 pp, $14.80. [ISBN: 0-387-90280-5; 3-540-90280-5] Aimed at nonscience students; a computer programming language, APL, is introduced in the development of the mathematics. APL notation is used throughout the text. Topics include elementary set theory, logic, matrices and determinants, functions and graphing, basic differential and integral calculus, probability and statistics, trigonometry. LCL

FINITE MATHEMATICS, T(13-14: 1). *Mathematics for Modern Management*. Sherman Chottiner. Har-Row, 1978, xvi + 576 pp, $16.95. [ISBN: 0-06-041265-8] Progressions, the mathematics of finance, linear, quadratic, exponential and logarithmic functions, matrices and linear programming, basic calculus, all with applications. Light, conversational presentation; close weaving of examples, applications, and exercises. LCL

CALCULUS, T(13: 2). *Calculus with Analytic Geometry, Third Edition*. Edwin J. Purcell. P-H, 1978, xiv + 945 pp, $19. [ISBN: 0-13-112052-2] A careful, precise and attractive presentation of the traditional topics of elementary calculus; changes incorporated in this edition to make the text more readable include: new illustrative examples, elimination of excessive algebraic manipulation, simplification of some proofs and the inclusion in each chapter of intuitive previews and review exercises. The exercises, however, remain unimaginative with only the traditional applications. JNC

CALCULUS, T(13-15: 1, 2). *Applied Calculus for Business and Economics with an Introduction to Matrices*. Gerald Alan Beer. Winthrop Pub, 1978, xiv + 478 pp, $14.95. [ISBN: 0-87626-039-3] A straightforward treatment of the basic ideas along with applications in economics and a brief introduction to matrix algebra. FLW

REAL ANALYSIS, T(17: 2), L. *Real and Functional Analysis*. A. Mukherjea, K. Pothoven. Math. Concepts and Methods in Sci. and Eng., V. 6. Plenum Pr, 1978, x + 529 pp, $25. [ISBN: 0-306-31015-5] A first-year graduate text designed for a rigorous one-year course either in measure and integration theory or in Banach spaces, Hilbert spaces, and spectral theory. A comprehensive text with many applications and exercises. TRS

DIFFERENTIAL EQUATIONS, P. *Potential Theory on Locally Compact Abelian Groups*. Christian Berg, Gunnar Forst. Ergebnisse der Math., B. 87. Springer-Verlag, 1975, vii + 197 pp, $25.40. [ISBN: 0-387-07249-7] An exposition of potential theory for transient convolution semigroups based on lectures given at the University of Copenhagen in 1973-74. JAS

NUMERICAL ANALYSIS, T(16-18: 1, 2), S, P, L. *Mathematical Modeling and Digital Simulation for Engineers and Scientists*. Jon M. Smith. Wiley, 1977, xii + 332 pp, $21. [ISBN: 0-471-80344-8] Presents techniques for the mathematical modeling of continuous and discrete systems. Numerical methods based on sampled-data and discrete system technology; developed from both the time-domain and frequency-domain viewpoints. Four parts: mathematical properties of continuous and discrete processes; linear system simulation on a digital computer; nonlinear system simulation on a digital computer; fast function evaluation. Appendices. Index. RJA

NUMERICAL ANALYSIS, P. *Padé and Rational Approximation, Theory and Applications*. Ed: E.B. Saff, R.S. Varga. Acad Pr, 1977, xiii + 491 pp, $19.50. [ISBN: 0-12-614150-9] Proceedings of a conference held December 15-17, 1976 in Tampa, Florida. This volume contains theory and applications, exposition and research papers. JAS

NUMERICAL ANALYSIS, T(15-16: 1), L. *Basic Numerical Mathematics, V. 2: Numerical Algebra*. John Todd. Int. Ser. Num. Math., V. 22. Birkhäuser, 1977, 216 pp, sFr. 48. [ISBN: 3-7643-0811-7] Numerical linear algebra. Matrix norms and induced norms. Direct and iterative methods for linear systems. Eigenvalue problems. Applications to 2-point boundary value problems and least squares. Exercises often illustrate convergence and roundoff problems. RWN

NUMERICAL ANALYSIS, P. *A Numerical Study of Stiff Two-Point Boundary Problems*. P.W. Hemker. Math. Centre Tracts, No. 80. Math Centrum, 1977, ix + 178 pp, Dfl. 22 (P). [ISBN: 90-6196-146-7] Includes essential theoretical background. Adaptation of finite difference methods. Global methods. Exponentially fitted methods. Comparisons of methods. Extension to nonlinear problems. RWN

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-584: Accélération de la Convergence en Analyse Numérique*. C. Brezinski. Springer-Verlag, 1977, 313 pp, $13.70 (P). [ISBN: 0-387-08241-7; 3-540-08241-7] Includes a review of convergence, summability and extrapolation. Features the ε-algorithm and several other methods for accelerating convergence, with extensions to matrices and Banach space. Applications to Taylor series, eigenvalue problems, nonlinear systems and continued fractions. RWN

NUMERICAL ANALYSIS, S(16-18), P. *Numerical Analysis of Spectral Methods: Theory and Applications*. David Gottlieb, Steven A. Orszag. CBMS Reg. Conf. in Appl. Math., No. 26. SIAM, 1977, 172 pp, $12.25 (P). Based on part of a series of lectures presented at the NSF-CBMS Regional Conference held at Old Dominion University, August 2-6, 1976. Appendices. References. Bibliography. Index. RJA

FUNCTIONAL ANALYSIS, T(16-17: 1), P, L. *Integral Equations*. Guido Hoheisel. Trans: A. Mary Tropper. Frederick Ungar Pub, 1968, 100 pp. An attractive introduction to nonsingular linear integral equations for scientists, engineers, and mathematicians. Basics on linear algebra and Hilbert space prepare the way for studying integral equations in operator-theoretic language. Text is similar in spirit and content to Widder's monograph. Exercises. TRS

FUNCTIONAL ANALYSIS, P. *Contractive Projections in C_1 and C_∞*. Jonathan Arazy, Yaakov Friedman. Memoirs No. 200. AMS, 1978, iv + 165 pp, $9.20 (P). [ISBN: 0-8218-2200-4] The authors characterize all the possible contractive projections and their ranges in the space C_1 of trace class operators on ℓ_2 and in the dual space C_∞ of compact operators on ℓ_2. TRS

FUNCTIONAL ANALYSIS, P. *Complex Fourier Transformation and Analytic Functionals with Unbounded Carriers*. J.W. de Roever. Math. Centre Tracts, No. 89. Math Centrum, 1978, xvii + 200 pp, Dfl. 24 (P). [ISBN: 90-6196-155-6]

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-639: Topological Vector Spaces, The Theory Without Convexity Conditions*. Norbert Adasch, Bruno Ernst, Dieter Keim. Springer-Verlag, 1978, 125 pp, $9 (P). [ISBN: 0-387-08662-5; 3-540-08662-5] Starting with the notion of a string (i.e., a sequence of vector space subsets which are balanced, absorbing and "summative"), the authors develop a general TVS theory which is free of duality theory and which includes many of the theorems of the locally convex theory. TRS

OPTIMIZATION, T*(15-17: 1), S*, P, L**. *Model Building in Mathematical Programming*. H.P. Williams. Wiley, 1978, xiv + 330 pp, $32; $14.50 (P). [ISBN: 0-471-99526-6; 0-471-99541-X] Highly readable guide for students and managers. Emphasis on the general principles of model building and interpretation of results as opposed to excessive devotion to algorithms. Concludes with a full discussion of twenty simplified but representative linear and integer programming problems from a wide variety of contexts. No exercises. LCL

OPTIMIZATION, T(15-17: 1). *Theory and Application of Mathematical Programming*. G. Mitra. Acad Pr, 1976, ix + 214 pp, $18. [ISBN: 0-12-500-450-8] Linear programming, alternate computational forms of the simplex method, duality theory, mixed and integer programming. General theory and algorithms for nonlinear programming. Concentrates on the fundamental theory. Examples. Exercises. RWN

OPTIMIZATION, S(16-17), P. *Lecture Notes in Economics and Mathematical Systems-152: Notwendige Optimalitätsbedingungen und ihre Anwendung*. Andreas Kirsch, Wolfgang Warth, Jochen Werner. Springer-Verlag, 1978, 157 pp, $9 (P). [ISBN: 0-387-08537-8; 3-540-08537-8] Let f be a mapping from V to W, where V and W are real vector spaces, and let $M \subseteq V$. These notes deal with necessary conditions that f take its "minimum" value on M at $\bar{X}$. JD-B

OPTIMIZATION, P. *Lecture Notes in Control and Information Sciences-6 and 7: Optimization Techniques*. Ed: J. Stoer. Springer-Verlag, 1978. *Part 1*, xiii + 528 pp, $21.50 (P) [ISBN: 0-387-08707-9; 3-540-08707-9]; *Part 2*, xiii + 512 pp, $21.50 (P). [ISBN: 0-387-08708-7; 3-540-08708-7] Over 100 papers (including 5 invited survey lectures) from the IFIP conference held in Würzberg, September 1977. Very diverse topics: world models, optimal control, differential games, immunology, environmental systems, mathematical programming, urban systems, economics, operations research, and computer networks. LAS

OPTIMIZATION, T(15-17: 2), S, L. *Introduction to Nonlinear Optimization, A Problem Solving Approach.* David A. Wismer, R. Chattergy. North-Holland, 1978, xii + 395 pp, $19.95. [ISBN: 0-444-00234-0] An introductory text which discusses most of the major topical areas of nonlinear optimization. Makes extensive use of examples, illustrative solved problems, and supplementary problems with answers. Includes a useful list of references. CEC

OPTIMIZATION, T(17-18: 1, 2), P. *Approximation et Optimisation.* Pierre-Jean Laurent. Hermann (US Distr: SMPF, 14 E. 60th St., NY 10022), 1972, xiii + 531 pp, $28. An advanced treatment for students and practitioners of applied mathematics. The presentation contains the necessary functional analysis but is very formal and contains essentially no indication of the usefulness of the hundreds of lemmas, theorems, and propositions which are proved. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-618: Extreme Eigen Values of Toeplitz Operators.* I.I. Hirschman, Jr., Daniel E. Hughes. Springer-Verlag, 1977, 145 pp, $8.30 (P). [ISBN: 0-387-07147-4; 3-540-07147-4] Systematic account of certain problems in the asymptotic behavior of eigenvalues. JEG

ANALYSIS, S(18), P. *Simple-Periodic and Non-Periodic Lamé Functions.* J.K.M. Jansen. Math. Centre Tracts, No. 72. Math Centrum, 1977, 104 pp, Dfl. 13 (P). [ISBN: 90-6196-130-0] This work is a result of antenna research carried out by the Numerical Mathematics group at the Technological University in Eindhoven. MU

ANALYSIS, P. *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, V. I.* Ed: R.J. Knops. Research Notes in Math., No. 17. Fearon-Pitman, 1977, 241 pp, $13.50 (P). [ISBN: 0-273-01128-6] This volume consists mainly of written versions of invited lectures given at two short symposia held in May and September 1976 at Heriot-Watt University. MU

ANALYSIS, T(18), S, P. *The Theory of Approximate Methods and Their Application to the Numerical Solution of Singular Integral Equations.* V.V. Ivanov. Trans: A. Ideh. Noordhoff Inter, 1976, xvii + 330 pp, Dfl. 80. [ISBN: 90-286-0036-1] A translation of a 1968 monograph on the theory, application, and basic structure of methods and algorithms for the approximate solution of singular integral equations. Many well-known results are cited without proof; however, references are given. The text is readable, but the printing is troublesomely faint. MU

ANALYSIS, T(17-18), S, P. *Homogeneous Banach Algebras.* Hwai-chiuan Wang. Lect. Notes in Pure and Appl. Math., V. 29. Dekker, 1977, vii + 204 pp, $19.75 (P). [ISBN: 0-8247-6588-5] An expository guide, with many examples, to homogeneous Banach algebras along the route from standard graduate level analysis courses to current research problems. The approach seems measured and accessible to advanced students. JAS

ANALYSIS, T(18), P. *Operational Calculus.* I.Z. Shtokalo. Trans: V. Kumar. Pergamon Pr, 1976, xi + 333 pp, $40. [ISBN: 0-85274-321-1] An advanced level introduction heavily laced with historical notes. Topics include applications to linear differential equations with constant, almost-periodic, quasi-periodic, and bounded coefficients, differential equations with deviating arguments, and stability of solutions. SG

ANALYSIS, S(16-18), L. *Problems and Theorems in Analysis, V. II: Theory of Functions, Zeros, Polynomials, Determinants, Number Theory, Geometry.* G. Pólya, G. Szegö. Trans: C.E. Billigheimer, Springer-Verlag, 1976, xi + 319 pp, $14.80 (P). [ISBN: 0-387-90291-0; 3-540-90291-0] Paperbound "Study Edition" of the 1976 hardcover English translation (TR, August 1976). LAS

ALGEBRAIC GEOMETRY, P. *Lecture Notes in Mathematics-620: Moduli Theory and Classification Theory of Algebraic Varieties.* Herbert Popp. Springer-Verlag, 1977, vii + 189 pp, $8.30 (P). [ISBN: 0-387-08522-X; 3-540-08522-X] A series of lectures on moduli theory and the relationship between "rough" classification and "fine" classification of algebraic varieties. SG

DIFFERENTIAL GEOMETRY, P. *Real and Complex Singularities, Oslo 1976.* Ed: P. Holm. Sijthoff & Noordhoff Inter, 1977, vii + 686 pp, Dfl. 230. [ISBN: 90-286-0097-3] The lectures given at the Nordic Summer School in Mathematics 1976, as well as the papers reported on in the coincident symposium with the exception of John Mather's work on the genericity and finite type of topologically stable maps. JAS

DIFFERENTIAL GEOMETRY, T**(15-17: 1, 2), S, L*. *Tensor Geometry, The Geometric Viewpoint and its Uses.* C.T.J. Dodson, T. Poston. Fearon-Pitman, 1977, xiii + 598 pp, $38.50. [ISBN: 0-273-00317-8] "The title of this book is misleading." So begins the preface to a most unusual presentation of elementary differential geometry as and with an introduction to relativity theory and the "geometry of the Universe." Although the contents are usually classified as "advanced," the presentation makes an apparently successful attempt to communicate both the mathematics and the mystique to undergraduates who know calculus and linear algebra. The careful work that has gone into the text, index, and numerous exercises is unfortunately somewhat offset by the unreasonable price for the typescript text even with its half-tone illustrations. However, this is a unique and worthwhile book. Grin, and have your library bear it. JAS

DIFFERENTIAL GEOMETRY, P. *Invariant Forms on Grassmann Manifolds.* Wilhelm Stoll. Annals of Math. Stud., No. 89. Princeton U Pr, 1977, ix + 113 pp, $12. Applies the technique of fibre integration to various well known results in the theory of holomorphic bundles, such as Bott-Chern and Matsushima. Unfortunately the technique is not well discussed here, and references are not readily available, thus the exposition loses some of its power. On the whole the results are interesting and the techniques are very close to the heart of the subject. TLS

DIFFERENTIAL GEOMETRY, P. *Global Variational Analysis: Weierstrass Integrals on a Riemannian Manifold.* Marston Morse. Princeton U Pr, 1976, iii + 255 pp, $6.50 (P). [ISBN: 0-691-08077-1] A systematic study of the properties of the integral $\int_a^b F(u(t),u'(t))dt$ and its extremals on a compact differentiable manifold with a Riemannian structure. JAS

GEOMETRY, S(15), P, L. *Stereographic Projection.* B.A. Rosenfeld, N.D. Sergeeva. Trans: Vitaly Kisin. MIR (Imported by: Imported Pub, 320 W. Ohio St., Chicago, IL 60610), 1977, 54 pp, $1.25 (P). Contents include: properties and history of stereographic projection; relation to inversions with respect to a circle and sphere; representation as a mapping of complex numbers and applications to astronomy and the Lobachevskian plane. JNC

GEOMETRY, S(15), P. *Lobachevskian Geometry.* A.S. Smogorzhevsky. Trans: V. Kisin. MIR (Imported by: Imported Pub, 320 W. Ohio St., Chicago, IL 60610), 1976, 71 pp, $1.25 (P). A brief but glowing biography of Lobachevsky is followed by a discussion of hyperbolic transformations, descriptions of basic curves and proofs of major theorems. No index. JNC

GEOMETRY, P. *Dancing Curves, A Dynamic Demonstration of Geometric Principles.* Merwin J. Lyng. NCTM, 1978, vi + 16 pp, $4.44 (P). [ISBN: 0-87353-124-8] Detailed, easy-to-follow instructions for the construction of the string model and slides required for the illustrated lecture "Dancing Curves"--a dynamic illustration of conic sections and other curves and surfaces. Four sample slides are included.JNC

GEOMETRY, S(13-18), L. *Geometry and the Liberal Arts.* Dan Pedoe. St. Martin's Pr, 1978, 296 pp, $10.95. [ISBN: 0-312-32370-0] This text can help to liberally educate and will be appreciated by the liberally educated. The development focuses on persons and events of historical significance in the development of geometry. Can be enjoyed by the general reader, yet there is much for the mathematical reader to gain. First sections on Vitruvius, Dürer, da Vinci. Then form in architecture, Euclid, Cartesian and projective geometry, curves, and space. Chapter exercises. Bibliography. Index. RJA

GEOMETRY, T(17-18: 2), S, P. *Konvexe Analysis.* Jürg T. Marti. Math. Reihe, B. 54. Birkhäuser, 1977, xi + 273 pp, sFr. 68. [ISBN: 3-7643-0839-7] An introduction to the theory of convex sets, with a chapter on convex functions. Written for students of mathematics and physics, it assumes a knowledge of calculus and of the elements of linear algebra, point-set topology and functional analysis. Some problems, and an extensive bibliography. JD-B

GEOMETRY, S?. *Topics from Triangle Geometry.* D. Moody Bailey (D. Moody Bailey, Princeton, WV 24740), 1972, vi + 258 pp. A collection of independent articles, many very computational. Presumes a course in plane geometry. No index. The majority of the references are previous articles by the author. JNC

GEOMETRY, P. *Lecture Notes in Mathematics-583: Invariant Manifolds.* M.W. Hirsch, C.C. Pugh, M. Shub. Springer-Verlag, 1977, 149 pp, $8 (P). [ISBN: 0-387-08148-8; 3-540-08148-8] A presentation of the authors' work which extends the fundamental theorem of normally hyperbolic invariant manifolds. JAS

GEOMETRY, T(15-16), S, L. *Geometrie für Lehrer und Studenten, Band 2.* Gerhard Holland. Hermann Schroedel, 1977, 177 pp, DM 19,80. A relatively intuitive presentation, though rather formal in style, of groups of linear transformations in the plane, in particular similarity transformations, and affine transformations. JAS

STATISTICS, T**(13-14: 1, 2), S**, L**. *Statistics.* David Freedman, Robert Pisani, Roger Purves. Norton, 1978, xv + 589 pp, $13.95. [ISBN: 0-393-09076-0]; *Instructor's Manual,* 135 pp, (P). [ISBN: 0-393-09041-8] An excellent introductory text which is "non-mathematical" and presents many fascinating applications. A very helpful instructor's manual is available for use with the text. FLW

STATISTICS, P. *Methoden zur Analyse von kurzen Zeitreihen.* Wolfgang Birkenfeld. Birkhäuser, 1977, 185 pp, DM 34 (P). [ISBN: 3-7643-0955-5] Analysis of short time series. Presentation of problems in which short time scores arise. Decision rules for analyzing specific series. Estimators for the frequency and time domains of stochastic processes; maximum likelihood estimation; Monte Carlo techniques. References, author and subject indices. RJA

STATISTICS, T(13: 1), S. *Basic Statistical Concepts, A Self-Instructional Text, Second Edition.* Jack I. Bradley, James N. McClelland. Scott F, 1978, 209 pp, $5.95 (P). [ISBN: 0-673-15075-5] A programmed text for self-study or supplemental use. Presupposes no college mathematics. FLW

COMPUTER SCIENCE, T(13-18: 1, 2), S, L. *Computers and Man, Second Edition.* Richard C. Dorf. Boyd & Fraser, 1977, xi + 487 pp, $8.95 (P). [ISBN: 0-87835-064-0] An abridged edition of *Introduction to Computers and Computer Science* by the same author. Missing from the present text are the chapters on programming languages, calculators, minicomputers, number systems, and computer arithmetic. Chapter problems and references. Glossary and index. RJA

COMPUTER SCIENCE, T(13-18: 1, 2), S, L. *A Practical Approach to Computing.* W.Y. Arms, J.E. Baker, R.M. Pengelly. Wiley, 1976, xii + 353 pp, $23.95; $11.95 (P). [ISBN: 0-471-03324-3; 0-471-99736-6] Focuses on data handling, file processing, and system software. Begins with the organization of a computer and machine language instructions. Text divided into three parts. (1) Data structures: arrays, linked lists, stacks, queues, trees, registers, indirect addressing, subroutines. (2) File processing: serial and dynamic files, magnetic tape sorting, random access devices. (3) Software: peripheral operation, supervisors, compiling. Chapter section questions. Appendix. Index. RJA

COMPUTER SCIENCE, T(13: 1). *Business Data Processing, Second Edition.* Mike Murach. SRA, 1977, x + 443 pp, $13.95. Introduction to various types of data processing equipment, together with discussions of flowcharting ("systems flow," "systems analysis and design") and programming. Modular design. LCL

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Jay E. Goldfeather, Carleton; Lorraine L. Keller, St. Olaf; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1224 Connecticut Avenue., N.W., Washington, D.C. 20036.

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1225 Connecticut Avenue, N.W., Washington, D.C. 20036.

PERSONAL ITEMS

State University College, Geneseo, New York: Professor Paul Schaefer, Geneseo, and Professor Raymond Killgrove, California State University at Los Angeles, are participating as exchange professors for the 1978-79 academic year. Assistant Professor Donald Trasher has been appointed Chairman of the Mathematics Department. Assistant Professor Jung Tsai has been promoted to Associate Professor.

Rensselaer Polytechnic Institute: Dr. Mark Holmes of the University of California at Los Angeles has been appointed Assistant Professor in the Department of Mathematical Sciences. Associate Professor Bobby F. Caviness has been granted tenure.

C. W. Post College, Greenvale, New York: Dr. Andrew M. Rockett has been appointed Assistant Professor. Assistant Professor Susan Andima has been promoted to Associate Professor. Associate Professor Elliott Bird has been named Chairman of the Mathematics Department.

Moorhead State University: Associate Professor and Chairman Milton Legg has been promoted to Professor. Assistant Professor James Hatzenbuhler has been promoted to Associate Professor.

John W. Palmer of Brown University has joined Daniel H. Wagner, Associates.

Professor Emeritus Robert B. Lyon, Arizona State University, died on October 21, 1977, at the age of 70. He was a member of the Association for twenty-six years.

Professor Harold Glander, Carroll College, died on June 21, 1977. He was a member of the Association for thirty years.

Mr. Glemm E. Engebretsen, an Administrative Assistant with the Los Angeles Police Department, died recently (no further information). He was a member of the Association for three years.

Former Professor Francis E. Johnston, George Washington University, died on March 2, 1978, at the age of 80. He was a member of the Association for fifty-five years.

Aaron S. Strauss, University of Maryland, died on April 14, 1978, at the age of 38. He was a member of the Association for six years.

Carol V. McCamman, former Managing Editor of the *Mathematics Teacher* for the NCTM in Reston, Virginia, died on March 16, 1978, at the age of 69. She was a (life) member of the Association for forty-two years.

THE 1978-79 SABBATICAL EXCHANGE INFORMATION SERVICE

Do you feel the need for a change of scene for a semester or a year, but are not eligible for a sabbatical leave? The MAA Sabbstical Leave Information Service (SEIS) may help you arrange for a "no-cost sabbatical" by exchanging positions with a faculty member with similar interests in another institution. We invite your participation in SEIS in 1978-79, looking toward an exchange in 1978 or later. Here is how SEIS works.

Many MAA members are in institutions not offering faculty members a program of sabbatical leaves. Even in institutions with sabbatical leave programs individual faculty members often find themselves ineligible for such leave at a time when the desire for one is strongest. The Association therefore suggests that an occasional exchange between two faculty members of similar interests, training, and experience at different institutions could be of great benefit to the individuals and also to their institutions.

It is often possible for two such faculty members to trade identities, so to speak, for a year. Such an exchange might involve trading teaching responsibilities, living quarters, and some departmental responsibilities. The extent of the exchange would depend on the individual circumstances. It is suggested, however, that salaries should not be exchanged or even discussed. Each faculty member would remain on the payroll of his permanent institution and receive all of his normal fringe benefits. Financially, his institution would not recognize the exchange at all.

The MAA proposes to become involved only to the extent of assisting in bringing together like-minded mathematics faculty members who are interested in an exchange. The information exchange will be accomplished by the annual publication by the Association in December of a list containing the names, addresses, and other pertinent information about members of the Association interested in

arranging a "Sabbatical Exchange" with a colleague in another institution. This list will be sent free of charge to all those on the list and to any other MAA member who requests it.

Members interested in being listed in December 1978 should write to "SEIS, The Mathematical Association of America, 1529 Eighteenth St., N.W., Washington, D.C. 20036," enclosing the following information about themselves:

1. Name
2. Institution
3. Department
4. Address
5. Rank
6. Major field of interest
7. Highest earned degree
8. Names of from one to five courses recently taught
9. Normal teaching load
10. Section of country preferred for visit: Northeast, Southeast, Northcental, Southcentral, Northwest, Southwest.
11. Period for which exchange is desired, e.g., all of the academic year 1979-80, or the first two quarters of 1979-80, or the second semester of 1979-80, etc.

Communications must reach the Washington office by December 1, 1978 for inclusion in the December 1978 list.

MINI-CONFERENCE ON PROGRAMMABLE CALCULATORS AND CALCULUS

Ohio State University is the recipient of a three-year NSF CAUSE grant to develop calculus curricula using hand-held programmable calculators. The conference will be held at the Ohio State University, Columbus, on November 17-18, 1978. The intent of the conference is to share OSU's experience to-date and to provide a forum for the interchange of ideas and experiences between mathematicians and engineers and scientists for whom the study of calculus is an essential requirement.

Registration will take place Friday morning, November 17, and workshops and panels are planned for Saturday, November 18. For further information contact Professor Harry Prince Allen, Department of Mathematics, Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210.

ANECDOTES WANTED

We wish to publish a collection of anecdotes about well-known mathematicians. If you are interested in contributing, please write to

Peter Borwein Department of Mathematics University of British Columbia Vancouver, B.C., Canada V6T 1W5	or	Maria Klawe Department of Computer Science University of Toronto Toronto, Ontario, Canada M5S 1A7

For each anecdote please include your source and your assessment of its truth (as a probability between 0 and 1).

INVENTORY OF PROGRAMS IN SCIENCE, MATHEMATICS AND ENGINEERING FOR WOMEN AND GIRLS

Washington, D.C. ... The National Science Foundation has asked the American Association for the Advancement of Science to survey programs in science for women and girls. The results will appear in a publication that describes all efforts made between 1966 and the present to improve the science, mathematics, and engineering education of girls and women in the United States and to increase their participation in science related careers.

Programs directed at any age level will be eligible for inclusion, as will work conducted by any type of organization or agency. Projects of direct benefit to women and girls and research on the topic will be surveyed.

Persons who know of projects which might be within the scope of this inventory are asked to contact Dr. Michele L. Aldrich, OOS-AAAS, 1776 Massachusetts Avenue N.W., Washington, D.C. 20036, 202/467-5431.

EMPLOYMENT REGISTER/ACM COMPUTER SCIENCE CONFERENCE

The Seventh Annual Computer Science Employment Register will be conducted at the Dayton Computer Science Conference, Dayton, Ohio, February 20-22, 1979. This Register, the only one of its kind, aids in matching computer scientists and data processing specialists with employer opportunities. Previous Registers have attracted thousands of applicants (experience individuals and new graduates) and employers (business, industry, government, and academic institutions) with listings distributed as follows: Eastern States 36%, Central States 39%, Western States 21%, and foreign 4%. In addition to being reviewed by conference attendees, after each conference many copies of the register books have been placed on display in libraries, computer science departments and other convenient locations.

The purpose of the Register is to provide a mechanism for establishing contact between applicant and employer in a professional manner. The Register operates as follows: the applicant completes a form giving identifying information, education, publications, experience, interests, references, position and salary desired. Provision is made for submission of an anonymous form if desired. The employer completes a similar form giving: identifying information; position available along with starting date, salary and benefits; and education, experience, and specialization requirements for the position.

Both applicants and employers must file their registration on official forms. Three different forms will be used: (1) applicant, (2) academic, and (3) business, industry, and government. Your request should specify which of the three forms is desired. These forms may be obtained from and completed forms should be returned to: Orrin E. Taulbee; ACM Computer Science Employment Register; Department of Computer Science; University of Pittsburgh; Pittsburgh, Pennsylvania 15260.

THE GREATER METROPOLITAN NEW YORK MATH FAIR

The FAIR is to be held on March 25, 1979 at Pace University. Students who have completed, or are currently taking, mathematics at the eleventh year level or higher in the public, private and parochial high schools in the New York City, Westchester, Putnam, Dutchess or Rockland Counties are eligible to participate.

The FAIR encourages a student to pursue in depth, some phase of mathematics to which he is drawn. The student will (1) research a topic in mathematics, (2) write a paper on this topic, (3) give a talk on the paper to a group of judges.

Further details and application forms may be obtained from Dr. John A. Chiaramonte, Mathematics and Computer Science Department, St. John's University, Jamaica, New York 11439.

SHOW AND TELL MICRO-COMPUTER CONFERENCE

The student chapters of ACM and of DPMA at the University of Oklahoma produced a novel and highly successful *SHOW & TELL micro COMPUTER CONFERENCE* on May 13, 1978 in Norman. Participants were encouraged to bring and demonstrate their own micro-computers at the day-long conference.

Three 30-minute talks were:

Where Micro-Computers are Going, by Charles Weddington
A History of Computer Circuitry, by David Ashbaucher
Floating-Point Arithmetic: Beauty or Beast? by Andrew Oldroyd

Twenty-one participants introduced their favorite "brag programs" in a 5 to 8 minute presentation and later demonstrated them to interested coparticipants in *SHOW & TELL* sessions. Other participants brought their systems and participated in *SHOW & TELL* without giving talks.

The brief presentations included:

A Text Formatter for use in Story Generation. Stephen Kenton (Altair 8800)
Hunt the Wumpus. Richard Todd (TRS-80)
30 Graphics. Mike Koss (Apple II)
Micros in Psychological Research. Charles Gettys (SWTP 6800)
Hexapawn: A Game you Play to Lose. Jim Trott (ITF)
The Game of Life. Mike Meyer (TRS-80)
Very Versatile Instrument. H. M. Bradbury (Altair 8080)
Trilogy. Rocky Rutter (Apple II)
Attendance Records. Dale Ernst (TRS-80)
Haikku, Sonnets, & Accts. Rec. Charles Coombs (Olivetti A-4)
Random Numbers. Julian Calderon (TRS-80)
Inventory Accounting System/Hamarabi. James Waldron
Level II BASIC. Ray Davis (TRS-80)
IC-Testing with a Microprocessor. Donald Walton (6502 Chip, EBKA Familiarizer) Proto Board
TRS-80 Architecture & Use of T-Bug. Donald Flower (TRS-80)
Demo of KIM-1 w TV interface. Stanley Turk (KIM-1)
First Order Differential Equations. William Stockwell (TRS-80)
4-Digit Prime Squares. David Vincent (ITF)
Hi-Low Game/Reverse. Bob Yarbrough (TRS-80)
Stock Plotting. Bill Winters (TRS-80)
Introducing the ICCD Journal. Harold Zallen

A Programming Contest was held with 23 entrants. Prizes were awarded to Mike Koss, Robbie Reid, and Donald Fowler. More than 200 registered participants attended what promises to become an annual event at the University of Oklahoma. Anyone interested in future participation in similar events should contact Dr. Richard V. Andree, Mathematics Department, University of Oklahoma, 601 Elm Street, Norman, Oklahoma 73019.

PROFESSIONAL SOCIETY ANNOUNCES ELECTION RESULTS

At the annual meeting of its Board of Directors in Newport Beach, California, The Society for Computer Simulation (SCS) announced the results of elections for fiscal year 1978-79.

Reelected to his second term as president of the international society, was Dr. Donald Martin, Chairman of the Department of Computer Science, North Carolina State University. Newly elected as Vice President was Dr. Stewart Schlesinger, Manager Information Systems, The Aerospace Corporation. In addition to Martin and Schlesinger, one new member—Dr. George Marr, President of Autodynamics, Inc.—was elected to the Executive Committee. Other members are: Dr. Roy Crosbie, University of Salford, England; Mr. Per Holst, Manager Technical Resources, Foxboro Co.; and Robert Gustafson, of Simulation Specialists, Inc.

Norbert Pobanz, Control Systems Engineering Supervisor, Bechtel, Inc. and Professor Ralph C. Huntsinger, Computer Science Department, California State University, Chico, were appointed National Treasurer and Secretary respectively.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ESTABLISHMENT OF THE ARCHIVES OF AMERICAN MATHEMATICS

At its meeting on January 5, 1978, in Atlanta, the Board of Governors of the MAA, on the recommendation of a Joint (AMS-MAA) Fact-Finding Committee on Archives, approved a contract with the Humanities Research Center (HRC) of the University of Texas, Austin, for the establishment at the HRC of THE ARCHIVES OF AMERICAN MATHEMATICS. This contract was approved by the Board of Regents of the University of Texas System on June 9, 1978. Dr. Albert C. Lewis, a member of the MAA, has been appointed by the University of Texas as Curator and Cataloguer for THE ARCHIVES OF AMERICAN MATHEMATICS.

It is hoped that establishment of these ARCHIVES will lead to a preservation effort on a national scale, not only by establishing a national archival center for history of mathematics, but also by encouraging mathematicians, mathematical institutions, and archivists to preserve papers and records at those local depositories qualified to take care of them. Past and present officers of the Association, committee chairpersons, and editors of Association publications are requested to send information regarding their non-current files of records and correspondence to the HRC at the address given below.

Personal archives relating to American mathematics thus far in the HRC include those of R. L. Moore, H. S. Vandiver, H. J. Ettlinger, and W. T. Reid. Other collections relating to mathematics are: the John Herschel papers and papers of Albert Einstein and, among the papers of the writer Christopher D. Morley, some mathematical correspondence of his father, Frank Morley, who was President of the American Mathematical Society, 1919-20.

The mathematical collections are complemented by other manuscript collections in history of science including the papers of the English physicist and Nobel Laureate O. W. Richardson and the American geophysicist Maurice Ewing, and the archives of the Royal Meteorological Society and the Botanical Soceity of America. The HRC also has approximately 25,000 books and journals and 24,000 reprints related to history of science, including most of the classics listed, for example, in H. D. Horblit's *One Hundred Books Famous in Science* (1964). One of the special sub-collections, developed by Professor I. Angelelli, is devoted to history of logic and has over 1,800 titles of early works. There is also the H. Warren Weaver collection of works by C. L. Dodgson, principally made up of his literary works, but also including manuscripts and books on symbolic logic. There are illustrated exhibit catalogues produced by the HRC relating to Kepler, Newton, and Herschel.

Inquiries regarding development of collections in history of mathematics at the HRC are invited and may be addressed to Dr. Albert C. Lewis, Humanities Research Center, Box 7219, Austin, Texas 78712.

HENRY L. ALDER, *President*

APRIL MEETING OF THE WISCONSIN SECTION

The forty-sixth annual meeting of the Wisconsin Section was held on the campus of the University of Wisconsin-Whitewater, located in Whitewater, Wisconsin, on April 28 and 29, 1978. There were 95 members and 9 students registered as participants in the meeting.

The program consisted of two invited addresses and eighteen contributed presentations. The invited addresses were *Roving Mathematicians of the Renaissance* by John Schumaker, Rockford College,

and *Combinatorial Problems with Surprising Solutions* by David Roselle, Virginia Polytechnic Institute and State University. In addition, several MAA films were presented on Friday evening.

The business meeting was held on Saturday afternoon. David Roselle gave an informative slide presentation on the new headquarters of the MAA. A motion was made and passed requiring the officers of the section to seek ways to improve the participation of the private colleges in the life of the Section. Other local issues were discussed.

The contributed papers included:

Preference Revealing Processes, by Philip Straffin, Beloit College

A Comparison of Hyperbolic, Euclidean and Spherical Geometries of 3-Space, by Michael L. Welcome, UW-Parkside

Dominating Sets in Trees, by David Bange, UW-LaCrosse

Combinatorial Group Theory, by Norman Frisch, UW-Oshkosh

A Call for the Revitalization of Pre-College Mathematics, by John D. Aceto, Racine Unified School District

A Trigonometric Proof of the Steiner-Lehmus Theorem, by Mike Walterman, UW-Rock County Center

A Trigonometric Proof of the Steiner-Lehmus Theorem, by Dennis Francis, UW-Rock County Center

L Functions, by Lannie Lipke, UW-Rock County Center

Computer Graphics in Numerical Analysis, by Daniel F. X. O'Reilly, Marquette University

Using a Computer Graphics Terminal in Mathematics Classes, by Martin Engert, UW-Whitewater

Generalizing Perfect Numbers, by Rudolph M. Najar, UW-Whitewater

2.7 Ways to Compute $\mathit{Log}_e\,(a)$, by Timothy V. Fossum, UW-Parkside

Random Arcs on the Circle, by Andrew F. Siegel, UW-Madison

Integrability Conditions and the Cohomology of Differential Forms, by Paul Wolfson, UW-LaCrosse

Recent Developments in Cryptanalysis, by Tom Renfrow, Beloit College

Solving Inconsistent Systems of Linear Equations, by Wayne Wallace, UW-Oshkosh

On Committee Decision Making: A Game-Theoretical Approach, by Prakash P. Shenoy, UW-Madison

Spirals of Idempotents in Regular Semigroups, by Karl E. Byleen, Marquette University

TOM RENFROW, *Secretary-Treasurer*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The annual meeting of the Pacific Northwest Section, MAA was held at the University of Oregon, June 15-16, 1978 in conjunction with the AMS and SIAM, with over 120 persons in attendance. Invited MAA addresses were:

Design of functions for hand-held calculators, by Joel Davis, Oregon State University

Solved and unsolved problems in tilings, by Ivan Niven, University of Oregon

Math for math avoiders, by Jerine Ridgeway, Bellevue Community College

Contact probabilities between convex figures, by William Firey, Oregon State University

Home grown video, by John Loughlin and Jim Snow, Lane Community College; Bob Finell, Portland Community College, and Ed Wright, Linn Benton Community College, led a panel discussion on *Computer literacy for technical students.*

Invited AMS addresses were:

The Boolean algebra of sentences of first order logic, by William P. Hanf, University of Hawaii

Brauer groups of fields, by Murray M. Schacher, University of California, Los Angeles

The AMS portion of the program included four special sessions on Commutative Harmonic Analysis. Each organization also had a session for contributed papers. Banquet speaker Arvid Lonseth, Oregon State University, presented *In the Wake of Galleons (Some Mathematical problems from the age of discovery).*

Unanimously elected as officers were Chairman-elect Norm Lindquist, Western Washington State University; Vice Chairman (4-year colleges) Afton Cayford, University of British Columbia; Vice-Chairman (2-year colleges) Norm Barton, Vancouver Community College. Larry Curnutt, Bellevue Community College becomes Chairman and John Herzog, Pacific Lutheran University, continues on as Secretary-Treasurer.

By-Laws changes were approved as follows:

The Pacific Northwest Section shall consist of members of the Mathematical Association of America residing in the states of Oregon (Zip Code 97000-97999), Washington (Zip Code 98000-98699 and 98800-99499), Idaho (Zip Code 83300-83399 and 83600-83899), Montana (Zip Code 59000-59714 and 59716-59999), Alaska (Zip Code 99500-99999) and the provinces of British Columbia, Alberta, Northwest Territory and Yukon. Membership for qualified residents of the province of Saskatchewan is optional.

The officers of this section shall be a Chairman, a Chairman-elect (even numbered calendar years), Past-chairman (odd numbered calendar years), a Vice-Chairman (universities and four year colleges), a Vice-chairman (two year colleges), and a Secretary-Treasurer.

The Chairman-elect shall serve for one year as Chairman-elect, the subsequent two years as Chairman followed by a year as Past-chairman, and shall not be eligible for reelection. The Vice-chairman shall be elected for one year terms and shall not be eligible for reelection. The Secretary-Treasurer shall be elected for a term of three years and shall be eligible for reelection.

JOHN O. HERZOG, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.
Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21–23, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday–Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Adelphi University, May 5, 1979.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April

NEW JERSEY, St. Peter's College, Englewood Cliffs, November 4, 1978.

NORTH CENTRAL, University of Saskatchewan, Saskatoon, October 20–21, 1978.

NORTHEASTERN, Bunker Hill Community College, Charlestown, Massachusetts, November 18, 1978.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Ohio Northern University, Ada, October 20–21, 1978.

OKLAHOMA–ARKANSAS, Oklahoma State University, Stillwater, March 30–31, 1979.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.

PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.

ROCKY MOUNTAIN, University of Denver, Denver, spring 1979.

SEAWAY, University of Rochester, New York, November 10–11, 1978.

SOUTHEASTERN, University of Tennessee, Chattanooga, spring 1979.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS'

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Chicago, January 3–8, 1979.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Stouffer's Greenway Plaza Hotel, Houston, Texas, October 10–14, 1978.

AMERICAN MATHEMATICAL SOCIETY, Biloxi, Mississippi, January 24–27, 1979.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton Park Hotel, Washington, D.C., December 4–6, 1978.

ASSOCIATION FOR SYMBOLIC LOGIC, Biloxi, Mississippi, January 24–25, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Bonadventure Hotel, Los Angeles, California, November 12–16, 1978.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Hotel Fort Des Moines, Des Moines, Iowa, November 2–4, 1978.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Hyatt Regency Hotel, Knoxville, Tennessee, October 30–November 1, 1978.

HOUGHTON MIFFLIN

UPDATE:

ARITHMETIC
An Applied Approach
Richard N. Aufmann and **Vernon C. Barker**
both of Palomar College
512 pages / paper / Instructor's Manual / 1978

Developmental mathematics skills text containing six modular units. Many worked examples, exercises, self-tests, applied problems with answers, metric measurements, and a section on consumer mathematics.

BASIC MATHEMATICS
Skills and Structure
John F. Haldi, Spokane Community College
397 pages / paper / Instructor's Manual / 1978

Large-format work text uses discovery approach to review arithmetic and to introduce elementary algebra, geometry, and right angle trigonometry. Abundant real-life examples and exercises with answers.

MODUMATH: Arithmetic
Miriam Hecht
Hunter College, City University of New York
Caroline Hecht
598 pages / paper / Instructor's Manual / 1978

Covers essential remedial topics: whole numbers, fractions, decimals, percent, measurement, and signed numbers. Many illustrations, examples, exercises with answers, pre- and post-tests.

APPLIED TECHNICAL MATHEMATICS
Merwin J. Lyng, Mayville State College
L. J. Meconi, University of Akron
Earl J. Zwick, Indiana State University
496 pages / Instructor's Manual / 1978

Thoroughly illustrated presentation of basic mathematics skills needed for technical careers. Provides over 2,000 job-oriented examples and exercises.

ESSENTIAL ALGEBRA AND TRIGONOMETRY
Doris S. Stockton
University of Massachusetts, Amherst
588 pages / Instructor's Manual / 1978

For students not necessarily preparing for calculus. Reviews basic algebra, covers topics in elementary functions and graphs, circular and triangle trigonometry, probability, and permutations and combinations. Detailed, readable explanations with abundant exercises.

ESSENTIAL PRECALCULUS
Doris S. Stockton
University of Massachusetts, Amherst
598 pages / Instructor's Manual / 1978

Designed especially for students who plan to take calculus. Emphasizes analytic geometry — with conic sections covered in detail — and elementary functions, including operations on functions and parametric equations. Extensive exercises ensure mastery of topics.

AN INTRODUCTION TO THE STATISTICAL ANALYSIS OF DATA
T. W. Anderson, Stanford University
Stanley L. Sclove
University of Illinois, Chicago Circle
704 pages / Solutions Manual / 1978

Comprehensive introduction that blends data analysis and statistical inference. Many examples, problems, and applications to social, biological, physical, and administrative sciences.

APPLIED NONPARAMETRIC STATISTICS
Wayne W. Daniel, Georgia State University
503 pages / Instructor's Manual / 1978

Nonmathematical treatment emphasizing applications and methods for the student/researcher. Worked-out examples for each technique; exercises based on real research.

INTRODUCTORY STATISTICS WITH APPLICATIONS
Wayne W. Daniel, Georgia State University
475 pages / Study Guide
Instructor's Guide with Solutions / 1977

STATISTICS STEP BY STEP
Howard B. Christensen, Brigham Young University
670 pages / paper / Instructor's Manual with Solutions / 1977

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1

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Contents:

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Part 1: (0 471 03285-9)
1978 934 pp. $17.50
Part 2: (0 471 03286-7)
1978 480 pp. $16.50
Combined: (0 471 74983-4)
1978 976 pp. $21.50

2.

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This book has been designed specifically as a lead-in to calculus. Here, you and your students will find coverage of only those topics in elementary mathematics necessary for understanding calculus—no more and no less. And the new second edition features—

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All topics are treated with an eye toward their usefulness in calculus: inequalities, absolute value, intervals, boundedness, symmetry, trigonometry, induction, polar coordinates, functions, etc. PRECALCULUS avoids all unnecessary sophistication and your students will find its clear direction easy to follow.

(0 471 03124-0) 1979 In Press

3.

ORDINARY DIFFERENTIAL EQUATIONS, 3rd Ed.

Garrett Birkhoff, *Harvard University,* **&**
Gian-Carlo Rota, *Massachusetts Institute of Technology*

The ideal text for easing your students' transition from elementary theory of differential equations to the study of advanced methods. The third edition presents a balanced account of key ideas in their simplest context, often that of second-order equations. Introductory chapters have been carefully reorganized for greater readability.

(0 471 07411-X) 1978
350 pp. $18.95

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(0 471 02985-8) 1978
approx. 576 pp. $19.95

5

ADVANCED ENGINEERING MATHEMATICS, 4th Ed.

Erwin Kreyszig, *University of Windsor*

This text presents the most important areas of mathematics for engineering and physics students, including ordinary differential equations, linear algebra and vector analysis, and complex analysis. Examples and problems illustrate concepts, methods, results, and their engineering applications.

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(0 471 02140-7)   1978
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6

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Contents:
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(0 471 50731-8)   1978
688 pp. $21.50

7

FINITE MATHEMATICS WITH APPLICATIONS, 3rd Ed.

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Abe Mizrahi, *Indiana University, Northwest,* &
Michael Sullivan, *Chicago State University*

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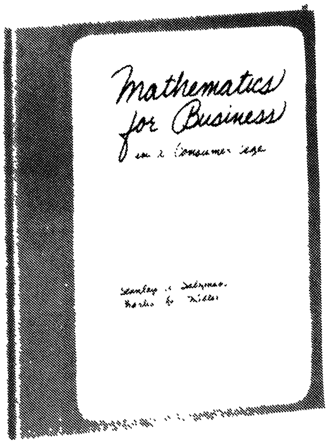

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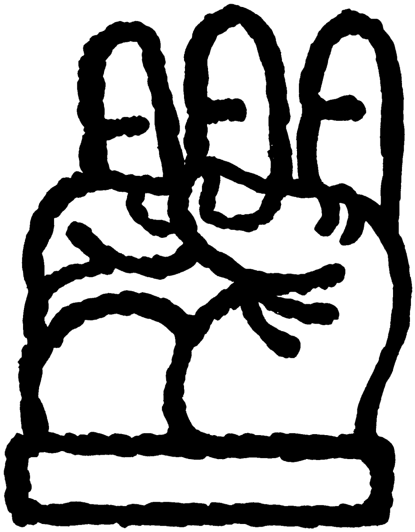

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THE AMERICAN MATHEMATICAL MONTHLY

THE MATHEMATICAL ASSOCIATION OF AMERICA

Volume 85, Number 9

Graduate Record Examination

EDUCATIONAL TESTING SERVICE

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OMIT	57	61	46	28	11
A	6	7	2	6	3
B	15	9	20	29	16
C	10	8	22	29	66
D	5	5	5	1	2
E	4	7	3	6	1
TOTAL	97	97	98	99	99

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THE GRE ADVANCED MATHEMATICS TEST

J. R. JEFFERSON WADKINS

The Advanced Mathematics Test of the Graduate Record Examinations (GRE) is a test that many current graduate students in mathematics, as well as a significant number of practicing mathematicians, have taken at some time in their careers. It is a test whose scores are used by many graduate mathematics departments to aid in the selection of those applicants who might best benefit from study in that particular graduate mathematics department; but it is probably safe to say that the U.S. mathematical community in general has only a vague understanding of how that test is constructed and the typical problems that arise in the process. The purpose of this article is to offer both mathematics faculty and prospective graduate students some background information about this test, its construction, its recent history, and some of the thinking that has gone into it—information with a depth and scope not appropriate for the descriptive booklets [3] which are sent to those who apply to take the GRE tests.

The information herein is anecdotal in form with a minimum of technical data. Those who wish to learn more about such aspects of the test as its content specifications, its current reliability, its effectiveness as a predictor of success in graduate mathematics, or the origin and early history of the GRE should consult [2], [3], [4], [5], or [6]. We concentrate here, for the most part, on providing information on those aspects of the Advanced Mathematics Test and its construction that would seem to be of most interest to mathematicians as mathematicians, information not elsewhere available in published form.

Test Responsibility

Responsibility for the GRE Advanced Mathematics Test is hierarchic: The Graduate Record Examinations Board determines the policies of the GRE program, and the Educational Testing Service (ETS) implements those policies. Internal ETS responsibility falls to the GRE Program Director and staff who are broadly responsible for all aspects of all GRE tests—including test development, production, administration, analysis, scoring, and reporting of scores. Coordination of test development aspects of all GRE tests is the responsibility of the GRE Test Development Coordinator. Responsibility for developing both the GRE Advanced Mathematics Test and the quantitative part of the GRE Aptitude Test belongs to S. Irene Williams in her role as Head of the Mathematics Group of the Division of Higher Education and Career Programs. Specific responsibility for actual construction of any test goes to its Primary Test Specialist, who—in the case of the GRE Advanced Mathematics Test—is the author of this article. Support for this effort is provided by the Secondary Test Specialist and other ETS staff members with appropriate mathematical training.

It is, however, with an external committee of five representatives from the U.S. mathematics community that—in a very real and practical sense—ultimate responsibility for the test lies. Members of this Committee of Examiners are appointed for two-year terms. At the time of this writing, the Committee consists of:

Israel N. Herstein, University of Chicago (Chairman)
Richard D. Anderson, Louisiana State University

J. R. J. Wadkins has been the Primary Test Specialist for the GRE Advanced Mathematics Test since joining Educational Testing Service in 1967. Before joining ETS he taught at the Marion Institute and at North Carolina Wesleyan College. He has done graduate work at the University of North Carolina, in part while holding an NSF Science Faculty Fellowship. He is interested in methods for explicit teaching of the mechanics of mathematical proof to beginning undergraduates and is pursuing this interest on a part-time basis at the Rutgers University School of Education.—*Editors*

Edwin Hewitt, University of Washington
Gloria Hewitt, University of Montana
David P. Roselle, Virginia Polytechnic Institute

Other people who have served on this committee during the past ten years are R. Creighton Buck, Samuel Eilenberg, Etta Z. Falconer, F. Burton Jones, Dorothy M. Stone, and Alfred B. Willcox—together with Robert P. Dilworth, who held the post of chairman from 1964 until 1969, and Ralph P. Boas, Jr., who took over as chairman in 1969 and served until 1975.

The sense in which this Committee of Examiners has the "real" responsibility for the test can best be seen by examining, step by step, the procedure for constructing a new edition of the test. (This procedure, described on the following pages, is essentially the one that has been followed annually with every new edition of the test that has been constructed since this writer came to ETS in 1967 as the Primary Test Specialist for the GRE Advanced Mathematics Test. Advanced tests in fields other than mathematics are produced under somewhat different procedures.)

The Test Construction Process

Approximately eighteen months before the new edition of the test is to be administered, each of the five members of the Committee mails to the ETS Test Specialist some twenty prospective questions that he or she would like to see considered for use. These questions, in multiple-choice format with five choices for each question, are given an initial review by ETS staff; some minor revisions are made in them; and an additional thirty to forty questions written by the Primary Test Specialist are added to the pool. Occasionally, questions written by other mathematicians are included. All the authors of these questions are aware of the content specifications for the test, but initially their charge is to write questions they like and think appropriate. Whenever this results in some areas of content not being adequately represented, additional questions in these areas are solicited. (For a detailed description of the content specifications, see [3].)

Rating the Questions. A copy of the pool of questions is sent to each Committee member, who rates the questions according to the scheme

A: Preferred
B: Acceptable but not preferred
C: Unacceptable

No one other than the staff at ETS knows who has written which questions, and such information is not generally discussed, although the question of authorship sometimes does arise at Committee meetings in connection with some question that is especially admired.

Several years ago, it was suggested that we keep a record of the success rate of each Committee member, based on the number of his questions that ended up on the final version of the test; but this policy was rejected by the Committee after one of its members—who probably had at that time the highest success rate of any Committee member—expressed his very strong opinion that such a policy would discourage creativity in the question-writing process and encourage adherence to conventional formats.

In rating the questions, each Committee member is asked to base his rating on the *idea* of the question, rather than the actual form of the question as it initially appears. (A question that appears on a final edition of the test in exactly the same form that its writer first submitted is a rare question indeed.) Comments and criticisms of each question, along with suggested revisions, are also to be offered.

During the time that the pool of questions is being reviewed and rated by the Committee, an "equating set" is being selected.

The Equating Process. An equating set is a collection of questions from a previous edition of a test and is used as a device to ensure comparability of the reported scores of students who take different editions of the test. In brief: The equating set appears in two different editions of a test, the two editions administered to two different samples of its intended population. The performance of the two different samples on the common equating set is used to estimate the way in which the second sample would have performed had it been given the edition taken by the first sample.

The exact mechanics of this equating process is a subject that comes up frequently during inquiries about the test from faculty members; and the short explanations of these mechanics, which seem dictated by the circumstances under which such questions arise, are inevitably unsatisfactory. Consequently, we digress briefly to outline the process of equating. (The casual reader should skip the remainder of this section.)

Let B represent a set of multiple-choice questions administered initially to a set $\mathfrak{B}$ of people. Call B the *base test*, and call $\mathfrak{B}$ the *base population*. Each member b of $\mathfrak{B}$ receives a raw score $s_B(b)$ representing the number of questions in B answered correctly by b—minus some penalty for guessing. (On five-choice questions, the penalty would be one-fourth of those marked incorrectly.)

The distribution of raw scores of the base population $\mathfrak{B}$ on the base test B has a mean m_0 and standard deviation σ_0.

The scaled scores $S_{B,\mathfrak{B}}(b)$ for members b of $\mathfrak{B}$ on test B are then obtained by defining

$$S_{B,\mathfrak{B}}(b)=500+\frac{s_B(b)-m_0}{\sigma_0}\cdot 100$$

e.g.,

a raw score of m_0 translates to 500,
a raw score of $m_0+\sigma_0$ translates to 600,
a raw score of $m_0-\sigma_0$ translates to 400,
a raw score of $m_0+x\sigma_0$ translates to $500+100x$.

This formula extends to the set $\mathfrak{B}'$ of all people b' who ever take base test B obtaining raw score $s_B(b')$, for we then assign to b' the scaled score

$$S_{B,\mathfrak{B}}(b')=500+\frac{s_B(b')-m_0}{\sigma_0}\cdot 100.$$

Next we construct another test T that is intended to test the same knowledge as that intended to be tested by B: To do this, we (1) choose a subset E of B, called *an equating set from test* B, so that E is a miniature of B in terms of content coverage, average difficulty of questions, and spread of difficulty of the questions, and (2) choose other questions for T so that E is also a miniature of T.

Finally we administer test T to a population $\mathfrak{T}$. As with the base test, each member t of $\mathfrak{T}$ receives a raw score $s_T(t)$ representing the number of questions in T answered correctly minus some penalty for guessing. The scaled score $S_{B,\mathfrak{B},T,\mathfrak{T}}(t)$ for members t of $\mathfrak{T}$ is then defined as an estimate of the score that t would have made on test B if B had been administered to t—the estimate being based on the performance of $\mathfrak{T}$ on the equating set E as compared to the performance of $\mathfrak{B}$ on E. More specifically, the estimate is made by (1) using equations that are set forth in **[1]** to estimate the mean and standard deviation of $\mathfrak{B}\cup\mathfrak{T}$ on B (alternatively, T) if each member of $\mathfrak{B}\cup\mathfrak{T}$ had taken B (alternatively, T) and (2) using the resulting estimates to derive a linear equation that yields the raw score $s_B(t)$ that each member t of $\mathfrak{T}$ is estimated to have made had t taken base test B instead of the test T actually taken by t. The scaled score $S_{B,\mathfrak{B},T,\mathfrak{T}}(t)$ (which is the only score that will be reported for t) is then computed as if t had actually taken test B and had actually made the score $s_B(t)$ that he is estimated to have made.

Even the process described above is somewhat oversimplified with respect to the GRE Advanced Tests because of the effort made to give some sort of comparison between scores in different subject areas, e.g., between scores on the GRE Advanced Mathematics Test and scores on the GRE Advanced History Test, Music Test, etc., and consequently instead of using the theoretical mean of 500 and standard deviation of 100 in the computation

$$S_{B,\mathfrak{B}}(b) = 500 + \frac{s_B(b) - m_0}{\sigma_0} \cdot 100,$$

we have rather

$$S_{B,\mathfrak{B}}(b) = m_{\mathfrak{B}} + \frac{s_B(b) - m_0}{\sigma_0} \cdot \sigma_{\mathfrak{B}}$$

where $m_{\mathfrak{B}}$ and $\sigma_{\mathfrak{B}}$ are numbers determined by the performance of the population $\mathfrak{B}$ on the GRE aptitude test as compared to the performance of other GRE populations (defined by whatever GRE Advanced Test was taken) on that same GRE Aptitude Test. Some $m_{\mathfrak{B}}$'s are below 500, others are above. The $m_{\mathfrak{B}}$'s for mathematics and the natural sciences are all above 500.

Theoretically, $S_{B,\mathfrak{B}}(b)$ assumes values on a continuous scale with no upper or lower bounds. Actually, scores on all GRE Advanced Tests are reported in multiples of 10 with a maximum range from 200 to 990.

This completes our outline of the equating process.

A First Draft. When the comments on the pool of questions are received from the Committee, the ratings, criticisms, and suggested revisions are carefully reviewed by the test assembler; and questions are selected for a draft test based on the ratings and on an outline of specifications that the Committee had previously approved. This selection is then adjusted in line with the equating set, and a first draft of the test is assembled.

The first draft is then given a review by an ETS mathematics staff person—a somewhat different review from the collection of reviews of individual questions previously done; for now the collection is reviewed as a whole with criticism given of content coverage, level and spread of difficulty, etc., as opposed to criticism of individual questions.

The first draft is then revised on the basis of the reviews and sent to the members of the Committee, for their individual perusal, some two to six weeks prior to a meeting of the Committee. Along with the draft, the Committee receives an additional list of the more highly rated questions that can be used as alternates, as well as a report on the makeup of the draft, a report that includes comments on (1) some particular revisions of individual questions that we feel require special attention from the Committee—either from lingering ambiguity or because of a concern that our attempt to avoid ambiguity may have changed the thrust of the question as originally intended by its writer, (2) the weaknesses we see in the draft and the list of alternates in relation to the outline of specifications previously approved by the Committee, and a corresponding request that the Committee members write and bring with them to the meeting questions that might reinforce those areas, and (3) any special issues or concerns to be discussed at the upcoming meeting—issues or concerns about the test that might have arisen since the preceding Committee meeting.

Committee Review of Past Editions. Normally in attendance at the Committee meeting, in addition to the Committee members, are (1) a representative of the GRE Program Direction staff, (2) the Primary Test Specialist, and (3) the Mathematics Group Head, who has been active in the assembly and review of the draft.

The meeting usually begins with a representative from the GRE Program Direction staff apprising the Committee of any overall GRE policy changes, reviewing ongoing policy, and answering any questions the members of the Committee might have about the GRE program in general. Next, the Committee examines a statistical analysis of the edition of the test that was constructed at the

Committee meeting the previous year. (That analysis has been prepared by the ETS Statistical Coordinator for the GRE Program, who also contributes advice from time to time on matters of concern in the equating process.) An important part of the information that is available is an individual analysis of each question, a facsimile of which is given in Figure 1 following the question on which the analysis is based.

For each $t>0$, define $\int_0^t (dx)^2$ to be the infimum (greatest lower bound) of all sums $\Sigma_{k=1}^n (x_k - x_{k-1})^2$ where $0 = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = t$. Then for each $t>0$, $\int_0^t (dx)^2$ is

(A) 0 (B) $(\int_0^t dx)^2$ (C) $\int_0^{t^2} dx$ (D) $\int_0^t 2x\,dx$
(E) not necessarily any of the above

FORM	BASE N	OMIT	A	B	C	D	E	M_{TOTAL}	Δ_E SCALE	Δ_E	CRITERION
RGR2	1015	425	239*	56	39	40	210	13.0	OGR1	17.7	IS065
TEST CODE	ITEM NO.	M_O	M_A	M_B	M_C	M_D	M_E	P_{TOTAL}	$P+$	Δ_O	r_{bis}
MATH	28	11.6	16.1	11.3	12.9	12.0	13.0	1.00	0.24	15.8	0.60

FIG. 1

The most important information available here, from the viewpoint of the Committee members, are (1) the observed difficulty of this question for this representative group of 1,015 GRE candidates, represented by the box labeled Δ_0, (2) an estimate of the difficulty of this question if it had been given to a certain reference group, represented by the box labeled Δ_E, (3) the biserial correlation between success on this question and success on this edition as a whole, represented by the box labeled r_{bis}, (4) the number of candidates not marking any answer to the question, represented by the box labeled OMIT, (5) the numbers marking choices A, B, etc., represented by the boxes labeled A, B, etc., and (6) the average achievement levels of the groups of candidates corresponding to OMIT, A, B, etc., which are represented by the boxes marked M_0, M_A, M_B, etc. The correct answer to the question is indicated by an asterisk in the appropriate box—in this case, choice A. Both the difficulty scale and the achievement scale have a mean of 13 and a standard deviation of 4. The difficulty scales are translatable as percent correct with 13 corresponding to 50% correct, 9 corresponding to approximately 84% correct and 17 corresponding to approximately 16% correct. (Because of the fact that random guessing would be expected to result in one correct answer out of five, it is generally assumed —for purposes of analysis—that a question with $\Delta 12$ means approximately half the candidates actually *knew* the correct answer even though 60% *marked* the correct answer.)

The achievement scale is based on the average percentile score of the group of people described, e.g., the 425 people who omitted this question had an average achievement level of 11.6, i.e., an average score of 1.4/4 standard deviations below the average percentile score of the entire group of 1,015 people. By contrast, the 239 people who marked the correct response, A, had an achievement level of 16.1, i.e., an average percentile score of 3.1/4 standard deviations above the average percentile score of the 1,015 people who took this edition of the test.

In addition to this analysis strip, an even more complete analysis is available to the Committee upon request, an analysis which shows how many candidates in each quintile of achievement marked a given answer, e.g., instead of just being able to tell the average achievement level of those marking various choices, one can also see the distribution of marks across OMIT, A, B, etc. by those in the top-scoring fifth of the group, those in the bottom fifth, etc.

The Committee also reviews a Test Analysis of the same edition of the test, an analysis that includes the distribution of raw scores, scaled scores, raw deltas (difficulty levels for this group of candidates), equated deltas (estimated difficulty levels for the reference group), and r-biserials (correlation between success on individual questions and success on the edition as a whole); it also

includes information on the speededness of the edition (percent marking an answer to the last question and percent marking an answer to some question beyond the one that is at the three-quarter point of this edition), the percentage whose scores fall in the chance range, and other statistical information.

After a thorough examination and discussion of this statistical analysis, the Committee decides what significance this information has for its present deliberations. For example, in recent years, there has been a deliberate attempt on the part of the Committee to increase gradually both the number and the depth of questions that involve abstract algebra. The statistical analysis has shown, however, that such questions tend to have a high number of OMIT's, sometimes with the achievement level of those people being surprisingly high, and a correspondingly low r-biserial (correlation between success on that question and success on the edition as a whole). Each time, the Committee has had to decide to what extent these results are to be allowed to influence their construction of the edition before them.

Discussion of the Draft Test. After the decisions made by the Committee during and following their consideration of the statistical analysis of the previous edition, the Committee next takes up the draft test, which they have reviewed and written comments upon prior to the meeting. Each question in the draft is discussed individually. Generally about half the questions at this stage are tentatively accepted with only brief discussion. Others are debated at length—with a number of controversial questions being put aside, because of time limitation, without agreement by the Committee as to acceptable wording or even as to whether some version of the question should be included or not. Such questions may be assigned to individual Committee members to think about overnight or may simply be left as a pool to be considered later in their deliberations if more acceptable questions on similar subjects are not found.

The task of finding acceptable wording for a question produces some very interesting discussions. Wording that would be completely acceptable in a classroom situation is not automatically acceptable in this test. In a classroom testing situation, the use of inconsistent conventions, indefinite antecedents, and ambiguous phraseology is understandable and excusable where the students have had the opportunity to acquaint themselves with the idiosyncratic language of individual teachers, and where the author of the test is available for explanations during the testing period. With the GRE Advanced Mathematics Test, we attempt to envision every interpretation that virtually anyone could reasonably be expected to make, and we try to adjust our wording with several things in mind. First, we must remember that the student reading the question cannot ask anyone what the intent of the question really is, and we must remember that the students taking the test come from a wide variety of institutions and have a widely varying idea of what conventional mathematical symbolism and terminology are. For example, it has been the opinion of the Committee that using f^{-1} to denote the inverse of f with respect to the operation of composition is not advisable. In some cases we have defined the symbol at the beginning of the question; in others we have postulated the existence of a function g with the property that $g(f(x))=x$ for all x in the domain of f (or $f(g(x))=x$, or both, as the case may be) and avoided the symbolism; while in still others we have devised some ad hoc solution to the predicament, e.g., if we wanted to give the graph of a function f and ask the candidate to choose which of five other graphs could be the graph of f^{-1}, we might refer to the former as the "graph of $y=f(x)$" and refer to the desired response as the "graph of $x=f(y)$." Other examples include: writing out "the empty set" instead of using $\varnothing$; avoiding the use of all logical shorthand such as $\exists$, $\forall$, $\wedge$, $\vee$, etc.; avoiding the construction "P only if Q"; using the explanatory phrases, "open interval," "closed interval," with symbols (a,b) and $[a,b]$, respectively, rather than relying solely upon the symbols themselves; and never using the somewhat standardized symbols of R, Q, and Z for the reals, rationals, and integers, respectively, without explicitly defining them.

Secondly, we must try not to penalize the student who knows something more than that which the writer of the question had in mind—the student should never be in a position of having to ask

himself, "Do I answer this question on the basis of what it actually says, or do I answer it on the basis of what I think the test maker intended?" As a somewhat picayune example, suppose that an examinee were confronted with the following question in a situation where no one is available to explain the intention of the test writer.

Which of the following is the set of real numbers on which the series Σx^n converges?

(A) The empty set
(B) 0
(C) $-1<x<1$
(D) $-1\leqslant x<1$
(E) None of the above

If the examinee has been under the influence of professors who especially emphasize the use of careful language, he may well be torn between choosing (C), which would probably seem to most people to be what the test writer had in mind, and (E)—which is somewhat defensible on a technical basis as the correct answer because it can be responsibly argued that "$-1<x<1$" is a statement (associated with a set) and does not really denote a set.

A somewhat less frivolous situation would present itself with the following question.

If f is a functon such that $f(x)=\left(\frac{1}{\sqrt{x}}-1\right)^{\sqrt{x}}$ for all x such that $0<x<1$ and if $f(0)=1$ and $f(1)=0$, which if the following must be true?

I. f is continuous on the interval $(0,1]$.
II. f is continuous on the interval $[0,1)$.
III. f is differentiable on the open interval $(0,1)$.

(A) II only (B) III only (C) I and II only (D) II and III only (E) I, II, and III

Since $\lim_{x\to 0^+}f(x)=1$ and $\lim_{x\to 1^-}f(x)=0$, it could be argued that the test writer probably intended the correct answer to be "(E) I, II, and III." However, one technical interpretation of the wording gives "(B) III only" as the correct answer because—even though a casual reading of the question seems to imply that the domain of f is uniquely defined—a careful reading of the question leads inescapably to the conclusion that the only thing postulated about the domain of f is that it *contains* the closed interval $[0,1]$, so there certainly exist extensions of the obvious candidate for f that are discontinuous at both 0 and 1. (We shudder to think that f could be defined throughout a region of the plane—raising questions about what "differentiable" means.) In such a case, what is the careful-language-conscious candidate to do? Should he answer the question on the basis of what it actually says or should he try to guess what the test writer had in mind?

Thirdly, there is the matter of maintaining technical correctness and avoiding the very real temptation that is always present among all committees of examiners to shrug off petty criticisms with "Oh, everybody will know what we mean." It is the duty of the Test Specialist to play the role of devil's advocate conscientiously in all such situations and to maintain that the everybody-knows-what-we-mean defense against trivial criticisms should be an infrequently used last resort and that too ready an acceptance of this defense can easily lead to an atmosphere of review in which questions are bound to slip by that are incorrect by any standards.

Finally, we must balance our desire to arrive at unambiguous and technically defensible terminology with our knowledge that such terminology is often abstruse and verbose and may require an inordinate amount of time to analyze sufficiently. For example, the following question is a carefully worded one that could hardly be criticized on grounds of ambiguity or technical incorrectness.

For all real x, let sgn(x) be $\frac{|x|}{x}$ unless $x=0$; let sgn(0)$=0$; and let $[x]$ denote the maximum of the set of all integers k such that $k \leqslant x$. If $\{x_i\}_{i=1}^n$ is a finite, strictly increasing sequence of real numbers such that there is a positive integer j with $1<j<n$ and $x_j=0$ and, furthermore, if $\Sigma_{i=1}^n \operatorname{sgn}(x_i)=0$, then $j=$

$$\text{(A)}\ \left[\frac{n-1}{2}\right] \quad \text{(B)}\ \left[\frac{n}{2}\right] \quad \text{(C)}\ \left[\frac{n}{2}+1\right] \quad \text{(D)}\ \left[\frac{2n+1}{2}\right] \quad \text{(E)}\ \left[\frac{n}{2}\right]-1$$

Compare this with the following version of the same question.

For all real x, let $\operatorname{sgn}(x)=\begin{cases} 1, & \text{if } x>0 \\ 0, & \text{if } x=0 \\ -1, & \text{if } x<0 \end{cases}$ and let $[x]$ be the greatest integer $\leqslant x$.

If $x_1<x_2<\cdots<x_j=0<x_{j+1}<\cdots<x$ and $\Sigma_{i=1}^n \operatorname{sgn}(x_i)=0$, then $j=$

$$(A)\ \left[\frac{n-1}{2}\right] \quad (B)\ \left[\frac{n}{2}\right] \quad (C)\ \left[\frac{n}{2}+1\right] \quad (D)\ \left[\frac{2n+1}{2}\right] \quad (E)\ \left[\frac{n}{2}\right]-1$$

While the second version is not, strictly speaking, completely unambiguous,* it is certainly much more easily comprehended than the more technically correct first version and would probably be accepted by the Committee in the second form. As the devil's advocate, a Test Specialist might argue that including "$-\infty<x_1$" and "$x_n<+\infty$" would scarcely add more verbiage and should be included. The argument against this inclusion would probably be that any candidate worrying about such possibilities has no answer to choose from and would be quickly forced to adjust his thinking along the intended lines. Such an argument would be settled strictly on the basis of the Committee's best judgment. ETS Test Specialists have no vote in such deliberations.

The Final Draft. Once the Committee has discussed each of the questions on the first draft, a count is made of those items tentatively accepted and a comparison is made with the table of content specifications. Keeping in mind those areas of content inadequately represented among the tentatively accepted items, the Committee then selects questions from the list of alternates after the same kind of discussions described previously. Inevitably, completely new questions—conceived during these discussions and brought forth on any available scrap of paper—are passed around for consideration, revision, and possible approval. If, at the time the Committee has finished its discussion of the list of alternates, there are still subject-matter areas inadequately represented, the Committee returns to those questions on which no agreement has been reached. Eventually this iterative process produces a collection of 66 questions that the Committee is willing to accept. However, this collection is not yet final. The following day, the Committee takes another hard look at the collection, after being away from it for some 12 hours, ordering the collection subjectively from easier questions to more difficult questions, inevitably throwing out questions here and there and inserting others in their place.

At some point in their deliberations—usually after they have first given tentative approval to the test and before ordering it by difficulty—the Committee holds a discussion of the broader aspects of the test and of the work of the Committee. Recommendations by the Committee for general changes in the program are communicated by the Test Specialist to the GRE Program Director.

Further Review. Following the meeting, the Test Specialist initiates the process that will eventually result, some six to ten months later, in printed booklets containing this most recent edition of the test

*The substitution of $-\infty$ for x_1, or $+\infty$ for x_n, in the line of inequalities is meaningful, but such substitution in the summation is not; and, does "$x_1<x_2<\cdots<x_j$" actually imply any more about x_3 than "$s_1=2, s_2=4, s_3=6, \ldots, s_j=16$" implies about s_4? (The sequence s could be, among other possibilities, a Fibonacci sequence.)

being distributed to candidates in GRE test centers in the United States and other parts of the world. Briefly, this process includes (1) a test review performed by ETS Mathematics Test Development staff members followed by discussions between reviewer and Primary Test Specialist, discussions that generally result in minor changes being made, (2) two reviews by ETS Test Editors to ensure consistency in style within the particular edition as well as with previous editions of the test, followed by discussions with the Primary Test Specialist and by the inevitable changes, (4) production of a typed copy—called a planograph—that includes figures drawn by professional artists, (5) proof reading and arrangement of the planograph in camera-ready form, (6) another complete test review of the planograph by an ETS Mathematics Test Development staff member followed by further discussions of the test and its format, (7) copies of the planograph made and sent to each member of the Committee who reviews it anew—not having seen it for several months and seeing for the first time the changes that resulted from internal ETS reviews, (8) consolidation by the Primary Test Specialist of the comments about the planograph from the various members of the Committee and reconciliation of any differences of opinion, usually in concert with the chairman of the Committee (again, the Committee has the final say on the form in which the questions appear, and the Committee has, on occasion, vetoed changes made as a result of internal ETS reviews), and (9) the usual processes involved in the actual printing of the test booklets.

Before Scores Are Reported. After any new edition of the test has been administered, a sample of the answer sheets is given a preliminary analysis before any actual scoring is done to determine whether or not there are any statistical peculiarities that might point to a faulty question having slipped by despite the lengthy and involved review process described above. Since 1967, when this writer came to ETS, there have been only two questions on the GRE Advanced Mathematics Test that have been thrown out because of this preliminary analysis—and in *neither* instance was the question in any way incorrect. Each time, we decided, in consultation with the Committee, not to use the particular question in scoring answer sheets because of a rather unusual set of circumstances to which the preliminary analysis of answer sheets seemed to point. Let us examine one of those two questions. The Committee's intention in asking this particular question was to reward those who approach any differential equation in a careful and thoughtful manner as opposed to those who approach the same equation in mechanical, cookbook fashion. The question was stated as follows.

The differential equation $(y')^2=y$ has how many solutions passing through $(0,0)$?

(A) One (B) Two (C) Three (D) A finite number greater than three (E) Infinitely many

The complete statistical analysis is given in Figure 2.

Two important things to note about this analysis are: (1) the r-biserial is -0.26, i.e., the correlation between success on this question and success on the test as a whole is -0.26; (2) the achievement levels of those marking (A), (B), (C), or omitting, are all higher than those marking the correct answer (E); and (3) the columns marked N_1, N_2, N_3, N_4, and N_5, indicating the quintiles of achievement, show that the number of people in each quintile marking the correct answer, (E), *decreases* monotonically as one moves from the lowest quintile to the highest quintile; in constrast, consider those marking choices (A) or (B); as one moves from those of lowest achievement to those of highest achievement, the number *increases* almost monotonically for both (A) and (B). Such an analysis would normally point to an incorrect key, i.e., one might normally suspect that the intended answer had been recorded incorrectly. Such is not the case with this question, of course; the facts appear to be that (1) relatively few candidates approached the problem in anything but cookbook fashion, (2) those who *were* more careful did not carry their thinking far enough, and (3) more important, there is a really naive line of thought that actually gives the correct answer for completely spurious reasons.

ITEM NO: TIS NO: TEST: FORM: BASE N: DATE TABULATED:

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ITEM ANALYSIS

RESPONSE CODE	LOW N_1	N_2	N_3	N_4	HIGH N_5
OMIT	50	42	59	52	34
A	56	63	48	101	115
B	23	27	40	28	79
C	9	2	8	8	5
D	12	4	2	1	2
E	49	47	34	32	21
TOTAL	199	185	191	222	256

* DENOTES CORRECT RESPONSE

FORM	BASE N	OMIT	A	B	C	D	E	M_{TOTAL}	Δ_E SCALE	Δ_E	CRITERION
RGR1	1425	237	383	197	32	21	183*	13.4	OGR1	19.0	ISO64
TEST CODE	ITEM NO.	M_O	M_A	M_B	M_C	M_D	M_E	P_{TOTAL}	P+	Δ_O	r_{bis}
MATH	63	12.6	14.2	14.8	12.7	9.9	11.9	0.74	0.17	17.2	-0.26

FIG. 2

One might expect the careful thinker to go through this problem in something like the following manner:

We look for the number of functions y such that

$$(y')^2=y \quad \text{and} \quad y(0)=0; \tag{1}$$

$$\begin{aligned}&\text{i.e., } 2y'y''=y' \text{ wherever } y' \text{ is differentiable and } y(0)=0;\\ &\text{i.e., } 2y''=1 \text{ wherever } y'' \text{ exists, } y' \text{ is nonzero, and } y(0)=0.\end{aligned} \tag{2}$$

Now, for each x,

$$\begin{aligned}2y''(x)=1 &\Leftrightarrow \left[y'(x)=\frac{1}{2}x+C_1, \text{ for some } C_1\right]\\ &\Leftrightarrow \left[y'(x)=\frac{1}{2}x\right] \text{ by (1)}\\ &\Leftrightarrow \left[y(x)=\frac{x^2}{4}+C_2 \text{ for some } C_2\right]\\ &\Leftrightarrow \left[y(x)=\frac{x^2}{4}\right] \text{ by (1).}\end{aligned} \tag{3}$$

Thus we need only consider those other functions y which have zero values for y', or for which y'' does not exist.

Now at any point x where $y'(x)=0$, the equation $(y')^2=y$ implies that $y(x)=0$, so the only functions left to consider are those which have the X-axis as the only possible horizontal tangent.

The zero function is an obvious example. (4)

Only slightly less obvious are two unions:

The non-negative X-axis with the left half of the parabola $y=\dfrac{x^2}{4}$; and (5)

the non-positive X-axis with the right half of $y=\dfrac{x^2}{4}$.

The question remains whether there are any other possibilities. Realizing that the conclusion $C_1=0$ drawn above depended on $(0,0)$ being on the graph of that function which was

eventually determined to be defined by $y(x)=\frac{x^2}{4}$, one is led to consider other functions—for example, those that have only zero values from $-\infty$ through some positive number c and that satisfy $2y''(x)=1$ for all $x>c$;

i.e., $y'(x)=\frac{x}{2}+C_1$ for all $x>c$;

i.e., $y'(x)=\frac{x-c}{2}$ for all $x>c$ since $y'(c)$ existing implies $y'(c)=0=y(c)$.

This gives us infinitely many solutions (and of course there are others, but this analysis is sufficient for answering the question correctly and validly).

Looking for an explanation for the performance of the candidates on this question, it seems clear —with hindsight—that the poorer candidates stopped at line (2) above, deducing that all functions of the form

$$y(x)=\frac{x^2}{4}+C_1x\ \left(\text{or even } y(x)=\frac{x^2}{4}+C_1x+C_2\right)$$

were solutions and drawing the conclusion—a correct conclusion reached in erroneous fashion—that the correct answer to the question is "infinitely many" (since C_1 and C_2 are supposedly arbitrary). Those who continued through (3) deduced that C_1 and C_2 are uniquely determined and, if they thought no further, went for choice (A). Those who went only as far as (4)—or who simply recognized the zero function as an example—would go for (B), and those whose thinking went through (5) but no further, would have chosen (C) or (D).

This is an example of a question that was too subtle for the candidates. In some cases such a question might be salvaged for other editions by eliminating the wrong choices that drew the best candidates—in this case, (A) and (B). That would tend to prod the better candidates to think a little further. However, when a very naive and incorrect approach yields the correct answer, as in this case, there is usually nothing that can be done except to discard the question.

This was an example of a question that was completely valid from a purely mathematical point of view, but was unacceptable because of the reaction of those who took the test. There exist questions at the other end of this spectrum as well, e.g., questions that appear to work exactly as the Committee intended them to work, but which come under fire from reviewers because of real ambiguity.

The following is an example of a question in which the Committee deliberately included an ambiguity; and the statistical analysis (see Figure 3) seems to indicate that the question worked rather much the way the Committee intended it. However, the question was criticized independently by two different mathematicians who had been sent copies of the test and asked for a critical review.

> A number x is said to be a limit point of the sequence $\{x_n\}$ of real numbers if and only if some subsequence of $\{x_n\}$ converges to x. If λ is the number of limit points for an arbitrary bounded sequence, then the set of possible values of λ is
>
> (A) $\{0\}$ (B) $\{1\}$ (C) $\{1,2,3,\dots\}$ (D) $\{2\}$ (E) $\{2,3,4,\dots\}$

The correct answer to the question posed above is "the set of all positive cardinal numbers not greater than the cardinal number of the reals and—if the continuum hypothesis be denied—not including any cardinal number properly between the cardinal number of the integers and the cardinal number of the reals." The Committee decided that the intended answer, (C), *could* be interpreted in that fashion and that the ambiguity was a desirable one. The statistical analysis seems to say that the Committee was correct. However, the reviewers maintained that the symbolism $\{1,2,3,\dots\}$ usually denotes the set of all positive integers, which does not include the two infinite cardinals that should be

EDUCATIONAL TESTING SERVICE

ITEM ANALYSIS

RESPONSE CODE	LOW N_1	N_2	N_3	N_4	HIGH N_5
OMIT	57	61	46	28	11
A	6	7	2	6	3
B	15	9	20	29	16
C	10	8	22	29	66
D	5	5	5	1	2
E	4	7	3	6	1
TOTAL	97	97	98	99	99

* DENOTES CORRECT RESPONSE

FORM	BASE N	OMIT	A	B	C	D	E	M_{TOTAL}	Δ_E SCALE	Δ_E	CRITERION	
OGR1	495	203	24	89	135*	18	21	13.0			IS70	REF
TEST CODE	ITEM NO.	M_O	M_A	M_B	M_C	M_D	M_E	P_{TOTAL}	$P+$	Δ_O	r_{bis}	
AD MATH	42	11.4	11.3	13.5	15.9	11.3	12.0	0.99	0.28	15.4	0.60	

FIG. 3

a part of the correct answer; and it might be further argued that the most natural extension of this symbolism would be to the class of all positive cardinal numbers rather than to the intended answer to this question. Although we did not feel that the criticism warranted changing any candidate's score, that particular question was not included in any subsequent editions of the test.

Insightful Questions

Prior to 1962, the GRE Advanced Test in mathematics was composed for the most part of fairly straightforward questions, many of which were hard questions dealing with subject matter usually encountered only by mathematics majors, but it was thought of strictly as a test of achievement. In 1962, the Committee began to discuss possible ways that might give the test more face validity—in the sense that the questions should appear to test more than knowledge of facts and standard techniques. Under the strong and enthusiastic leadership of Professor Dilworth, the Committee began an organized attempt to include questions that might test the kind of thinking that seems to be necessary for success in mathematics, thinking that is not merely step-by-step deduction, but thinking that includes intuitive leaps across what seem to be chasms of difficulty either unbridgeable by the familiar logic at hand or strewn with algebraic or computational obstacles that appear unsolvable within the time available. These kinds of questions are often described as "insightful."

In the late 1950's, the Committee had tried to avoid questions requiring no more than pre-calculus knowledge; but, in time, it was apparent that many of the most insightful questions required little in the way of background knowledge. It is probably best not to try to define precisely an "insightful question," but the category certainly includes all those questions for which there are at least two avenues of approach, one of which appears obvious, mechanical, computational, and time-consuming and the other being less obvious, not mechanical, less computational, and quick. These kinds of questions can require little knowledge other than simple facts about numbers or geometry, or they may require more sophisticated background.

For examples of insightful questions, the classic is the old chestnut:

> Two locomotives are traveling toward each other at a constant speed of 50 mph each. A hypothetical bee, which always flies at a constant speed of 100 mph, flies back and forth from one locomotive to the other, beginning when the locomotives are 100 miles apart and ending when the locomotives crash, smashing the bee between them. What is the distance traveled by the bee?

The hard way to answer the question is to sum the infinite series whose terms represent the

distances flown from one locomotive to the other. The insightful way is to observe that the trains traveled exactly 50 miles, at 50 mph, so the total flying time of the bee is one hour—ergo, the bee flew exactly 100 miles.

One example of a type of question that might have conceivably appeared on the test and that might be termed insightful is the following one, which should illustrate the kind of nonmechanical, noncomputational thinking that an insightful question is supposed to evoke.

$\int_{-1}^{1} \cos^{-1} x\,dx =$

(A) $2\cos 1 - 2\sin 1$ (B) $\frac{5\pi}{6}$ (C) π (D) $\frac{5\pi}{4}$ (E) 2π

Of course, if one remembers that the indefinite integral of $\cos^{-1}x$ is $x\cos^{-1}x - \sqrt{1-x^2}$, the question is quite straightforward; most GRE examinees would not; so another approach would have to be found. One fairly straightforward approach would be to substitute $\cos t$ for x in the expression $\int \cos^{-1} x\,dx$, evaluate the result in terms of t, convert back to x, and then evaluate that antiderivative from -1 to 1. Another relatively straightforward approach would be to use integration by parts with $u=\cos^{-1}x$ and $dv=dx$. A much simpler method than either of these is to look at the obvious region whose area equals $\int_{-1}^{1}\cos^{-1}x\,dx$ (see Figure 4), and observe that the part of that region that is in the first quadrant has the same shape and size as that part of the indicated rectangle (in the second quadrant of the picture) that is *not* in the shaded region. Thus the area we seek is the same as the area of that rectangular region, i.e., π.

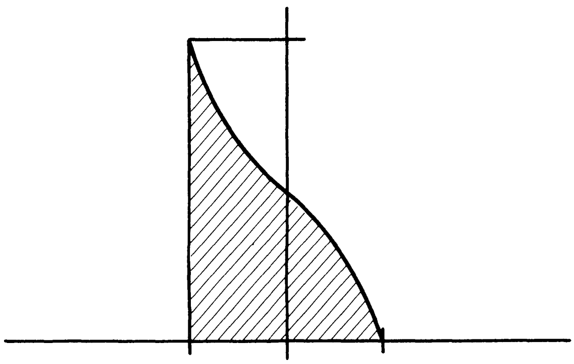

FIG. 4

The preceding question actually requires no knowledge of calculus other than the concept that the definite integral of any nonnegative function is the area of a certain region determined by the function and the limits of integration. Another question, this one actually requiring some standard calculus knowledge even for an insightful solution, is the following one—which will in addition illustrate some of the frustrating difficulties one encounters in trying to write questions that might pick out people who have mathematical insight.

$$\text{For}\quad 0<c<1,\quad \int_0^c \log\frac{1}{x}\,dx =$$

This question would be quite straightforward for anyone who remembers the antiderivative of log, e.g.,

$$\int_0^c \log\frac{1}{x}\,dx = \int_c^0 \log x\,dx = \lim_{t\to 0^+}\int_c^t \log x\,dx = \lim_{t\to 0^+} t(\log t - 1) - c(\log c - 1)$$

and the correct answer is now apparent using L'Hospital's Rule. However, many examinees for this

test would not remember that antiderivative and, for them, this question might be called insightful based on the following line of thinking (see Figure 5):

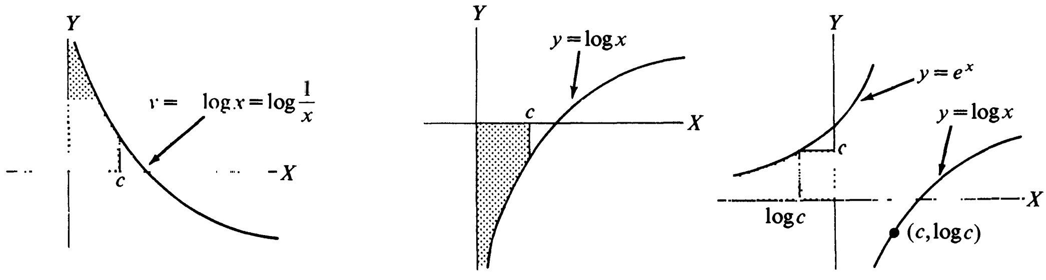

FIG. 5

We look for the area of the shaded region of the figure on the left above, which is the same as the area of the shaded region of the figure in the center, and which is the same as the area of the shaded region of the figure on the right, which is

$$\int_{-\infty}^{\log c} e^x\,dx + c\log c, \text{ i.e.,}$$
$$c + c\log c, \text{ i.e., } c(1-\log c).$$

The claim that this latter approach is more "insightful" than the former is of course based on the assumption that the evaluation of $\int e^x\,dx$ is better known than is the evaluation of $\int \log x\,dx$.

Despite the seemingly insightful nature of this question, it could not be used on the test because the naive application of the formula

$$\int \log x\,dx = x(\log x - 1)$$

to

$$\int_0^c \log\frac{1}{x}\,dx = -\int_0^c \log x\,dx = \int_c^0 \log x\,dx$$

yields formally

$$0(\log 0 - 1) - c(\log c - 1)$$

and, cavalierly, we have the correct answer

$$c(1-\log c)$$

arrived at in fallacious fashion.

Here we have an example of a question that would probably work beautifully and would undoubtedly rate being called "insightful" if we could count on having no one remember a certain integration formula; but, since we obviously cannot count on such a thing, we have merely an interesting question that would better be discarded.

Finally, lest the reader get the mistaken impression that all the questions we call insightful involve drawing pictures, let us consider:

$$\lim_{x\to e} \frac{1+\ln\dfrac{1}{x+e}}{e-x} =$$

(A) $-\infty$ (B) $-\mathrm{e}$ (C) 0 (D) $\frac{1}{e}$ (E) 1

A legitimate and straightforward approach to a solution of this problem is to recognize that L'Hospital's rule is applicable and proceed in that fashion. A quicker solution is available to anyone who recognizes that the quotient has the form

$$\frac{f(e)-f(x+e)}{e-x},$$

where $f(x)=\log x$, so the limit as $x \to e$ is $f'(e)=1/e$.

Although the inclusion of "insightful" questions on the test is still thought to be a very desirable goal, some interviews with high-scoring candidates conducted several years ago pointed to the possibility that the Committee might sometimes be aiming too high in this regard. These candidates—interviewed in two groups, one with scores from 800 to 850 and the other with candidates all of whose scores were 990—all seemed to indicate that the more subtle questions, which the Committee thought would *really* pick out people who were most insightful, had apparently missed the mark at least with respect to everyone being interviewed. They had all taken the more pedestrian approach on the most subtle questions; many had also followed the more mechanical path on the less subtle ones; and, in some cases where the obvious route appeared too formidable, a few actually omitted the question. It seemed that the strategy most often employed by these candidates was to follow their first inclination. If that inclination was along the insightful path, they arrived quickly; but otherwise they either took the more well-beaten path and, once started, did not look back to see if they might have missed a much shorter and easier route, or decided that the obvious route would take too much time, and they went on to another question. Some of the Committee's most "insightful" questions went completely over the heads of all those interviewed—although most of the 990-scoring candidates had eventually answered these questions by using more standard, time-consuming procedures.

The lesson for the Committee seemed to be that "insightful" questions are desirable, but that there is a point on the scale of subtlety easily reached by the Committee but above which all is lost on the candidates who actually take the test under timed conditions.

Historical Notes

The length of the test, as well as its form, has changed somewhat through the years. In 1939, there were 60 questions to be answered in 90 minutes; the answers were to be handwritten in the examination booklet; some were completely free-response, some were three-choice questions, and others were four-choice. In 1947, all questions were five-choice and answer sheets were now in use. There were still 60 questions to be answered in 90 minutes. In 1953, the test had increased to 75 questions to be answered in three hours, and all questions had eight possible answers to choose from. In 1963, the number of choices was reduced to five; in 1965, the number of questions was reduced to 70, and then to 65 in 1967. During all this time, there was just a single section of the test. In 1970, the GRE Board authorized a study to determine the desirability of changing all Advanced Tests in the 19 subject areas from a single three-hour examinaton to a five-part examination, each part separately timed, with a total time limit of three hours fifteen minutes. Although the Committee for the Advanced Mathematics Test was not among the early proponents of this change, the Committee soon became convinced that separately timed sections could work to its advantage. It was decided that there should be two 60-minute sections and three 25-minute sections. The Committee decided to put into the first 60-minute section 30 questions that might reasonably be expected to be answered quickly or quickly omitted—the latter on the basis of a lack of necessary knowledge of subject matter, e.g., one might expect to see such questions as the following one in the first section.

For each complex number z, let C_z denote the path described by $w(t)=z+e^{it}$, $0 \leqslant t \leqslant 2\pi$ and let $f(z)=\int_{C_z}\frac{dw}{w-z}$. Then f is

(A) the zero function
(B) a constant real-valued function but not zero

(C) a constant function but not real-valued
(D) a nonconstant entire function
(E) a function with one and only one simple pole

If the candidate has the background to answer this question, he will quickly recognize that (C) is the desired response; otherwise, he will even more quickly recognize that he should move on to another question.

In the second 60-minute section, the Committee put twenty questions that would be expected to take somewhat longer either to comprehend or to perform the manipulations necessary to arrive at the intended answer.

In the first 25-minute section, the Committee placed eight of what they considered to be their most insightful questions and the ones which previous experience had shown were most likely to be omitted by candidates using the previously very effective strategy of "If I can't see how to do it immediately, I'll skip it." The results using the first few editions of the test under the separately timed sections format seemed to indicate that the Committee had succeeded, in the sense that the per cent of "Omits" on such items decreased and both the per cent marking answers correctly and the r-biserial increased. It seemed that when better students had been forced to think a little longer about the insightful questions they were able to deal with them effectively.

In the second 25-minute section, the Committee placed eight other questions that also would be expected to draw a high percentage of "Omits"—this time, however, because each of them dealt with applications and generally appeared more formidable than they actually were. These questions were generally of three types: (1) Given a "real-life" situation, which of five offered mathematical models matched that situation better than any of the others, (2) given a mathematical model, which of five offered "real-life" situations matched that model better than any of the others, or (3) given a "real-life" problem—normally much simpler than those of types (1) or (2) above—answer a question about that situation by devising an appropriate model.

Again, the per cent of "Omits" on such questions generally went down and the per cent of correct answers generally went up, although the r-biserials for such questions did not appear to have changed significantly.

The third 25-minute section was for experimental questions that did not count toward the candidate's score. In many testing programs conducted by ETS, no question is ever used in an operational section of a test until it has first been tested experimentally. That had never been done with the Advanced GRE Tests. The advantages of such pretesting seem obvious upon reflection. The difficulty level of the test is much easier to control if the performance on these questions by appropriate candidates has been analyzed and the resulting statistics have been made available to the test assemblers before the questions are chosen for the operating sections of the test; furthermore, any questions with unacceptably low r-biserials can be discarded beforehand, and any distractors (wrong choices) that do not appear to be distracting anyone can be rewritten.

These advantages did not actually accrue, however, for the GRE Advanced Mathematics Test. Mathematics seems to be a discipline in which questions can be written with rather consistently high r-biserials and with more predictable difficulty levels than in most other fields. Individual questions may sometimes fool us, but we have been able to put together examinations of untested mathematics questions that, in the aggregate, behave rather much the way we had expected them to behave. The Committee selected questions for the experimental section that generally had received mixed praise and criticism beforehand, questions that we would certainly have hesitated to include in the operational sections; and the results were as we thought they would be, generally low r-biserials, high per cent of "Omits," and low per cent of correct answers.

The committees for GRE Advanced Tests in other fields, although not quite as successful as mathematics in such predictions, had similar experiences. For this reason—and the fact that separately timed sections were causing some administrative problems, including higher operational costs—the GRE Board authorized a return to a single-section test. The GRE Mathematics Committee

was less happy about this return than any of the other GRE committees because of what the Committee considered to be the salutary effects of forcing candidates to consider at some length the questions they would normally tend to omit after a quick reading. However, the decision of the GRE Board was accepted by the Committee as being in the best interests of the GRE program as a whole, and today the test consists of 66 questions to be answered in two hours, fifty minutes.

Upon his retirement from the Committee in 1969, Professor Dilworth wrote the following comments in a letter to Richard L. Burns, who was then Program Director for GRE.

> In the not-too-distant past, the GRE Advanced Mathematics Examination consisted almost entirely of problems whose solution required only the knowledge of the facts and techniques of the standard underclass mathematics courses. The content of the problems was primarily that of the basic freshman and sophomore calculus courses. Yet this examination was expected to provide criteria for admission to Graduate School and for graduate Fellowship awards. It is not surprising that, at that time, the examination was not held in high regard in the mathematical community.
>
> There appears to be general agreement that the graduate schools in mathematics are seeking students who can exhibit insight, understanding, and ingenuity in approaching mathematical problems. There is furthermore general agreement that the GRE Advanced Mathematics Examination should test the ability of the students to exercise these aspects of mathematical thinking. However, there are still many members of the mathematical community who feel that these aspects can only be tested by using elaborate essay-type questions. During the past few years the Committee has concentrated upon the construction of items which require for their solution some of the individual insights involved in the analysis of much more comprehensive essay problems. It has been our feeling that a suitable variety of such items will be just as effective in testing for insight, understanding, and ingenuity as an elaborate essay question. Such evidence as we have seen seems to indicate that this is the case. It was interesting to note that, at the meeting of the Committee in New Orleans, even [those] guests [some leading mathematicians invited by the Committee to examine and discuss a recent edition of the Test] who were unsympathetic to objective testing admitted that most of the items in the recent copy of the examination would require significant mathematical insight for a solution. I would also note that the former criticism of the GRE Examination is now heard hardly at all from either students or faculty.

Professor Dilworth was concerned that the favorable reception of the test by leading members of the mathematical community (a number of whom had just taken the opportunity to discuss the test in depth) should not be jeopardized. We feel that the gradual rotation of members of the Committee, which occurs with regularity, has been accomplished with that end very much in mind.

Under very restrictive conditions, copies of the test can be viewed by chairmen of departments that offer the Ph.D. in mathematics and may be examined, in the chairman's presence, by such other graduate faculty as the chairman may designate. In addition, selected faculty members are asked by ETS from time to time to examine a copy of the test and offer a written evaluation.

The Committee continues to welcome comments, queries, and suggestions about the test from the mathematical community at large.

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EVOLUTION OF THE TOPOLOGICAL CONCEPT OF "CONNECTED"

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In memory of Edwin W. Miller and Paul M. Swingle

Introduction. The purpose of this paper is to trace the evolution of one of the most basic concepts in Topology, viz., that of *connected* (not to be confused with "simply connected"). Like many other mathematical concepts of a fundamental nature (e.g., continuous function), it had only an intuitive meaning (such as "connected figure" in geometry) until the increasingly subtle demands of Analysis and Topology forced formulation of a satisfactory definition. The latter was not achieved, as one might expect, until a number of definitions had been proposed—each sufficient within its mathematical context but quite insufficient as the configurations studied became more general and abstract.

We try to clear up, incidentally, the existing confusion regarding the actual authorship of the definition ultimately adopted. Not surprisingly, we uncover a "multiple." For several years, European topologists considered F. Hausdorff to be the prime originator of the definition, apparently because their knowledge of set theory and fundamental topological notions was usually derived from his classic "Grundzüge der Mengenlehre" published in 1914 [5]. However, by the time of publication of his 1944 "Mengenlehre" [6], which was a third edition of the "Grundzüge," Hausdorff had discovered Lennes's earlier version of the same definition (see below), and called attention thereto in a note at the end of his book.

Thereafter the definition was commonly called the "Lennes–Hausdorff definition" of connected. Many modern textbooks on Topology seem to have adopted the term "Hausdorff–Lennes Separation Condition," or "Hausdorff–Lennes condition" for the type of separation involved in the definition. Possibly this received stimulus from the use of the term by S. Lefschetz in his American Mathematical Society Colloquium volume entitled *Algebraic Topology* [10]. On page 15, Lefschetz speaks of the "so-called Hausdorff–Lennes separation condition."

In his classic work on Topology [9], Kuratowski states in a footnote (p. 127) that the definition of connected "originates from" C. Jordan's *Cours d'Analyse* of 1893, and also cites Lennes's work. A justification for Kuratowski's statement is offered below.

W. Sierpiński, in the 1952 English edition of his work on general topology [19], attributes the definition to Hausdorff. However, in his *Foundations of Point Set Theory* [13, p. 378], R. L. Moore attributes the definition to Lennes. In my own book, *Topology of Manifolds* [20], I cited Schoenflies, Lennes, and Hausdorff, the Schoenflies definition being the same, although independently arrived at, as the definition of Jordan which was cited by Kuratowski.

Without further citing of literature, it seems fair to conclude that little attention was paid to United States journals during the early part of the present century, since Lennes' definition was published in both the *Bulletin of the American Mathematical Society* and the *American Journal of Mathematics* in 1906 and 1911, respectively. Perhaps, too, the same should be said about the Hungarian journals, for nowhere in the topological literature cited above (nor in any other, so far as I have observed) is the name of F. Riesz mentioned in connection with the definition of connected, although the same definition as that given by Lennes was published by him in 1906 (in Hungarian) and in 1907 (in German).[1]

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His main interests are topology, foundations of mathematics, and the cultural history of mathematics. His books include *Lectures in Topology* (edited with W. L. Ayres); an AMS Colloquium Volume, *Topology of Manifolds; Introduction to the Foundations of Mathematics*; and *Evolution of Mathematical Concepts.—Editors*

The evolution. Unquestionably the roots of the concept of connected lie in the notion of the continuous, but more specifically the linear continuum, which goes back as far as the Greeks, who struggled to clarify the notion in the light of Zeno's paradoxes. The history of this, so far as it is known, is already adequately covered in the literature. Similar remarks hold for the contributions of the medieval mathematicians and philosophers, especially of the scholastic tradition, whose influence on both Bolzano and Cantor have been widely commented upon.

Bolzano's contribution. Although the theory of proportion given by Eudoxus (and reproduced in Euclid's *Elements*) has been credited by some as the equivalent of Dedekind's definition of the real continuum, it seems not to have figured in the analysis of the early part of the nineteenth century. During the latter period, the growing stress for a proper basis for establishing the "location theorem"[2] of algebra, using only arithmetic (as opposed to geometric) means, led Bernard Bolzano to offer a proof of the theorem in 1817 [**1**]. A casual reading of Bolzano's works convinces one that he had a remarkable intuitive knowledge of the structure of the real continuum[3] as it is understood today. Along with this, he evidently conceived of the notion of a general continuum. Consider the following definition (given in his *Paradoxien* [**2**, p. 129]): "... a continuum is present when, and only when, we have an aggregate of simple entities (instances or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbor. When this does not obtain, when so much as a single point of the aggregate is not so thickly surrounded by neighbors as to have at least one at each individual and sufficiently small distance from it, then we call such a point *isolated*, and say for this reason that our aggregate does not form a continuum."

Curiously, the motivation for this definition, according to Bolzano's own testimony, lay in the paradoxes that plagued the philosophical and mathematical conceptions of time, space, and "substance." Bolzano reasoned that, by establishing a suitable characterization of the abstract structural pattern common to all these concepts, the paradoxes could be explained. The analogy with the Greek dilemma and the efforts to resolve it is striking.

Now the *Paradoxien* was written toward the end of Bolzano's life and published posthumously in 1851, while the proof of the "location theorem," cited above, was published some 34 years previously. But the motivation for the latter was strictly mathematical in that it was to free analysis of its notorious reliance on the geometric aspects of continuity. There can be little doubt, however, that Bolzano's development of his intuition of the continuous in the latter work was contributory to his philosophical conception of time, space, and substance as continua. And it seems to represent the first attempt at a mathematical formulation of the topological notion of connected. Since, as was to be the case for over a half-century thereafter, the definition of "connected" was tied to that of "continuum," it would perhaps be more proper to term it the mathematical progenitor of the notion of continuum. However, the time for consideration of point sets having no compactness properties had not arrived in mathematics, and there is little doubt that the intuitive notion which Bolzano (and after him Cantor) was trying to make precise was equivalent, in its context, to that which led later to the "unrestricted" notion of topological connectedness.

Cantor's contribution. Cantor, who was familiar with Bolzano's work, saw clearly that the property used by Bolzano was insufficient to make precise the intuitive notion of continuum. In a paper often called the *Grundlagen* [**3**, §10], he pointed out, for example, that sets consisting of several separated continua satisfy Bolzano's condition. Moreover, he recognized intuitively that the compactness properties now associated in topology with the notion of continuum had not been required in Bolzano's definition, and pointed out that the complement of an "isolated" point set in n-dimensional coordinate space, E^n, $n \geqslant 2$, is a continuum according to Bolzano.[4] He also rejected enlisting the concepts of time or space as aids in exploring the mathematical notion of continuum, deeming the relationship quite the reverse.

According to Cantor, a continuum in E^n must possess *two* basic properties, namely, the property of being *perfect* and that of being *connected.* In modern terms, a point set in E^n is perfect if it is closed and dense-in-itself (i.e., each of its points is a limit point of it).[5] In this connection he pointed out the insufficiency of requiring a point set to be only perfect in order that it be a continuum by giving his classical example of a totally disconnected perfect set—the "Cantor ternary set," now often called simply the "Cantor set." (See footnote 11, p. 590, *loc. cit.*) He then defined *connected* as follows: A point set T is connected if for every two of its points t and t', and arbitrary given positive number ϵ, there always exists a *finite* number of points $t_1, t_2, \dots, t_n$ of T such that the distances $tt_1, t_1t_2, \dots, t_nt'$ are all smaller than ϵ (*loc. cit.*, 575–576). Then any perfect and connected subset of E^n is a *continuum*, according to Cantor, who pointed out in a footnote (#12, p. 590, *loc. cit.*) that no special dimension was implied in the definition; a line, surface, solid, etc., are all continua. Incidentally, for bounded subsets of E^n, this is equivalent to the modern definition of a continuum.

The most important aspect of Cantor's definition of continuum is his separation of the two concepts *perfect* and *connected*, thus identifying for the first time the latter as an independent property. At the time, however, Topology was virtually nonexistent as a field of study, and it could not be expected that sets having the sole property of connectedness would receive any attention. And as already implied above, Cantor's definition of connected was quite adequate for the study of continua.

C. Jordan's contribution. The next noteworthy step in the evolution of the concept of connected is found in C. Jordan's *Cours d'Analyse.*[6] Apparently Jordan was not familiar with Cantor's definition of ten years earlier, since he makes no mention of it. Following a discussion of closed sets,[7] establishing the notion of distance ("écart") between them, and defining sets as *separated* when the distance between them is greater than zero, he gives a definition of what he calls "un seul tenant"—in modern terms "component"[8]—of a bounded, closed set, to wit: a bounded and closed set of points has a single component if it cannot be decomposed into several closed separated sets. "One sees easily that the distinctive character of such a set is the following: 'For arbitrary ϵ, one can intercalate, between any two of its points p, p', a chain of intermediate points of the set such that the distance between consecutive points is less than ϵ.'" It is this statement that Jordan italicized, not the preceding definition.

This coincides, of course, with Cantor's definition of connected and is proved a necessary and sufficient condition for a bounded and closed set to consist of a single component (*loc. cit.*, p. 26). It is then simple to prove (*loc. cit.*, p. 27) that a subset of the real line which forms a single component and contains two numbers, a and b, must contain every number between a and b. This corollary of Bolzano's theorem seems to have been the chief motive for Jordan's definition of component.

Schoenflies' contribution. In 1904, A. Schoenflies published the first of his fundamental researches into the *topological* aspects of point set theory **[18]**. He was aware of Cantor's definition of connected, which he cited (*loc. cit.* pp. 208–209), but went on to comment that even though the concept of *distance* formed a primitive geometric notion for the axiomatic basis of his work, it was preferable to give a purely set-theoretic definition of connected, whereupon he gives the following: A perfect set is called *connected* if it is not decomposable into [at least two nonempty][9] subsets each of which is perfect.

This is, for bounded sets, the equivalent of Jordan's definition of *un seul tenant* which, according to the accompanying remarks, became known to Schoenflies only after he had announced his own version. Stating that Jordan introduced the definition only to derive Cantor's formulation of the concept (with which he operated thereafter), Schoenflies observes that "connectedness is an important and fundamental property for Analysis Situs as a whole." This statement represents an important step forward in the evolution of the connectedness concept. Whereas Cantor only separated the notion from the other properties of a continuum, Schoenflies now elevated it to the position of a *fundamental*

property of Topology, and went on to prove its invariance and to study the property especially in the context of plane topology.

Despite Schoenflies' recognition of the fundamental character of connectedness, his view was still limited, in that he expressed the opinion that, while the definition was formulated for perfect sets, it could equally well be stated for (merely) closed sets; but "since for closed sets which are not perfect, connectedness cannot come into question, it is sufficient to limit the definition to perfect sets" (*loc. cit.*, p. 173)! Thus, while making an important step forward, Schoenflies made another step backward.

The work of W. H. and G. C. Young. Although chronologically the work of W. H. and G. C. Young virtually coincides with that of Lennes and Riesz to be discussed below, it is interpolated here as a kind of capstone to the work already described, as well as of intrinsic interest for its adumbration of later work in the theory of connectedness.

The classic book **[22]** of W. H. and G. C. Young[10] introduces a definition of connected in terms of regions:[11] "A set of points such that, describing a region in any manner round each point and each limiting point of the set as internal point, these regions always generate a single region, is said to be a *connected* set provided it contains more than one point. Hence if a set is connected the set got by closing it is connected, and *vice versa*."[12]

From this definition, the Youngs prove: A connected set cannot be divided into closed components (= subsets) without common points. Conversely a set which cannot be divided into closed components without common points is, if closed, a connected set. We recall that this proposition was used by Jordan and others to define connected in the case where the set in question is closed.

The Lennes and Riesz definitions. It is remarkable that throughout the period discussed above—from the time of Bolzano to 1905, over half a century—the notion of connected was confined to *closed* sets; and this in spite of the fact that Cantor divorced the notion from closure in his definition of continuum. On the other hand, it is not surprising, since attention was devoted exclusively either to the real continuum or to the subsets of euclidean space (usually the plane), and the only non-closed sets of importance were of a special character, such as the set of rationals or open segments on the real line, and the circular or triangular regions of the plane.

Of course, Jordan, and following him Schoenflies, had proposed definitions of connected which virtually begged for generalization to non-closed sets. And this step was finally taken in 1905–06 by both N. J. Lennes and F. Riesz. Lennes gave his definition at a meeting of the American Mathematical Society in December 1905, and it was published in the abstract of his talk the following year in the Bulletin of that Society **[11]**. Riesz's definition was presented to the Hungarian Academy of Sciences on January 22, 1906, and published later the same year **[15]**. Here was clearly a "multiple" —a case of independent invention by more than one investigator.

Lennes' definition reads (*loc. cit.*): *A set of points is connected if in every pair of complementary subsets at least one subset contains a limit point of points in the other set.* This is stated in such a fashion that it is meaningful in any space in which limit point is defined (although undoubtedly the author's thinking, like that of most topologists of the time, was of euclidean spaces).

Riesz's definition has several remarkable features. In the first place, it is given in the context of an essay **[16]** devoted to the relations between the "physical continuum" and the "mathematical continuum."[13] In defining the physical continuum, Riesz uses the relation "unterscheidbar" between space points, not the topological notion of limit point. The definition proceeds as follows: "Das physikalische Kontinuum heisst zusamenhängend, wenn es nicht in zwei Teilmengen zerlegt werden kann, dass jedes Element der einen Teilmenge unterscheidbar sei von jedem Elemente der anderen Teilmenge." Notice the striking resemblance to the Jordan–Schoenflies definition, although Riesz makes no reference to the latter. However, there is conclusive evidence in previous papers of Riesz' that he was familiar with Jordan's *Cours d'Analyse* and hence, probably, with Jordan's definition.[14]

In the second place, the definition of *connected* for topological spaces (he does not use the latter

term) is given initially for what he calls a "mathematical continuum," which, in modern terms, is an abstract topological space defined by four axioms adumbrative of such later systems as were given by Hausdorff and Kuratowski. The definition reads as follows: "Das mathematische Kontinuum heisse zusammenhängende wenn es nicht in zwei offene Teilmengen zerlegt werden kann, die Komplementarmengen für einander sein." For *subsets* of such a space he then distinguishes two degrees of connectedness: A set is called *connected* if it cannot be decomposed into two subsets whose closures are disjoint; it is called *absolutely connected* if for every decomposition of it into two [nonempty] subsets, there exists at least one element which belongs to one subset and is a limit point of the other. It is the second of these, of course—i.e., absolutely connected—that is the modern definition of connected.

If Riesz had been familiar with the modern device of relativizing the topological notions of "closed" and "open," he would, presumably, have identified the definitions of connectedness for a mathematical continuum and that of absolute connectedness.

Actually, the multiple which occurred when these two definitions were given was joined by a third, viz., F. Hausdorff's definition. Apparently when giving his definition in his book of 1914 (*loc. cit.*), Hausdorff was unaware of either Lennes' or Riesz' definitions. On the other hand, Hausdorff did proceed, in this book, to study some of the properties of connected sets as topological concepts in their own right.

However, the first paper devoted to the study of connected sets was not published until 1921; we refer here to the classic paper *Sur les ensembles connexes* of B. Knaster and C. Kuratowski **[8]**. This paper was significant in the evolution of the concept of connectedness because: (1) it established the fact that connected spaces lacking compactness properties have a variety of interesting topological properties; (2) it gave impetus to a host of studies, both in Topology and in the logical foundations of set theory; (3) it gave the ultimate emphasis to Schoenflies' statement, quoted above, concerning the fundamental character of connectedness.[15]

Concluding remarks. From an evolutionary point of view, the development of the concept of connectedness proves to be a revealing "case study." Its roots, as in the case of many other mathematical concepts, are embedded in the contemplation of physical time, space, and "substance." At the hands of Cantor it finally split off from philosophical and physical considerations to become a part of mathematical theory. But it was not easily divorced from the concept of *continuum* within which it was first formulated—a consequence of its mathematical environment, which consisted chiefly of the study of curves and surfaces, examples of what Cantor called *continua* in the mathematical sense. This was a case of the operation of "environmental stress," in that the mathematical environment worked to confine the notion within a restricted area.

It failed to find its proper place in mathematical theory until Schoenflies pointed out its invariance under topological transformations, as well as its independent status as a topological property. But although Schoenflies, who discovered essentially the same definition that Jordan had given over a decade earlier, made an important step forward, Topology had still not grown much beyond the study of configurations whose compactness properties made the Jordan–Schoenflies definition quite adequate. Indeed, so much so, that when the Youngs wrote their classic "The Theory of Sets of Points" during the decade between Jordan and Schoenflies, they seem to have deliberately phrased their definition of *region* (allowing a region to include boundary points freely) so that the Cantor definition would be preserved (see the remark above concerning the Young definition).

Lennes's generalization of these definitions was apparently a result of his consideration of non-closed sets. In the paper **[12]** giving in detail the results announced in the 1906 abstract (*loc. cit.*), he first defined connectedness for open sets (in euclidean space) by using broken lines, an open set U being connected if every pair of points a, b in U are joined by a broken line lying wholly in U (*loc.*

cit., p. 293). He observed that by the Cantor definition of connected, the union of the interior and exterior of a planar circle forms a connected set; moreover, that "if from the ordinary continuum in space of any dimensions any set whatever which is nowhere dense is removed, the residue would form a connected set" (*loc. cit.*, 303, footnote). He thereupon gave the form of the definition now generally accepted, remarking that it "applies in cases where the former does not." (In other words, it renders sets connected which our intuition tells us should be connected and rules out those that, like the complement of the circle in the plane, should not be termed connected.)

One of the remarkable features of Riesz' definition, as we have already noticed, is that it was given in the context of an abstract topological space. This aspect of Riesz' work seems also to have been generally unnoticed for some time, despite the fact that Fréchet called attention [**4**, Note B] to Riesz' abstract space axioms as they were later presented at the International Congress in Rome, 1908 [**17**]. In any event, his definition, although agreeing with Lennes', achieves thereby its most general character, freed from all metric considerations.

Although the lack of diffusion from one country to another, which characterized earlier periods in mathematics, had begun to subside, the period during which the topological concept of connectedness was developed still shows considerable lack of diffusion. Lennes' and Riesz' work, both published in reputable journals during the first decade of the century, was generally unknown until the journal *Fundamenta Mathematicae* commenced publication in 1920. The occurrence of a three-member multiple during the first quarter of the present century is quite noteworthy.

One further comment: One of the noteworthy features of the Knaster and Kuratowski article referred to above was its presentation of paradoxical examples of connected sets having no compactness properties. I have pointed out elsewhere [**21**] the contribution that paradox can make to the development of mathematical concepts. The examples given by Knaster and Kuratowski (*loc. cit.*) proved a great stimulation to the study of connectedness. There ensued a sizable literature devoted to the concept, and in recent years this has engendered interesting questions in the Foundations of Set Theory.

Notes

1. My attention was first called to Riesz' work in this connection by Professor C. E. Aull, to whom I am indebted for references thereto.

2. That is, if a real polynomial $f(x)$ is negative for $x=a$ and positive for $x=b$, then it is zero at some value between a and b. (Bolzano stated, namely, that if $f(x)$ and $\varphi(x)$ are continuous real functions over an interval $a\leqslant x\leqslant g$, and $f(a)<\varphi(a)$, $f(b)>\varphi(b)$, then there exists a real number c such $a<c<b$ and $f(c)=\varphi(c)$.)

3. The notion of a countable dense subset (e.g., the rationals) escaped him, to be sure. But Bolzano did not encounter the same sort of problem in real analysis that forced Cantor's formulation of the notion.

4. It is curious that Cantor did not point out that the set of rational points in E^1, the real line, is also a continuum by Bolzano's definition.

5. Cantor's precise definition utilized the notion of *derivative* of a point set, a point set P being *perfect* if it coincides with its first derivative P'.

6. We refer here to the second edition, 1893, vol. 1, pp. 24–28 [7]. This is the edition in which the classical concept of *continuous curve* was first defined; the latter was to receive much attention when its "space-filling" character was discovered.

7. Jordan used the word "perfect" ("parfait") instead of "closed." We use the latter term here to avoid confusion with Cantor's use of the term "perfect." Modern terminology conforms to the latter.

8. Notice that Jordan did not use the term "connected." The assumption of "bounded" was necessitated by the defining of "separated" in terms of distance.

9. The bracketed condition is clearly implied in Schoenflies's statement, although not explicitly stated by him.

10. This book was intended as the "first attempt at a systematic exposition" of Cantor's ideas on the theory of sets. See Preface, *loc. cit.*

11. In modern terms, a *region* was a connected open set with or without an arbitrary set of its boundary points. The Youngs defined it as generated by successively overlapping triangles. The definition of connected is on p. 204, *loc. cit.*

12. If the words "and each limiting point" are omitted from this definition and "region" restricted to *interiors* of the regions as defined by the Youngs, then the above definition is equivalent to the "simple chain definition" of connected. See R. L. Wilder, *loc. cit.*, p. 34, Corollary 12.5.

13. "Ich suche nur den Weg, der von den räumlichen Vorstellung zu dem [mathematischen] Raumbegriffe führt" (*loc. cit.*).

14. See, for instance, in the collected works [14], paper A2 (1905), in which he mentions Jordan's "écart" on the first page; paper A3 (1905), references to the *Cours d'Analyse*; and paper A5 (1905), in the first sentence of which he defines "d'un seul tenant," the same term that was used by Jordan.

15. Of course, Schoenflies was not strictly speaking of the type of connectedness exploited by Knaster and Kuratowski, which was the Lennes–Riesz definition now generally accepted.

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MATRICES, EIGENVALUES, AND COMPLEX PROJECTIVE SPACE

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Let me pose a straightforward problem in linear algebra. Recall that the eigenvalues of an $(n \times n)$ complex matrix $M=(m_{ij})$ are the roots of the characteristic polynomial:

$$p(t)=\text{determinant }(tI-M)$$

where I is the identity matrix. So p is a complex polynomial of degree n; there are thus n eigenvalues, possibly with some duplications. The problem I want to pose is sort of an inverse to that of finding eigenvalues. Suppose we are given n complex numbers $\{\xi_1, \xi_2, \ldots, \xi_n\}$, possibly with some duplications, and a matrix M. Can we, by adjusting only the *diagonal* elements of M, cause the adjusted matrix to have precisely the eigenvalues $\{\xi_1, \ldots, \xi_n\}$?

This is the additive inverse eigenvalue problem. The general subject of inverse problems (of both mathematical and more vulgar natures) is the topic of a recent article in this journal by J. B. Keller **[8]**. Usually inverse problems arise from a particular application, but here we want to consider the problem we have posed in its own right. I daresay it is not at all clear on the face of it what the answer should be. The answer is in fact "yes"; indeed, generically (whatever that means) there are $n!$ different solutions. After several people had found partial answers, the general result was established by S. Friedland. He has given two proofs. The first [5] is totally algebraic; the second [7] uses methods from algebraic geometry. Here we will develop machinery to give a third proof based on topological degree (see [1]). Degree theory is a powerful and general method, often used to prove existence results in analysis, as well as being easy to apply once the problem is set correctly. I hope to illustrate this here. Also, the present proof gives me the occasion to introduce a nice family of manifolds (complex projective spaces) which the reader may not have had the opportunity to meet before.

A particular case. Before taking on the general problem, let us tackle a particular example in a straightforward way. If $n=1$, the problem is trivial, so let M be the two-by-two matrix

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

Let D be the diagonal matrix with entries (d_1, d_2). Changing the diagonal elements of M amounts to adding a diagonal matrix D to M. We are thus interested in determining D so the characteristic polynomial of $M+D$ has prescribed roots. Let us make the following simplifying observation: specifying the roots amounts to the same thing as specifying the characteristic polynomial, since the roots and the polynomial determine each other. So given α_1, α_2, we want to solve

$$\text{determinant }(tI-M-D)=t^2-\alpha_1 t+\alpha_2$$

for d_1, d_2. Expanding and equating coefficients, we get the equations

$$d_1+d_2=\alpha_1$$
$$d_1 d_2=\alpha_2+2.$$

Thus d_1, d_2 are the roots of the quadratic

$$d^2-\alpha_1 d+\alpha_2+2=0.$$

By the quadratic formula, the d's are

$$\frac{1}{2}\left(\alpha_1 \pm \sqrt{\alpha_1^2-4\alpha_2-8}\,\right)$$

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(in this particular case, it doesn't matter which d is d_1 and which is d_2). Note that there are two answers unless $\alpha_1^2-4\alpha_2-8=0$, in which case there is one.

A little experimentation will show that this straightforward algebraic approach is going to get sticky (the characteristic polynomial of $M+D$ for M a general 2-by-2 matrix is

$$t^2-t(m_{11}+m_{22}+d_1+d_2)+(m_{11}m_{22}-m_{21}m_{12}+d_1d_2+d_1m_{22}+d_2m_{11})),$$

so we will proceed to develop a more abstract approach that avoids explicit equations.

Complex projective space. As we observed above, specifying the eigenvalues of a matrix is the same thing as specifying its characteristic polynomial, and we will consider the problem in this latter form. This avoids the complication of dealing with the unordered set of roots and allows us to deal with the ordered set of coefficients of the polynomial. (An unordered set is harder to coordinatize than an ordered one.) Let

$$p=a_0t^n+a_1t^{n-1}+\cdots+a_n$$

be an arbitrary complex polynomial of degree n. We can write $p=(a_0,\dots,a_n)$. The only polynomial we want to eliminate is the zero polynomial, so the only restriction we put on the a's is that they are not all zero. We want to identify two polynomials if their roots are equal—i.e., if the polynomials are proportional—so we set

$$(a_0,\dots,a_n)\equiv(za_0,\dots,za_n)$$

for any non-zero complex number z, and denote the quotient space $\mathbf{C}P^n$. The notation might stand for "*c*omplex *p*olynomials of degree n" but the space is usually called complex projective space of dimension n. (The analogous real 2- or 3-dimensional projective space is where classical projective geometry is done.) It is in fact a compact complex analytic manifold of complex dimension n (real dimension $2n$). To prove that is straightforward, well known, and not very interesting, but for completeness we have included a proof in the appendix. An element of $\mathbf{C}P^n$ will be written $[a_0,\dots,a_n]$. We will abuse terminology and sometimes call elements of $\mathbf{C}P^n$ polynomials.

Note that we allow the lead coefficients of p to possibly be zero, so $\mathbf{C}P^n$ actually consists of polynomials of degree $\leqslant n$. Of course, a polynomial of degree $m<n$ has only m roots. We can artificially give such a polynomial n roots by giving it $n-m$ roots equal to ∞. This small complication is the price we pay to have a compact manifold.

Consider the case $n=1$. If $a_0\neq0$, the polynomial $p(t)=a_0t+a_1$ has the single root $-a_1/a_0$. As a_0 goes to 0, the absolute value of the root becomes large. The correspondence

$$p=[a_0,a_1]\leftrightarrow \text{root } -a_1/a_0$$
$$p=[0,a_1]\leftrightarrow \text{root } \infty$$

is a homeomorphism between $\mathbf{C}P^1$ and the one-point compactification of the complex numbers $\mathbf{C}$, i.e., the two-sphere S^2. In fact, as a complex analytic manifold, $\mathbf{C}P^1$ is the Riemann sphere. The $\mathbf{C}P^n$ for larger n are not such familiar spaces.

We might also note that the subspace of $\mathbf{C}P^n$ of points with $a_0\neq0$ (i.e., the polynomials of actual degree n) is diffeomorphic to $\mathbf{C}^n$. Thus $\mathbf{C}P^n$ is a compactification of $\mathbf{C}^n$. Being a complex analytic compactification of $\mathbf{C}^n$ is one of the properties that makes $\mathbf{C}P^n$ of such paramount importance in mathematics. The one-point compactification S^{2n}, for example, is not a complex analytic manifold if $n>1$. In particular, the reader is invited to look in any book on algebraic geometry or several complex variables (where $\mathbf{C}P^n$ is called a closure of $\mathbf{C}^n$) to see how important—and classical—$\mathbf{C}P^n$ is.

Setting the problem. As above, let D be the diagonal matrix with entries $(d_1,\dots,d_n)$. Thus we consider $D\in\mathbf{C}^n$. Fix M. We define a continuous map $f_M^0:\mathbf{C}^n\to\mathbf{C}^n$ by $f_M^0(D)=(a_1,\dots,a_n)$ if

$$\text{determinant }(tI-M-D)=t^n-a_1t^{n-1}+\cdots+(-1)^na_n. \tag{*}$$

We write this as $\det(tI-M-D)=(a_1,\dots,a_n)$. If f_M^0 is surjective, the answer to the inverse eigenvalue problem for M is "yes."

Degree theory works for maps between compact manifolds, so we proceed to compactify the map f_M^0. Since CP^n is a compactification of C^n, we obviously have a map f_M^1: $C^n \to CP^n$. Using (*), we can write $f_M^1(D)=[1,a_1,\ldots,a_n]$. We compactify the domain differently. We compactify each copy of C in C^n to a Riemann sphere S^2. (Put informally, we allow each d_i to take on ∞ as a value.) We claim that $f_M^1: C^n \to CP^n$ extends to a continuous map $f_M:(S^2)^n \to CP^n$. (If such an extension exists, it is unique, since C^n is dense in $(S^2)^n$.) To prove the claim, we observe

$$f_M^1(d_1,\ldots,d_n)=\det\begin{bmatrix} t-m_{11}-d_1 & -m_{12} & \cdots & -m_{1n} \\ -m_{21} & t-m_{22}-d_2 & \cdots & -m_{2n} \\ \vdots & \vdots & & \vdots \\ -m_{n1} & -m_{n2} & \cdots & t-m_{nn}-d_n \end{bmatrix}$$

$$=d_1\det\begin{bmatrix} \frac{t}{d_1}-\frac{m_{11}}{d_1}-1 & -\frac{m_{12}}{d_1} & \cdots & -\frac{m_{1n}}{d_1} \\ -m_{21} & t-m_{22}-d_2 & \cdots & -m_{2n} \\ \vdots & \vdots & & \vdots \\ -m_{n1} & -m_{n2} & & t-m_{nn}-d_n \end{bmatrix}.$$

In CP^n, two polynomials are equal if one is a constant multiple ofthe other. Therefore $f_M^1(D)$ is equal to the last expression without the multiplier d_1. Now suppose d_1 goes to ∞. The determinant goes to

$$\det\begin{bmatrix} -1 & 0 & \cdots & 0 \\ -m_{21} & t-m_{22}-d_2 & \cdots & m_{2n} \\ \vdots & \vdots & & \vdots \\ -m_{n1} & m_{n2} & \cdots & t-m_{nn}-d_n \end{bmatrix}$$

$$=-\det\begin{bmatrix} t-m_{22}-d_2 & \cdots & m_{2n} \\ \vdots & & \vdots \\ m_{n2} & \cdots & t-m_{nn}-d_n \end{bmatrix}.$$

Therefore $f_M(\infty,d_2,\ldots,d_n)$ is well defined as this last expression. The situation is handled exactly the same if any collection of the d_i go to ∞. Note that if k of the d_i are ∞, then $f_M(d_1,\ldots,d_n)$ is a polynomial of degree $n-k$. Therefore if f_M is surjective, so is f_M^0.

Degree theory. Let N, N' be two compact connected oriented manifolds without boundary of the same dimension. Degree theory assigns to each continuous map $f:N\to N'$ an integer $\deg f$ satisfying the following axioms:

(1) (Homotopy invariance) If $F:N\times I\to N'$ is a continuous map (where I is the closed unit interval $[0,1]$), and $f_i=F|N\times\{i\}:N\to N'$ for $i=0,1$, then deg $f_0=$deg f_1. (That is, if f_0 can be continuously deformed to f_1, their degrees are the same.)

(2) (Localizability) If $n'\in N'$ is a regular value of f, then $\deg f=\Sigma\deg_n f$ where the sum is over all $n\in f^{-1}(n')$.

Let me explain this last axiom. A point n' in N' is *regular* if f is differentiable at each $n\in f^{-1}(n')$ and the derivative is non-singular at each of those n. Thus the Jacobian of f at each such n has non-zero determinant and the sign of this determinant (± 1) is the local degree $\deg_n f$ of f at n. (It is here that the orientation comes into play.) Since N is compact, the inverse image of a regular point is finite; hence the sum in (2) is well defined. In particular, suppose f is not surjective. If $n' \notin$ image f, it is

vacuously a regular value. The sum in axiom (2) is also vacuous, so $\deg f = 0$. So we can append a third property:

(3) If $\deg f \neq 0$, then f is surjective.

A considerable amount of machinery of algebraic topology is developed to prove the degree exists; but, granting the existence, it is easy to calculate using the axioms. Here is an elementary example to illustrate (2). Let $N = N' = S^1$, as in Figure 1, and f be radial projection in. The point α is a regular point. Its inverse image is the point a and the local degree of f at a is $+1$. Thus $\deg f = +1$. Similarly β is a regular point with inverse image points b_1, b_2, b_3. The local degrees at b_1, b_3 are $+1$. At b_2, the orientation is reversed and the local degree is -1. Again, we see $\deg f = +1$. The points γ, δ are not regular since the derivative is singular at the points c, d.

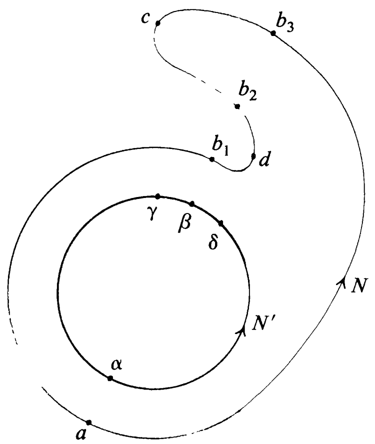

FIG. 1

It is well known that the Brouwer Fixed Point Theorem can be proved by a degree argument, so let's make a small digression to show how to do it. The fixed point theorem is known to be equivalent to the statement that there does not exist a map f of the unit n-dimensional disk D^n to its boundary sphere S^{n-1} which is the identity on S^{n-1}. Suppose such f existed. Extend f to a map of all of real n-space R^n to itself by mapping by the identity outside of the disk. This clearly extends to a map of the one-point compactification to itself. This extension has a degree. The inverse image of any point outside the unit disk is one point, and the local degree is $+1$, since the map is the identity there. Hence the degree of the map is $+1$. On the other hand, if D^n maps to its boundary, the map is not surjective and has degree 0. This is a contradiction and f cannot exist.

Completion of the problem. We propose to show that the degree of $f_M: (S^2)^n \to \mathbf{C}P^n$ is $n!$. This will finish the problem. Using axiom 1, we deform f_M to a more standard map. The space of all complex $(n \times n)$ matrices is homeomorphic to $\mathbf{C}^{n^2}$. The homeomorphism is effected by the components of the matrices. Hence the path τM for $\tau \in I$ is a "straight line" from M to the zero matrix Z. Let $F: (S^2)^n \times I \to \mathbf{C}P^n$ be

$$F(D, \tau) = \det(tI - \tau M - D).$$

This is a homotopy between f_M and f_Z, so $\deg f_M = \deg f_Z$. But f_Z is a well-known map:

$$f_Z(d_1, \dots, d_n) = \det(tI - D) = (t - d_1)(t - d_2) \cdots (t - d_n).$$

(If some of the $d_i = \infty$, the corresponding factors are omitted.) Thus a point in the inverse image under f_Z of a polynomial $p(t)$ is a list of the n roots of p in some order. If p has no multiple roots, there are $n!$ ways of listing its roots. If such a p is a regular point of f_Z and the local degree at each of

its inverse image points is ± 1, the degree of f_Z is $n!$ These are more or less standard facts (at least they should seem reasonable), and we will consider the proof done. However, for the insistent reader, we have included proofs of them in the Appendix.

Some concluding remarks. We should point out that degree theory also works for manifolds with boundary if the maps take the boundary to the boundary. In this case the degree of the map on the whole manifold is equal to the degree of the map restricted to the boundary. The Brouwer Fixed Point Theorem yields easily to this degree also. Degrees are also defined for maps between non-compact manifolds if all maps and homotopies are proper. We could have worked directly with $f_M^0: \mathbf{C}^n \to \mathbf{C}^n$. We would have had to do the estimates to show that f_M^0 and the homotopy between f_M^0 and f_Z^0 are proper. We would have ended up doing essentially the same work we did and the course we chose allowed us to introduce the $\mathbf{C}P^n$.

The Fundamental Theorem of Algebra (that any non-zero complex polynomial p has a root) is easily proved using degree theory, and the reader is invited to do it. Consider p as a map from $\mathbf{C}$ to $\mathbf{C}$. Show that as $|z| \to \infty$, so does $|p(z)|$, and thus p extends to a map $S^2 \to S^2$. If p is of algebraic degree n, homotop p to the polynomial z^n by linearly damping the lower degree terms to zero. Show that the homotopy extends to a homotopy over S^2. Finally, show that $z \to z^n$ has degree n by considering the regular point 1 in the range. (Incidentally, what is the degree of a real polynomial as a real map?)

For a nice application of the inverse eigenvalue problem, the reader is referred to [3]. It is well known that a square complex matrix X with zero trace can be written as a commutator $X = AB - BA$. The authors show in [3] that the eigenvalues of the two matrices A and B can be specified so long as the specified eigenvalues of one of the two are all different.

There is another inverse eigenvalue problem, the multiplicative one, where one tries to specify the eigenvalues of MD instead of $M + D$. The multiplicative problem is not always possible. A complete answer has been given by Friedland [6].

The reader who knows about Sard's theorem (that the regular values of a smooth map are generic) and something about the degrees of complex analytic maps (see the end of the Appendix) can easily prove the more precise statement that there are at most $n!$ solutions to the additive inverse eigenvalue problem and generically there are exactly $n!$

When this paper was presented as a talk, a member of the audience asked what is so special about the diagonal. Why not choose some other n positions in the matrix, say with no two in the same row or column, and by deforming those, match some given eigenvalues? It is not always possible. For example, it is not possible to deform the two off-diagonal elements of the 2-by-2 zero matrix so that the resulting matrix has two positive eigenvalues. The reader is invited to investigate what can be said.

Finally there is the question—important in applications—of actually finding the diagonal matrix D computationally. There is a method, more efficient than a search, called the continuation method, which starts from the known solution for the zero matrix and follows how D changes as the matrix is homotoped to M. In fact, the proof presented in this paper was developed precisely to show that the continuation method always works for this problem. See [2], [3], [9], [10].

Appendix. A couple of times in the body of the paper, we referred to standard facts. In order to make the paper more self-contained, we sketch proofs for these facts.

We referred to the fact that $\mathbf{C}P^n$ is a compact complex analytic manifold. Let $\Psi_i: \mathbf{C}^n \to \mathbf{C}P^n$ $(i = 0, \ldots, n)$ be the map

$$\Psi_i(z_1, \ldots, z_n) = [z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n].$$

This is a homeomorphism between $\mathbf{C}^n$ and the points $[a_0, \ldots, a_n]$ in $\mathbf{C}P^n$ with $a_i \neq 0$. The inverse map is

$$[a_0, \ldots, a_n] \to \left(a_0 a^{-1}, \ldots, a_{i-1} a_i^{-1}, a_{i+1} a^{-1}, \ldots, a_n a_i^{-1}\right).$$

The images of the Ψ_i are coordinate charts for CP^n; to show CP^n is an analytic manifold, we must show that $\psi_j^{-1}\psi_i$ is analytic on ψ_i^{-1} (image ψ_j). But $\psi_j^{-1}\ \psi_i$ is just the map

$$(z_1,\dots,z_n)\rightarrow\left(z_1z_j^{-1},\dots,z_{i-1}z_j^{-1},z_j^{-1},z_{i+1}z_j^{-1},\dots,z_nz_j^{-1}\right)$$

(omitting the term $z_jz_j^{-1}$), which is clearly analytic. To show CP^n compact, we observe that the continuous map $S^{2n+1}\rightarrow CP^n$ given by $(a_0,\dots,a_n)\rightarrow[a_0,\dots,a_n]$ is surjective where S^{2n+1} is the unit sphere in $\mathbf{C}^{n+1}$. Since S^{2n+1} is compact (being closed and bounded), so is CP^n. This also shows CP^n is connected.

We also needed the degree of the map $f_Z:(S^2)^n\rightarrow CP^n$. We restrict ourselves to the open sets of polynomials with no infinite roots and consider the map $\mu:\mathbf{C}^n\rightarrow\mathbf{C}^n$ given by

$$\mu(z_1,\dots,z_n)=(a_1,\dots,a_n)\quad \textit{if}$$

$$(t+z_1)(t+z_2)\cdots(t+z_n)=t^n+a_1t^{n-1}+\cdots+a_n.$$

We show the complex Jacobian $J_{\mathbf{C}}\mu(z_1,\dots,z_n)=\Pi_{j>i}(z_i-z_j)$. Note that a_k is the kth elementary symmetric function of the z's (the sum of all k-fold products of different z's). Let $\sigma_k=\sigma_k(z_1,\dots,z_n)$ denote the kth elementary symmetric function and let $\sigma_k^i(k=0,\dots,n-1;\ i=1,\dots,n)$ denote the kth elementary symmetric function of the z's with z_i then set equal to zero (let $\sigma_0^i\equiv1$). The matrix of partials $\partial a_k/\partial z_i$ is easily seen to be $(\sigma_{k-1}^i)i,k=1,\dots,n$. The determinant is a homogeneous polynomial of degree $(n^2-n)/2$ in the z's. If $z_i=z_j$, the ith and jth columns of the matrix are equal and the determinant is zero. Therefore the determinant is divisible by (z_i-z_j). Since $\Pi_{j>i}(z_i-z_j)$ is also a homogeneous polynomial of degree $(n^2-n)/2$, the determinant $J_{\mathbf{C}}$ must be a scalar multiple of $\Pi_{j>i}(z_i-z_j)$. Checking the main diagonal, we find the scalar multiple is $+1$.

(Here is an exercise for the reader generalizing this result. Let

$$p_1(t)=t^{n_1}+\alpha_1t^{n_1-1}+\cdots+\alpha_{n_1},$$

$$p_2(t)=t^{n_2}+\beta_1t^{n_2-1}+\cdots+\beta_{n_2}$$

be monic polynomials of degree n_1,n_2 respectively with $n_1+n_2=n$. Consider the map $\mathbf{C}^{n_1}\times\mathbf{C}^{n_2}\rightarrow\mathbf{C}^n$ given by

$$\mu(\alpha_1,\dots,\alpha_{n_1},\beta_1,\dots,\beta_{n_2})=(a_1,\dots,a_n)\quad\text{if}$$

$$p_1(t)p_2(t)=t^n+a_1t^{n-1}+\cdots+a_n.$$

Find the complex Jacobian of this map. Express the answer in terms of the roots of p_1 and p_2, but remember the coordinates are the coefficients. The cases of more than two polynomials can also be done easily.)

A map $f:\mathbf{C}^n\rightarrow\mathbf{C}^n$ can also be considered a map $f:R^{2n}\rightarrow R^{2n}$. We want to show that the complex and real Jacobians are related by the formula $J_R=|J_{\mathbf{C}}|^2$. Consider first the case $n=1$. Suppose $w=u+iv$ is a function of $z=x+iy$. Then

$$\frac{\partial w}{\partial z}=\frac{\partial u}{\partial x}+i\frac{\partial w}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

and

$$J_R=\text{determinant}\begin{bmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{bmatrix}=\left|\frac{\partial w}{\partial 2}\right|^2.$$

For larger n, we put the matrix of complex partials in upper diagonal form (say Jordan normal form). The Jacobian is then the product of the diagonal terms and the result follows from the case $n=1$.

To return to the degree of f_Z, we note that the coordinates of the image are polynomials in the coordinates of the domain and hence f_Z is analytic. If $p(t)=t^n+a_1t^{n-1}+\cdots+a_n$ is any nth degree polynomial in $\mathbf{C}P^n$ with distinct roots, at each of the $n!$ preimages, the complex Jacobian is non-zero so the real Jacobian is positive. Thus the local degree at each preimage point is $+1$. Hence the degree of f_Z is $n!$.

(Continuing the previous exercise, note that multiplying two polynomials together defines a map $\mathbf{C}P^{n_1}\times\mathbf{C}P^{n_2}\to\mathbf{C}P^n$. Calculate the degree of this map.)

The local degree of a complex analytic map f at any preimage of a regular point is always $+1$. Thus the number of points in the preimage of a regular point of f is always exactly $\deg f$. In the real case, there can be some cancellation (as in the example given when we discussed degree), and the degree gives only a lower bound on the number of points in the preimage of a regular point (even for real analytic maps). For this reason, when it is important to have control on the number of points in the preimage of an analytic map, complexifying the problem is a standard technique.

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MISCELLANEA

15. The calculating power alone should seem to be the least human of qualities, and to have the smallest amount of reason in it; since a machine can be made to do the work of three or four calculators, and better than any one of them.

Oliver Wendell Holmes, *The Autocrat of the Breakfast Table*, 1857.

CYCLOTOMIC POLYNOMIALS AND FACTORIZATION THEOREMS

SOLOMON W. GOLOMB

1. Introduction. One of the motives for this paper was to exhibit the methodology for answering such questions as for what positive integer values of r is $f(a^r)$ prime, for certain polynomials f (specifically, cyclotomic polynomials), and integers a. The cyclotomic polynomial $\Phi_n(x)$ has as its roots the primitive nth roots of unity, and so is defined by

$$\Phi_n(x)=\prod_{\substack{d=1\\(d,n)=1}}^{n-1}\left(x-e^{2\pi id/n}\right).$$

For example, when $f(x)=x^2+x+1=\Phi_3(x)$ and $a=2$, we find from Theorem 1 that a necessary condition for 4^r+2^r+1 to be prime is that r be a power of 3. A similar result, which has been applied by Liu, Reed, and Truong [1] based on $f(x)=x^2-x+1=\Phi_6(x)$, and $a=2$, is that a necessary condition for 4^r-2^r+1 to be prime is that $r>1$ be a multiple of 4 with no prime factor greater than three, i.e., that $r=2^\alpha 3^\beta$ with $\alpha\geqslant 2$, $\beta\geqslant 0$. When r is of this form, it can be shown (Theorem 4) that every prime q which divides 4^r-2^r+1 is of the form $6rk+1$. Thus, in attempting to factor $4^{12}-2^{12}+1=16{,}773{,}121$, it suffices to look only for prime factors of the form $q=72k+1$. (In fact, the factors are $q_1=433=72\times 6+1$ and $q_2=38{,}737=72\times 538+1$, where 433 is only the second prime number in the sequence $72k+1$.)

The techniques introduced in this paper involve the factorization of $\Phi_n(x^r)$ over the rational field, the factorization of $\Phi_n(a)$ over the integers, and, for comparison and completeness, the factorization of $\Phi_n(x)$ and $\Phi_n(x^r)$ over the integers modulo q.

2. Factorization of $\Phi_n(x^r)$ over prime fields.

THEOREM 1. *Let $\Phi_n(x)$ be the cyclotomic polynomial of order n. Then $\Phi_n(x^r)$ is irreducible over the rational field if and only if every prime factor of r is also a prime factor of n.*

Proof. Since the roots of $\Phi_n(x)$ are the primitive nth roots of unity, the roots of $\Phi_n(x^r)$ are the rth roots of these, which are (rn)th roots of unity, including all the primitive (rn)th roots of unity. Thus $\Phi_{rn}(x)$ divides $\Phi_n(x^r)$, and $\Phi_n(x^r)$ can only be irreducible if it is equal to $\Phi_{rn}(x)$. On the other hand, since all cyclotomic polynomials are irreducible over the rational field, $\Phi_n(x^r)$ is irreducible if and only if it equals $\Phi_{rn}(x)$. Since $\Phi_{rn}(x)$ divides $\Phi_n(x^r)$, these two polynomials are equal if and only if they have the same degree. Since the degree of $\Phi_m(x)$ is Euler's function $\varphi(m)$, the condition becomes $\varphi(rn)=r\varphi(n)$. Since $\varphi(m)=m\Pi_{p|m}(1-1/p)$, the condition is satisfied if and only if r has no prime factors not contained in n.□

An alternative proof of Theorem 1, suggested by the referee, can be based on the identity

$$\Phi_n(x^r)=\Phi_{rn}(x) \text{ if every prime dividing } r \text{ also divides } n;$$

$$\Phi_n(x^r)=\Phi_n(x^w)\Phi_{np}(x^w) \text{ if } r=pw \text{ and the prime } p\nmid n.$$

Applying Theorem 1 to $\Phi_3(2^r)$ yields the result that a necessary condition for 4^r+2^r+1 to be prime

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is that $r>0$ be a power of 3. Applying Theorem 1 to $\Phi_6(2^r)$ yields the result that a necessary condition for 4^r-2^r+1 to be prime is that $r>0$ have no prime factor greater than 3. Since, for odd r, $4^r-2^r+1\equiv 1+1+1\equiv 0 \pmod 3$, it is also necessary that r be even, in which case $4^r-2^r+1\equiv 1 \pmod 3$. Furthermore, r must be divisible by 4 because, if $r=2H$, when $H=2K-1$,

$$4^r-2^r+1=(2^{2H}+2^{H+K}+2^H+2^K+1)(2^{2H}-2^{H+K}+2^H-2^K+1).$$

(The referee has pointed out that this result was published by Aurifeville in 1878.)

Another type of problem is to show [2] that certain polynomials, such as x^4+1 and x^4-x^2+1, while irreducible over the rationals, factor modulo q for every prime q. The polynomials in question are $\Phi_8(x)=x^4+1$ and $\Phi_{12}(x)=x^4-x^2+1$. The general result is:

THEOREM 2. *The cyclotomic polynomial $\Phi_n(x)$ factors modulo q for every prime q, unless n equals* 1, 2, 4, *p^k or $2p^k$, where p is an odd prime and k is any positive integer, in which case $\Phi_n(x)$ remains irreducible modulo q for infinitely many primes q, while factoring modulo q if $n>2$ for infinitely many other primes q.*

Proof. It is a basic result of Galois theory [3] that an irreducible polynomial over a finite field must have a cyclic Galois group. (The Galois group of a polynomial is the automorphism group of the roots.) For $\Phi_n(x)$, the Galois group is isomorphic to the multiplicative group modulo n. It is a classical result [4] that this group is cyclic if and only if $n=1$, 2, 4, p^k, or $2p^k$. In all other cases, since the Galois group is non-cyclic modulo q, $\Phi_n(x)$ factors modulo q for every prime q.

Since $\Phi_1(x)=x-1$ and $\Phi_2(x)=x+1$ are linear polynomials, they remain irreducible modulo q for all q. If $n=p^k$ or $2p^k$, with $p>2$ and $k\geqslant 1$, then the multiplicative group mod n is cyclic, with $\varphi(p^k)=\varphi(2p^k)=(p-1)p^{k-1}$ elements, of which $\varphi(\varphi(n))=\varphi(p-1)\varphi(p^{k-1})\geqslant 1$ are primitive. Similarly, with $n=4$, $\varphi(n)=2$, $\varphi(\varphi(n))=1$, and there is one primitive root. Let g be such a primitive root. By Dirichlet's theorem on primes in arithmetic progressions, since $(g,n)=1$, there are infinitely many primes q such that $q\equiv g \pmod n$, and modulo such a prime $\Phi_n(x)$ remains irreducible. Conversely, for every h with $(h,n)=1$ where h is not a primitive root modulo n, there are also infinitely many primes q with $q\equiv h \pmod n$, and modulo every such prime q, $\Phi_n(x)$ will factor.□

For example, $\Phi_6(x)=x^2-x+1$, where $6=2\cdot 3^1$. The multiplicative group modulo 6 consists of 1 and 5, in which 5 is primitive and 1 is non-primitive. Thus $\Phi_6(x)$ remains irreducible modulo q whenever $q\equiv 5 \pmod 6$, whereas $\Phi_6(x)$ factors into two linear factors modulo q whenever $q\equiv 1 \pmod 6$. Thus, x^2-x+1 is irreducible modulo 5, while $x^2-x+1=(x+2)(x-3) \pmod 7$.

These two theorems can be combined to yield the following:

THEOREM 3. *$\Phi_n(x^r)$ is reducible modulo q, for every prime q, except in the following cases*:

$$n=1, r=1;\ n=2, r=1,2;\ n=4, r=1;\ n=p^k, r=p^l\ (l\geqslant 0);$$

$$n=2p^k, r=p^l\ (l\geqslant 0).$$

In these exceptional cases, $\Phi_n(x^r)$ is again cyclotomic, and factorization modulo q is as specified in Theorem 2.

Proof. In order for $\Phi_n(x^r)$ to be irreducible modulo q, it must be irreducible over the rationals, which imposes the condition of Theorem 1 on r, and $\Phi_n(x)$ must be irreducible modulo q, which imposes the conditions of Theorem 2 on n. The final requirement is that $\Phi_n(x^r)=\Phi_{rn}(x)$ must have a cyclic Galois group modulo q. Combining these conditions yields the indicated result.□

For example, $x^{2r}+x^r+1$ is irreducible over the rationals if and only if $r=3^k$. Now, all the polynomials $x^{2\cdot 3^k}+x^{3^k}+1$ remain irreducible modulo 2, as well as modulo any prime q with $q\equiv -1 \pmod 6$, while factoring modulo 3 and modulo q for every prime q with $q\equiv 1 \pmod 6$.

3. Factorization of $\Phi_n(a)$ over the integers. The theory of the factorization of $\Phi_n(a)$ over the integers was elegantly summarized by Birkhoff and Vandiver [5].

Throughout this section, let a and b be integers, $a>b\geqslant 1$, $(a,b)=1$, and define $\Phi_n(a,b)=b^{\varphi(n)}\Phi(a/b)$, a homogeneous polynomial of degree $\varphi(n)$ in a and b. Clearly $a^n-b^n=\Pi_{d|n}\Phi_d(a,b)$, for all $n\geqslant 1$. We are interested in the prime factors of $\Phi_n(a,b)$. The basic result is as follows:

THEOREM 4. *The prime factors of $\Phi_n(a,b)$ may be of two types: those which do not divide any $\Phi_d(a,b)$ with $d|n$ and $d<n$, called the primitive prime factors; and those which divide some $\Phi_d(a,b)$ with $d|n$ and $d<n$, called the* ***imprimitive*** *prime factors. If q is a primitive prime factor, then $q\equiv 1 \pmod n$. If q is an imprimitive prime factor, then $n\equiv 0 \pmod q$, as well as $\Phi_d(a,b)\equiv 0 \pmod q$ for some $d<n$, $d|n$.*

Proof. See [5], Theorems I–IV, which include a more precise characterization of the imprimitive prime factors.□

For example, $\Phi_{18}(2,1)=2^6-2^3+1=57=3\cdot 19$, where the primitive factor 19 satisfies $19\equiv 1$ (mod 18), while the imprimitive factor 3 divides both 18 and $\Phi_6(2,1)=3$, where $6|18$.

When $r=2^\alpha 3^\beta$, we have $\Phi_6(2^r,1^r)=\Phi_{6r}(2,1)=2^{2r}-2^r+1$, where the only possible imprimitive prime factors (by Theorem 4) are the prime divisors of $6r$, namely 2 and 3. However, 2 is obviously not a factor of $\Phi_{6r}(2,1)$; and we showed, in the example following Theorem 1, that 3 is a factor of $\Phi_{6r}(2,1)$ if and only if r is odd. Hence, for $r=2^\alpha 3^\beta$, $\alpha\geqslant 2$, $\beta\geqslant 0$, every prime factor q of $\Phi_{6r}(2,1)$ is primitive, and therefore satisfies $q=6rk+1$ for some $k\geqslant 1$. Clearly, the general result is:

THEOREM 5. *If r satisfies the condition of Theorem 1 (i.e., $r|n^r$), then every prime factor q of $\Phi_n(a^r,b^r)$ is either the largest prime factor of n, or of the form $q=nrk+1$.*

Note that if n is odd, the primitive prime factors must in fact be of the form $q=2nrk+1$.

For example, every prime factor q of $4^r+2^r+1=\Phi_3(2^r,1^r)$, when r is a power of 3, must be of the form $q=6rk+1$. The smallest prime factor for each such value of 4^r+2^r+1, with $r\leqslant 729$, if $<10^6$, is listed in Table 1. Similarly, Table 2 shows the smallest prime factor, if $<10^6$, of 4^r-2^r+1 for $r=1$ and for $r=2^\alpha 3^\beta$, $\alpha\geqslant 2$, $\beta\geqslant 0$, $r\leqslant 1024$. These tables were compiled by Martin J. Cohen, using the theorems contained herein, on a Texas Instruments SR-56 hand-held programmable calculator. (This limited the search range to $q<10^6$. Some larger values of q were supplied by the referee.)

For completeness, we mention the following remarkable result due to A. S. Bang in 1886.

THEOREM 6. *For $n\neq 2$, $\Phi_n(a,b)$ always contains at least one primitive prime factor, with the single exception of $\Phi_6(2,1)=3$.*

Proof. This is equivalent to the result that, for $n\neq 2$, $V_n(a,b)=a^n-b^n$ always contains at least one primitive prime factor, with the single exception of $V_6(2,1)=2^6-1=63$. In this form, a detailed proof is given in Theorem V of [5].□

TABLE 1: The Smallest Prime Factor of 4^r+2^r+1, $r=3^k$

r	Smallest Prime Factor
1	7*
3	73*
9	262657*
27	2593
81	487
243	80191
729	39367

*In these cases, 4^r+2^r+1 is prime.

TABLE 2: The Smallest Prime Factor of $4^r - 2^r + 1$, $r = 2^\alpha 3^\beta$

r	Smallest Prime Factor	r	Smallest Prime Factor
1	3*	144	? (*not* prime)
2	13*	192	145143857
4	241*	216	10369
8	97	256	? (*not* prime)
12	433	288	? (*not* prime)
16	193	324	26232337
24	577	384	18433
32	18446744069414584321*	432	?
36	33975937	512	?
48	1153	576	?
64	769	648	?
72	209924353	768	101377
96	3457	864	331777
108	1297	972	139969
128	?(*not* prime)	1024	112289

*In these cases, $4^r - 2^r + 1$ is known to be prime.
?In these cases, there is no prime factor less than 10^6.

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THE MORLEY TRISECTOR THEOREM

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The great algebraic geometer Frank Morley (1860–1937) came to this country from England to teach at Haverford College in 1887. In 1900 he moved on to Johns Hopkins to head their department of mathematics. (For *in memoriam* résumés of his life and works see [33], [92].) In 1900 his brilliant paper "On the Metric Geometry of the Plane n-line" appeared in the first issue of the *American*

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Mathematical Society Translations **[78]**. In it he proved several very general theorems about the behavior of n-lines in the plane and their characteristic constants. In passing, it is interesting to note that in this important memoir not once is use made of the now customary format: *Theorem…Proof.*

But among these unannounced theorems there is a very, very special case of one which has intrigued mathematicians for the past three-quarters of a century. It is now simply known as **Morley's trisector theorem.**

The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle.

It is one of the most astonishing and totally unexpected theorems in mathematics and, jewel that it is, for sheer beauty it has few rivals. The simplest case involves only the interior angles. In **[13]**, **[81]**, **[82]**, you will find Morley's thoughts on how the theorem arises quite naturally from his own contributions to what has now become known as the theory of Clifford chains. They are coded CC in the references.

There have appeared in print many proofs of this theorem. We believe references to most of them will be found in our verified list. With popularization and proliferation of proofs, it is understandable that some uncertainties and errors of fact concerning the origin, statement, and earliest printed proofs have crept into the literature. Perhaps this note can set much of the record straight. To begin with, it is *Frank* Morley, not *John*, as stated in **[31]**, **[44]**.

Morley, of course, was well aware of the unique characteristics of his theorem and its ramifications. Indeed, his theory accounted for all 18 cases of Morley equilateral triangles, but it pleased him to indicate that he had not bothered to make a big song and dance about it since it was only a small part of his general theory. And so he never enunciated, in print, just the simple theorem, nor did he ever publish a direct verification of it.

However, he had not been slow in communicating it to his friends, such as Richmond at Cambridge and Whittaker at Edinburgh, and by 1904 it had become public. See Morley's letter to G. Loria in **[71]**.

The earliest printed statements of the theorem we have found are those of E. J. Ebden, who apparently was so taken with the problem that he introduced it, simultaneously, in the British Isles and on the continent. In 1908 it appeared in *The Educational Times*, London **[42]**, **[101]**, as problem 16381, and in *Mathesis*, Brussels **[36]**, as problem 1655—in both instances without the benefit of Morley's name. But this is not surprising, because for some years the theorem seemingly floated around in search of an author; and as late as 1913 Taylor and Marr read a paper on the subject before the Edinburgh Mathematical Society without knowing the authorship of the theorem. An acknowledgment is in their paper **[108]**, which, by the way, was the first to give the complete solution.

The solution to Ebden's problem 16381 is given by Satyanarayana in **[101]** and to his problem 1655 by Delahaye and H. Lez in **[36]**. The elegant proof of the latter consists in finding the length of a side of the Morley triangle. Let the given triangle be ABC with interior angles $A=3\alpha$, $B=3\beta$ and $C=3\gamma$. Let O be the center and $OC=r$ be the radius of the circumcircle and let the Morley triangle be DEF (see Fig. 1). In our notation they found $EF=8r\sin\alpha\sin\beta\sin\gamma$, which, by symmetry, proves that DEF is equilateral. This seems to be the first occurrence of this formula, although Kaven **[61]** states that Hofmann **[54]** was the first to use it.

To the best of our knowledge, the next earliest proof is in **[83]** by M. T. Naraniengar, *Mathematical Questions and Solutions*, from *The Educational Times*, with many papers and solutions in addition to those published in *The Educational Times*, London, New Series, 15 (1909) 47. We have spelled out the reference in detail here because confusion does arise: this paper, apparently, did not appear in *The Educational Times*. It occurs in the *Mathematical Questions and Solutions*, which is often referred to as the "Reprints" from *The Educational Times* (misspelled "Repruits" in **[61]**, causing more confusion).

By 1920 the problem had aroused so much interest that it was set in the St. John's group of Entrance Scholarships.

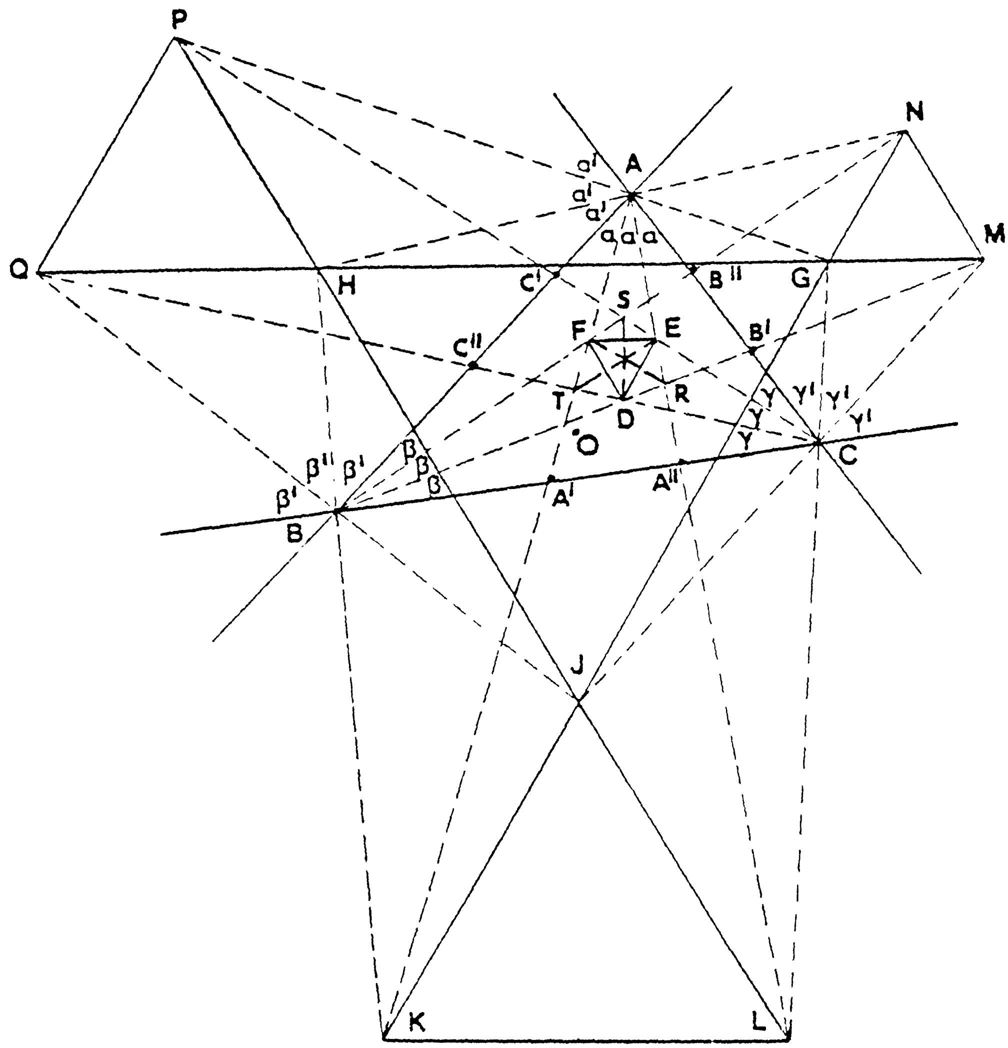

FIG. 1.

Specializations of Morley's general theory of 1900 hold (see Fig. 1) for (i) interior angles, yielding triangle DEF, (ii) exterior angles, yielding triangle GHJ, and (iii) a mixture of two exterior angles and one interior angle resulting in three Morley triangles, JKL, GMN, and PQH. Both Satyanarayana and Delahaye–Lez treat all three cases as do **[44]**, **[69]**, **[85]**.

But there are still further generalizations since each angle A, B, C can be trisected in three distinct ways by using A, $A+2\pi$, $A+4\pi$, and similarly for B and C. When this is done, there arise 27 triangles, 18 of which are equilateral. These 18 include the ones previously noted and constitute the complete solution (CS). The figure for the complete solution is complicated. See **[38]**, **[55]**, **[108]**. All Morley triangles have parallel sides, and the length of a side of each is of the form

$$8r\sin(\alpha+\theta)\sin(\beta+\phi)\sin(\gamma+\psi)$$

where each of θ, ϕ, ψ is some particular arrangement of 0, $\pi/3$, $\pi/6$ **[71]**.

We have included reference materials, coded R, *related* material, because, as you might expect, Morley's theorem has connections with many notable point-line-plane-circle-polygon configurations bearing such names as Apollonius, Brianchon, Ceva, Desargues, Feuerbach, Hesse, Lemoine, Menelaus, Pascal, Ptolemy, Simson, Spieker, Steiner, etc. Indeed, some of the proofs begin with a named theorem. See, for example, **[48]**, where the beginning cites Desargues and Menelaus. Again, in **[85]**, Neuberg credits Ad. Mineur with a proof which first notes that hexagon $A'A''B'B''C'C''$ is Pascal and hexagon $DRESFT$ is Brianchon (see Fig. 1). In the footnote of **[85**, p. 363] correct $A'C$ to read AC'. And make the same correction in **[74]**, **[109]** where the error has been repeated, possibly through careless editing.

For the interested reader, we recommend the following papers for both the variety and ingenuity they offer: **[32]**, **[48]**, **[71]**, **[88]**, **[106]**, **[108]**, **[113]**.

All of the proofs that we have seen use only elementary mathematics, but few of them can be said to be simple. Some polish off their proofs with a two- or three-line flourish *after* starting with as many as three lemmas, usually involving somewhat complicated trigonometric identities. By finding the length of a side of the Morley triangle, a number of papers follow the general pattern of **[36]**, but none so succinctly as **[66]**. Because of its brevity we forthwith give **[66]** in its entirety (with only a change in notation to fit Figure 1).

$$AF=\frac{c\sin\beta}{\sin(\alpha+\beta)}=\frac{2r\sin\beta\sin 3\gamma}{\sin\left(\frac{\pi}{3}-\gamma\right)}=8r\sin\beta\sin\gamma\sin\left(\frac{\pi}{3}+\gamma\right).$$

Similarly

$$AE=8r\sin\beta\sin\gamma\sin\left(\frac{\pi}{3}+\beta\right)$$

But

$$\overline{EF}^2=\overline{AE}^2+\overline{AF}^2-2\,\overline{AE}\cdot\overline{AF}\cdot\cos\alpha.$$

And it follows that $EF=8r\sin\alpha\sin\beta\sin\gamma$. By symmetry this proves the theorem.

Come on now and be a good sport: fill in the trigonometry—and time yourself!

It occurred to us that some readers might be interested in having a more personal look at the man who so nonchalantly tossed off this everlasting geometric gem. Accordingly, we asked the youngest of the Morley sons, Frank V. Morley, who had worked with his father in geometry, to share with us some of his thoughts. Here are his comments.

"I was a school-boy when my father, who was almost forty years older than I was, sketched for me, free-hand, a pencilled diagram of the simplest form of the above-discussed theorem in plane geometry.

"I tested it at once with my own drawing instruments. No matter what the shape of the original triangle I started with, there in its midriff was an equilateral triangle, picked out by the trisectors. It was wizard, it was weird—and it was True!

"Always, to the eye at least, the theorem, if you drew accurately, proved itself. What caused me considerable annoyance was that I could not for a long time comprehend what purblind examiners might accept as a valid proof (*demonstratio mirabilis sane*). But before I could prove the theorem, I went on drawing diagrams, and a secondary wonder emerged: how the simplest diagram had remained a secret until my father spotted it. People had been toying with ruler and compasses and poring over the geometry of the triangle in their many generations—at least since the time of Euclid—how come nobody broke the taboo on trisecting angles—how come nobody had drawn the trisectors and *seen* the equilateral triangle inside?

"Now my father did not lack warmth for any geometrical property so simple and startling as this one. I never asked him outright the question, though it is a proper one, that Professor Oakley now asks me: namely, why at the time of discovery my father kept his cool about promoting the

'gem'—there might have been some bit of hoo-ha if he had removed the cover and sent it to the show-room as a separate static cut stone. I think the way the theorem is presented in the book *Inversive Geometry* [**13**] may answer the question. Attention to the detached theorem was not, for him, to interfere with the pleasure of watching his 'mobile' of cardioids and their tangents: it was the cardioids which led him to, and provided for him the most elegant proof of, the trisector theorem. Proof and theorem were pleasing in their togetherness. Isolate the theorem if you wish, but for him the cardioids in their happy behaviour were 'in beauty surpassing the Princes [*sic*] of Troy.'

"Nevertheless the ease with which simple diagrams of trisectors of a triangle's angles may be drawn certainly suggested that someone, somewhere, might have visualized and commented on the theorem before my father's contemplation of the n-line brought him to it. Hence, I think, my father's quiet, semi-private mentions of the theorem to expert colleagues, as occasion offered. He was not informed by any colleague he tried, in the U.S.A., or Britain, or European countries, of any prior knowledge of the existence of the 'gem.' His permission for publication of the theorem in Japan elicited no prior knowledge of it in the Far East either. I think he would have agreed that by now you could put his name to it.

"As to portraiture of that Frank Morley, author of the above-discussed theorem and others, I did the best I could some years ago in a small book called *My One Contribution to Chess* (originally published by B. W. Huebsch, New York, 1945). But copies of that, if they now exist, must be rare."

Acknowledgments: The authors would like to thank Suzanne Newhall, curator of the science library, Haverford College, for her indefatigable search for hard-to-find references, and also Prof. H. S. M. Coxeter for his helpful suggestions during the preparation of the manuscript. Thanks are due to Prof. Jan van de Craats, Leiden University, for supplying us with some articles we had been unable to locate, and to Prof. Leroy F. Meyers for his editorial help. And very special thanks go to Dr. Frank V. Morley for his sensitive sketch of his father.

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References

The following letter-coding of the reference numbers should be clear and, we trust, useful. They give some indication of the mathematical nature of the references.

- B. Book
- CC. Mathematics associated with Clifford chains
- CS. Complete solution (for all 18 Morley triangles)
- CV. Proof using complex variables
- G. Proof by geometry
- IP. Indirect proof
- PG. Proof by projective geometry
- PP. Proposed problem (Morley, or related)
- PPS. Proposed problem solved
- R. Related material
- T. Proof by trigonometry

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In October, 1977, we sent a preliminary copy of the above list of references to Professor H. S. M. Coxeter for his comments. When he replied, he told us that Charles W. Trigg, Professor Emeritus, Los Angeles City College, had prepared a similar list, of approximately the same length, for publication in *Eureka*, a monthly mathematics journal published by Algonquin College, Ottawa, Canada (Editor: Léo Sauvé, Algonquin College, 281 Echo Drive, Ottawa, K1S 1N3). Through Professor Coxeter's good offices, and with the informal cooperation of *Eureka* and this MONTHLY, it was agreed to combine the two lists of references in the following way. Our list of 116 coded items (above) would be published in both *Eureka* and this MONTHLY. Professor Trigg's items *not* in our list would follow in both journals so that, in effect, the complete list of references would be equivalent to one of joint authorship. The two reference lists, numbered consecutively, were printed in *Eureka*, 3, No. 10, (Dec. 1977) 281–290, along with the following Morleyana items:

1. Presenting the Morley issue of *Eureka*, p. 272.
2. On the intersections of the trisectors of the angles of a triangle, F. Morley, p. 273–275 (item [**81**] in our list).
3. Notes on Morley's proof of his theorem on angle trisectors, Dan Pedoe, pp. 276–279.
4. Robson's proof of Morley's theorem, pp. 280–281, (item [**96**] in our list).
5. An elementary geometric proof of the Morley theorem, Dan Sokolowsky, pp. 291–294. To our knowledge, this is the only paper on the Morley theorem that considers not only the interior angles 3α, etc., and the exterior angles $\pi - 3\alpha$, etc., but also the reflex angles $\pi + 3\alpha$, etc.
6. The beauty and truth of the Morley theorem, Leon Bankoff, pp. 294–296.

Supplementary List of References to the Morley Theorem

(Prepared by Charles W. Trigg, Professor Emeritus, Los Angeles City College)

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MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

Beginning January 1, 1979, this section will be edited by Deborah Tepper Haimo and Franklin Tepper Haimo. Material for this department should be sent to Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

A CHARACTERIZATION OF PERMUTATION POLYNOMIALS OVER A FINITE FIELD

L. CARLITZ AND JO ANN LUTZ

1. Let $F = GF(q)$ denote the finite field of q elements, where $q = p^n$, p prime, $n \geqslant 1$. A polynomial $P(x) \in F[x] = GF[q, x]$ is called a **permutation polynomial** if

$$\{P(c) | c \in F\} = \{c | c \in F\}. \tag{1.1}$$

Dickson, in his thesis [**2**], made an extensive study of permutation polynomials of low degree. For permutation polynomials of arbitrary degree not a great deal is known. A number of necessary conditions have been obtained. For example [**2**, p. 67], the degree of a non-linear permutation polynomial may not divide $q-1$.

As a result of an entirely different kind, we note that it has been proved [**1**] that, for $q > 2$, all permutations are generated by the special permutations x^{q-2} and

$$\{ax + b | a, b \in F, a \neq 0\}.$$

Dickson [**3**, p. 59] has proved the following criterion:

THEOREM A. *Let $P(x)$ be a polynomial with coefficients in F and put*

$$(P(x))^k = B_k(x)(x^q - x) + A_k(x), \quad \deg A_k(x) < q \qquad (1 \leqslant k < q-1).$$

Also let

(i) $\deg A_k(x) < q-1 \qquad (1 \leqslant k < q-1)$

and

(ii) *the equation $P(x) = 0$ has exactly one solution in F.*

Then $P(x)$ is a permutation polynomial.

In this note we shall prove the following variant of Theorem A:

THEOREM B. *Let $P(x)$ be a polynomial with coefficients in F and put*

$$(P(x))^k = B_k(x)(x^q - x) + A_k(x), \quad deg\, A_k(x) < q \qquad (k = 1, 2, 3, \ldots). \tag{1.2}$$

Also let

(i) $\deg A_k(x) < q-1 \qquad (1 \leqslant k < q-1)$

and

(ii) $\deg A_{q-1}(x) = q-1$.

Then $P(x)$ is a permutation polynomial.

2. To prove Theorem B, we note first that, by (1.2),

$$(P(b))^k = A_k(b) \qquad (b \in F, k = 1,2,3,\ldots). \tag{2.1}$$

We recall that (since $0^0 = 1$)

$$\sum_{c \in F} c^m = \begin{cases} 0 & (0 \leqslant k < q-1) \\ -1 & (k = q-1). \end{cases} \tag{2.2}$$

Hence by (i) and (2.1) we have

$$\sum_{b \in F} (P(b))^k = 0 \qquad (1 \leqslant k < q-1). \tag{2.3}$$

For $k = q-1$, by (ii)

$$A_{q-1}(x) = a_0 x^{q-1} + a_1 x^{q-2} + \cdots + a_{q-1}, \quad a_0 \neq 0.$$

Thus it follows from (2.2) that

$$\sum_{b \in F} (P(b))^{q-1} = -a_0. \tag{2.4}$$

Next since, for arbitrary $c \in F$, $c^q = c$, it is evident that

$$(P(b))^k = (P(b))^{k+q-1} = (P(b))^{k+2(q-1)} = \cdots \ (k \geqslant 1),$$

$$(P(b))^{q-1} = (P(b))^{2(q-1)} = (P(b))^{3(q-1)} = \cdots .$$

Hence (2.3) and (2.4) imply

$$\sum_{b \in F} (P(b))^k = \begin{cases} 0 & (k = 0 \ \text{ or } \ q-1 \nmid k) \\ -a_0 & (q-1 \mid k, \quad k > 0). \end{cases} \tag{2.5}$$

Put

$$Q(x) = \prod_{b \in F} (x - P(b)),$$

so that $Q(x)$ is monic and $\deg Q(x) = q$. Then, by (2.5),

$$\begin{aligned} \frac{Q'(x)}{Q(x)} &= \sum_{b \in F} \frac{1}{x - P(b)} \\ &= \sum_{k=0}^{\infty} \frac{1}{x^{k+1}} \sum_b (P(b))^k \\ &= -a_0 \sum_{k=1}^{\infty} \frac{1}{x^{(q-1)k+1}} = -\frac{a_0}{x^q - x}, \end{aligned}$$

so that

$$-a_0 Q(x) = Q'(x)(x^q - x). \tag{2.6}$$

Since $\deg Q(x)=q$ and $Q(x)$ is monic, we get $Q(x)=x^q-x$. Since

$$x^q-x=\prod_{c\in F}(x-c),$$

it is clear that $P(x)$ satisfies (1.1).

This evidently proves Theorem B.

REMARK. The argument above shows incidentally that $a_0=1$.

3. Let $a_1,a_2,\ldots,a_q$ denote a fixed numbering of the elements of F and let $c_1,c_2,\ldots,c_q$ denote any q numbers (not necessarily distinct) of F. Then by the Lagrange interpolation formula there exists a unique polynomial $R(x)$ with coefficients in F and of degree less than q such that

$$R(a_j)=c_j \qquad (j=1,2,\ldots,q).$$

Indeed

$$R(x)=-\sum_{j=1}^{q}\frac{x^q-x}{x-a_j}c_j=\sum_{j=1}^{q}\left\{1-(x-a_j)^{q-1}\right\}c_j. \tag{3.1}$$

Thus Theorem B can be restated in the following form:

THEOREM B′. *Let $c_1,c_2,\ldots,c_q$ be any q numbers of F such that*

(i) $\sum_{j=1}^{q} c_j^k=0 \qquad (1\leqslant k<q-1)$

and

(ii) $\sum_{j=1}^{q} c_j^{q-1}\neq 0.$

Then the c_j are distinct.

However we can do better.

THEOREM C. *Let $c_1,c_2,\ldots,c_m$ be any m numbers of F, $m\leqslant q$, such that*

(i) $\sum_{j=1}^{m} c_j^k=0 \qquad (0\leqslant k<q-1)$

and

(ii) $\sum_{j=1}^{m} c_j^{q-1}=-a_0\neq 0.$

Then $m=q$ and the c_j are distinct.

The proof is like the proof of Theorem B. Put

$$Q(x)=\prod_{j=1}^{m}(x-c_j).$$

Then, exactly as in the proof of (2.6), we have

$$-a_0Q(x)=Q'(x)(x^q-x).$$

It follows that $Q(x)=x^q-x$.

REMARK. Note that hypothesis (i) of the theorem requires that $p|m$.

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THE ASYMPTOTIC BEHAVIOR OF DERIVATIVES

R. P. BOAS, H. POLLARD, AND D. V. WIDDER

An elementary theorem of Hardy and Littlewood, and a major tool for classical analysts, reads: if f has a derivative f' on $(0, \infty)$, and if f' is absolutely continuous, then the conditions

$$f(x)=o(1), \qquad x\to\infty, \tag{1}$$

and

$$f''(x)=O(1), \qquad x\to\infty, \tag{2}$$

together imply that

$$f'(x)=o(1), \qquad x\to\infty. \tag{3}$$

The condition (2) cannot be omitted altogether; for example, let $f(x)=x^{-1}\sin x^2$. But it can be weakened to $f''(x)<O(1)$. (See **[1]** for an account of theorems of this kind.)

We shall show that condition (2) can be replaced by suitable hypotheses on the *average* behavior of $f''(x)$. For example, (2) can be weakened to

$$\int_x^{x+1}|f''(x)|^p\,dx=O(1), \qquad x\to\infty, \tag{4}$$

provided that $p>1$, but not if $p=1$. To see that $p=1$ cannot work, define $f'(x)$ on each interval $(n, n+1)$ by an isosceles triangle of height 1 on a subinterval of length $2/n^2$ centered at $n+\frac{1}{2}$; otherwise let it be zero. Then the function $f(x)$ defined as $-\int_x^\infty f'(u)\,du$ meets conditions (1) and (4) with $p=1$, but (3) obviously fails.

The condition (4) is included in (5) of the following theorem.

THEOREM. *Let θ be positive, increasing and convex on* $(0, \infty)$. *If $f(x)$ satisfies*

$$\int_x^{x+1}\theta(|f''(u)|)\,du=O(1), \qquad x\to\infty, \tag{5}$$

then the conclusion (3) *holds, provided that*

$$\theta^{-1}(x)=O(x), \qquad x\to\infty. \tag{6}$$

The counterexample of the preceding paragraph shows that $o(x)$ cannot be replaced by $O(x)$, for then $\theta(x)\equiv x$.

To prove the theorem start with the exact form of Taylor's formula:

$$f(x+h)-f(x)-hf'(x)=\int_x^{x+h}(x+h-y)f''(y)\,dy.$$

Since $x+h-y$ is largest at $y=x$, it follows that

$$|f(x+h)-f(x)-hf'(x)|\leqslant h\int_x^{x+h}|f''(y)|\,dy,$$

provided that $h>0$. Divide by h and let $x\to\infty$. According to condition (1)

$$\varlimsup_{x\to\infty}|f'(x)|\leqslant\varlimsup_{x\to\infty}\int_x^{x+h}|f''(y)|\,dy. \tag{7}$$

According to Jensen's inequality (**[2**, pp. 150–152])

$$\frac{1}{h}\int_x^{x+h}|f''(u)|\,du\leqslant\theta^{-1}\left(\frac{1}{h}\int_x^{x+h}\theta(|f''(u)|)\,du\right)\leqslant\theta^{-1}\left(\frac{M}{h}\right),$$

for $0<h<1$, where M is a constant. It is at this stage that we have used (5). Then (7) becomes

$$\varlimsup_{x\to\infty}|f'(x)|\leqslant h\theta^{-1}\left(\frac{M}{h}\right).$$

Now let $h \to 0+$. According to (6) the right-hand side approaches zero. This completes the proof.

Observe that although condition (4) does not work as it stands when $p=1$ it can still be used if $O(1)$ is replaced by $o(1)$. This is a consequence of (7).

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PROOF OF A SPECIAL CASE OF FERMAT'S LAST THEOREM

BARRY POWELL

Fermat's Last Theorem asserts that the equation

$$x^n + y^n = z^n \tag{*}$$

has no solution in nonzero integers x, y, and z if $n \geqslant 3$. Most of the work that has been done on this problem has dealt with prime exponents, because if Fermat's Last Theorem is true for odd prime exponents then it is true for all exponents n (since it is known to be true for $n=4$). However, in spite of all the research that has been done on this subject, it is yet to be proved that Fermat's Last Theorem holds for infinitely many prime exponents (see [6]).

Proofs that (*) has no solution for a particular prime exponent n are usually subdivided into two cases. In the first case, an assumed solution (x, y, z) has the property that the exponent is relatively prime to each of x, y, and z. At present it is not even known whether there exist infinitely many prime exponents for which Fermat's Last Theorem holds in the first case. Since Fermat's equation (*) is known to have no solutions in the first case when the exponent is a Mersenne prime [4], and $p = 2^{19,937} - 1$ is the largest known Mersenne prime [5], the first case of (*) has no solutions when the exponent n is $2^{19,937} - 1$. This seems to be the largest prime exponent for which the first case of Fermat's Last Theorem is known to hold. This evidences our lack of knowledge about prime exponents and motivates the consideration of composite ones.

It has already been proved by Ankeny [1] and Ankeny and Erdös [2] that the density of exponents for which the first case of Fermat's Last Theorem is false is zero. The result that is proved here, namely, that the first case holds for infinitely many relatively prime exponents n, is weaker than that of Ankeny and Erdös. However, the proof is much shorter and simpler, not requiring the use of density theory or analytic number theory.

THEOREM. *There exist infinitely many relatively prime exponents n such that the equation* (*) *has no solutions with* $(n, xyz) = 1$.

Proof. Suppose that $n_1, n_2, \ldots, n_k$ are any positive integers. We prove that there exists a positive integer n_{k+1} which is relatively prime to each of $n_1, \ldots, n_k$, and for which the first case of Fermat's Last Theorem holds. This immediately implies the existence of the infinite set of the theorem.

By Dirichlet's Theorem [3] there exists an odd prime p such that $p \equiv -1 \pmod{4n_1n_2 \cdots n_k}$. We set $n_{k+1} = p(p-1)/2$. Since $(p(p-1)/2, (p+1)/2) = 1$, we have $(n_{k+1}, 4n_1n_2 \cdots n_k) = 1$.

If $(x, p(p-1)/2) = 1$, then $x^{(p-1)/2} \equiv \pm 1 \pmod{p}$. Therefore if $(xyz, p(p-1)/2) = 1$, then

$$x^{p(p-1)/2} + y^{p(p-1)/2} \neq z^{p(p-1)/2},$$

since the left side is congruent to 0 or $\pm 2 \pmod p$, while the right side is congruent to 1. Therefore n_{k+1} satisfies all the required conditions.

Note. Any integer n of the form $p(p-1)/2$, where p is a prime, satisfies the first case of Fermat's Last Theorem. It may also be shown, by similar methods, that infinitely many relatively prime integers n, each of the form $(p-1)/2$, p a prime, satisfy the above theorem.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The Univeristy of Calgary, Calgary, Alberta, Canada, T2N 1N4.

HOW MANY *i–j* REDUCED LATIN SQUARES ARE THERE?

GARY L. MULLEN

There is a large literature [**2**] on latin squares, i.e., $n \times n$ squares with each of the numbers $1, 2, \ldots, n$ in each row and column. A latin square is **reduced** if the numbers $1, 2, \ldots, n$ are in natural order in the first row and first column and we call it ***i–j* reduced** if the first i rows and j columns are cyclic permutations of the first row and column. Thus a latin square of order n is i–j reduced if it has the following form

$$\begin{array}{ccccccc}
1 & 2 & \cdots & j & \cdots & n-1 & n \\
2 & 3 & \cdots & j+1 & \cdots & n & 1 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
i & i+1 & \cdots & j+i+1 & \cdots & i-2 & i-1 \\
\vdots & \vdots & & \vdots & & & \\
n-1 & n & \cdots & j-2 & & \multicolumn{2}{c}{\boxed{}} \\
n & 1 & \cdots & j-1 & & &
\end{array}$$

Thus a 1–1 reduced latin square is reduced and an n–n reduced square is a **cyclic** latin square. We can also allow $i=0$ or $j=0$ so that a general latin square is 0–0 reduced. As an illustration, the following square is a 2–1 reduced latin square of order 5:

1	2	3	4	5
2	3	4	5	1
3	1	5	2	4
4	5	2	1	3
5	4	1	3	2

We now study several properties of i–j reduced latin squares. Let $L(i,j,n)$ denote the number of i–j reduced latin squares of order n, so that the total number of latin squares is $L(0,0,n)=L_n=n!(n-1)!R_n$ where $R_n=L(1,1,n)$ is the number of reduced latin squares of order n. If $n\leqslant 9$, the value of R_n is known, see [**2**, p. 144]. It is clear that $L(1,0,n)=L(0,1,n)=(n-1)!R_n$ and that $L(i,j,n)=L(j,i,n)$, and not hard to prove that $L(i,0,n)=L(0,i,n)=(n-i)!L(i,1,n)$ and that $L(i,j,n)=1$ if $i+j\geqslant n-1$.

These results were used along with an IBM 370/168 computer to determine the values of $L(i,j,n)$ for $n<9$. The computer was programmed to fill cells sequentially by the usual back-track method. If $n\leqslant 3$ then clearly $L(i,j,n)=1$ for all $1\leqslant i,j\leqslant n$. If $n\geqslant 4$ we summarize the results in the following tables where the entry in row i and column j of each table shows the prime factorization of $L(i,j,n)$. Only $n-3$ rows are given since all other entries are 1, and only $[(n-2)/2]$ columns, since the tables are symmetric. The entries $L(2,1,8)=L(1,2,8)$ are omitted as they would have needed too much time to compute.

$n=4$	$n=5$	$n=6$		$n=7$		$n=8$		
2^2	$2^3 7$	$2^6 3\cdot 7^2$	$2^3 3\cdot 7$	$2^{10} 3\cdot 5\cdot 1103$	$2^6 3^2 5\cdot 19$	$2^{17} 3\cdot 1361291$		$2^7 3\cdot 863$
	$2\cdot 3$	$2^3 3\cdot 7$	$2\cdot 7$	$2^6 3^2 5\cdot 19$	$2^2 3\cdot 73$		$2^6 3^2 1259$	$2^6 3\cdot 23$
		$2^2 3$	1	$2^3 3\cdot 19$	$2\cdot 19$	$2^7 3\cdot 863$	$2^6 3\cdot 23$	$2^3 19$
				$2\cdot 11$	1	$2^3 5\cdot 31$	$2\cdot 5\cdot 11$	1
						$2^2 11$	1	1

This table leads us to generalize a question posed by Alter [**1**]: If 2^k is the highest power of 2 dividing $L(i,j,n)$, is k always positive when $i+j<n-1$, and is k then strictly monotonic in i,j, and n (increasing for n, decreasing for i and j)?

It is tempting to seize on some of the divisibilities that appear for these small values of n, but it seems difficult to formulate any good conjecture.

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MISCELLANEA

16. ELEGANCE (AN EXAMPLE)

$$\frac{a}{b}=\frac{b}{a+b}>0 \quad \text{iff} \quad \frac{a}{b}=\frac{-1+\sqrt{5}}{2}$$

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THE JORDAN AND SCHOENFLIES THEOREMS IN AXIOMATIC GEOMETRY

H. Guggenheimer

The Jordan theorem for polygons is one of the basic theorems of the topology of the plane, and so is the Schoenflies theorem, which asserts that every homeomorphism of a Jordan polygon onto a triangle can be extended to a homeomorphism of the plane onto itself. Both theorems are valid in every plane geometry which satisfies Hilbert's axioms of incidence and order, and the proof is not difficult. It is astonishing that none of the textbooks of elementary axiomatic geometry gives a proof. To remedy the situation, I offer here a simple proof adapted from work of N. J. Lennes (1903/1911) and Max Dehn (1899). A detailed exposition of the work of Lennes, Dehn, and others is given in **[2]**. The proof is not far from a proof of R. H. Bing **[1]** which uses the full structure of the Cartesian plane over the reals.

Hypothesis: *Given a geometry which satisfies Hilbert's plane axioms of incidence and order.*

The main consequence of the order axioms needed here is:

Lemma 1. *Finitely many points on a line can always be ordered linearly with a first and a last point.*

Notation: The order relation is denoted by (ABC). The open segment of endpoints A,B is (AB). Half-open and closed segments are denoted by $[AB)$ and $[AB]$, respectively. The halfplane defined by the line AB which contains the point $P \notin AB$ is $\pi(AB|P)$, and the complementary halfplane is $\pi(AB|\sim P)$.

Definitions. A *Jordan polygon* P is the union of finitely many open segments and their endpoints such that

(i) P is a connected set and

(ii) every point of a segment belongs to exactly one segment, every endpoint is endpoint of exactly two segments, and these are not collinear.

The endpoints of the segments are the vertices of the polygon. A vertex V_i of the polygon $P = V_1 \dots V_N V_1$ is *principal* if no vertex of P is in the interior of the triangle $V_{i-1}V_iV_{i+1}$ or on $(V_{i-1}V_{i+1})$.

Lemma 2. *A Jordan polygon always has a principal vertex.*

Proof. A vertex V_j of P either is principal or there are finitely many vertices V_s in the interior of $V_{j-1}V_jV_{j+1}$ or on $(V_{j-1}V_{j+1})$. If there are vertices of P in the interior of $V_{j-1}V_jV_{j+1}$, we disregard the points on $(V_{j-1}V_{j+1})$. In that case, the lines $V_{j-1}V_s$ intersect (V_jV_{j+1}) in a finite number of points that can be ordered: $V_jQ_1 \dots Q_rV_{j+1}$. Let V_t be the last vertex of P on $(V_{j-1}Q_1)$ before Q_1. If there are no vertices in the interior of $V_{j-1}V_jV_{j+1}$, let V_t be the last vertex of P in $(V_{j-1}V_{j+1})$ before V_{j+1}. In both cases, $P \cap (V_jV_t) = \varnothing$. Each of the two polygons $V_jV_{j-1} \dots V_tV_j$ and $V_jV_{j+1} \dots V_tV_j$ has fewer vertices than P. The polygons are Jordan polygons unless V_{t-1}, V_t, V_j are collinear. In the latter case,

V_t is deleted from the two polygons which are now Jordan polygons. One may now continue with either of the two Jordan polygons. After a finite number of steps, one obtains either a principal vertex or a triangle. The remaining vertex of the triangle is principal.

COROLLARIES. *A Jordan polygon can be triangulated by diagonals. A Jordan polygon has at least two principal vertices (it has two principal vertices that are exposed points of its convex hull* [3]).

LEMMA 3. *Let V_i be a principal vertex of a Jordan polygon P. There exists a Jordan quadrilateral $V_{i-1}RV_{i+1}QV_{i-1}$ whose interior (defined by Pasch's axiom) contains the interior of $V_{i-1}V_iV_{i+1}$ and $(V_{i-1}V_i]\cup[V_iV_{i+1})$ but no other point of P. The points R and Q are points in the plane not on P.*

Take $A\in(V_{i-1}V_{i+1})$. The lines joining V_{i-1} and V_{i+1} to all vertices of P in $\pi(V_{i-1}V_{i+1}|{\sim}V_i)$ intersect V_iA at most at a finite number of points. Let B be the first of these points starting from A (by Lemma 1) and choose $Q\in(AB)$. Take the ray r of V_iA that starts at V_i and does not contain A. If P intersects r in a finite number of closed segments, replace each segment by its two endpoints. Then the points of intersection of r and P together with the points of intersection of r and the lines that join V_{i-1} to the vertices of P in $\pi(V_{i+1}V_i|{\sim}V_{i-1})$ and V_{i+1} to the vertices of P in $\pi(V_{i-1}V_i|{\sim}V_{i+1})$ are finite in number. By Lemma 1, the points of intersection can be ordered with a first point C, starting from V_i. For $R\in(V_iC)$, the quadrilateral $V_{i-1}RV_{i+1}QV_{i-1}$ has the desired properties.

We are now ready for the main proof.

DEHN'S THEOREM. *There exists a one-to-one map of the plane onto itself which is a homeomorphism in the topology induced by the open halfplanes and which maps a given Jordan polygon P onto a triangle.*

The main tool in the construction is Dehn's order-preserving map of a segment onto a subsegment. If $(AB)\subset(AC)$ and $A\neq B\neq C$, choose one halfplane of AC and in it points O_2, D, E such that (AO_2D) and (CO_2E). Then $O_1=BE\cap(AO_2)$ exists by Pasch's axiom for BE and AO_2C; $O_1\in(BE)$ by Pasch for AD and BEC. For $X\in(AC)$ define $Y=XO_2\cap(ED)$, $F(X)=YO_1\cap(AB)$. Since $X\in\pi(AO_2,C)\cap\pi(CO_2,A)$, the ray of O_2X which starts at O_2 and does not contain X is in $\pi(AO_2,{\sim}C)\cap\pi(CO_2,{\sim}A)$ and Y exists. A similar argument for O_1 and the lines EO_1, DO_1 shows that $F(X)$ exists. By construction, the map $X\to F(X)$ is one-to-one. The halfplane argument also shows that (CXX') implies (EYY') and $(BF(X)F(X'))$, the map is order-preserving and a homeomorphism in the relative topology, i.e., the topology generated on the line by open segments. (Fig. 1)

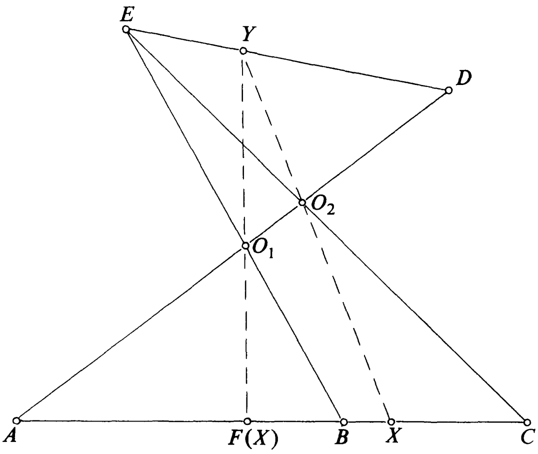

FIG. 1

Let now P be a Jordan polygon with principal vertex V_i and quadrilateral $V_{i-1}RV_{i+1}QV_{i-1}$ as constructed in Lemma 3. Put $V_{i+1}=O_2$. On QO_2 choose D_0 such that (QO_2D_0), on RO_2 choose E

such that (RO_2E). For $Z \in [RO_2)$ put $Y=(QZ)\cap[V_iV_{i+1})$, $X=(QZ)\cap(V_{i-1}V_{i+1})$. Then $D=V_iV_{i+1} \cap D_0E \in (D_0E)$ by Pasch; by the same axiom, $O_1(X)=(DX)\cap(EO_2)$ and $O_1'(X)=(DX)\cap(QO_2)$ do exist. Dehn's map for $O_2, O_1(X), D, E$ maps (ZY) onto (ZX); the same construction for $O_2, O_1'(X), D_0, E$ maps (YQ) onto (XQ). If we define $F(A)=A$ for every point $A \notin (QZ]$ on the line QZ, we obtain an order-preserving map of the line onto itself that maps $[YQ]$ onto $[XQ]$. For $Z' \in (V_{i-1}R)$, the same construction will do if D, E are replaced by $D'=Y'O_2\cap(D_0E)$, $E'=(D_0E)\cap Z'O_2$. Together we obtain an order-preserving map of the plane onto itself that maps P onto $V_1 \ldots V_{i-1}V_{i+1} \ldots V_NV_1$. The counter-image of every open halfplane in the map is open. After a finite number of steps, we obtain a map F of the plane onto itself that maps P onto a triangle formed by three vertices of P. (Fig. 2)

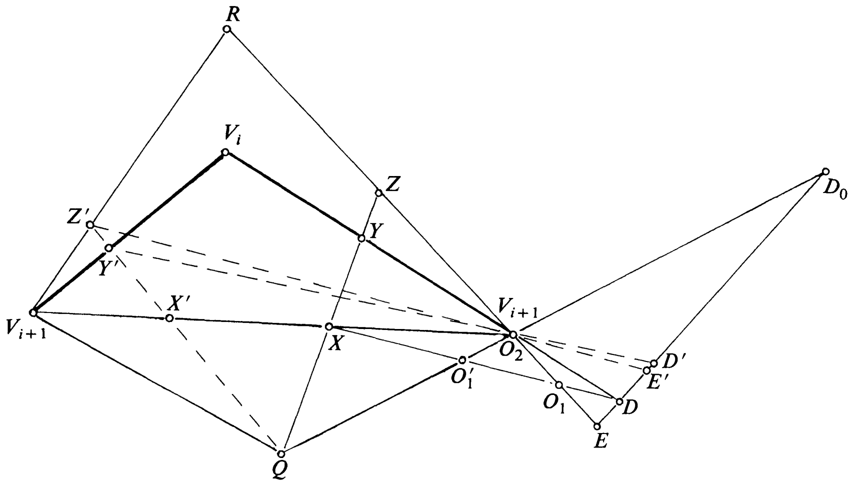

FIG. 2

JORDAN'S THEOREM. *A Jordan polygon separates the remaining points of the plane into two domains, interior and exterior. Any two points in the same domain can be joined by an arc in the domain; any polygonal arc joining points in different domains must meet the polygon.*

Jordan's theorem for triangles is an immediate consequence of Pasch's axiom. For a general Jordan polygon, interior and exterior are defined as counter-images of interior and exterior of the triangle in Dehn's map. In each step of Dehn's map, the counter-image of a polygonal arc is not necessarily a polygonal arc. Since Dehn's map preserves rays, the counter-image of a polygonal arc in every step intersects a diagonal of the triangulation at most in a single point or a single interval. For the purposes of the proof, any such interval is replaced by one of its points. The counter-image of a polygonal arc is replaced by the polygonal arc connecting by straight segments the starting point, successive points of intersection with diagonals of the triangulation, and the endpoint. Since this construction works in both directions, the validity of the Jordan theorem for polygons is equivalent to its validity for triangles.

SCHOENFLIES'S THEOREM. *Every homeomorphism of a Jordan polygon onto a triangle can be extended to a homeomorphism of the polygon and its interior onto the triangle and its interior.*

The proof is based on an auxiliary construction. (This part is neither in Dehn nor in Lennes.) For a triangle ABC, we take an interior point O and points X, Y for which (ABX) and (BCY) hold. For $P \in (AO)$, define $P'=(PX)\cap(OB)$, $P''=(P'Y)\cap(OC)$. It follows from Pasch's axiom that for a point Q with (PQO) the construction gives $(P'Q'O)$ and $(P''Q''O)$, i.e., the triangle $QQ'Q''$ is in the interior of triangle $PP'P''$. (Fig. 3).

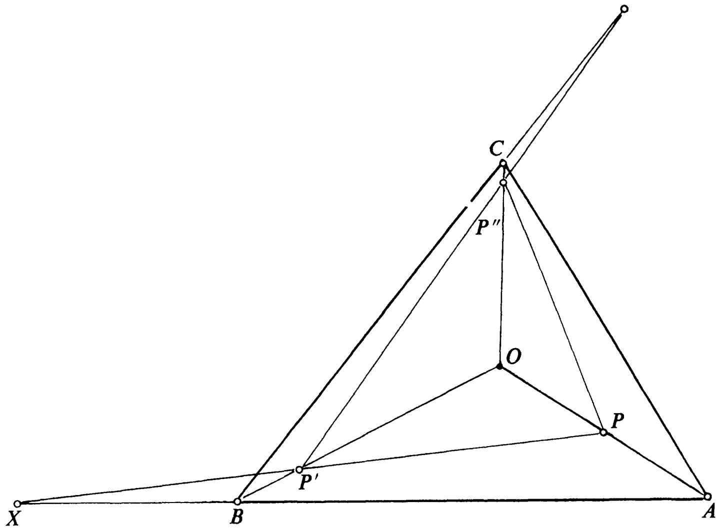

FIG. 3

By Dehn's theorem, Schoenflies's theorem holds if it holds for automorphisms of a triangle. Let f be a one-to-one order-preserving map of ABC onto itself. Let M be a point on triangle ABC. Every point of (OM) is on some triangle $PP'P''$. The mapping f is extended to the interior of ABC by

$$f(O)=O$$

$$f[(OM)\cap PP'P'']=(Of(M))\cap PP'P''.$$

The extended mapping satisfies the conditions of the theorem.

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A SIMPLE PROOF FOR A THEOREM OF KRONECKER

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Joel Spencer [1] gave an elementary proof for the following theorem of Kronecker. Let us propose a simpler one:

KRONECKER'S THEOREM. *Every algebraic integer w with $|w|=1$ and $|w'|=1$ for all conjugates w' of w is a root of unity.*

Proof. w is a root of an irreducible polynomial

$$f=X^n+\sum_{i=0}^{n-1}a_iX^i \qquad \text{with all } a_i\in\mathbf{Z}.$$

Let us write $M_n(\mathbf{Z})$ and $M_n(\mathbf{C})$ for the rings of $n\times n$ matrices M with entries from $\mathbf{Z}$ and $\mathbf{C}$ respectively; $|M|$ stands for the matrix we get by replacing every entry of M by its absolute value. The eigenvalues of

$$A:=\left[\begin{array}{ccccc|c} & & 0 & & & -a_0 \\ \hline 1 & & & & 0 & -a_1 \\ & 1 & & & & -a_2 \\ & & 1 & & & -a_3 \\ & & & \ddots & & \vdots \\ 0 & & & & 1 & -a_{n-1} \end{array}\right]\in M_n(\mathbf{Z})$$

are exactly the n roots of f. Hence A is similar to a diagonal matrix $D\in M_n(\mathbf{C})$ such that $D_{1,1}=w$ and $D_{2,2},\dots,D_{n,n}$ are the conjugates of w. Choose an invertible $U\in M_n(\mathbf{C})$ with $A=UDU^{-1}$. As $|D|$ is the identity-matrix, we have $|UD^t|=|U|$ for all $t\in\mathbf{N}$. The subset

$$\{A^t=UD^tU^{-1}|t\in N\}$$

of $M_n(\mathbf{Z})$ is therefore bounded, hence finite. We see: There are $s,\ t\in\mathbf{N}$ with $A^t=A^{t+s}$; thus $D^t=D^{t+s}$, so that $w^s=1$.

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EXPONENTIAL DECAY IN SOME LINEAR DELAY DIFFERENTIAL EQUATIONS

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If b and k are positive constants, it is easy to show that every solution of

$$y''(t)+by'(t)+ky(t)=0 \tag{1}$$

tends to zero exponentially as $t\to\infty$. One simply writes down the general solution and looks at it.

But it is not so easy to do as much for the solutions of the delay differential equation

$$y''(t)+by'(t)+qy'(t-r)+ky(t)=0, \tag{2}$$

where $r>0$, $b>0$, $k>0$, and $|q|<b$.

It *is* true that all solutions of equation (2) decay exponentially; and this fact *is* of interest, since such equations arise naturally in certain feedback control systems. See Minorsky [**6**, Chap. 21].

In fact, delay differential equations are encountered frequently in engineering, physics, the biological sciences, and economics. Yet such equations are rarely discussed in undergraduate differential equations courses because of the apparent need for a knowledge of complex function theory and functional analysis.

The easiest proofs of asymptotic stability of the trivial solution of equation (2) (never mind the specific exponential decay) have used suitable Lyapunov functionals. But, even then, a fair amount of effort and sophistication is required. See Krasovskiĭ [**5**, Chaps. 6 and 7].

One of the reasons for the difficulty can be seen by considering the analogous argument for the simple ordinary differential equation (1). The most natural Lyapunov function is the "energy" of the

system:

$$v(t)=\tfrac{1}{2}ky^2(t)+\tfrac{1}{2}y'^2(t).$$

Then, if y is any solution of equation (1),

$$v'(t)=ky(t)y'(t)+y'(t)y''(t)=-by'^2(t)\leqslant 0.$$

It follows that "small" initial conditions at $t=0$ must imply a "small" solution for all $t\geqslant 0$. Specifically,

$$ky^2(t)+y'^2(t)\leqslant ky^2(0)+y'^2(0) \quad \text{for} \quad t\geqslant 0.$$

So the trivial solution of equation (1) is stable. But, since $-by'^2$ is not "negative definite," this calculation is inadequate for proving *asymptotic* stability with the basic Lyapunov theorems.

To get around the difficulty in this case, one can use either the more sophisticated Barbašin–Krasovskiĭ theorem of 1952 or a more artificial Lyapunov function

$$v(t)=\tfrac{1}{2}ky^2(t)+cy(t)y'(t)+\tfrac{1}{2}y'^2(t)$$

with some suitable value of c. See, for example, Krasovskiĭ [**5**; p. 67] and Brauer and Nohel [**2**, p. 202], or see [**3**, pp. 370, 371].

Here is a third and simpler procedure. To show that every solution of equation (1) tends to zero exponentially, introduce

$$z(t)=e^{\delta t}y(t), \tag{3}$$

where δ is some positive constant to be selected. Then equation (1) yields

$$z''(t)+(b-2\delta)z'(t)+(k-b\delta+\delta^2)z(t)=0.$$

Clearly, if $\delta>0$ is sufficiently small, the coefficients in this equation will be nonnegative. So z will be bounded and $y(t)=e^{-\delta t}z(t)$ will decay exponentially toward zero as $t\to\infty$. As a matter of fact, one can put $\delta=b/2$ if $b^2\leqslant 4k$ or $\delta=[b-(b^2-4k)^{1/2}]/2$ if $b^2>4k$; and these are exactly the known sharp values for the exponential decay coefficient. [When $\delta=b/2$, equation (3) is the usual change of variables for eliminating the first-derivative term from equation (1).]

To see how this same type of simple transformation can work for certain delay differential equations, consider the following examples.

Example 1. The first-order delay differential equation

$$y'(t)=ay(t)+by(t-r) \quad \text{for} \quad t\geqslant 0, \tag{4}$$

where $r>0$, arises in elementary models for population growth and for mixing of fluids [**3**]. One usually seeks to solve equation (4) together with initial data

$$y(t)=\phi(t) \quad \text{on} \quad [-r,0]. \tag{5}$$

If ϕ is a given continuous function it is easily shown, by the "method of steps" that equations (4) and (5) have a unique solution on $[-r,\infty)$. See, for example, [**1**, p. 45], [**3**, p. 227], or [**4**, p. 208]. And, thanks to Krasovskiĭ [**5**], it is also easy to show that if

$$a<0 \quad \text{and} \quad |b|<|a|, \tag{6}$$

(or merely $a\leqslant 0$ and $|b|\leqslant|a|$), then the solution is bounded. Simply define

$$v(t)=y^2(t)+|a|\int_{t-r}^{t}y^2(s)\,ds,$$

and compute for $t\geqslant 0$

$$\begin{aligned}v'(t)&=(2a+|a|)y^2(t)+2by(t)y(t-r)-|a|y^2(t-r)\\&\leqslant(-|a|+|b|)\left[y^2(t)+y^2(t-r)\right]\leqslant 0.\end{aligned}$$

Thus $|y(t)|\leqslant[v(0)]^{1/2}=[\phi^2(0)+|a|\int_{-r}^{0}\phi^2(s)\,ds]^{1/2}$ for $t\geqslant 0$.

But from this result, we can easily show that $y(t)$ actually tends to zero. Define $z(t)=e^{\delta t}y(t)$ as before. Then

$$z'(t)=(a+\delta)z(t)+be^{\delta r}z(t-r) \quad \text{for} \quad t\geqslant 0$$

with

$$z(t)=e^{\delta t}\phi(t) \quad \text{on} \quad [-r,0].$$

For sufficiently small $\delta>0$,

$$a+\delta\leqslant 0 \quad \text{and} \quad |be^{\delta r}|\leqslant|a+\delta|,$$

in which case z is bounded and $y(t)=e^{-\delta t}z(t)\to 0$ exponentially as $t\to\infty$.

Example 2. The solution of the second-order equation

$$y''(t)+by'(t)+qy'(t-r)+ky(t)+py(t-r)=0 \quad \text{for} \quad t\geqslant 0, \tag{7}$$

where $r>0$, together with a given continuously differentiable initial function

$$y(t)=\phi(t) \quad \text{on} \quad [-r,0] \tag{8}$$

also exists and is unique on $[-r,\infty)$. [We are requiring y and y' to be continuous on $[-r,\infty)$.] Moreover, if

$$b>0,\ k>0,\ p>-k, \quad \text{and} \quad |q|+r|p|<b, \tag{9}$$

then the solution is bounded. To show this (again finding the clues in Krasovskiĭ's book) one defines

$$v(t)=(k+p)y^2(t)+y'^2(t)+\int_{t-r}^{t}[b-|p|(t-s)]y'^2(s)\,ds.$$

Then for $t\geqslant 0$

$$v'(t)=2py'(t)[y(t)-y(t-r)]-by'^2(t)-2qy'(t)y'(t-r)$$
$$-(b-|p|r)y'^2(t-r)-\int_{t-r}^{t}|p|y'^2(s)\,ds.$$

But

$$|2py'(t)[y(t)-y(t-r)]|=\left|p\int_{t-r}^{t}2y'(t)y'(s)\,ds\right|$$
$$\leqslant|p|\int_{t-r}^{t}\left[y'^2(t)+y'^2(s)\right]ds.$$

Thus, for $t\geqslant 0$,

$$v'(t)\leqslant(-b+|q|+r|p|)\left[y'^2(t)+y'^2(t-r)\right]\leqslant 0.$$

It follows that $v(t)$ is bounded for $t\geqslant 0$. Hence $y(t)$ and $y'(t)$ are bounded for $t\geqslant 0$.

Now, once again, introduce z as defined by (3). Then if δ is a sufficiently small positive number, one readily finds that z satisfies an equation like (7) with coefficients satisfying (9). Thus z is bounded and $y(t)\to 0$ exponentially.

The assertion made initially about equation (2) is a special case of this.

Example 3. Perhaps of greater interest is a nonhomogeneous equation with a periodic forcing term such as

$$x''(t)+bx'(t)+qx'(t-r)+kx(t)+px(t-r)=a\cos\omega t. \tag{10}$$

It follows, just as for linear ordinary differential equations, that if $\tilde{x}$ is any particular solution of equation (10), then every other solution of (10) has the form

$$x=\tilde{x}+y,$$

where y is a solution of the associated homogeneous equation (7). Moreover, if conditions (9) are

satisfied, one can show by substitution that equation (10) has a particular solution of the form

$$\tilde{x}(t) = A\cos\omega t + B\sin\omega t$$

valid for all t. Since $y(t) \to 0$, every solution of equation (10) must become close to the periodic "steady state" solution $\tilde{x}$ as $t \to \infty$, regardless of the initial function.

Simple tricks such as the change of variables (3) make it possible to include a useful discussion of applicable delay differential equations in even the most elementary course on ordinary differential equations [4].

Using other Lyapunov functionals together with transformation (3), one can verify further sufficient conditions, besides (6) and (9), for exponential decay; and one can extend the results to equations with several delays.

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MATHEMATICAL EDUCATION

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PSI IN COLLEGE MATHEMATICS

Matthew J. Hassett and Richard B. Thompson

Introduction. The use of Personalized Systems of Instruction (PSI) in college and university mathematics courses has become quite common in recent years. However, there seem to be very few published evaluations of PSI that are readily available to mathematicians.

Prompted by our own experiences with PSI and by the lack of literature on the subject, we organized a conference on Innovative Teaching Methods in Introductory College Mathematics. This was held in the spring of 1976 under the sponsorship of the Rocky Mountain Consortium, which has published the proceedings of the conference. (They may be obtained by writing to the Rocky Mountain Mathematics Consortium, Department of Mathematics, Arizona State University, Tempe, AZ 85281.) Our purpose is to summarize the results of the conference and to make the mathematical community aware of several recent survey papers in nonmathematical journals.

Conference Report. The Conference had 128 participants, representing 72 colleges and universities in 21 states and provinces. Reports were received on at least 30 PSI programs operated by those at the conference.

A striking initial impression was that a wide range of courses could be handled by PSI. Although the majority of courses reported were algebra skill courses on the intermediate-college-algebra level, courses in remedial arithmetic and algebra, statistics, calculus, and linear algebra were all represented.

Studies quoted by participants on the comparison of PSI with the lecture method fell into three main classes: (1) complete statistical studies, in which experimental and control groups were pretested to assure no significant difference in background and ability, and final data were subjected to some appropriate statistical test; (2) experimental-control studies with no statistical analysis, but random assignment of students to experimental and control groups; (3) comparison of available data without designation of experimental and control groups.

In the eleven papers to appear in the conference proceedings, the following results were presented:

Final examination. Nine of the eleven papers mentioned final exam comparisons. One found no significant difference and eight favored PSI. Five of the eight were complete statistical studies, and the remaining three were careful experimental-control studies with no statistical tests of significance. (Mean final exam scores on a 100-point final favored the PSI groups by 12, 13, and 15 points on the last three studies.)

Transfer. Four follow-up studies were discussed. One complete statistical study found significant grade-point advantages of one full letter-grade or more in five out of seven follow-up courses, and no significant difference in the remaining two follow-up courses. One experimental-control study found a full letter-grade advantage for PSI students in a follow-up math course; here there was no statistical analysis, but experimental and control groups were large ($N=400$). The two remaining studies consisted of a statistical analysis in which a slight letter-grade advantage for PSI students was found to be nonsignificant and an analysis of calculus pass rates that favored PSI students over "all other students."

Attitude. Seven papers described attitude improvements due to PSI or questionnaire responses favoring PSI. None of these was subjected to statistical analysis. One asserts that, when asked, students favored PSI over lectures by a ratio of 5 to 1; the remaining claims of improvement rely upon similar assertions.

Cost-effectiveness. Six papers discussed cost. Four authors presented evidence that their programs provided significant savings for their institutions. One author simply indicated that costs were satisfactory, and one stated that his costs were somewhat higher than lecture costs for his institution. Neither of the last two papers presented data to support its statements.

The participants furnished useful information on course demographics and the environment in which PSI currently exists. For example, there are successful and cost-effective programs which handle 2000–3000 students per semester. The smallest of the seven algebra programs for which student numbers are given in our articles handles 200 students per quarter.

Although the administrative procedures by which the programs are run differ widely from one institution to another, one common element appears in all of the methods. The material to be learned is divided into relatively short units which are studied in sequence. A student progresses to a new unit only after he or she has demonstrated mastery over the preceding unit by achieving a predetermined level of performance on a unit quiz. This basic principle, called "mastery unit testing," stands almost alone as the intersection of the various successful programs.

PSI program directors at the conference displayed great interest in placement testing and determination of prerequisite skill levels. This is an expected result of a system in which one constantly tests students and meets them face to face. A surprising new element is the interest of PSI mathematics-course directors in reading skills. Many students read at elementary-school levels and cannot learn from written materials.

The most successful PSI programs discussed by conference participants used available commercial

textbooks or workbooks. It is not necessary to write your own complete set of materials for a PSI course. When you find a book the students can read, you can write study guides for that text.

Survey papers. Most of the reports on PSI programs that first appeared in mathematical publications were descriptive rather than concerned with evaluation. Although evaluative studies have begun to appear (cf. [5], [7], [11]), we have not seen any surveys similar to our survey for mathematics. However, there are several recent survey papers that report on the evaluation of PSI programs in a wide variety of disciplines. Of the survey articles listed, four appear in vol. 13, no. 1 of *Programmed Learning and Educational Technology*, published in 1976. Other articles in the same issue will be of help to those who wish to start PSI programs.

The strong general trend of the articles discussed in the surveys is to support the effectiveness of PSI in comparison with other methods of instruction. Perhaps the strongest statement in support of PSI is that of Taveggia [12]. In 1968, Taveggia and Dubin [2] published a survey of the effectiveness of methods of instruction used in college teaching from 1924 to 1965—*excluding PSI.* In this study they concluded that "data demonstrate clearly and unequivocally that there is no measurable difference among truly distinctive methods of college instruction when evaluated by student performance on final examinations." In 1976, after a study of the literature on PSI which had been published since 1965, Taveggia concluded: "When evaluated by average student performance on course content examinations, the Personalized System of Instruction has proven superior to the conventional teaching methods with which it has been compared."

In an attempt to analyze the advantages of PSI, Kulik et al. [6] indicate that in final examinations, transfer, retention, and attitude, PSI has out-performed traditional lecture approaches in an overwhelming majority of those published studies which meet generally accepted standards of statistical validity. For example, of 39 end-of-course comparisons, 34 find a significant difference favoring PSI, and only one a non-significant difference favoring lectures. The courses surveyed by Kulik et al. were not in mathematics; but as we have seen, the Kulik survey is quite consistent with the results reported at our conference.

Which of the basic ingredients of PSI are responsible for these performance results? Kulik et al. conclude: "PSI seems to work well because it involves: (1) small units of work; (2) immediate and specific feedback at every step; and (3) a requirement of mastery at every step. Other features seem to be less crucial: interactions with proctors, self-pacing, and absence of regular lectures." These three essential features all come under the heading of unit-mastery learning, with the remaining features being important only insofar as they support the objective of unit mastery. Thus one could classify PSI as a specific means of attaining the objective of nearly total mastery of subject matter for all students; this is the mastery learning theory of Bloom [1]. In simplest terms, a PSI course forces students to perform better because its key objective is to have students learn course material completely. There has been a great deal of confusion about this, and many people still describe PSI as "self-paced instruction." However, courses designed with self-pacing, and not mastery, as their objective may achieve exactly that—self-pacing without mastery. Schoen [9] has written a survey on "self-paced instruction"; his conclusions are quite negative.

Many of the issues discussed above are of concern to individuals who do not wish to use PSI. Taveggia [12] points out that the adoption of PSI is only one of many possible responses to the studies on PSI. "A second, less obvious option suggested by the explanation developed above for the superiority of PSI is to reorganize one's conventional courses, building in the unit-perfection, forced pacing, and monitored progression features of PSI. The available evidence suggests that these probably are the features which account for the superiority of PSI over conventional methods. Thus to the degree that these are incorporated into conventional courses, student mastery of course content material probably will be enhanced." Some of our colleagues suggest that the features mentioned above have always been a part of good teaching by any method. Perhaps current investigators are

merely rediscovering the key elements of effective teaching, elements which have been used implicitly for decades by master teachers.

Conclusions. It is interesting that the reports on PSI in mathematics courses given at our conference are in close agreement with the articles from various other fields. There seems to be little doubt that PSI, or some instructional system that incorporates unit-mastery testing, can be used to improve the effectiveness of our mathematics instruction.

We feel that the studies we have discussed are only preliminary research on teaching methods. These studies are a first step which provides justification for individuals to try new ways of inducing students to learn. Once this first step is taken, the intelligent thing to do is to undertake further studies designed to identify the key variables which make various systems work for various individuals. We hope to see studies on such questions as mastery testing vs. normal curve testing, proctors vs. no proctors, self-pacing vs. forced pacing. We can thank the initial investigators of PSI for getting us started on this program and for making the point that it really is possible to do something to improve instruction.

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MISCELLANEA

17. If elementary mathematics is to continue to furnish the best possible preparation for the study of advanced mathematics, it is evident that it has to adapt itself to the rapid changes which are going on in the different branches of mathematics. A need is thus created for elementary text-books which meet the new requirements, and we are happy to be able to state that such books are being produced in our midst. How radical such changes may become cannot be foretold.... It is to be hoped that our inherited habits will not furnish an insurmountable barrier to progress in this direction.

G. A. Miller, this MONTHLY, 7 (1900) 97.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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Beginning in January, 1979, this Department will be edited by A. P. Hillman.

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

No solutions (except those accompanying proposals) should be sent to Professor Hillman.

*An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.*

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate with double spacing) and should be mailed before March 31, 1979. Please enclose a self-addressed card or label (for acknowledgment).

E 2737. *Proposed by Robert Ross Wilson, California State University, Long Beach*

Define a sequence of polynomials by $P_0 = 1$, $P_1 = x+1$, and $P_{n+1} = P_n + xP_{n-1}$ ($n \geqslant 1$). Show that all roots of each P_n are real.

E 2738. *Proposed by Michael W. Ecker, City University of New York*

Let σ be a permutation of the digits $0, 1, \ldots, 9$. Let $f:[0,1] \to [0,1]$ be the "extension" of σ, i.e., $f(x)$ is obtained from x by applying σ to each digit in the decimal expansion of x. (For uniqueness of decimal expansions, we do not allow expansions with all but finitely many digits equal to 9.)

(1) Find the points where f is continuous (or differentiable).
(2) Show that f is Riemann integrable and compute the integral.

E 2739. *Proposed by Marvin C. Papenfuss, Loras College, Iowa*

Prove that

$$x\sec^2 x-\tan x \leqslant \frac{8\pi^2x^3}{(\pi^2-4x^2)^2} \qquad (0\leqslant x<\pi/2).$$

E 2740. *Proposed by Victor Pambuccian, Bucharest, Romania*

Show that if P is a convex polyhedron, one can find a square all of whose vertices are on some three faces of P, as well as a square whose vertices are on four different faces of P.

E 2741. *Proposed by H. S. Witsenhausen, Bell Laboratories, Murray Hill, New Jersey*

Given a complex square matrix A, show that there exists a unitary matrix U such that U^*AU has all diagonal entries equal. If A is real, U can be taken real orthogonal.

E 2742. *Proposed by P. M. Gibson, University of Alabama at Huntsville*

In a ring with 1, find two matrices such that only the scalar matrices commute with both.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Simple Identity

E 2652 [1977, 295; Correction 1977, 567]. *Proposed by Jeffrey L. Rackusin, California State University at Northridge*

Let $A=(a_{ij})$ be a row-stochastic $n\times n$ matrix. Show that

$$\sum_{\sigma\in S_n}\prod_{i=1}^{n}\left(a_{i,\sigma(i)}\Big/\sum_{j=i}^{n}a_{i,\sigma(j)}\right)=1.$$

Solution by D. Ž. Djoković, Associate Editor. We claim that this identity holds for all $n\times n$ matrices (provided that no sum in the denominator is zero). Proof is by induction on n; the case $n=1$ is trivial, so let $n\geqslant 2$. The expression $E(A)$ on the left-hand side of the proposed identity can be written as

$$E(A)=\sum_{k=1}^{n}\left\{\frac{a_{1k}}{a_{11}+\cdots+a_{1n}}\sum_{\tau\in F_k}\prod_{i=2}^{n}\frac{a_{i,\tau(i)}}{a_{i,\tau(i)}+\cdots+a_{i,\tau(n)}}\right\}$$

where

$$F_k=\{\tau\in S_n|\tau(1)=k\}.$$

Let A_{1k} be the submatrix of A obtained by omitting the first row and kth column of A. Then it is clear that

$$E(A_{1k})=\sum_{\tau\in F_k}\prod_{i=2}^{n}\frac{a_{i,\tau(i)}}{a_{i,\tau(i)}+\cdots+a_{i,\tau(n)}}.$$

By the induction hypothesis, $E(A_{1k})=1$ for each k and so

$$E(A)=\sum_{k=1}^{n}\frac{a_{1k}}{a_{11}+\cdots+a_{1n}}=1.$$

Also solved by the proposer.

Minimum Subcover of a Cover of a Finite Set

E 2654 [1977, 386]. *Proposed by D. E. Daykin, Reading University, England*

Let $A=\{0,1,2,\ldots,n-1\}$. For $m\in A$ let $f(m,n)$ be the least integer k with the following property. If F is a family of subsets of A such that every $i\in A$ belongs to more than k members of F then A can be covered by $n-m$ members of F. Evaluate $f(m,n)$ for $2m\leqslant n$.

Solution by the proposer (revised by the editor). It is clear that $f(0,n)=0$. We shall prove that for $1\leqslant m\leqslant n/2$ we have $f(m,n)=2^{m-1}$.

Consider the family F which contains all subsets of A that meet $\{0,1,2,\ldots,n-m\}$ in exactly one element. Each element of A belongs to at least 2^{m-1} members of F, but A is not the union of any $n-m$ members of F. Hence $f(m,n)\geqslant 2^{m-1}$. In order to prove the opposite inequality we shall use the following graph-theoretical result.

LEMMA. *If G is a finite graph (without loops or multiple edges) on n vertices with minimum degree δ, then G has a matching of cardinality $k=\min\left(\delta,\left[\frac{n}{2}\right]\right)$.* (A matching is a set of pairwise disjoint edges.)

Proof of the Lemma. Let $M=\{u_1v_1,u_2v_2,\ldots,u_rv_r\}$ be a maximum matching in G. Assume that $r<k$. Let W be the set of vertices not incident with edges in M. Then $|W|=n-2r\geqslant n-2(k-1)\geqslant 2$. Hence we can choose distinct vertices w_1,w_2 in W. By maximality of M there is no edge in G having both endpoints in W. Hence if d_i is the number of edges with one end in $\{w_1,w_2\}$ and the other in $\{u_i,v_i\}$ then $d_1+d_2+\cdots+d_r\geqslant 2\delta\geqslant 2k$. Consequently, at least one d_i, say d_1, is $\geqslant 3$. Hence, we may assume that u_1 and w_1 are joined and also v_1 and w_2 are joined. But then $\{u_1w_1,v_1w_2,u_2v_2,\ldots,u_rv_r\}$ is also a matching, which contradicts our hypothesis about M.

Returning to our problem, let F be any family of subsets of A. Let G be the graph with vertex set A in which $\{i,j\}$ $(i\neq j)$ is an edge if and only if $\{i,j\}$ is a subset of some member of F. If each element $i\in A$ is in more than 2^{m-1} members of F, then it is easy to see that the minimum degree δ of G is $\geqslant m$. Since $m\leqslant n/2$, the Lemma implies that G has a matching M of size m. The $2m$ vertices in M can be covered by m members of F, and the remaining $n-2m$ vertices can be covered by $n-2m$ members of F. Thus A can be covered by $n-m$ members of F, i.e., $f(m,n)\leqslant 2^{m-1}$.

An Inequality for Positive Real Numbers

E 2656 [1977, 386]. *Proposed by G. Tsintsifas, Thessaloniki, Greece*

Let $a_2,a_3,\ldots,a_n$ be positive real numbers and $s=a_2+a_3+\cdots+a_n$. Show that

$$\sum_{k=2}^{n}a_k^{1-1/k}<s+2\sqrt{s}\ .$$

Solution by Allen Stenger, American Express Company. Phoenix, Arizona. Clearly, we may restrict attention to those a_k with $0<a_k<1$. Then by the Cauchy–Schwarz inequality we have

$$\sum\left(a_k^{1-1/k}-a_k\right)=\sum a_k^{1/2}\left(a_k^{1/2-1/k}-a_k^{1/2}\right)\leqslant\sqrt{sA}$$

where

$$A=\sum\left(a_k^{1/2-1/k}-a_k^{1/2}\right)^2.$$

It remains to show that $A<4$.

By the Mean-Value Theorem,

$$a_k^{1/2-1/k}-a_k^{1/2}=-\frac{1}{k}a_k^y\log a_k$$

for some y satisfying $1/2-1/k<y<1/2$.

Considering $a^y\log a$ as a function of a, it is easy to show that

$$-(ey)^{-1}\leqslant a^y\log a<0\qquad(0<a<1).$$

Hence, for $k\geqslant 3$ we have

$$0<-\frac{1}{k}a_k^y\log a_k\leqslant(eyk)^{-1}<\frac{2}{e(k-2)},$$

and so

$$A<1+\frac{4}{e^2}\sum_{k=3}^{\infty}\frac{1}{(k-2)^2}=1+\frac{2\pi^2}{3e^2}<4.$$

Also solved by David Bloom, Robert Breusch, L. E. Mattics, St. Olaf Problem Group, and the proposer.

Forcing a Quasigroup to Be a Group

E 2659 [1977, 486]. *Proposed by Arthur L. Holshouser, Charlotte, North Carolina*

The sequence a,b,c,d can be parenthesized in five ways. Equating these two at a time we obtain the following identities:

(1) $(ab)(cd)=a(b(cd))$, (2) $(ab)(cd)=a((bc)d)$,
(3) $(ab)(cd)=((ab)c)d$, (4) $(ab)(cd)=(a(bc))d$,
(5) $a(b(cd))=a((bc)d)$, (6) $a(b(cd))=((ab)c)d$,
(7) $a((bc)d)=(a(bc))d$, (8) $((ab)c)d=(a(bc))d$,
(9) $a(b(cd))=(a(bc))d$, (10) $a((bc)d)=((ab)c)d$.

Which of these identities is such that a quasigroup satisfying it is necessarily a group?

Solution by Anders Bager, Hjørring, Denmark. We claim that a quasigroup satisfying any one of (1)–(8) is associative and hence a group, but this is not true for (9) or (10). The problem can be reduced by recognizing pairs of equations which are mirror images of each other: (1)↔(3), (2)↔(4), (5)↔(8), (9)↔(10), while (6) and (7) are self-dual.

(1) Choose d so that $cd=c$ gives $(ab)c=a(bc)$.

(2) Choose a so that $ba=b$. Then for all x we have $b(xb)=(ba)(xb)=b((ax)b)$ and consequently $ax=x$. Using this (2) gives $b(cd)=(bc)d$.

(5) Canceling a in (5) gives $b(cd)=(bc)d$.

(6) Choose e so that $be=b$. Then for all x we have $be(e(xe))=((be)e)(xe)=(be)(xe)=b(xe)$. Canceling b we get $e(e(xe))=xe$. Applying (6) to e,e,x,e we get $((ee)x)e=xe$, and so $(ee)x=x$. Taking $a=ee$ in (6) gives $b(cd)=(bc)d$.

(7) Choosing c so that $bc=b$ gives $a(bd)=(ab)d$.

(9) (The following is taken from the proposer's solution.) Let σ be an automorphism of order 2 of an abelian group $(A,+)$ and define $xy=x+\sigma(y)$ for $x,y\in A$. Then $(A,\cdot)$ is a quasigroup satisfying (9) but it is not a group.

Also solved by Floyd Barger, Theodore Bolis, K. W. Heuer & G. A. Heuer, Norman Roth, and the proposer. Partial solutions by Thomas Elsner and by Peter Lindstrom.

Editor's comments. (i) Let $(G,\cdot)$ be a quasigroup satisfying (9). Fix $e\in G$ and define $x*y=x(ey)$ for $x,y\in G$. Then $(G,*)$ is also a quasigroup. Since $(x*y)*z=(x(ey))(ez)=x(e(y(ez)))=x*(y*z)$, $(G,*)$ is in fact a group.

(ii) J. Stasheff remarks that (1), (3), (5), (7), (8) are obtained by a single application of the associative law. This leads to the subject of combinatorial and polyhedral structure of bracketing (bibliography available from William Butler, McGill University, Montreal).

Minimal Solution of a System of Diophantine Equations

E 2664 [1977, 487]. *Proposed by Robert L. Bishop, Massachusetts Institute of Technology*

For a fixed $n\geqslant 3$, describe how one can construct all solutions of the system of Diophantine equations

$$\left(\sum_{i=1}^{n} x_i\right)-x_j=y_j^2, \qquad 1\leqslant j\leqslant n.$$

For $n=10$, find a solution such that the x_i's are distinct positive integers and $x_1+\cdots+x_{10}$ is minimal.

Solution by Allan Wm. Johnson, Jr., Washington, D.C. Let $S=x_1+\cdots+x_n$. By adding the given equations we find

$$S=\frac{1}{n-1}\sum_{i=1}^{n} y_i^2,$$

and so

$$x_j=\frac{1}{n-1}\sum_{i=1}^{n} y_i^2-y_j^2, \qquad 1\leqslant j\leqslant n. \tag{1}$$

Thus the solutions are given by (1) where $y_1,\ldots,y_n$ are integers such that $n-1$ divides $y_1^2+\cdots+y_n^2$.

Now let $n=10$. If $x_1,\ldots,x_n$ are distinct positive integers then $y_1^2,\ldots,y_n^2$ are distinct and we may take $y_1>y_2>\cdots>y_{10}\geqslant 0$. Let $d_i=y_1-y_{i+1}$ $(1\leqslant i\leqslant 9)$ and put

$$D_1=\sum_{i=1}^{9} d_i, \qquad D_2=\sum_{i=1}^{9} d_i^2.$$

Then we have

$$9S=10y_1^2-2D_1y_1+D_2, \qquad 9x_1=9S-9y_1^2=(y_1-D_1)^2+D_2-D_1^2>0. \tag{2}$$

Since $D_1-d_9<\sqrt{D_1^2-D_2}$, we must have

$$y_1>D_1+\sqrt{D_1^2-D_2}\ . \tag{3}$$

From (2) we have

$$D_1^2-D_2\equiv 0,\ 1,\ 4, \text{ or } 7 (\text{mod } 9).$$

Using a computer, we sought solutions of (4) among 220 nine-element subsets of $\{1,2,\ldots,12\}$ as $d_1,\ldots,d_9$. For each of the 96 resulting solutions we computed the smallest y_1 satisfying (3) and such that $(y_1-D_1)^2\equiv D_1^2-D_2$ (mod 9). Among all these solutions the minimal S is 8656 occurring for

d_i		1	2	3	4	5	6	7	9	11
y_i	93	92	91	90	89	88	87	86	84	82
x_i	7	192	375	556	735	912	1087	1260	1600	1932

For non-examined solutions we would have $d_9\geqslant 13$ and

$$D_1^2 - D_2 = 2\sum_{i<j} d_i d_j \geqslant 1092 + 72\cdot 13 = 2028.$$

Hence, by (3) $y_1 > 49 + \sqrt{2028} > 94$, $S > y_1^2 \geqslant 95^2 = 9025 > 8656$.

Also solved by J. C. Binz (Switzerland), Peter de Buda, and the proposer. Partially solved by R. J. Stroeker (Netherlands), and L. Kuipers (Switzerland).

Partial Checkerboards

E 2665 [1977, 567]. *Proposed by Sidney Penner, Bronx Community College, CUNY*

A *partial checkerboard* is a checkerboard from which squares have been removed so that

(a) it is impossible to place even one domino on the remaining board, and

(b) the replacement of a single deleted square, regardless of its location, makes it possible to place a domino on the board. (A domino covers two squares having a common side).

It is easy to see that, for an 8×8 partial checkerboard, the minimum number of deleted squares is 32. What is the maximum number?

Solution by Michael W. Ecker, City University of New York (*revised by the editor*). Experimenting suggests that the maximum number is 48 (see diagram):

$$\begin{matrix} 0&1&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&1&0\\ 0&1&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&1&0 \end{matrix}$$

Let x be the number of squares left in the outer layer of the board, y in the next layer, then z, and finally u in the innermost layer. It is clear that we must have

$$(1)\ 3x+y\geqslant 28,\qquad (2)\ x+3y+z\geqslant 20,$$
$$(3)\ y+3z+u\geqslant 12,\qquad (4)\ z+3u\geqslant 4.$$

Clearly, $u=0$, 1, or 2. If $u=0$ then

$$4(x+y+z)=(3x+y)+(x+3y+z)+3z\geqslant 28+20+12=60$$

and so $x+y+z\geqslant 15$. If $x+y+z=15$ then we must have equalities in (1), (2), and (4). But this system has no integral solutions. Hence we must have $x+y+z\geqslant 16$.

If $u=1$ then

$$7(x+y+z)=2(3x+y)+(x+3y+z)+2(y+3z)\geqslant 2\cdot 28+20+2\cdot 11=98,$$

i.e., $x+y+z\geqslant 14$. If $x+y+z=14$ then we must have equalities in (1), (2), and (3). But this system has no integral solutions. Hence we must have $x+y+z\geqslant 15$ and so $x+y+z+u\geqslant 16$.

If $u=2$ then

$$7(x+y+z)=2(3x+y)+(x+3y+z)+2(y+3z)\geqslant 2\cdot 28+20+2\cdot 10=96$$

and so $x+y+z\geqslant 14$, $x+y+z+u\geqslant 16$.

Thus in all cases at least 16 squares must remain.

Also solved by Beloit College Solvers, George Berzsenyi, Alan Frank, Jim Fickett, Carl Hurd, Elgin Johnston, Terry Lawson & the University of Miami Problems Group, L. E. Mattics, David Montana, Harry Nelson & Scot Nelson, Edward Nordhaus & Stephen Wilson, Victor Pambuccian (Romania), Henry Thomas, Thomas Vining, and the proposer.

Partially solved by Fred Buckley, Thomas Elsner, Mark Johnson, Sidney Kravitz, and Carolyn MacDonald.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor R. C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before March 31, 1979.

6234*. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Canada*

Let D_n and M_n denote the derangement number and the ménage number respectively. Prove or disprove that the sequence $\{M_n/D_n\}$, $n=4,5,6,\cdots$ is monotonically increasing and $\lim_{n\to\infty}(M_n/D_n)=1/e$. (For definitions see, e.g., H. J. Ryser, *Combinatorial Mathematics*, Chap. 2, Sec. 3 and Chap. 3, Sec. 2.)

6235. *Proposed by Robert J. Anderson and M. Ram Murty, Massachusetts Institute of Technology*

Let $M(x)=\sum_{n\leqslant x}\mu(n)$, where μ is the Möbius function. H. Gupta conjectured that $\sum_{n\geqslant x}M(n)=O(x\log x)$. (See Journal Indian Math. Soc., 1949.) He also gave numerical evidence to support this conjecture. Settle this conjecture.

6236. *Proposed by Antal E. Fekete, Memorial University of Newfoundland*

We say that two endomorphisms of the complex vector space C^n are of the same type if there is a bijection between their respective sets of eigenvalues which maps the Jordan normal form of one endomorphism into that of the other. Find a formula determining the number of different endomorphism types of C^n. Define what is meant by an endomorphism type of the real vector space R^n and determine their number.

6237. *Proposed by Emeric Deutsch, Polytechnic Institute of New York*

Show that every zero z of the complex polynomial

$$f(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_{n-1}z+a_n$$

satisfies $-\beta\leqslant \operatorname{Re} z\leqslant\alpha$, where α and β are the unique positive roots of the equations

$$x^n+(\operatorname{Re}a_1)x^{n-1}-|a_2|x^{n-2}-|a_3|x^{n-3}-\cdots-|a_{n-1}|x-|a_n|=0,$$

$$x^n-(\operatorname{Re}a_1)x^{n-1}-|a_2|x^{n-2}-|a_3|x^{n-3}-\cdots-|a_{n-1}|x-|a_n|=0,$$

respectively.

6238*. *Proposed by F. David Hammer, Santa Cruz, California*

To see if a binary operation on a set with n elements is associative, one might think it necessary to verify directly n^3 instances of the associative law. Often, however, for instance if the operation is commutative and has an identity, considerably fewer need be verified. Is there a set of n elements and an operation on them for which all n^3 verifications are necessary?

6239. *Proposed by F. David Hammer, Santa Cruz, California*

Is the following conjecture true? Let $p(x,y)$ be any polynomial in x and y; then $|x^y-y^x|\leqslant|p(x,y)|$ has only finitely many solutions (x,y) in unequal integers $\geqslant 2$.

SOLUTIONS OF ADVANCED PROBLEMS

Iteration of a Continuous Map

6133 [1977, 140]. *Proposed by J. Cano and A. Gruebler, Universidad Simon Bolivar, Venezuela*

Let $f:[a,b]\to[a,b]$ be continuous and denote $P(f)=\{x:f^n(x)=x \text{ for some } n=1,2,\dots\}$, $C(f)=\{x:f^m(x)\in P(f) \text{ for some } m=1,2,\dots\}$, and L_x the set of limit points of the sequence $\{x,f(x),f^2(x),\dots,f^n(x),\dots\}$. Here $f^n(x)$ is the nth iterate of $f(x)$. Prove that for each $x\in[a,b]$, $L_x\subset\overline{C(f)}$. Is this true in R^2?

Solution by Robert B. Israel. University of British Columbia. Let S be the complement of $C(f)$ and let y be an interior point of S. Let J be the connected component of S containing y. Since $f^n(S)\subseteq S$, $f^n(J)$ is contained in a connected component of S. If more than one $f^n(x)$ is contained in J, we must have $f^m(J)\subseteq J$ for some m. Suppose, without loss of generality, $f^m(y)>y$. Then, since J is connected and contains no fixed points of f^m, we must have $f^m(s)>s$ for all $s\in J$. Take $\epsilon>0$ so that $f^m(s)>y+\epsilon$ whenever $|s-y|<\epsilon$, and $(y-\epsilon,y+\epsilon)\subseteq J$. Then for such s we have $f^{km}(s)>y+\epsilon$ for all positive integers k. Thus for any $x\in[a,b]$ and each $j=0,1,\dots,m-1$, there is at most one k such that $|f^{km+j}(x)-y|<\epsilon$. Therefore y can not be a limit point of $\{f^n(x)\}$.

If $[a,b]$ is replaced by a disk in $\mathbf{R}^2$, the result is false. Let f be a rotation of the disk by an irrational multiple of π. Then $C(f)$ consists only of the center of the disk, while L_x is a circle.

Also solved by the proposers.

Chain Conditions In Rings

6134 [1977, 141]. *Proposed by Barbara Osofsky, Rutgers University*

Let R be a ring, not necessarily with identity, and let R^n be the subring generated by n-fold products of elements of R. Prove: If R has descending chain condition (d.c.c.) on right ideals, then so does R^n. Does this result hold if d.c.c. is replaced by ascending chain condition (a.c.c.)?

Solution by the proposer. A stronger result actually holds. Let R be a ring, not necessarily with identity. Let M_R be a right R-module. Then M_R has a.c.c. (d.c.c.) on R-submodules if and only if M has a.c.c. (d.c.c.) on R^n-submodules.

Proof. We note that any R-submodule of M is *a fortiori* an R^n-submodule, so a chain condition on R^n-submodules will imply the same chain condition on R-modules.

Now let M have a chain condition on R-submodules. We shall use induction on $n>1$.

If $n=2$, let $I_1,I_2,\dots,I_k$ be an ascending (descending) chain of R^2-submodules of M. Then $\{I_k+I_kR\}$ is a chain of R-submodules of M and so terminates, say at $k=N_1$. Moreover for all $x\in I_k\cap I_kR$, $xR\subseteq I_kR$ and $xR\subseteq(I_kR)R=I_kR^2\subseteq I_k$, so $xR\subseteq I_k\cap I_kR$. Thus $\{I_k\cap I_kR\}$ is a chain of R-submodules of M and so also terminates, say at $k=N_2$. For $k\geqslant\max(N_1,N_2)=e$

$$I_k\cap I_kR=I_e\cap I_eR,$$
$$I_k+I_kR=I_e+I_eR, \quad\text{and}\quad I_k\subseteq I_e.$$

(In the case of d.c.c., reverse the roles of k and e.)

Let $x\in I_e\subseteq I_k+I_kR$. Then $x=y+zr$, y and $z\in I_k$, and $x-y=zr\in I_kR\cap I_e\subseteq I_eR\cap I_e=I_kR\cap I_k\subseteq I_k$. Since $y\in I_k$, x is also in I_k, and $I_k=I_e$. This shows that M has the appropriate chain condition on R^2-submodules.

Induction step. If $n>2$, there exists a $k>1$ with $2^k>n$, $2^{k-1}<n$. By the induction hypothesis, M

has chain condition on $R^{2^{k-1}}$-submodules. Using the demonstrated result for $n=2$, M has chain condition on $(R^{2^{k-1}})^2=R^{2^k}$-submodules and hence on R^n-submodules.

Since chain conditions are inherited by submodules and R^n is an R-submodule of R considered as a right module over itself, the result stated in the problem immediately follows from this result.

Polynomials in Two Variables

6136 [1977, 141]. *Proposed by H. L. Montgomery, University of Michigan*

Let $P(z, w)=\Sigma c_{mn}z^m w^n$ be a polynomial in $\mathbf{C}[z, w]$. Suppose that $Q(z, w)=P(z, w/z)$ is also a polynomial: that is, $c_{mn}=0$ whenever $n>m$. Show that

$$P=\{P(z, w):|z|<1, |w|<1\}=\{Q(z, w):|z|<1, |w|<1\}=Q.$$

Solution by Adam Riese, Wright State University, Dayton, Ohio. For $|z|\leqslant 1$, $P_z=\{P(z, w):|w|<1\}\subset\bar{P}$ and since P_z is also open, $P_z\subset P$. Similarly $\{P(z, w):|z|<1\}\subset P$ for all $|w|\leqslant 1$. Thus

$$\bar{P}\backslash P=\{P(z, w):|z|=|w|=1\}, \qquad \bar{Q}\backslash Q=\{Q(z, w):|z|=|w|=1\}.$$

Now, for $|z|=|w|=1$, we have

$$P(z, w)=Q(z, zw)\in\bar{Q}\backslash Q, \qquad Q(z, w)=P(z, w/z)\in\bar{P}\backslash P,$$

i.e., $\bar{P}\backslash P=\bar{Q}\backslash Q$. The desired result now follows because two bounded open sets with identical boundaries have to be identical.

Also solved by O. P. Lossers (Netherlands), and by the proposer.

Abundant Numbers of the Form $p_ip_{i+1}p_{i+2}\cdots p_{i+n}$

6138 [1977, 221]. *Proposed by Harry D. Ruderman, Hunter College Campus School*

Let $p_1, p_2, \cdots$ be consecutive primes with $p_1=2$. Then

(1) for every n there is a k for which $\prod_{i=n}^{n+k}p_i$ is an abundant number.

(2) Find an upper bound for k in terms of n.

Solution by Ernst Trost, Zürich, Switzerland. That k exists follows from $\sigma(n)/n=\prod_{i=n}^{n+k}(1+1/p_i)$ and the divergence of Σp_i^{-1}.

We put $P(u,v)=\prod_{i=u}^{v}(1-p_i^{-1})^{-1}$, $Q(u,v)=\prod_{i=u}^{v}(1+p_i^{-1})$ and make use of the following facts:

$$p^{-1}(1,\infty)Q(1,\infty)=6\pi^{-2}=0.60793 \tag{1}$$

$$P(1,v)>\log p_v \tag{2}$$

$$P(1,v)=e^C(1+\delta)\log p_v, \qquad C=0.5772\ldots, \qquad \delta=o(1) \tag{3}$$

$$\pi(m)<1.255m(\log m)^{-1} \tag{4}$$

$$m\log m<p_m<m(\log m+\log\log m), \qquad m\geqslant 6. \tag{5}$$

From (1) we infer $p^{-1}(6,\infty)Q(6,\infty)=0.97775$ and therefore

$$0.977P(6,v)<Q(6,v)<(168/169)P(6,v), \qquad v>6 \tag{6}$$

(2) yields $P(6,r)>(16/77)\log p_r$ and from (3) follows

$$P(6,s)=0.370(1+\delta)\log p_s, \qquad s>6.$$

For $r,s>6$ (6) gives

$$Q(6,r)>0.203\log p_r,\ Q(6,s)<0.368(1+\delta)\log p_s. \tag{7}$$

$a=p_sp_{s+1}\cdots p_r$ is abundant if $\delta(a)=aQ(s,r)>2a$ or $Q(s,r)>2$. From (7) we get, with $t=s-1$

$$Q(s,r)=Q(6,r)Q^{-1}(6,t)>0.552(1+\delta)^{-1}\log p_r(\log p_t)^{-1}.$$

This proves the first assertion of the problem.

Now $Q(s,r)>2$ implies $\log p_r > 3.623(1+\delta)\log p_t$ or

$$p_r > p_t^{3.623+\varepsilon}, \qquad \varepsilon = 3.623\delta = o(1). \tag{8}$$

Let p_r be the least prime satisfying (8), then we infer from (4) and (5)

$$r-1=\pi\left(p_t^{3.623+\varepsilon}\right) < \frac{1.255}{3.623+\varepsilon} p_t^{3.623+\varepsilon}(\log p_t)^{-1}$$
$$< \frac{1.255}{3.623+\varepsilon} t^{3.623+\varepsilon}(\log t + \log\log t)^{2.623+\varepsilon}.$$

Putting $s=n$, $r=n+k$ we get

$$k<t\left[\frac{1.255}{3.623+\varepsilon}\{t\log(t\log t)\}^{2.623+\varepsilon}-1\right], \qquad t=n-1. \tag{9}$$

The table of the exact values of k begins as follows

n	1	2	3	4	5	6	7	8	9	10	11
k	3	5	11	21	35	61	75	97	130	167	204

In this range, (9) with $\varepsilon=-1$ gives a reasonable upper bound for $n \geqslant 5$.

Also solved by David Anderson, Charles Cable, Robert Dressler, Ernst Heppner & Wolfgang Schwarz (Germany), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), James Ridley (South Africa), David Rusin, and Manuel Scarowsky (Canada),

Notes. (1) Rusin's calculations lead to the asymptotic result: minimum value of $k \sim -n+(e/2)\,n^2\log n$.

(2) Calculation by Heppner and Schwarz yields the theorem: Let be $0<\varepsilon\leqslant 0.85$, and $n_0 \geqslant \max(100, 0.75254 \log(1+\varepsilon/2), -0.64096\log(1-\varepsilon/2))$. Then $a_{n,k}$ is abundant if $n\geqslant n_0$ and $k \geqslant n^{2+\varepsilon}$, and not abundant if $n \geqslant n_0$ and $k \leqslant n^{2-\varepsilon}$. Here $a_{n,k}=\prod_{i=n+1}^{n+k} p_i$. For example, if $\varepsilon=0.85$ one may choose $n_0=2036$, and for $\varepsilon=0.5$ one may choose $n_0=2\cdot 10^7$.

Functions with Prescribed Discontinuities

6142 [1977, 222]. *Proposed by L. O. Chung, North Carolina State University*

Find a function $f:[0,1]\to[0,1]$ which is continuous everywhere except on two countable dense subsets D_1, D_2 of rationals such that on D_1 f is right continuous but not left continuous, and on D_2 f is left continuous but not right continuous.

Solution by Ellen Hertz, Paramount Design Company, New York City. Let D_1, D_2 be any pair of disjoint sets of rationals that are dense in $(0,1)$. Enumerate the elements of D_1 as $a_1, a_2, \ldots$, and the elements of D_2 as $b_1, b_2, \cdots$. Let $s_1, s_2, \ldots$, be a series of positive terms such that $\Sigma s_k = 1/2$.

For $0 \leqslant x \leqslant 1$, set $f_1(x)=\Sigma_{n:a_n\leqslant x} s_n$. Then $f_1(0)=0$, $f_1(1)=1/2$, and f_1 is continuous except on D_1 where it is right continuous. Set $f_2(x)=\Sigma_{n:b_n<x} s_n$. Then f_2 is left continuous on D_2, continuous elsewhere. Then $f=f_1+f_2$ is as required.

Also solved by George Akst, Kenneth Andersen, Charles Belna, Eric Chandler, Michael Ecker, T. E. Gantner, Marguerite Gerstell, Eric Grinberg, Gustaf Gripenberg (Finland), I. I. Kotlarski, Joel Levy, O. P. Lossers (Netherlands), J. G. Mauldon, Gene Ortner, N. Fowler, Walter Stromquist, University of Wyoming Problem Group, and the proposer.

Note. For functions of the nature required, Gantner and Ortner refer us to Hewitt and Stromberg, *Real and Abstract Analysis*, p. 113; Andersen and Belna refer to Rudin, *Principles of Mathematical Analysis*, p. 84.

Dividing the Pie Fairly

6143 [1977, 222]. *Proposed by A. L. Macdonald, Eastern Michigan University*

The familiar method of fair division of a pie by passing a knife over it until someone is satisfied

suggests the problem: Let $\pi_1, \pi_2, \ldots, \pi_n$ be non-atomic probability measures on a set X. Then there are pairwise disjoint sets $B_1, B_2, \ldots, B_n$ with $\pi_i(B_i) \geqslant 1/n$.

Solution by T. Sekiguchi, University of Arkansas. The proof is by induction: for $n=1$, the result is obvious. Assume the result for some $n \geqslant 1$.

Let $\pi_1, \pi_2, \ldots, \pi_n, \pi_{n+1}$ be $n+1$ non-atomic measures on X. By the induction hypothesis, there exists a partition $\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n$ such that

$$\pi_i(\tilde{B}_i) \geqslant 1/n \quad \text{for } i=1,2,\ldots,n.$$

Now partition each $\tilde{B}_i$ into $B_{i1}, B_{i2}, \ldots, B_{i,n+1}$ with equal probabilities with respect to measure π_i, that is

$$\pi_i(B_{ij}) = \frac{1}{n+1}\pi_i(\tilde{B}_i), \qquad i=1,2,\ldots,n; \qquad j=1,2,\ldots,n+1.$$

Now arrange the second subindex with B_{i1} such that

$$\pi_{n+1}(B_{i1}) = \max_{1<j<n+1} \pi_{n+1}(B_{ij}), \qquad i=1,2,\ldots,n.$$

Define

$$B_i = \bigcup_{j=2}^{n+1} B_{ij}, \qquad i=1,2,\ldots,n$$

and

$$B_{n+1} = \bigcup_{i=1}^{n} B_{i1}.$$

For

$$1 \leqslant i \leqslant n, \qquad \pi_i(B_i) = \sum_{i=2}^{n+1} \pi_i(B_{ij}) = n\frac{1}{n+1}\pi_i(\tilde{B}_i) \geqslant \frac{1}{n+1}$$

and

$$\pi_{n+1}(B_{n+1}) = \sum_{i=1}^{n} \pi_{n+1}(B_{i1}) \geqslant \sum_{i=1}^{n} \frac{1}{n+1}\pi_{n+1}(\tilde{B}_i) = \frac{1}{n+1}\pi_{n+1}\left(\bigcup_{i=1}^{n} \tilde{B}_i\right) = \frac{1}{n+1}.$$

Now induction is complete.

Also solved by Leslie Arnold, Ethan Bolker, David Cantor, L. E. Clarke (England), Robert Field & Martin Ortel, Ellen Hertz, O. P. Lossers, (Netherlands), J. G. Mauldon, R. M. Norton, Henry Ricardo, Andrew Siegel, Stanford Statistics Problem Solving Group, J. G. Wendel, and the proposer.

Notes. (1) Mauldon shows with an example the necessity of the assumption that the measures π_i have the same family of measurable sets.

(2) The fair division problem and some recent references appear in A. M. Fink, *A note on the fair division problem*, Mathematics Magazine, vol. 37, p. 341.

(3) Bolker shows that there is a partition for $\pi_j(B_i) = 1/n$, $i=1,2,\ldots,n$; $j=1,2,\ldots,n$ so that "not only is each person satisfied, but each considers the whole partition fair."

(4) Ricardo points out that a generalization appears as corollaries 1.1, 1.2 in Dubins and Spanier, this MONTHLY, vol. 68, 1961, pp. 1 ff.

(5) Thurmon Whitley writes that the problem is a special case of Lemma 2 of *Relations among certain ranges of vector measures*, a paper by A. Dvoretsky, A. Wild, and J. Wolfowitz, which appeared in the Pacific Journal of Mathematics, 1951, p. 66. In fact, this lemma actually shows that the "greater than or equal" required in Problem 6143 can actually be "equality." Related results also appear in Whitley's Master's Thesis, "Some applications of vector-valued measures," University of North Carolina, 1966.

TELEGRAPHIC REVIEWS

Edited by J. Arthur Seebach, Jr. and Lynn A. Steen
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Collaborating Editor for Films: Seymour Schuster, Carleton College

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Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print.*

GENERAL, S, L**, *A Bibliography of Recreational Mathematics, V. 4.* William L. Schaaf. NCTM, 1978, xii + 172 pp, $9 (P). [ISBN: 0-87353-128-0] Update of *Volume 3* (TR, May 1974), following the same organization, including primarily references between 1972 and 1977. Includes a complete chronological list (Dec. '56-Aug. '77) of Martin Gardner's column in *Scientific American,* and a supplementary glossary of recreational terms. LAS

GENERAL, S(18), P, *Panorama des mathématiques pures. Le choix bourbachique.* Jean Dieudonné. Gauthier-Villars (US Distr: SMPF, 14 E. 60th St., NY 10022), 1977, xv + 302 pp, 150 F. [ISBN: 2-04-010012-1] An introductory (though high level) survey of contemporary mathematics as seen through the eyes of the Seminar Bourbaki. This volume offers capsule descriptions of many major achievements of contemporary mathematics, with notes on the principal discoverers and bibliographic information with particular emphasis on recent expository conferences. A good deal of effort has gone into indicating the relations among the achievements presented and between these achievements and science. This survey is in turn surveyed in the author's "Present trends in pure mathematics" in *Advances in Mathematics* 27 (1978) 235-255. JAS

GENERAL, P, *Transactions of the Moscow Mathematical Society, 1978, Issue 1.* AMS, 1978, iii + 280 pp, $48 (P). Translation of Volume 33 (1976). LAS

PRECALCULUS, T(13: 1), *College Algebra, Third Edition.* Thurman S. Peterson, Charles R. Hobby. Har-Row, 1978, xii + 384 pp, $13.95. [ISBN: 0-06-045161-0] Succinct and direct treatment of standard topics plus a chapter each on trigonometry, complex numbers, progressions, counting, probability, and statistics. (First Edition, TR November 1974.) LCL

PRECALCULUS, T(13: 1, 2), *Essential Algebra and Trigonometry.* Doris S. Stockton. HM, 1978, xi + 651 pp, $14.50. [ISBN: 0-395-25413-2]; *Instructor's Manual,* viii + 178 pp, $.70 (P). [ISBN: 0-395-25414-0] This text, a reworking of *Essential Precalculus,* provides a treatment of college algebra and trigonometry "for students who are not necessarily planning to take calculus." However, the book remains primarily a precalculus text. It is abstract in tone with relatively few practical examples and exercises. MU

PRECALCULUS, T(13: 2), *Lessons in College Algebra and Trigonometry.* Robert A. Nowlan. Har-Row, 1978, xv + 733 pp, $12.95. [ISBN: 0-06-044856-3] A larger-than-average text aimed at two audiences--precalculus students and more applied "terminal" students of economics and business. There are optional sections on rate of change if one has the luxury of a full year of college algebra and trigonometry. LLK

PRECALCULUS, T(13: 1), *Plane Trigonometry, A New Approach, Second Edition.* C.L. Johnston. P-H, 1978, xiii + 351 pp, $11.95. [ISBN: 0-13-677666-3] Usual topics in trigonometry with little to set it apart from the myriads of texts on the market. LLK

PRECALCULUS, T(13: 1), *Fundamentals of Trigonometry, Fourth Edition.* Earl W. Swokowski. Prindle, 1978, ix + 338 pp, $15.95. [ISBN: 0-87150-254-2] Fourth Edition (first three editions, TR October 1969, February 1972, and December 1975) of an unquestionably successful text. Rewriting reflects the trend toward more informal approach, additions include more exercises, applied problems, and optional exercises for hand-held calculators. LLK

PRECALCULUS, T(13), S, *Trigonometry, A Modern Approach.* Jack Ceder. HR&W, 1978, v + 223 pp, $11.95. [ISBN: 0-03-020901-3] Starts with a full discussion of the trigonometry of triangles, with many applications, then proceeds to introduce the circular function and its role. A brief chapter on logs and exponents. Finishes with vectors and complex numbers, stressing trigonometry applications. Abuncance of exercises, though almost all are rehash of examples. TLS

PRECALCULUS, T(13-14: 1), *Trigonometry.* Cameron B. Douthitt, Joe A. McMillian. McGraw, 1978, xi + 324 pp, $9.95 (P). [ISBN: 0-07-017670-1]; *Instructor's Manual to Accompany Trigonometry,* 118 pp, $3.50 (P). [ISBN: 0-07-017671-X] Each chapter begins with a list of objectives for the student together with correlated section numbers. Many examples and diagrams in a clear, attractive format. Large selections of problems at various levels of difficulty. Chapter review exercises. Three appendices: algebra review, logarithms, tables. Index. *Instructor's Manual* teaching suggestions, two forms of chapter tests, supplementary test questions, solutions to tests. RJA

FOUNDATIONS, P, *Algebra of Proofs.* M.E. Szabo. Stud. in Logic and Found. of Math., V. 88. North-Holland, 1978, xii + 297 pp, $43.50. [ISBN: 0-7204-2286-8] A study of algebraic properties of the proof theory of intuitionist first-order logic in a categorical setting. Based on confluence of ideas and techniques from proof theory, category theory and combinatory logic. GHM

FOUNDATIONS, T(15-16: 1). *Introduction to Set Theory.* Karel Hrbacek, Thomas Jech. Pure and Appl. Math., V. 45. Dekker, 1978, vi + 190 pp, $12.50. [ISBN: 0-8247-6570-2] Axiomatic (but not formalistic) development of the most rudimentary properties of sets and ordinal and cardinal numbers. Illustrates role of set theory in math by defining and deriving simple properties of integers, rational and real number systems. Similar in scope to classic text of Rotman and Kneebone, but a more leisurely, informal style. Very brief discussion of recent consistency and independence results. GHM

FOUNDATIONS, T(16-18: 1, 2), S, L. *Ensembles, Structures, Catégories, Faisceaux.* Serge Vasilach. Masson (US Distr: SMPF, 14 E. 60th St., NY 10022), 1977, x + 315 pp, $28.60 (P). [ISBN: 2-225-46849-4] A text for a sophisticated foundations or set theory course which reflects the current status of "foundations" by presenting more than set theory. The presentation is quite formal but thorough. It starts with an introduction to mathematical logic, then does set theory, structures, algebraic structures, categories, and then returns to a large block of set theory before touching on sheaves. JAS

FOUNDATIONS, T??(14-15: 1), S??. *A New Approach to Logic.* Robert Katz (240 Rockport, MA 01966), 1977, 336 pp, $12 (P). Republication of first twelve lessons in author's *Axiomatic Analysis* (Heath, 1964). A muddled hybrid of formal logic and practical advice about English, based on numerous ambiguous or meaningless definitions. Perhaps enjoyable as a form of poetry for self-indulgent (and uncritical) logicians or philosophers, but not as a serious text. GHM

COMBINATORICS, P. *Algorithmic Aspects of Combinatorics.* Ed: B. Alspach, P. Hell, D.J. Miller. Annals of Discrete Math., No. 2. North-Holland, 1978, vii + 245 pp, $41.50. [ISBN: 0-7204-1043-6] The papers presented (all invited) at the conference held in Vancouver, British Columbia, emphasizing graph-coloring algorithms and presenting a number of research questions. JAS

NUMBER THEORY, P. *Lecture Notes in Mathematics-627: Modular Functions of One Variable VI.* Ed: J.-P. Serre, D.B. Zagier. Springer-Verlag, 1977, 339 pp, $14.30 (P). [ISBN: 0-387-08530-0; 3-540-08530-0] The second and final volume of proceedings of the Conference on Modular Forms held in Bonn in July 1976. The first volume appeared as Lecture Notes 601. (Parts I, II, III, and IV of this "series" appeared as Lecture Notes 320, 349, 350, and 476.) JAS

LINEAR ALGEBRA, T*(14-16: 1, 2), L. *Applied Linear Algebra, Second Edition.* Ben Noble, James W. Daniel. P-H, 1977, xvii + 477 pp, $15.95. [ISBN: 0-13-041343-7] This book is a major rewriting of the 1969 first edition (TR, October 1969) which preserves the spirit of the first edition. However, the material on rank and generalized inverses has been sharply cut and combined with the material on applications of eigenvectors. The material on abstract vector spaces also has been cut and brought together earlier in the book. The chapter on "Norms and Error Estimates" appears to have disappeared altogether. The result is a noticeably shorter book which still emphasizes applications. JAS

ALGEBRA, P. *Lecture Notes in Mathematics-608: Groupes et Anneaux Réticulés.* Alain Bigard, Klaus Keimel, Samuel Wolfenstein. Springer-Verlag, 1977, xi + 334 pp, $14.30 (P). [ISBN: 0-387-08436-3; 3-540-08436-3] Self-contained development of the theory of lattice ordered groups and rings. Extensive bibliography. GHM

CALCULUS, T(13-14: 1-3). *Calculus with Analytic Geometry.* Mustafa A. Munem, David J. Foulis. Worth, 1978, xi + 1004 pp, $21.95. [ISBN: 0-87901-087-8]; *Study Guide*, ix + 368 pp, $3.95 (P). [ISBN: 0-87901-091-6] Standard contents. Much of material developed via hundreds of worked examples. Many excellent problems, a fair number giving applications to a wide variety of sciences. Considerable attention (too much?) given to precise statements of definitions and theorems, though in lax moments reverts to traditional muddle ("let x denote a variable number"). Two-tone printing, nice layout, many figures and graphs. Programmed supplementary workbook available. A pleasing text. GHM

CALCULUS, T(13-14: 1, 2). *Introduction to Calculus for the Biological and Health Sciences.* Rodney D. Gentry. A-W, 1978, xiv + 686 pp, $17.95. [ISBN: 0-201-02477-2] Distinguished from traditional "mainstream" calculus texts by greater emphasis on numerical techniques and discrete models, considerable attention to difference equations, and examples and exercises almost exclusively biologically or medically oriented. Integration is nicely motivated by applied problems whose solutions can be approximated by Riemann sums. (One section on areas comes only later.) Final chapters introduce differential equations and probability and statistics. GHM

REAL ANALYSIS, T(16-18: 1, 2), P, L. *The Bochner Integral.* Jan Mikusiński. Math. Reihe, B. 55. Birkhäuser, 1978, xii + 233 pp, sFr. 74. [ISBN: 3-7643-0865-0] A thorough investigation of integration theory of Banach-space valued functions based on a strikingly simple definition of the Lebesgue integral which employs stacks of "bricks" rather than the traditional horizontal slices. This definition, which avoids measure theory and convergence a.e., is simple enough to introduce in advanced calculus courses yet it extends immediately to the Bochner integral. LAS

REAL ANALYSIS, T(14-15: 1), L. *Introductory Mathematical Analysis.* I.J. Maddox. Pergamon Pr, 1977, xii + 327 pp, $18.75. [ISBN: 0-08-040000-0] Classical analysis (infinite series, continuity, differentiability, and integrability) including some discussion of numerical methods. The text integrates the language and basic notions of abstract structural mathematics (e.g., groups, linear spaces, homeomorphism, etc.). Detailed proofs, many illustrations, examples, and exercises (with hints). LCL

COMPLEX ANALYSIS, T(17: 1, 2). *Introduction to Functions of a Complex Variable.* J.H. Curtiss. Pure and Appl. Math., V. 44. Dekker, 1978, xv + 394 pp, $16.50. [ISBN: 0-8247-6501-X] An introductory text for a very rigorous course in complex analysis. Cauchy theory first developed for star domains. Extensive exposure to Runge's Theorem and its polynomial specializations, including applications of conformal mapping to polynomial approximations. Exercises. TRS

COMPLEX ANALYSIS, P. *Théorie des Fonctions Algébriques, Tome II: Fonctions Automorphes.* Paul Appell, Edouard Goursat, Pierre Fatou. Chelsea, 1978, xiv + 521 pp, $25. [ISBN: 0-8284-0299-X] Reprint, on long-life paper, of the 1930 original edition. The reprint of *Tome I* appeared earlier (TR, November 1977). LAS

DIFFERENTIAL EQUATIONS, P. *Sliding Modes and Their Application in Variable Structure Systems.* V.I. Utkin. Trans: A. Parnakh. MIR (US Rep: Four Continent Book Corp., 156 Fifth Ave., NY 10010), 1978, 257 pp. An exposition of the treatment of sliding modes as limits of differential equations with continuous coefficients. Over half the book is devoted to solving problems with the techniques developed in the first part. Contains an extensive bibliography. JAS

DIFFERENTIAL EQUATIONS, T(15-16: 1, 2), L. *Problemas de Ecuaciones Diferenciales Ordinarias.* M. de Guzman, Il Peral, M. Walias. Editorial Alhambra, 1978, x + 213 pp, (P). [ISBN: 84-205-0389-4] An introductory "How to Solve It" chapter inspired by George Pólya's work leads to two parallel parts: problems and outlines of solutions. The entire book is keyed to the theory presentation by the first author in his book "Ecuaciones diferenciales ordinarias. Teoria de estabilidad y control." The problems start with the basics but quickly get into substantial work at the level of fixed point theory and the Ascoli-Arzela theorem. JAS

DIFFERENTIAL EQUATIONS, T(14-15: 1), *Ordinary Differential Equations.* Tyn Myint-U. North-Holland, 1978, xii + 295 pp, $18.50. [ISBN: 0-444-00233-2] A solid treatment of the fundamental theory of ordinary differential equations: first and second order equations, Stürm separation theorems, Legendre and Bessel equations, linear systems, Green's functions, Stürm-Liouville theory, stability, Laplace transform and numerical methods. Few applications. Requires calculus and some matrix algebra. TRS

DIFFERENTIAL EQUATIONS, T(17-18: 1), S, P. *Lecture Notes in Mathematics-629: Dichotomies in Stability Theory.* W.A. Coppel. Springer-Verlag, 1978, 98 pp, $9 (P). [ISBN: 0-387-08536-X; 3-540-08536-X] Nine lectures given at the University of Florence in 1977 concerning the use of dichotomies in the study of the asymptotic behavior of nonautonomous differential equations. LCL

DIFFERENTIAL EQUATIONS, T(16-18: 1, 2), S, P. *Finite Element Galerkin Methods for Differential Equations.* Graeme Fairweather. Pure and Appl. Math., V. 34. Dekker, 1978, ix + 263 pp, $19.75 (P). [ISBN: 0-8247-6673-3] Methods for the approximate solution of two-point boundary value problems. Approximation properties of the most commonly used piecewise polynomial functions and elliptic boundary value problems. Parabolic and hyperbolic partial differential equations. Special emphasis on time-dependent problems. Chapter references. Index. RJA

DIFFERENTIAL EQUATIONS, T(17-18: 1, 2), P. *Hilbert Space Methods for Partial Differential Equations.* R.E. Showalter. Fearon-Pitman, 1977, xii + 196 pp, $25.75. [ISBN: 0-273-01012-3] Topics: Hilbert space, distributions, Sobolev spaces, boundary value problems, first order, second order and implicit evolution equations, optimization and approximation methods. For beginning graduate students of mathematics, engineering and physical science. Exercises. TRS

NUMERICAL ANALYSIS, T(14-17: 1, 2), S. *Numerical Analysis.* Richard L. Burden, J. Douglas Faires, Albert C. Reynolds. Prindle, 1978, ix + 579 pp, $19.95. [ISBN: 0-87150-243-7] Incorporates both traditional and current methods. "Real life" problems used to motivate techniques contained in specific chapters. Solutions of equations in one variable; interpolation and various approximations; numerical differentiation and integration. Linear and nonlinear systems of equations; matrix algebra. Integral- and boundary-value problems for ordinary differential equations, partial differential equations. Attractive format. More than 700 exercises. Selected answers. Bibliography. Index. RJA

NUMERICAL ANALYSIS. *Mathematical Models and Numerical Methods.* Ed: A.N. Tikhonov, *et al.* PWN, 1978, 391 pp. Papers by the participants in the special semester at the Stefan Banach International Mathematical Center, February to June 1975. Mostly in Russian. JAS

NUMERICAL ANALYSIS, T(14-17: 1, 2), S, L. *Numerical Analysis for Computer Science.* Irving [illegible]. North-Holland, 1978, xviii + 618 pp, $18.95. [ISBN: 0-444-00238-3] Begins with a thorough discussion of error analysis. Content is built around nine concerns: deriving a formula from a table of values and vice versa, solving equations and systems of equations, matrix problems, summing efficiently, numerical integration, numerical solutions of ordinary differential equations and systems of ordinary differential equations. Details mathematical background for each problem; emphasis on most accurate computer methods; some complete Fortran programs. Problem sections throughout each chapter. Tables. Bibliography. Answers to numerical problems. Index. RJA

NUMERICAL ANALYSIS, T(15-16: 1, 2), S, L. *Applied Numerical Analysis, Second Edition.* Curtis F. Gerald. A-W, 1978, x + 557 pp, $17.95. [ISBN: 0-201-02696-1] Emphasis on applications. Wide selection of topics including elliptic, parabolic, and hyperbolic partial differential equations; also curve-fitting, splines, and approximation of functions. Extensive exercise sets grouped according to text sections plus special sets of applied problem/projects. Selections of Fortran programs are grouped together at the end of each chapter. Topic treatment is extensive but informal. Nice layout and diagrams. Bibliography: appendices; answers to selected exercises; index. RJA

FUNCTIONAL ANALYSIS, S(16-17), P, L. *Integral Transforms and Their Applications.* B. Davies. Appl. Math. Sci., V. 25. Springer-Verlag, 1978, xii + 411 pp, $14.80 (P). [ISBN: 0-387-90313-5; 3-540-90313-5] Intended to serve as introductory and reference material for the application of integral transforms to mathematical problems in physical science and engineering. Detailed studies of the Laplace, Fourier, Mellin and Hankel transforms and of transforms generated by Green's functions. Complex variable methods are stressed. Exercises. TRS

FUNCTIONAL ANALYSIS, P. *Kernels and Integral Operators for Continuous Sums of Banach Sapces.* Irwin E. Schochetman. Memoirs No. 202. AMS, 1978, v + 120 pp, $8 (P). [ISBN: 0-8218-2202-0]

FUNCTIONAL ANALYSIS, P, *Locally Solid Riesz Spaces.* Charalambos D. Aliprantis, Owen Burkinshaw. Pure and Appl. Math., V. 76. Acad Pr, 1978, xii + 198 pp, $19.50. [ISBN: 0-12-050250-X] A study aimed at exposing the relationships between topological structure and order structure in vector lattices. Exercises. TRS

FUNCTIONAL ANALYSIS, P, *Saks Spaces and Applications to Functional Analysis.* James Bell Cooper. Math. Stud., V. 28. North-Holland, 1978, x + 325 pp, $30.50 (P). [ISBN: 0-444-85100-3] A monograph which provides a synthesis of two studies in functional analysis: (1) Saks spaces, i.e., vector spaces endowed with a norm and with a related locally convex topology, and (2) various topologies on spaces of functions and operators. TRS

OPTIMIZATION, P, *Optimal Control and Differential Equations.* Ed: A.B. Schwarzkopf, Walter G. Kelley, Stanley B. Eliason. Acad Pr, 1978, xi + 335 pp, $17. [ISBN: 0-12-632250-3] Papers from a conference held in March 1977 at the University of Oklahoma, including several of an historical or survey nature. LAS

OPTIMIZATION, P, *Studies in Integer Programming.* Ed: P.L. Hammer, *et al.* Annals of Discrete Math., No. 1. North-Holland, 1977, vii + 562 pp, $48.95. [ISBN: 0-7204-0765-6] The proceedings of the Workshop on Integer Programming held at Bonn, September 8-12, 1975. JAS

ANALYSIS, S(16), L, *Exercices résolus d'analyse.* Jacqueline Lelong-Ferrand. Dunod (U.S. Distr: SMPF, 14 E. 60th St., NY 10022), 1977, vi + 301 pp, 59F (P). Problems and solutions in analysis, including sequences, topology of metric spaces, derivatives, integrals, series and differential equations. The solutions immediately follow each problem. JEG

ANALYSIS, P, *On Representation of Functions by Means of Superpositions and Related Topics.* A.G. Vitushkin. L'Enseignement Math, 1978, 68 pp, Frs. 18 (P). Summary of a series of lectures at UCLA, May 1977. First chapter gives extensive history of the problem, with many references, and notes current problems. Later chapters discuss various problems arising in smooth and continuous functions and with linear superpositions. TLS

ANALYSIS, P, *Nonlinear Analysis: A Collection of Papers in Honor of Erich H. Rothe.* Ed: Lamberto Cesari, Rangachari Kannan, Hans F. Weinberger. Acad Pr, 1978, xiii + 238 pp, $24.50. [ISBN: 0-12-165550-4]

ANALYSIS, P, *The Motion of a Surface by Its Mean Curvature.* Kenneth A. Brakke. Princeton U Pr, 1978, 239 pp, $9 (P). The study of families of surfaces parametrized by time such that each point at each time is moved with a velocity equal to the mean curvature vector of the surface at that point at that time. JEG

ANALYSIS, P, *Lie Theories and Their Applications.* Ed: A.J. Coleman, P. Ribenboim. Pure and Appl. Math., No. 48. Queen's U, 1978, 577 pp, (P). The short courses, seminar, lectures and research announcements from the June 20 to July 8, 1977 seminar of the Canadian Mathematical Congress held at Queen's University in Kingston, Ontario. JAS

ANALYSIS, P, L, *Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer Programs.* Harold Exton. Wiley, 1978, 316 pp, $37.50. [ISBN: 0-470-26342-3] Seven chapters of general theory and examples of applications in statistics and mathematical physics, followed by about 100 pages of tables of the formulae and about 50 pages of computer programs (Fortran) to evaluate the integrals. Large bibliography. LCL

DIFFERENTIAL GEOMETRY, T(16-17: 1), S, L, *A Course in Differential Geometry.* Wilhelm Klingenberg. Trans: David Hoffman. Grad. Texts in Math., V. 51. Springer-Verlag, 1978, xii + 178 pp, $14.80. [ISBN: 0-387-90255-4; 3-540-90255-4] A mathematically mature introduction to curves and surfaces. It is only at the end that homotopy theory and point set topology are needed at all but multivariable analysis, linear algebra, and Euclidean geometry are assumed. The translator, David Hoffman, is also responsible for a more relaxed format compared to the "tightly constructed" German edition which was an outgrowth of the author's one-semester lecture course. JAS

DIFFERENTIAL GEOMETRY, S(18), P, *Lie Groups: History, Frontiers and Applications, V. V: Symplectic Geometry and Fourier Analysis.* Nolan R. Wallach. Math Sci Pr, 1977, xvii + 436 pp, (P). "Lightly edited" notes from a 1975 course at Rutgers which presents recent developments inspired by the growing mathematization of quantum mechanics, in particular the work of Kirillov. JAS

DIFFERENTIAL GEOMETRY, S(16-18), *The Minkowski Multidimensional Problem.* Aleksey Vasil'yevich Pogorelov. Trans: Vladimir Oliker, Louis Nirenberg. Wiley, 1978, v + 106 pp, $13.75. [ISBN: 0-470-99358-8] A beautiful monograph, showing the tremendous power of geometric techniques. The book gives a complete exposition of the Minkowski problem (on the existence of a hypersurface with prescribed Gaussian curvature) with both generalized and regular solutions, then discusses various off-shoots of the problem, i.e., replacing the Gaussian curvature with other curvature forms. Seems ideal for outside reading in a Riemannian geometry course. TLS

DIFFERENTIAL GEOMETRY, T(16-17), S, *Differentialgeometrie.* Rolf Walter. Bibliographisches Institut, 1978, iii + 278 pp, (P). [ISBN: 3-411-01543-8] An introduction to modern differential geometry. Assumes some knowledge of differentiable manifolds but not of fiber bundles. Some problems. JD-B

GEOMETRY, P, *Geometry of Electromagnetism.* Robert de Boer (6041 Charles St., Halifax, Nova Scotia, Canada B3K 1K9), 1977, 81 pp, $10 (P). The geometry is that of a homography transformation on the category of projective spaces with two homogeneous coordinates taken from an arbitrary associative ring with 1. The development of these geometries is shown to be relevant to the study of Maxwell's equations. JAS

STATISTICS, T(13: 1), *Elementary Statistical Methods, Revised Edition.* James Lumsden. U of Australia Pr (US Distr: ISBS, Inc., P.O. Box 555, Forest Grove, OR 97116), 1974, xiv + 166 pp, (P). [ISBN: 0-85564-031-6] Reprinted version of 1974 edition (TR, November 1976) with minor alterations. See also review of first edition (TR, April 1972). LCL

STATISTICS, T(14-15: 1, 2), *Introduction to Probability and Statistics, Fourth Edition.* B.W. Lindgren, G.W. McElrath, D.A. Berry. Macmillan, 1978, xii + 356 pp, $13.95. [ISBN: 0-02-370900-6] Content is similar to that of the 1969 *Third Edition*, but the material has been greatly reorganized and almost completely rewritten. Includes optional chapters on the Poisson process, Bayesian inference, and sequential testing. Although the use of calculus is minimal, mathematical sophistication is required. RSK

STATISTICS, S**(13-18), *Statistics Tables for Mathematicians, Engineers, Economists and the Behavioural and Management Sciences.* H.R. Neave. Allen & Unwin, 1978, 87 pp, $3.75 (P). [ISBN: 0-04-001001-5] A very useful and reasonably priced collection of tables to use in statistics courses. FLW

STATISTICS, T(13-14: 1, 2), L, *Statistical Techniques in Business and Economics, Fourth Edition.* Robert D. Mason. Irwin, 1978, xx + 596 pp, $15.50. [ISBN: 0-256-02025-6] Extensive coverage of descriptive statistics, index numbers, time series analysis; probability, inferences concerning means, analysis of variance, simple and multiple regression and correlation, nonparametrics. Coherent and thorough; well-chosen, meaningful examples and exercises. LCL

STATISTICS, T(16-18), S*, P, L, *Specification Searches: Ad Hoc Inference with Nonexperimental Data.* Edward E. Leamer. Wiley, 1978, xiii + 370 pp, $24.95. [ISBN: 0-471-01520-2] A Bayesian-based "metastatistical" theory of inferences actually drawn from data, especially in those common situations in which empiricists employ "number crunching" programs to search one model after another for a hypothesis that produces "statistically significant" results. Special emphasis on the linear regression model and economic examples. Many references, but very few problems. LAS

STATISTICS, S, *Introduction to the Use of Computer Packages for Statistical Analyses.* Richard W. Moore. P-H, 1978, xii + 115 pp, $9.95 (P). [ISBN: 0-13-480970-X] Simple non-statistical introduction to the use of three major statistical computer packages: Statistical Analysis System (SAS), Statistical Package for the Social Sciences (SPSS), and BMD Biomedical Computer Programs (BMD). Designed for the apprehensive who have some knowledge of statistics but no computer experience. RSK

STATISTICS, T?(13: 1), *Basic Statistics for Health Science Students.* David S. Phillips. Freeman, 1978, xiii + 185 pp, $10; $5.50 (P). [ISBN: 0-7167-0051-4; 0-7167-0050-6] Listing, with scanty elaboration, of major definitions used in statistics together with cookbook descriptions of special inferential procedures. Twenty-six routine exercises. No pretense to developing even minimal understanding of concepts. LCL

STATISTICS, P, *Markov Decision Theory.* Ed: H.C. Tijms, J. Wessels. Math. Centre Tracts, No. 93. Math Centrum, 1977, 220 pp, Dfl. 26 (P). [ISBN: 90-6196-160-2] Proceedings of a seminar held at the University of Amsterdam, September 13-17, 1976. JAS

COMPUTER PROGRAMMING, T*(13-18), S*, L, *A FORTRAN Coloring Book.* Roger Emanuel Kaufman. MIT Pr, 1978, [illegible] pp, [illegible].95 (P). [ISBN: 0-262-61026-4] A lively approach to seduce,/In manner quite simlar to Seuss./ Handwritten, with drawings and lots of guffawings/Disposed to make Fortran transluce. LCL

COMPUTER PROGRAMMING, T(13), S, *Programming for Poets: A Gentle Introduction Using PL/1.* Richard Conway. Winthrop Pub, 1978, xiv + 347 pp, $10.95 (P). [ISBN: 0-87626-724-X] A book without programming for the typical "computer appreciation" course, which gets at the what, how, and why of computer programs by supervising the reading of selected programs rather than the writing of programs. LCL

COMPUTER PROGRAMMING, T, S*, L, *Etudes for Programmers.* Charles Wetherell. P-H, 1978, viii + 200 pp, $12.95 (P). [ISBN: 0-13-291807-2] Programming projects of intermediate difficulty for beginners--including computer simulation, game-playing and artificial intelligence. Each etude contains background information, careful problem description, and suggestions for solution, including references. LCL

COMPUTER PROGRAMMING, T(13-18: 1), *Basic, Second Edition.* Robert L. Albrecht, LeRoy Finkel, Jerald R. Brown. Wiley, 1978, ix + 325 pp, $4.95 (P). [ISBN: 0-471-03500-9] A self-teaching text. Each chapter begins with a list of specific objectives. Material is presented in short frames, each of which concludes with a question or task to be performed and is followed immediately by the answer or appropriate solution. Chapters conclude with self-tests and their solutions. Contains a final self-test with solutions and an index. The content is similar to Dartmouth Basic. RJA

COMPUTER PROGRAMMING, S, *Basic and the Personal Computer.* Thomas Dwyer, Margot Critchfield. A-W, 1978, x + 438 pp, $12.95 (P). [ISBN: 0-201-01589-7] Informal presentation of many topics for an interested person with no previous experience with computers. Project ideas are included. Topics include microcomputer hardware, microcomputer programming in Basic and extended Basic, computer graphics, word processing, data structures, sorting algorithms, computer games, computer art, simulations, and color graphics. LLK

COMPUTER SCIENCE, T(13-18: 1, 2), S, L, *Introduction to Computers and Computer Science, Second Edition.* Richard C. Dorf. Boyd and Fraser, 1977, xi + 650 pp, $13.95. [ISBN: 0-87835-061-6] Aim is to introduce the use of algorithms and computers to solve problems. Emphasis on algorithmic languages, Basic and Fortran. Development of computers and computer organization. Data processing; data banks and information retrieval; simulations and games. Computers in government and in the arts. Cybernetics, artificial intelligence, and the social impact of computers. Chapter problems and references. Glossary and index. RJA

COMPUTER SCIENCE, S(13-18). *Microprocessor Programming for Computer Hobbyists*. Neill Graham. TAB Books, 1977, 382 pp, $12.95; $8.95 (P). [ISBN: 0-8306-7952-9; 0-8306-6952-3] Discusses techniques of programming and data structuring valuable in using a microprocessor in a variety of application areas. Text consists of six parts: number systems, programming in an extension of PL/M, arithmetic using micros, data structures, searching, and sorting. All topics are oriented toward the microprocessing environment. Short bibliography. Index. RJA

COMPUTER SCIENCE, T(15-17: 1). *Automata Theory: An Engineering Approach*. Igor Aleksander, F. Keith Hanna. Crane Russak, 1975, xi + 172 pp, $15.50. [ISBN: 0-8448-0657-9] Intended as a pragmatic approach to automata theory. Sequential machines, state partitions, introductory identification, control and diagnostic techniques. Learning automata. Relationships with mathematical linguistics. No exercises. RWN

COMPUTER SCIENCE, T(15-18: 1, 2), S, P, L. *Problems, Programs, Processing, Results: Software Techniques for Sci-Tech Programs*. Pál Quittner. Crane Russak, 1977, 381 pp, $27.50. [ISBN: 0-8448-1085-1] Directed toward application programmers and systems analysts. Written from the viewpoint that a serious programmer needs more than a "black box" understanding of hardware and system software. Examples of important computer applications; the translating and loading process; computer organization and machine code. Assembly languages and assemblers; high-level programming languages and compilers; loaders; operating systems. Program testing; documentation; data organization; data structures. Excellent chapter references. Glossary. Index. RJA

COMPUTER SCIENCE, T(13-18: 1), S. *A Step by Step Introduction to 8080 Microprocessor Systems*. David L. Cohn, James L. Melsa. Dilithium Pr (Exclusive Dist r: ISBS, P.O. Box 555, Forest Grove, OR 97116), 1977, xii + 169 pp, $7.95 (P). [ISBN: 0-918398-04-5] A concise, easy-to-read introduction to a very popular microprocessor. Many basic computer organization and system software concepts are presented in the specific context of the 8080 system. Text takes the myriad of details usually associated with any computer system and organizes them in an attractive, straightforward style. Contains a comparison with other micros. Exercises. Appendices. Index. RJA

COMPUTER SCIENCE, T(17-18), P. *Denotational Semantics: The Scott-Strachey Approach to Programming Language Theory*. Joseph E. Stoy. MIT Pr, 1977, xxx + 414 pp, $19.95. [ISBN: 0-262-19147-4] First book-length exposition of the "denotational" (or "mathematical" or "functional") approach to the formal semantics of programming languages (in contrast to "operational" and "axiomatic" approaches). Treats various kinds of languages, beginning with the pure λ-calculus and progressing through languages with states, commands, jumps and assignments. This somewhat discursive account is a valuable compilation of results not otherwise available in a single source. GHM

COMPUTER SCIENCE, T(15-18: 1, 2), S, P, L. *Content Addressable Parallel Processors*. Caxton C. Foster. Van N-Rein, 1976, xiii + 233 pp, $11.95. [ISBN: 0-442-22433-8] A Content Addressable Parallel Processor (CAPP) is a computer (1) capable of determining simultaneously all memory cell containing data that matches specific criteria broadcast from the central control unit and (2) capable of processing simultaneously all memory cells that matched the criteria. Includes overview of the potential of CAPPs; examples, theory, and applications of CAPPs. Algorithms for parallel processing; distributed hardware functions; STARAN. Bibliography. Index. RJA

COMPUTER SCIENCE, S(16-18), P. *Program Behavior: Models and Measurements*. Jeffrey R. Spirn. Elsevier Sci Pub, 1977, x + 277 pp, $9.95 (P); $17.95. [ISBN: 0-444-00220-0; 0-444-00219-7] Primary theme is that models require measurement for validation and measurement requires models for interpretation and application. Heavy emphasis on the methodology of measurement and statistical characterization of the program behavior model. Includes the principle of locality, paging systems, LRU stack model, spectral analysis, semi-Markov page fault model, Markovian distance string model, summary of various models and measurement methods. Appendix. Bibliography. Index. RJA

COMPUTER SCIENCE, S(13-18). *Programming Microprocessors*. M.W. Mcmurran. TAB Books, 1977, 279 pp, $6.95 (P). [ISBN: 0-8306-6985-X; 0-8306-7985-5] Microprocessor organization; programming micros; arithmetic in micros. Specific configurations of the Motorola M6800, Intel 8080, and Rockwell PPS-8. Characteristics and fabrication of MOS and I^2L components. Bibliography. Several appendices of tables. Index. RJA

COMPUTER SCIENCE, S(13-18). *The "Computator" Book--Building Super Calculators & Minicomputer Hardware with Calculator Chips*. R.P. Haviland. TAB Books, 1977, 320 pp, $10.95; $7.95 (P). [ISBN: 0-8306-7975-8; 0-8306-6975-2] Contains detailed information on the design of calculator chips. A storehouse of ideas and actual plans for using calculator chips in diverse application areas: display building, large calculator designs, counting, keeping time, measure frequencies, dial telephones, electronic locks, signal converting, etc. Many detailed circuit drawings included. Very useful chapter references. Index. RJA

COMPUTER SCIENCE, T(16-17: 1), S, P. *Modern Methods for Computer Security and Privacy*. Lance J. Hoffman. P-H, 1977, xiii + 255 pp, $17.95. [ISBN: 0-13-595207-7] Methods for authentication, authorization, maintenance of logs and encryption. Privacy transformations. Influence of architecture and operating system design on security. "Statistical" data banks, math models, legislation and areas for research. RWN

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; John Dyer-Bennet, Carleton; Jay E. Goldfeather, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Milton Ulmer, Carleton; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D.C. 20036.

AMS SHORT COURSE ON GAME THEORY AND ITS APPLICATIONS

The American Mathematical Society in conjunction with its eighty-fifth annual meeting in Biloxi, Mississippi, will continue its short course series with a course entitled "Game Theory and its Applications." The one and one-half day course will be held on Monday and Tuesday, January 22 and 23, 1979, at the Holiday Inn in Biloxi.

Game theory is a collection of mathematical models designed to study situations involving conflict and/or cooperation. It allows for a multiplicity of decision makers who may have different preferences and objectives. Such models involve a variety of different solution concepts concerned with strategic optimization, stability, bargaining, compromise, equity and coalition formation. This short course will be primarily concerned with the n-person theory ($n \geq 3$). It will emphasize the cooperative models, but the fundamental aspects of the noncooperative approach will also be covered. The applications will include auctions, bidding, and market equilibria in economics; measurements of power in political science; as well as some multiperson and equity considerations in operations research.

The program is under the direction of William F. Lucas of the Center for Applied Mathematics and School of Operations Research and Industrial Engineering at Cornell University, and will consist of six seventy-five minute lectures.

Participants may preregister for the short course until December 22, 1978 for $18; a special preregistration fee of $3 has been set for students and unemployed individuals. After the preregistration deadline, the fees will be increased to $20 and $5, respectively. Time schedules and further information about registration and accommodations can be found in the October and November 1978 and January 1979 issues of the Notices, or may be obtained by writing or calling the American Mathematical Society's Meeting Arrangements Department, P. O. Box 6887, Providence, Rhode Island 02940; Telephone (401)272-9500, Ext. 239.

AAAS ANNUAL MEETING

Washington, D.C. ... The 145th national meeting of the American Association for the Advancement of Science (AAAS) will be held 3-8 January 1979 in Houston, Texas. Built around the theme "Science and Technology: Resources for our Future," the meeting will feature 140 symposia looking at science and technology from numerous perspectives.

Ten public lectures dealing with popular aspects of science and the AAAS Science Film Festival, featuring over 40 of the best short science films produced over the last year, will be included in the meeting program.

The fifth annual SCIENCE INTERNATIONAL exhibit of scientific instruments and publications will again be a part of the meeting, opening the second day of the AAAS meeting (4 January) and running through Sunday (7 January).

Continuing its commitment to the handicapped in science, the AAAS will again make its meeting accessible to people in wheelchairs and other persons with physical disabilities. The AAAS Project on the Handicapped in Science will operate a "Handicapped Resource Center," where handicapped meeting attendees can get assistance.

Further information about the meeting is available from the AAAS Meetings Office, 1776 Massachusetts Avenue, N.W., Washington, D.C. 20036. A preliminary program for the meeting will appear in the 29 September issue of SCIENCE, the weekly AAAS journal.

ROBERT W. FLOYD TO RECEIVE 1978 A.M. TURING AWARD FROM ASSOCIATION FOR COMPUTING MACHINERY

Robert W. Floyd of Stanford University has been named the recipient of ACM's Turing Award for 1978. Professor Floyd will deliver the A.M. Turing Lecture at the opening session of the 1978 ACM Annual Conference on December 4 at the Sheraton Park Hotel in Washington, D.C.

The Turing Award is presented in commemoration of Dr. A.M. Turing, an English mathematician who made many important contributions to the field of computing. It is ACM's most prestigious award for technical contributions to the computing community and carries a remuneration of $2,000.

CONFERENCE ON NUMERICAL ANALYSIS

Conference on the Numerical Analysis of Semiconductor Devices (NASECODE 1) June 27-29, 1979. Venue - Trinity College, University of Dublin, Dublin, Ireland. Subject - The use of numerical methods and computational techniques for the modelling, analysis and design of semiconductor devices. Contributed papers (3 page maximum) and 200-word abstracts, 3 copies of each to either Prof. J.J.H.

Miller or Dr. B. Browne, Numerical Analysis Group, Trinity College, Dublin, Ireland. Deadline 16 March 1979.

ASSOCIATION FOR PROMOTION OF MATHEMATICS EDUCATION OF GIRLS AND WOMEN

A new organization concerned with increasing the participation of girls and women in the study of mathematics was formed during the National Council of Teachers of Mathematics annual meeting in April, 1978. The major goal of the Association for the Promotion of the Mathematics Education of Girls and Women (APMEG&W) is to take positive action aimed at reversing the trend of avoidance of mathematics among girls and women in elementary and secondary schools. Membership dues are $2.00. The address is: APMEG&W, c/o Education Department, George Mason University, 4400 University Drive, Fairfax, VA 22030.

SHORT COURSE ON MATH ANXIETY

The mathematical community has given considerable attention to the problems caused by the "fear" of taking mathematics courses. Some institutions have directed their attention to dealing with this problem through panel discussions, student counselling projects, and symposia of various types.

One such approach was a short course, entitled "Dealing With Math Anxiety," conducted at the University of Nebraska at Omaha (UNO), September 26-October 31, 1978. The instructors were Dr. John Konvalina, Assistant Professor of Mathematics and Computer Science at UNO, and Dr. Kay Hood, Director of Women's Support Programs at UNO.

The course had the following objectives:

(1) Establish a non-threatening and supportive environment to learn and do basic mathematics.
(2) Discuss the nature and causes of math anxiety and avoidance.
(3) Reduce math anxiety by increasing confidence level.
(4) Establish a firm foundation in basic mathematics.
(5) Improve skills in arithmetic and beginning algebra.

The content consisted of six modules:

MODULE I *The psychological aspects of math anxiety*
MODULE II *Whole Numbers—basic properties of whole numbers*
MODULE III *Integers—basic properties of signed numbers*
MODULE IV *Rationals—fractions and mixed numbers*
MODULE V *Some Consumer Math—applications of decimals and percent*
MODULE VI *Introduction to Algebra—solving equations*

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

AMS-MAA-SIAM CONGRESSIONAL SCIENCE FELLOWSHIP FOR 1979-80

Applications are invited from candidates in the mathematical sciences for a Congressional Science Fellowship to be supported jointly by the American Mathematical Society, the Mathematical Association of America and the Society for Industrial and Applied Mathematics for the twelve-month period begining 1 September 1979. The AMS-MAA-SIAM Fellow will serve, along with several Fellows selected by the American Association for the Advancement of Science and around a dozen Fellows sponsored by other scientific societies, under an annual program coordinated by AAAS. The stipend for the 1979-80 AMS-MAA-SIAM Fellowship is $18,500, which may be supplemented by a small amount toward relocation and travel expenses. It may also be supplemented by sabbatical salary or other employer contribution in the case of a person on sabbatical leave for the 1979-80 year.

The AMS-MAA-SIAM Congressional Science Fellowship was awarded for the first time in 1978-79, to Dr. Edmund Gregory Lee, Assistant Professor of Mathematics at Fordham University, Bronx, N.Y. Dr. Lee, who is 30 years old, is a graduate of Reed College and received his doctorate in mathematics from the Massachusetts Institute of Technology in 1975. The 1 September 1978 issue of *Science* gives a brief description of the overall Congressional Science Fellow Program and the Fellows for 1978-79. As indicated there, the Fellows spend their fellowship year working on the staff of an individual congressman or a congressional committee or in the congressional Office of Technology Assessment, the objective of the program being to enhance science-government interaction, the effective use of science in government, and the training of persons with scientific background for careers involving such use. Based on information on available congressional staff positions gathered by the AAAS during the summer, each Fellow's assignment is worked out by the Fellow and the congressional office concerned following an intensive two-week orientation and interview procedure organized by the AAAS during which

the Fellows encounter many facets of Congress, the Executive Branch, and people and organizations on the Washington scene. The AAAS provides advice and assistance during the process and remains in frequent and regular contact with all the Fellows throughout the fellowship year. More detailed information about the program as a whole may be requested from AAAS Congressional Science Fellow Program, 1776 Massachusetss Avenue, N.W., Washington, D.C. 20036; telephone (202) 467-4475.

The AMS-MAA-SIAM Congressional Science Fellowship is to be awarded competitively to a mathematically trained person at the postdoctoral to mid-career level without regard to sex, race, or ethnic group. Selection will be made by a panel of the AMS-MAA-SIAM Joint Projects Committee for Mathematics, a nine-member committee consisting of three representatives from each of these organizations, with the cooperation and advice of the overall AAAS Program. *Applications should be sent to the Conference Board of the Mathematical Sciences, 1500 Massachusetts Avenue., N.W., Suite 457-458, Washington, D.C. 20005. The deadline for receipt of applications is 15 February, 1979,* and it is anticipated that the award will be made by around 1 April 1979.

In addition to demonstrating exceptional competence in some areas of the mathematical sciences, an applicant for the AMS-MAA-SIAM Congressional Science Fellowship should have a rather broad scientific and technical background and a strong interest in the uses of the mathematical and other sciences in the solution of societal problems. He or she should also be articulate, literate, flexible and able to work effectively with a wide variety of people. An application should state why the applicant wants to be a Congressional Science Fellow, should summarize his or her qualifications, and should be accompanied by a resumé. Also, CBMS should receive by 15 February 1979 three letters from knowledgeable persons about the applicant's competence and suitability for the award.

SUGGESTIONS FOR PROGRAMS FOR ANNUAL MEETINGS

Although the request made in this article may be somewhat late for the coming annual meeting, and even though a letter went to all members of the Association asking for input, the following letter from Professor K. B. Reid bears publication for its significance in future years as well as for the present:

Dear Colleagues:

President Alder has appointed me to serve as Chairman of the Program Committee for the Sixty-Second Annual Meeting of the Association to be held in Biloxi, Mississippi from Friday, January 26, through Sunday, January 23, 1979. The Committee has received a few suggestions for the program, but we would like to solicit more in view of the fact that good suggestions not utilized will be passed on to future Program Committees. We are seeking to identify exceptionally stimulating apeakers, perhaps ones who stand out in your memory from recent sectional meetings. In particular, we seek mainly speakers who have not had much national exposure, but have shown to be speakers comparable to past Association speakers. Of course, we expect more suggestions than we can use for the Biloxi meeting, but we will give careful consideration to your responses. I would like to ask that you don't merely send names and addresses, but communicate some of your knowledge concerning the person's qualities as a lecturer and make suggestions for possible topics for their lecture topics, mathematical or educational, which would be appropriate for talks or series of talks at Biloxi or subsequent meetings. Are there gaps in the content of recent MAA programs that we should try to fill? Please mail your suggestions to

K. B. Reid
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803

CALENDAR OF FUTURE MEETINGS

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.
Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21–23, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.
FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.
ILLINOIS, first Friday/Saturday in May.
INDIANA
INTERMOUNTAIN
IOWA, third weekend in April. Deadline for papers February 1.
KANSAS, Johnson County Community College, Overland Park, April 7, 1979
KENTUCKY, early April. Deadline for papers 6 weeks before meeting.
LOUISIANA–MISSISSIPPI, Friday–Saturday before February 20. Deadline for papers 3 months before meeting.
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.
METROPOLITAN NEW YORK, Adelphi University, May 5, 1979.
MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.
MISSOURI, late March/early April. Deadline for papers January 31.
NEBRASKA, April.
NEW JERSEY, St. Peter's College, Englewood Cliffs, November 4, 1978.
NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.
NORTHEASTERN, Bunker Hill Community College, Charlestown, Massachusetts, November 18, 1978.
NORTHERN CALIFORNIA, first or second Saturday in February.
OHIO, Miami University, Middletown, April 20–21, 1979.
OKLAHOMA–ARKANSAS, Oklahoma State University, Stillwater, March 30–31, 1979.
PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.
PHILADELPHIA, Millersville State College, Millersville, Pennsylvania, November 18, 1978.
ROCKY MOUNTAIN, University of Denver, Denver, spring 1979.
SEAWAY, University of Rochester, New York, November 10–11, 1978.
SOUTHEASTERN, University of Tennessee, Chattanooga, spring 1979.
SOUTHERN CALIFORNIA, first or second Saturday in March.
SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.
TEXAS, Friday and Saturday in early April. Deadline for papers March 1.
WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Houston, Texas, January 3–8, 1979.
AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES
AMERICAN MATHEMATICAL SOCIETY, Biloxi, Mississippi, January 24–27, 1979.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION
ASSOCIATION FOR COMPUTING MACHINERY, Sheraton Park Hotel, Washington, D.C., December 4–6, 1978.
ASSOCIATION FOR SYMBOLIC LOGIC, Biloxi, Mississippi, January 24–25, 1979.
ASSOCIATION FOR WOMEN IN MATHEMATICS
CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS, Washington, D.C., August 13–16, 1979.
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.
OPERATIONS RESEARCH SOCIETY OF AMERICA, Bonaventure Hotel, Los Angeles, California, November 12–16, 1978.
PI MU EPSILON
SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Hotel Fort Des Moines, Des Moines, Iowa, November 2–4, 1978.
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Hyatt Regency Hotel, Knoxville, Tennessee, October 30–November 1, 1978.

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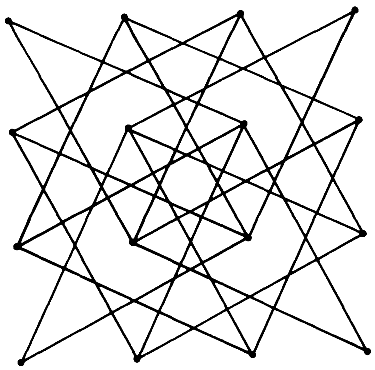

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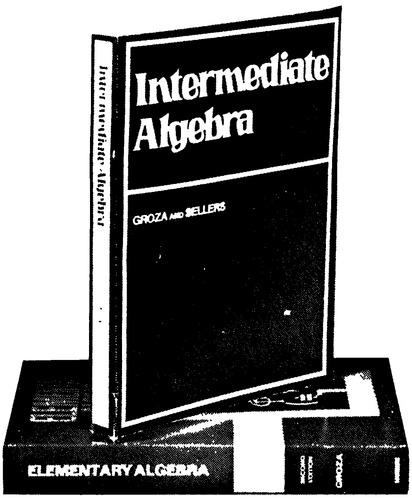

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p. 231, Efron, second line. For "mitigate," read "militate."

p. 354, after "U.S. Military Academy, at West Point," add "with additional financial support from the Office of Naval Research."

p. 473, second line from below. Read "$4M/L$" for "$4ML$."

p. 474, fifth line. Read "n^2M/L" for "n^2ML."

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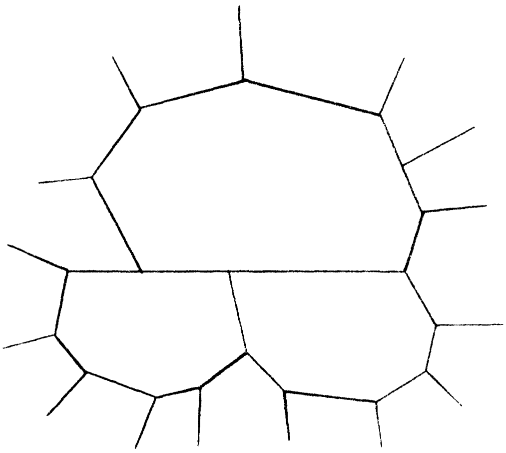

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1978

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CONVEX POLYGONS THAT CANNOT TILE THE PLANE

IVAN NIVEN

1. Introduction. The primary purpose of this paper is to prove the following result.

THEOREM. (a) *If α and β are any positive real numbers, it is impossible to tile the plane with any collection of convex polygons each of which has 7 or more sides, area greater than α, and perimeter less than β.* (b) *If any one of the conditions on the polygons is removed, then it is possible to tile the plane with polygons meeting the other requirements.*

It should be noted that in part (a) there is no congruence requirement whatsoever on the polygons. By "any collection" we mean just that, allowing for any set of convex polygons whatsoever that meets the specifications. In particular, this implies that it is impossible to tile the plane with any collection of polygons that consists of congruent copies of a finite number of distinct convex polygons, each having 7 or more sides. In part (b) there are four conditions referred to: convexity; 7 or more sides; area bounded below; perimeter bounded above.

Tiling the plane with polygons means covering it without gaps or overlaps, except for the common boundaries of the polygons.

The proof of part (a) of the theorem given in §3 shows something more:

COROLLARY 1. *It is impossible to cover a square of side*

$$4\beta+32\beta^3\alpha^{-1} \tag{1}$$

and its interior with polygons satisfying the conditions of the theorem.

The word "cover" is used here to allow the possibility that the polygonal covering extends beyond the boundary of the square. Thus a covering of a square is to be distinguished from a dissection of a square, where the problem is to subdivide the square into regions meeting certain specified conditions. Corollary 1 can be improved by reducing the value (1) for the length of the side of the square, but I have not tried to make any such improvement. Other consequences of the basic theorem are these results.

COROLLARY 2. *Given any tiling of the plane by convex polygons having areas bounded below and perimeters bounded above, there must be infinitely many polygonal tiles having six or fewer sides.*

COROLLARY 3. *Any tiling of the plane by a collection of convex polygons whose areas are bounded below and whose perimeters are bounded above must contain infinitely many polygons whose edges meet edges of not more than six other polygons.* (Here "edges meet edges" is to be interpreted in the sense that the two edges have more than just a vertex point in common.)

The proofs of these three corollaries are given in §3 following the proof of the theorem.

In the proof in §3 of part (a) of the theorem we use the Euler theorem that $v+f=s+1$, where v, f and s are respectively the numbers of vertices, polygons, and sides (or edges) in a polygonal network in the plane having a finite number of polygons. The only other result that is used is the isoperimetric theorem in the plane that, among all regions bounded by a simple closed curve of given length, the circle has the largest area. This is used to give an upper bound β^2 on the area

The author was a student of L. E. Dickson's at Chicago; he has taught at Illinois, Purdue, and since 1947 at Oregon. He has been active in the affairs of our Association; in particular, he was Hedrick Lecturer in 1960 and First Vice-President in 1974–75. His main fields of interests are number theory and combinatorics; his publications include the Carus Monograph, *Irrational Numbers*; two volumes, *The Mathematics of Choice* and *Numbers: Rational and Irrational* in the New Mathematics Library; a textbook on calculus; and, at a higher level, *The Theory of Numbers* (with H. S. Zuckerman) and *Diophantine Approximations*. — *Editors*.

of each polygon. The isoperimetric theorem, although well known, is not easy to prove except in the restricted form where a solution is assumed to exist. So we point out that we can bypass the isoperimetric theorem, if we wish, by the following simple argument. Any polygon in the plane of perimeter less than β, having (say) one vertex at $(0,0)$, cannot have any point in common with the boundary of a circle of radius $\beta/2$ having center at the origin. Hence the polygon lies entirely inside this circle of area $\pi\beta^2/4$, and it follows that such a polygon has area less than β^2.

The proof given of part (b) of the theorem in §4 is just a description of counterexamples, using only the simplest of background ideas.

A brief outline, not intended to be comprehensive, of various results related to the theorem and its corollaries is given in §2.

2. Related results. Because the theorem proved in the paper can be stated so simply, it seemed to the writer that it might well be in the literature already, especially since the proof reveals that the result, although non-trivial, is not very deep. Correspondence with several experts in the field did turn up related results. L. Fejes Tóth [**5**, §7] pointed out that "without proof in a slightly different form" he has stated a similar result. B. Grünbaum observed that in the dissertation of K. Reinhardt [**17**] there are "several similar results but he obtains them under stronger assumptions."

The theorem is well known in the special case where the tiling uses only one polygon and congruent replicas thereof. This case seems to be regarded as a "folk theorem." Although the literature does not abound with proofs, we cite two. First, H. S. M. Coxeter [**3**, p. 56] proves the special case that "if a convex polygon can be repeated by translations to fill the plane, the number of its sides cannot exceed six." R. B. Kershner [**12**, pp. 5, 6] proves that if a convex polygon can be used to tile the plane it cannot have more than six sides.

Tiling the plane with polygons having six or fewer sides can be done in a variety of ways. A standard question here is the tiling of the plane by a single polygon and its congruent images. First of all, the case of regular polygons is readily disposed of. It is easy to prove that there are only three kinds of regular polygons that can be used to tile the plane, namely the equilateral triangle, the square, and the regular hexagon. This follows because the interior angle of a regular n-gon is $\pi(1-2/n)$ radians. So if such a polygon tiles the plane, then from angle considerations 2π must be an integral multiple of $\pi(1-2/n)$, say

$$2\pi = k\pi(1-2/n) \quad \text{or} \quad (k-2)(n-2)=4.$$

The only solutions in integers are $n=3$, 4, and 6, giving the three types mentioned. This result was known to Pappus, as is clear from direct quotations cited in Heath's *History of Greek Mathematics* [**10**, p. 390].

It is an almost obvious assertion that the plane can be tiled by any triangle and its congruent images. Similarly, the plane can be tiled by any quadrilateral, convex or not. One standard way to do this [**2**, p. 56] is to start by rotating the quadrilateral through a half-turn (through an angle π) about a midpoint of any one of its sides. The original quadrilateral and its congruent replica by a half-turn form a hexagon that can be used to tile the plane by translations alone.

The situation is more complicated for pentagons and hexagons, some of which tile the plane, and some of which do not. Starting with the easier case, the class of convex hexagons each of which tiles the plane was characterized by Reinhardt [**17**]. R. B. Kershner [**13**] "solved" the analogous problem for convex pentagons, but unfortunately the characterization was not complete. Kershner's paper does turn up some new classes, previously unknown, of pentagons that tile the plane. But his listings are not exhaustive. The gap in the Kershner paper was turned up in the following way. Martin Gardner [**6**], in an article on tiling in his regular column in the July 1975 *Scientific American*, included a listing of the classes of convex pentagons that tile the plane, based on Kershner's article. Richard James [**11**] found a new tiling not in the list and reported it to Gardner. Several others were subsequently found by another of Gardner's readers, Marjorie Rice. It is still an unsolved problem whether all convex pentagons that tile the plane

are known. Also, there is no complete analysis of which non-convex pentagons and which non-convex hexagons tile the plane. One of the most comprehensive summaries of what is and what is not known has been given by Schattschneider [**18**]. For a nicely illustrated treatment of tilings and the related groups of geometric transformations in two-dimensional crystallography, see Coxeter [**2**, Chapter 4].

The above discussion of polygonal tilings by n-gons with $n \leqslant 6$ is limited to so-called monohedral tilings, namely, by one tile and congruent replicas thereof. Wider classes of tilings are discussed in a recent survey article, giving a great deal of historical background and an extensive list of references, by Grünbaum and Shephard [**8**]. In sections 4 and 7 of another recent paper by the same authors [**7**], there are further historical observations about attempts to enumerate and classify plane tilings.

Special tilings with metric properties have also been studied. For example, J. H. Conway [**1**] pointed out that the plane can be tiled using exactly one triangle from each congruence class of rational-sided triangles. That is, every triangle with rational sides appears in the tiling, and no two triangles in the tiling are congruent. A constructive proof of this was given by R. B. Eggleton [**4**]. Eggleton later raised the question of whether there is such a tiling by rational-sided triangles that is "strict," meaning that any point in common to two of the tiles is a vertex of both or a vertex of neither. This was answered in the affirmative by Pomerance [**16**]. (A strict tiling is called an "edge-to-edge" tiling by some writers.)

3. Proof of part (a) of the theorem. Suppose that the plane can be tiled by a collection of convex polygons each of which has seven or more sides, area greater than α, and perimeter less than β, where α and β are any preassigned positive real numbers. On the basis of this assumption, we establish a contradiction. In fact, a contradiction is obtained if we assume only that a square (plus its interior) of side equal to or greater than the expression (1) can be covered by a tiling using a collection of such polygons. However, it is simpler to assume that the entire plane is tiled; at the end of the proof, it is adapted to the finite square region.

It is important to note in this proof that the theorem is not restricted to tilings that are edge to edge. An edge-to-edge tiling of the plane by polygons is one in which every two polygonal tiles have (i) no point in common, or (ii) exactly one point in common which is a vertex of each of the two polygonal tiles, or (iii) a line segment in common that is a complete edge (or side) of each of the tiles. We are allowing for the possibility of tilings that are not edge to edge, as illustrated in Figure 1. Here we have three tiles, T_1, T_2, and T_3, and three of the vertices of the polygonal network formed by the tiling marked for special attention, A, B and C.

We will refer to each of the polygons in the tiling in two senses, as a "tile" and as a "network polygon." The first sense is the one intended in the statement of the theorem. The second sense,

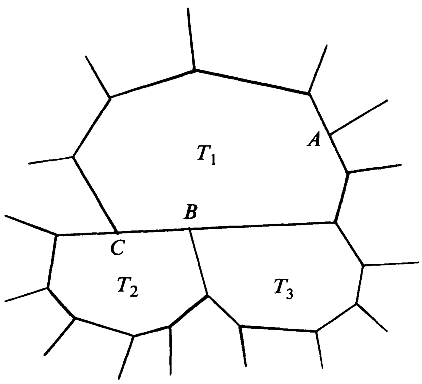

FIG. 1

the network polygon, may have more vertices and sides than the tile. For example the tile T_1 has 7 vertices and 7 sides, but as a network polygon it has 9 vertices and 9 sides. The vertices A and B are included in the analysis of T_1 as a network polygon, but not as a tile. Similarly, the tile T_2 has 7 vertices and 7 sides, but as a network polygon it has 8 vertices and 8 sides, because of the inclusion of the vertex C. The tile T_3, by contrast, has 7 vertices and 7 sides, with no increase when viewed as a network polygon. Each network polygon has the property, just as does each tile, that the number of vertices and the number of sides are equal. Whether a polygon is viewed as a tile or as a network polygon, its area is the same of course, and so is its perimeter.

However, what may change as we consider a polygonal tile in its corresponding version as a network polygon is the number of vertices and sides, as we have seen. At each vertex added in the process, for example, A and B in tile T_1 in Figure 1, an additional "interior angle" of measure π is introduced. Thus the sum of the interior angles of the network polygon T_1 is 7π, because T_1 has 9 sides. In general, the sum of the interior angles of any network polygon with n sides is $(n-2)\pi$, and this is always at least 5π, since (by our assumption) the number of sides of a network polygon is 7 at least.

With a coordinate system imposed on the plane let $S(r)$ denote the square plus its interior containing all points (x,y) satisfying $|x|\leqslant r$, $|y|\leqslant r$, where the positive real number r will be chosen large enough to give a contradiction. Let N_1 be the finite network of polygonal tiles in the plane that just covers $S(r)$. To be more specific, the polygons in N_1 include all polygonal tiles having at least one point in common wth $S(r)$. But also, we want the network N_1 to have no "holes," so that the polygons plus their interiors in N_1 constitute the topological equivalent of a disk. So in case there is a polygonal tile T that has no point in common with $S(r)$ but is itself surrounded by tiles each of which has at least one point in common with $S(r)$, include T in the network N_1. The lower bound α on the area of each polygon guarantees that the network N_1 is a finite collection of polygons.

Let f_1 be the number of polygons in the network N_1, and let v_1 be the total number of vertices, each counted only once, in these f_1 *network* polygons. Because the perimeter of each polygon is less than β, it follows that the entire network N_1 lies entirely in the interior of the closed square region $S(r+\beta)$, which is the square plus interior with vertices $(\pm(r+\beta),\pm(r+\beta))$.

Let N be the finite network of polygonal tiles that just cover the closed square region $S(r+\beta)$. Thus N stands in relation to $S(r+\beta)$ just as N_1 stands in relation to $S(r)$, as explained above. Let f be the number of polygons in the finite network N, and let v and s denote the number of vertices and sides respectively, each counted just once, of these f *network* polygons. By Euler's theorem on a network of polygons [**15**, pp. 98–100] we know that

$$v+f=s+1, \quad v+f>s. \tag{2}$$

Consider the interior angles in the polygons in the network N_1. Since there are f_1 such polygons, and since the sum of the interior angles in a polygon with 7 or more sides is at least 5π radians, the sum of these angles is at least $5\pi f_1$. But these network polygons have v_1 vertices, and the total angle sum at each vertex is at most 2π; the angle sum is exactly 2π at interior vertices in the network N_1, but less than 2π at vertices on the exterior boundary of the network. It follows by this comparison of these estimates for the total sum of all interior angles of the polygons in the network N_1 that

$$2\pi v_1 \geqslant 5\pi f_1, \quad 2v_1 \geqslant 5f_1. \tag{3}$$

Next we examine the v vertices of the polygons in the network N. Since these network polygons have s sides in all, and each side links two vertices, we see that $2s$ counts each vertex more than once. Each vertex is attached to at least two sides, so each vertex is counted at least twice by $2s$. Moreover, any interior vertex belongs to at least three network polygons, because the polygonal tiles are convex by hypothesis, and so at least three tiles meet at each vertex. Hence each interior vertex is counted at least three times by $2s$.

Now each vertex of the network N_1 is an interior vertex of the network N, so there are at least v_1 interior vertices in the network N. Counting each of these 3 times and counting the other

vertices, $v-v_1$ in number, twice we get the inequality

$$2s \geqslant 3v_1 + 2(v-v_1) = 2v + v_1. \tag{4}$$

But $2v+2f>2s$ by (1), and this with (4) gives $2f>v_1$. Multiplying this by 2 and using (3) we get

$$4f > 2v_1 \geqslant 5f_1. \tag{5}$$

Defining $f_2 = f - f_1$, we see that f_2 is the number of polygons in N which are not in the network N_1. This with (5) gives

$$4f = 4f_1 + 4f_2 > 5f_1, 4f_2 > f_1. \tag{6}$$

Each polygon has perimeter less than β. By the isoperimetric theorem in the plane, or by the argument in §1 bypassing this theorem, each polygon has area less than β^2. Hence the f_1 polygons in the network N_1 have total area less than $\beta^2 f_1$. But these polygons cover the closed square region $S(r)$ with area $4r^2$, and so we conclude that

$$\beta^2 f_1 > 4r^2. \tag{7}$$

Next consider the f_2 polygons that belong to the network N but not to N_1. Each of these polygons has area greater than α, so the total area of all the f_2 polygons exceeds αf_2. Now these polygons lie completely inside the closed square region $S(r+2\beta)$; the reason for this is that since the perimeter of each polygon is less than β, the entire network N lies inside the closed square region $S(r+2\beta)$, which is the square with vertices

$$(\pm(r+2\beta), \pm(r+2\beta)).$$

Also the f_2 polygons belonging to N but not N_1 lie entirely outside the closed square region $S(r)$. The area of the region lying inside $S(r+2\beta)$ but outside $S(r)$ is $4(r+2\beta)^2-4r^2$. This is an upper bound for the total area of the f_2 polygons, and so we have

$$4(r+2\beta)^2 - 4r^2 \geqslant \alpha f_2. \tag{8}$$

Inequalities (6), (7), and (8) are contradictory if r is sufficiently large, as we now prove. Rearranging (8) and multiplying by $4\beta^2$ we get

$$64\beta^3(r+\beta) \geqslant 4\beta^2\alpha f_2 > \beta^2\alpha f_1 > 4\alpha r^2,$$

by use of (6) and (7). Dividing by $4\alpha r$, this implies

$$r < 16\beta^3\alpha^{-1}(1+\beta/r). \tag{9}$$

It is easy to verify that this is false if $r=\beta+16\beta^3\alpha^{-1}$. Also if (9) fails for one positive value of r, it fails for all larger values of r. Thus we have a contradiction if

$$r \geqslant \beta + 16\beta^3\alpha^{-1}, \tag{10}$$

and so part (a) of the theorem is proved.

This contradiction was arrived at by having the closed square region $S(r+\beta)$ covered by the network of polygons N. This square has sides of length $2r+2\beta$. In view of (10), we have proved therefore that it is impossible to cover a square plus interior of side $4\beta+32\beta^3\alpha^{-1}$ or greater with polygons of the type described in the theorem. This is Corollary 1.

Corollary 2 follows at once. For if there were only finitely many polygonal tiles having 6 or fewer sides, it would be possible to locate an arbitrarily large region of the plane tiled only by convex polygons with 7 or more sides, contrary to Corollary 1.

Corollary 3 follows from the proof of the theorem by the following observation. Since the inequalities (2), (3), (4), (5), and (6) in the proof of the theorem are argued on the network polygons rather than the original polygonal tiles, Corollary 2 is valid with the stronger conclusion that there must be infinitely many network polygons having 6 or fewer sides. This implies Corollary 3.

4. Proof of part (b) of the theorem. There are four conditions on the polygons in part (a) of

the theorem: convexity; 7 or more sides; area greater than α; perimeter less than β. To prove part (b) we want to show that there are tilings of the plane that satisfy all but one of these conditions.

First, the plane can be tiled with non-convex polygons having 7 or more sides. Figure 2 illustrates such a tiling with a 7-sided polygon and congruent replicas thereof.

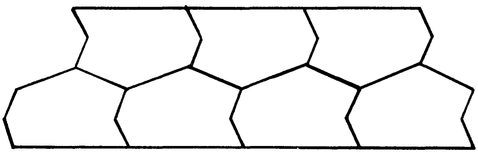

FIG. 2

Second, the condition that there must be 7 or more sides is essential because, as noted in §2, there are many tilings by n-gons with $n=3,4,5,6$. For a rich variety of examples, see Grünbaum and Shephard [**8**]. (Of course the plane can be tiled by a collection of heptagons and triangles, and various other combinations, with infinitely many of each.)

Finally, we set forth two tilings of the plane by convex heptagons, the first having polygons with perimeter less than β, the second having polygons with area greater than α. These two tilings are constructed on the same basic pattern, by using convex 7-gons stacked radially outward from the origin. The vertices of the polygons, given in polar coordinates, will lie on concentric circles of radii $\rho_1, \rho_2, \rho_3, \ldots$ as follows.

First define the polygon Q by the vertices, listed in counterclockwise order, $(0,0)$ and $(\rho_1, k\pi/10)$, $k=0,1,2,3,4,5$, where ρ_1 is to be specified. Three more polygons are obtained by rotating Q successively through angles $\pi/2$, π, and $3\pi/2$, with the origin as the center of rotation. Thus we have four convex heptagons enclosing the origin, with 20 vertices (in all) lying on the circle of radius ρ_1, center at the origin.

For $n=1,2,3,\ldots$ define the polygon P_n by the vertices

$$(\rho_n, \pi/(10\cdot 4^{n-1})), (\rho_n, 0), (\rho_{n+1}, k\pi/10\cdot 4^n), \quad k=0,1,2,3,4, \tag{11}$$

listed counterclockwise in order, where ρ_n and ρ_{n+1} are to be specified. Note that the first and last vertices listed in (11) are collinear with the origin, as also are the vertices $(\rho_n, 0)$ and $(\rho_{n+1}, 0)$. Hence a ring of polygons encircling the origin is obtained by rotating p_n about the origin repeatedly through the angle $\pi/10\cdot 4^{n-1}$. The ring is closed after the rotation is repeated $20\cdot 4^{n-1}-1$ times. (The word "ring" is used here in the sense of a geometric pattern, not in the technical algebraic sense.) See Figure 3 for an illustration of Q, P_1 and P_2.

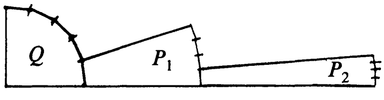

FIG. 3

Moreover, the outer vertices of the ring of polygons formed by rotating P_n are identical with the inner vertices of the ring of polygons formed by rotating P_{n+1} about the origin. These common vertices are

$$(\rho_{n+1}, k\pi/(10\cdot 4^n)), \quad k=0,1,2,\ldots,20\cdot 4^n-1.$$

It follows that the convex polygons $Q, P_1, P_2, P_3, \ldots$, together with their congruent replicas formed by the rotations about the origin, cover the entire plane without gap or overlap, provided that ρ_n tends to infinity with n.

Now we specify the values of $\rho_1, \rho_2, \rho_3, \ldots$ in two ways, first to get a set of polygons satisfying the condition that each has perimeter $<\beta$, and second to get a set of polygons each with area

$>\alpha$. The first purpose can be achieved by defining

$$\rho_n = n\beta/8, \quad n=1,2,3,\ldots, \tag{12}$$

so that ρ_n tends to infinity with n. We now verify that the polygon P_n has perimeter $<\beta$. The side joining the first two vertices listed in (11) has length less than the arc of the circle $r=\rho_n$ through these points. This arc length is

$$\pi\rho_n/(10\cdot 4^{n-1}) = n\beta\pi/(20\cdot 4^n), \tag{13}$$

and this is clearly less than $\beta/7$. The side of P_n joining the vertices $(\rho_n,0)$ and $(\rho_{n+1},0)$ has length

$$\rho_{n+1}-\rho_n=\beta/8<\beta/7.$$

One other side of P_n also has length $\rho_{n+1}-\rho_n$, and the 4 remaining sides, those joining the last five vertices in (11), have lengths of the type (13) with n replaced by $n+1$. Thus all 7 sides of P_n have length $<\beta/7$, and so the perimeter is less than β. It is easy to verify that the polygon Q has the same property.

Thus we have a tiling of the plane by convex heptagons each with perimeter less than β. By part (a) of the theorem, it follows that the areas of these heptagons cannot be bounded below by any positive constant α. Without citing part (a) of the theorem, it is not difficult to verify directly that the area of the heptagon P_n, with ρ_n as specified in (12), approaches 0 as n tends to infinity. The key to this can be seen in (13), because $n/4^n$ tends to 0 as n tends to infinity.

Finally, to get a tiling with convex heptagons each with area greater than α, we use a definition of ρ_n different from (12). Of course in this case the perimeters of the heptagons will not be bounded above by any constant β, because of part (a) of the theorem. First we want to choose ρ_1 so that the heptagon Q has area greater than α. The area of any convex heptagon certainly exceeds the area of the triangle formed by three of its vertices, so consider the triangle formed by

$$(0,0),\ (\rho_1,0) \text{ and } (\rho_1,\pi/2).$$

This triangle has area $\rho_1^2/2$, and we can certainly choose ρ_1 so that this exceeds α.

We proceed recursively by defining, for $n=1,2,3,\ldots,$

$$\rho_{n+1}=n+\rho_n+2\alpha\rho_n^{-1}\operatorname{cosec}(\pi/10\cdot 4^{n-1}). \tag{14}$$

Note that ρ_n tends to infinity with n, so this definition does give a tiling of the entire plane. It is also straightforward to verify that each of the heptagons $P_1,P_2,\ldots,P_n,\ldots$ has area greater than α. To prove this, it suffices to consider the general case P_n. This heptagon has area larger than that of the triangle formed by the first three vertices listed in (11), namely,

$$(\rho_n,\pi/10\cdot 4^{n-1}),(\rho_n,0),(\rho_{n+1},0).$$

The area of this triangle is, from the base-altitude formula,

$$\tfrac{1}{2}(p_{n+1}-\rho_n)\rho_n\sin\pi/(10\cdot 4^{n-1}).$$

This area exceeds α, as is readily seen by the use of (14). Also, since one side of the heptagon P_n has length $\rho_{n+1}-\rho_n$, and since this exceeds n, it is clear that the perimeters of the tiling polygons are not bounded above. This completes the proof of the theorem.

Sherman Stein, in correspondence with the author, pointed out the following equivalent form of the theorem: if the plane is tiled with convex polygons each having 7 or more sides, and if r is the ratio of the perimeter of a tile to its area, then the set of values of r over the polygonal tiles is unbounded.

I am grateful to my colleagues Theodore Palmer and Allan Sieradski for observations about delicate aspects of the argument.

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APPORTIONMENT METHODS AND THE HOUSE OF REPRESENTATIVES

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ABSTRACT. The seemingly straightforward task of assigning seats to states according to population runs into several politically unacceptable complications. It is shown that this is a serious problem in the sense that these complications will occur for most population densities. The mathematical reasons for these complications are discussed, and one of them is related to flows on a higher dimensional torus. Finally, a simple apportionment method is suggested.

1. Introduction. According to the Constitution of the United States of America, the apportionment of congressional seats to the individual states must be according to population. That is, if there are n congressional seats to be apportioned and if p_i denotes the percentage of the total U.S. population residing in the ith state, $i=1,2,\ldots,s$, then the ith state is entitled to np_i representatives. The problem of apportionment is to reconcile the conflict between having an integer number of representatives for each state and the fact that np_i is, in general, not an integer.

Stated in terms of vectors, this becomes a problem of integer programming. That is, we wish to find the family of integer s-vectors which best approximates a given family of non-integer s-vectors. The theoretical solution is trivial. Associate with each vector of the given family the closest integer vector, where the distance between vectors is determined by some given norm or metric. However, this obvious approach possesses certain characteristics which make it undesirable and/or controversial when it is used in the problem of apportionment. Historically, these characteristics forced, in the name of political reality, some additional side constraints on the process of apportionment. What adds interest to this story is that the method adopted and currently employed by the United States to satisfy these constraints gives rise to a new set of difficulties which can be more inequitable than the original approach!

Because of the political importance of this apportionment problem, it has received the attention of both politicians and mathematicians. Several of the names associated with this problem are well known; for example, T. Jefferson, A. Hamilton, J. von Neumann, etc. For a witty, informative discussion of the political and mathematical history of this problem, I recommend the paper by Balinski and Young [1] (to be referred to as BY). Indeed, I learned about this problem by reading BY after hearing a lecture by H. P. Young at Northwestern, January, 1977.

In the literature (at least that portion with which I am familiar) the complications mentioned above, and discussed later in this paper, are explained by means of examples. This raises the question as to whether these complications are serious and general, or merely anomalies which are illustrated in terms of contrived examples. To answer this question, and to find the mathematical explanation for these difficulties, we depart from the usual description of this problem, which is in terms of iterative processes. Instead, the apportionment process will be described in terms of elementary dynamics and geometry. At the same time, we use this geometrical description to reexamine a portion of the history of this problem. In one case the geometry suggests that the additional structure (ranking function) imposed upon the problem wasn't necessary. At the end of the paper, we use this observation and the geometric description to design a simple apportionment procedure.

The mathematical representation given here relates this question of apportionment to well-known mathematical constructs. For example, one of the political complications discussed can be understood by equating the problem to a flow on an s-torus. In the case of the present-day United States, this is a flow on a 50-torus! In particular, as we shall see, a study of this flow shows there is a close relationship between denseness, complete irrationality, and some of the methods of congressional apportionment.

Mathematically, the integer approximation problem is as follows. Assume the population of the ith state, $1 \leqslant i \leqslant s$, is g_i and $g=\Sigma_{i=1}^{s} g_i$ is the total population. If there are n seats to be apportioned according to population, then the problem is to find $\boldsymbol{k}=(k_1,k_2,\ldots,k_s)$, k_i a non-negative integer and $\Sigma k_i = n$, which best approximates $\boldsymbol{np}=(np_1,\ldots,np_s)$ where $p_i = g_i/g$. That is, the seats are apportioned in a manner which approximates the ideal apportionment, usually a non-integer valued vector, which is determined by the actual population. For several years the United States used what amounted to a norm-minimizing approach where, of all the integer vectors satisfying $\Sigma k_i = n$, the solution vector was the one closest to $\boldsymbol{np}$. The distance was determined by the sup norm $\|\boldsymbol{x}\| = \max_{1 \leqslant i \leqslant s} |x_i|$. It is a simple exercise to see this is equivalent to the usual description of the Hamilton method where first the ith state receives the integer portion of np_i, that is, $[np_i]$, and then any remaining seats are allocated to the states based on the magnitude of the fractional part of np_i, denoted by $(np_i) = np_i - [np_i]$. Notice that, except possibly when the fractional part of two or more components of $\boldsymbol{np}$ are the same, the choice of $\boldsymbol{k}$ is unique.

A complication leading to the abandonment of this method is known as the Alabama paradox (BY).

Definition. *Let $\boldsymbol{k}^{(n)}$ and $\boldsymbol{k}^{(n+1)}$ be integer solutions for $\boldsymbol{np}$ and $(n+1)\,\boldsymbol{p}$ respectively. If some component of $\boldsymbol{k}^{(n+1)}$ is less than the corresponding component of $\boldsymbol{k}^{n}$, then we say that $\boldsymbol{p}$ has an Alabama paradox at n.*

In other words, when the size of the House of Representatives is n, some state, say the ith, has k_i representatives. However, when an additional seat is *added* to the total house size, the ith state *loses* a representative. Mathematically $\boldsymbol{k}^{(n+1)}$ would be the "best" approximation, but politically this solution would be difficult to implement, particularly if the representatives from the ith state have anything to say about it.

Since this paradox bears the name Alabama, it is reasonable to suspect it can and has occurred. Not only can it occur, but in the next section it will be shown that if a norm-minimizing method is used, then for "most" choices of $\boldsymbol{p}$ ("most" will be defined later) there is an n

giving rise to the Alabama paradox. This is true for *any* norm, so the choice of distance function is not at fault! We will isolate the dynamic which causes this behavior.

Politically, this is intolerable! Therefore, a political side constraint on choice of the family of apportionment vectors is that it does not admit an Alabama paradox. Such a family is called house monotone. (Although the family depends upon $\boldsymbol{p}$, our notation will not reflect this dependency.)

DEFINITION. *Let* $\{\boldsymbol{k}^{(n)}\}_{n=0}^{\infty}$ *be a family of integer vectors where* $\boldsymbol{k}^{(n)}$ *gives the apportionment for* $n\boldsymbol{p}$. *That is, the components of* $\boldsymbol{k}^{(n)}=(k_1^n,\ldots,k_s^n)$ *are non-negative integers which sum to* n. $\{\boldsymbol{k}^{(n)}\}$ *is said to be house monotone if for all* $i=1,\ldots,s$ *and all* $n\geqslant 0$, $k_i^{n+1}\geqslant k_i^n$.

The problem is to find a method of selecting a house-monotone family which approximates the ideal family $\{n\boldsymbol{p}\}$. While the actual story is much longer and more involved, the adopted approach is essentially as described below. Presumably, a norm or distance approach doesn't work, so the idea was to replace it with some other measure of equal apportionment. For example, at the ideal apportionment $p_i/k_i=p_j/k_j$ for all $i,j=1,2,\ldots,s$. Notice that gp_i/k_i is the density of population per representative, so a high density corresponds to an underrepresented state.

With the adoption of some "measure" of equality, such as the density of population per representative, we can discuss the "amount of inequality" [5] in the representation between states. Then, the goal of any apportionment method should be to minimize this "amount of inequality." That is, when the house size is increased from n to $n+1$, the method should award the extra representative to a state which would minimize this inequality. The resulting method gives a ranking depending on np_i and the current apportionment. Since for increasing n, this method *awards* extra seats to states, rather than computing a general redistribution of the seats, the method is house monotone.

Of course, the actual choice of measure and the method of computing the amount of inequality is open to debate. For example, should we use the density of population per representative, the density of representatives per 1000 population, or perhaps something else? Since other choices can lead to different apportionments, this is a very real political question. We will dodge this issue by keeping only the properties of the ranking methods.

DEFINITION. *Let* $l:(0,\infty)\times[0,\infty)\to R\cup(\infty)$ *be a smooth function on* $(0,\infty)\times(0,\infty)$ *such that for fixed* $x, l(x,y)$ *is a monotonically decreasing function in* y. *Such an* l *will be called a ranking function.*

We intend the first variable to represent either the population or the percentage of population, and the second variable to represent either the apportionment or percentage of apportionment.

A related function is defined in BY which has less restrictions on it. Yet all the ranking functions seriously considered satisfy the constraints listed here. It can be shown that if a ranking function doesn't satisfy these requirements, then pathology may be introduced; and this can be equated with political controversy.

The idea is essentially as described above. When $n=0$, $\boldsymbol{k}^{(0)}=(0,0,\ldots,0)$. When going from house size n to house size $n+1$, the additional seat is apportioned to the state which maximizes $l(np_j,k_j^{(n)})$. Should there be a tie, then some sort of tie-breaking scheme is introduced. Since we are not reapportioning the seats, but rather we are adding to previous apportionments, this is house monotone. Currently the United States uses $l(x,y)=x/(y(y+1))^{1/2}$.

A natural question is whether this approach may introduce a different form of pathology; and the answer is yes. What happens is that this method admits the possibility that a state may receive more than its fair share of representatives. If [ ] is the greatest integer function, then the ith state should receive either $[np_i]$ or $[np_i]+1$ representatives. If the ranking function does not satisfy a certain diagonal condition, it turns out that this method *must* eventually deviate from this quota for most choices of p. This will be shown in Section 3. In Section 4 a simple method which avoids all these difficulties will be discussed.

2. Alabama paradox and flows on a torus. Let $P_n=\{x\in\mathbf{R}^s|\Sigma x_i=n, x_i\geqslant 0\}$. Then $p\in P_1$ and both np and $k^{(n)}\in P_n$. The difference is that $k^{(n)}$ must be an integer lattice point of this simplex. Notice that these lattice points define simplexes which cover P_n. Now suppose $\{k^{(n)}\}_{n=0}^{\infty}$ is a house-monotone family giving the apportionment for $\{np\}_{n=0}^{\infty}$. This means that when the house size is changed from n to $n+1$, the additional seat is added to some one state. That is, *one* of the components of $k^{(n)}$ is increased by unity, while the others remain the same. There are only s choices of unit integer vectors which can be added to $k^{(n)}$ to obtain $k^{(n+1)}$, and they are all represented as vertices of P_1. Therefore, $k^{(n+1)}$ is one of the vertices of $(k^{(n)}+P_1)\cap P_{n+1}$. Thus, the vertices of this copy of P_1 on P_{n+1} represent the only eligible apportionments which permit the method to be house monotone.

On the other hand, the norm-minimization approach selects the lattice point of P_{n+1} which is closest to $(n+1)p$. This is the same as first finding the appropriate simplex on P_{n+1} which contains $(n+1)p$, and then selecting the nearest vertex of this simplex. Now, the direction of p may be such that these two descriptions are not compatible. The following statement asserts that this is a common occurrence.

THEOREM. *Let $s>2$. There exists an open dense set $\mathfrak{B}\subset P_1$ such that if $p\in\mathfrak{B}$ and if $\{k^{(n)}\}$ is a family of integer vectors selected by a norm-minimization process to approximate $\{np\}$, then $\{k^{(n)}\}$ will have an Alabama paradox for some n.*

It is a simple exercise to show that the Alabama paradox cannot occur if $s=2$.

In the proof we use the non-integer part of a number, so recall our notation $(x)=x-[x]$.

Proof. We first prove the theorem with respect to the sup norm. Notice that $\Sigma_{i=1}^{s}(np_i)$ is always an integer. This is because $\Sigma np_i=n$ and $\Sigma[np_i]$ is an integer.

CLAIM: *If state α satisfies the following three conditions for some positive integers δ, n, then it will suffer the Alabama paradox.*

1. If $\Sigma(np_i)=j_1\leqslant s-1$, then (np_α) is one of the largest j_1 terms.
2. $0<(np_\alpha)<1-\delta p_\alpha$.
3. If $\Sigma((n+\delta)p_i)=j_2\leqslant s-1$, then $((n+\delta)p_\alpha)$ is *not* one of the largest j_2 terms.

In what follows we shall restrict attention to $\delta=1$. Imposing the additional constraints that (np_α) is strictly larger than the (j_1+1)st largest term in the set $\{(np_i)\}$ and $((n+1)p_\alpha)$ is strictly smaller than the j_2 largest term in the set $\{((n+1)p_i)\}$ makes this an open condition.

With the sup norm, the allocation of representatives goes as follows. First the ith state is assigned $[np_i]$ representatives. This accounts for $\Sigma[np_i]=n-\Sigma(np_i)=n-j_1$ representatives. The last j_1 representatives are then assigned according to the magnitude of (np_i). Therefore, the ordering given in condition 1 implies that state α receives $[np_\alpha]+1$ representatives when the house size is n. Condition 2 implies that $[(n+1)p_\alpha]=[np_\alpha]$, and Condition 3 implies that state α receives $[(n+1)p_\alpha]=[np_\alpha]$ representatives when the house size is $n+1$, a loss of one representative with the increase in house size.

Define $\mathfrak{B}=\{p\in P_1|$ for some choice of $n>1$, the strict inequalities given in Conditions 1, 2, and 3 are satisfied$\}$. The proof that $\mathfrak{B}$ is an open set is an immediate consequence of the continuity of the mapping $h_N: P_1\to P_N$ where $h_N(p)=Np$. If $p\in\mathfrak{B}$, then for some n, (np) satisfies the three open conditions. Since h_n is continuous, this means there is an open neighborhood $\mathcal{V}$ of p in P_1, such that if $q\in\mathcal{V}$ then $h_n(q)$ also satisfies these three open conditions. Therefore, $\mathcal{V}\subset\mathfrak{B}$, which establishes the fact that $\mathfrak{B}$ is an open set.

We shall show that $\mathfrak{B}$ is non-empty at the same time we show that $\mathfrak{B}$ is dense. Let $p^*\in P_1$ be such that $p_i^*\neq p_j^*$ for $i\neq j$. For any $\varepsilon>0$ we will show there exists $p\in\mathfrak{B}$ such that p is at most distance ε from p^*. Actually we shall prove much more. We will prove there exists $p\in\mathfrak{B}$ and integer n so that j_1 is equal to unity and j_2 is equal to 2.

Assume without loss of generality that $p_1^*>p_2^*>\cdots>p_s^*>0$, so $0<p_s^*<s^{-1}$. Next choose

rational numbers x_i so that

$$\sum_{i=1}^{s} x_i = 1, x_i < x_s < 2/s \quad \text{for} \quad i = 1,2,\ldots,s-1; x_1 < 1 - p_1^*;$$
$$x_2 < 1 - p_2^*; x_s < x_1 + 3(p_1^* - p_s^*)/4, x_s \leqslant x_2 + 3(p_2^* - p_s^*)/4.$$

Such x_i's can always be selected if $s > 2$. Now let n be the first multiple of 10 greater than $\max(10\varepsilon^{-1}, ((p_1^* - p_s^*)/10)^{-1}, ((p_2^* - p_s^*)/10)^{-1})$. Define

$$\boldsymbol{p}' = ([np_1^*], [np_2^*], \ldots, [np_s^*])n^{-1}.$$

By adding or subtracting n^{-1} to some of the components of $\boldsymbol{p}'$, it can be adjusted so that the sum of the components equals $1 - n^{-1}$. Assume this has been done. Define $\boldsymbol{p} = \boldsymbol{p}' + (x_1, x_2, \ldots, x_s)n^{-1}$. By construction $\boldsymbol{p} \in P_1$, $(np_i) = x_i$, and $\boldsymbol{p} \in \mathfrak{B}$. Also, $\boldsymbol{p}$ is within distance ε of $\boldsymbol{p}^*$.

Now assume $\boldsymbol{p}^* \in P_1$ is arbitrary. If some of the components are equal, approximate $\boldsymbol{p}^*$ with some vector within distance $\varepsilon/2$ where the components all differ. The above now applies. The proof is now completed for the sup norm.

There are two parts to this proof. The first is the choice of $\boldsymbol{p}$: this corresponds to the dynamics. As we shall see, this is nothing more than an adaptation of the "irrational flow on a torus" for this simplex model, where the details are carried out above for the sup norm. The part of the dynamics which makes this proof work is the denseness of the image of $((np_1), (np_2), \ldots, (np_s))$ if the components of $\boldsymbol{p}$ satisfy some sort of rational independence condition. This means we can get the image to enter any open set. Consequently, the proof can be modified to hold for any norm, since the open unit ball for any norm is a convex open set.

This leads to the second part of the proof, which is the construction of a non-empty open set which would permit the Alabama paradox to occur for some n. Since we choose $j_1 = 1$ and $j_2 = 2$, the choice of this open set or, equivalently, the restriction on the choice of the x_i's, turns out to be a discussion about subsets of P_1 and P_2. The geometry of this construction for an arbitrary norm will now be discussed.

The first problem with a given norm is that even with the assumption $p_s < p_i$ state s need not be a victim of the Alabama paradox. This is because the norm may favor state s. An example of this would be

$$|||\boldsymbol{x}||| = \sum_{i=1}^{s-1} |x_i| + (10)^{-1}|x_s|.$$

Nevertheless, a "victim" state is one of the states which has a "small" population with respect to the given norm. That is, for $\boldsymbol{p}$, *a* "victim" state γ could be one for which the distance from $\boldsymbol{p}$ to $\boldsymbol{e}_\gamma$ is strictly larger than the distance from $\boldsymbol{p}$ to any other vertex of P_1. With the possible exception of ties, such a state can always be found. Now let open set $U_1 \subset P_1$ be all points which are strictly closer to vertex $\boldsymbol{e}_\gamma$ than any of the other $s-1$ vertices of P_1. Set U_1 isolates values of $(\boldsymbol{np})$ for which the extra seat is awarded to state γ. Let $U_2 \subset P_2$ be the set of points which are strictly closer to a vertex of P_2 which has a zero for the γth coordinate than to a vertex which has a non-zero γth coordinate value. Clearly, U_2 is open on P_2. Set U_2 isolates values of $(\boldsymbol{np})$ for which neither of the two extra seats would be awarded to state γ.

The next step is to relate the two sets by means of the dynamics. Because

$$U_1 + \boldsymbol{p} = \{\boldsymbol{x} + \boldsymbol{p} \mid \boldsymbol{x} \in U_1\}$$

is an open set of P_2, set $U_3 = U_2 \cap (U_1 + \boldsymbol{p})$ is also an open set. If U_3 is non-empty, then the subset of U_1 which gets mapped onto U_3, that is, set $U_3 - \boldsymbol{p}$, is a non-empty open set of P_1. This is the set we need for our construction. This is because a seat is assigned to state γ when $(\boldsymbol{np})$ lies in $U_3 - \boldsymbol{p}$, but this seat is taken away from state γ for house size $n+1$, $((\boldsymbol{np} + \boldsymbol{p}) \in U_2)$.

The fact that U_3 is non-empty follows from our choice of γ. The easiest way to see this is to consider the equidistant point $\boldsymbol{x}^*$ which is a boundary point of U_1. The definition of γ implies that $\boldsymbol{x}^*+\boldsymbol{p}\in U_2$. By using continuity properties of the norm, we can perturb $\boldsymbol{x}^*$ to obtain a point in $U_3-\boldsymbol{p}$ which will be mapped into U_2. Thus U_3 is non-empty.

The proof of the theorem for an arbitrary norm now follows from the above discussion of the "dynamics" of $(n\boldsymbol{p})$ which show that arbitrarily close to any $\boldsymbol{p}^*\in P_1$ there is a $\boldsymbol{p}$ which satisfies the rational independence condition. This means that $\{(n\boldsymbol{p})\}_{n=0}^{\infty}$ is dense, so it will eventually enter the set U_3 corresponding to this $\boldsymbol{p}$.

In the above proof, no attempt was made to make an economical choice of j_i, n, or $\boldsymbol{p}$. Indeed, for the given choice of $\boldsymbol{p}$, an Alabama paradox may have already taken place for a much smaller value of n. However, since the three conditions given in the proof for the sup norm characterize the paradox, it is easy to see that the closer the populations are to the equidistant point of P_1, the larger n need be before a paradox can occur (if it will!).

In the proof, the real culprit of these problems for apportionment methods was isolated. Namely,

$$\{(n\boldsymbol{p})\}_{n=0}^{\infty}=\{((np_1),(np_2),\ldots,(np_s))\}_{n=0}^{\infty}$$

can be dense in a certain union of simplexes. This means that should anything go wrong for a small open set, it probably goes wrong for a dense open set. Because it plays a central role in the discussion, we will re-prove this fact.

Since the simplex is $s-1$ dimensional, to locate a point $(n\boldsymbol{p})$ on the simplex we need only consider $((np_1),(np_2),\ldots,(np_{s-1}))$. The value of (np_s) is found in the following fashion. Let $\beta=\Sigma_{i=1}^{s-1}(np_i)$. If β is an integer, then $(np_s)=0$ since $\Sigma_{i=1}^{s}(np_i)$ is an integer and $0\leqslant(np_s)<1$. If β is not an integer, then the same reasoning gives us that $(np_s)=[\beta]+1-\beta$. The vector $(n\boldsymbol{p})$ can enter certain regions of $P_1,P_2,\ldots,P_{s-1}$. Namely $(n\boldsymbol{p})$ is restricted to $\mathfrak{D}$ where $\mathfrak{D}$ is obtained in the following fashion. Let

$$\mathfrak{D}_i=\{\boldsymbol{x}\in P_i|0\leqslant x_j<1, j=1,2,\ldots,s-1,\quad\text{and}\quad \Sigma x_j>i-1\}.$$

Let

$$\mathfrak{D}=\left(\bigcup_{i=1}^{s-1}\mathfrak{D}_i\right)\cup\{\boldsymbol{0}\}.$$

Notice that in an obvious fashion, $\mathfrak{D}$ can be identified with an $s-1$ dimensional unit cube. To see this, take the $s-1$ dimensional unit cube $C_{s-1}=\{y\in\mathbf{R}^{s-1}|0\leqslant y_i\leqslant 1\}$, and divide it into the following parts:

$$\boldsymbol{0}, C^i=\left\{y\in C_{s-1}|i-1<\sum y_i\leqslant i\right\}$$

for $i=1,2,\ldots,s-1$. The homeomorphism between C^i and $\mathfrak{D}_i$ is the obvious one.

Assume that $p_1,p_2,\ldots,p_{s-1},1$ are rationally independent positive numbers such that $\Sigma_{i=1}^{s-1}p_i<1$. (That is, if $\{a_i\}_{i=1}^{s}$ are rational numbers such that $a_s+\Sigma_{i=1}^{s-1}a_ip_i=0$, then all the a_i's must equal zero.) Let $\boldsymbol{p}'=(p_1,p_2,\ldots,p_{s-1},1-\Sigma^{s-1}p_i)$.

THEOREM. $\{(n\boldsymbol{p}')\}_{n=0}^{\infty}$ *is dense in* $\mathfrak{D}$. *Furthermore,* $t(p_1,\ldots,p_{s-1},1)$, $t\in R$, *can be identified with a completely irrational flow on an s-torus.*

Proof. In $\mathbf{R}^s$, identify in the obvious fashion the s cubes defined by the integer lattice points. That is, consider $\mathbf{R}^s/\boldsymbol{Z}^s$. With this identification, the differential equation $dx_i/dt=p_i$, $i=1,2,\ldots,s-1$, $dx_s/dt=1$ defines a flow on the unit cube where opposite faces are identified. This identification of faces defines an s-torus. The rationally independent condition means this is a completely irrational flow on an s-torus, which in turn implies that the flow is dense. In particular, the intersection of the flow with the $s-1$ dimensional cube corresponding to the face $x_s=0$, denoted by C_{s-1}, is dense. Since $dx_s/dt=1$, each point here corresponds to an integer

multiple of $(p_1,\ldots,p_{s-1},1)$ followed by the identification of each coordinate with its non-integer part. This means that the terms $\{((np_1),\ldots,(np_{s-1}),0)\}_{n=0}^{\infty}$ are dense in C_{s-1}. The conclusion follows from the obvious topological identification of C_{s-1} with $\mathfrak{D}$.

Adding to this denseness statement is the fact that the vectors $\boldsymbol{p}\in P_1$ which satisfy this rational independence condition are dense in P_1. Most of our concern is with rational entries of P_1. However, using continuity of the differential equations with respect to small perturbations of the vector field (over compact intervals of time) and the denseness of the rationals, a rational point can be selected to exhibit most of the pathology the rationally independent points do. A political corollary of this is: if anything can go wrong, it probably will—and densely!

Finally we note that since the set of completely irrational points is of full Lebesgue measure, so is set $\mathfrak{B}$.

3. Dynamics and house monotone methods. The easiest way to see what can go wrong with house monotone methods is to project everything onto P_1. Namely, instead of comparing vectors $n\boldsymbol{p}$ and $\boldsymbol{k}^{(n)}$, we compare unit vectors $\boldsymbol{p}$ and $n^{-1}\boldsymbol{k}^{(n)}$. This means the ranking function needs to be redefined as a smooth function r: $(0,1)\times[0,1]\to\mathbf{R}$ such that for fixed x, $r(x,y)$ is monotonically decreasing. For additional flexibility and to make the function r correspond to the five methods seriously considered (see BY) we could allow r to depend upon n, but the analysis is essentially the same, so we do not.

So, a ranking method works as follows. When the house size changes from n to $n+1$, where the apportionment at n is given by $\boldsymbol{k}^{(n)}$, then a state which maximizes $r(p_i,k_i^{(n)}/n)$ receives the additional seat. This can also be described in terms of the simplexes. Corresponding to house size n, P_1 contains smaller copies of P_1 with edge dimension $\sqrt{2}/n$ and the vertices are at rational points with denominator equal to n. The numerators correspond to the apportionment. Call this a $1/n$ copy of the simplex. Each vertex of a $1/n$ copy is in the interior of a $1/(n+1)$ copy of the simplex, and it is the only vertex in the interior. All of this means that a house monotone method with vertex at $\boldsymbol{k}^{(n)}/n$ selects some vertex from the $1/(n+1)$ copy of the simplex containing $\boldsymbol{k}^{(n)}/n$, where the dynamic is described by r.

If a state is to receive its "quota" of representatives, then for house size n, the ith state should receive either $[np_i]$ or $[np_i]+1$ representatives, where the latter figure is considered *only* if $(np_i)\neq 0$. An apportionment family satisfying this constraint for all n is said to respect quota. It turns out that in order to respect quota the choices of the apportionment at house size n are restricted to the numerators of some vertex of the $1/n$ copy of the cube containing $\boldsymbol{p}$. This follows immediately from the fact that the quota constraint restricts attention to the vertices of the s cube where the ith coordinate is either $[np_i]/n$ or $([np_i]+1)/n$. The claim follows by taking the intersection of this cube with P_1.

THEOREM. *If a ranking method respects quota for all $\boldsymbol{p}$ in the interior of P_1, then there exists C such that $r(x,x)=C$ for all $x\in(0,1)$.*

The geometric idea behind the proof follows. Define $B(\boldsymbol{p},\boldsymbol{x})=\max_i\{r(p_i,x_i)\}$. B can be viewed as the altitude of a bowl over $\boldsymbol{x}$. The dynamics are designed to choose iterates which will cause the value of B to decrease; that is, to slip down the side of the bowl to the center. (Notice the similarity between this bowl and a geometric description of a Liapunov function for the stability of differential equations. In both, a "bowl" is selected for which the dynamics will tend to the bottom.) However, in order for the center to correspond to $\boldsymbol{p}$, $r(p_i,p_i)=r(p_j,p_j)$ for $i\neq j$.

Proof. Assume there does not exist C such that $r(x,x)=C$ for all $x\in(0,1)$. Let $\boldsymbol{p}$ be an element in the interior of P_1 such that $r(p_1,p_1)>r(p_i,p_i)$ for $i=2,\ldots,s$. By the continuity of r, there is an open neighborhood about $\boldsymbol{p}$ such that $B(\boldsymbol{p},\boldsymbol{x})=r(p_1,x_1)$. According to the ranking method, for any apportionment in this neighborhood state 1 gets any additional seat due to increase in house size. At the nth stage, this is a change from $\boldsymbol{k}^{(n)}n^{-1}$ to $(\boldsymbol{k}^{(n)}+\boldsymbol{e}_1)/(n+1)$, or a step size of a constant multiple of $1/(n+1)$ along the line connecting $\boldsymbol{k}^{(n)}n^{-1}$ and $(1,0,0,\ldots,0)$.

Since the harmonic series is divergent, this means that the iterates due to the ranking method must eventually leave this neighborhood about $\boldsymbol{p}$.

On the other hand, the iterates for a quota respecting apportionment are on the vertices of the $1/n$ copy of the cube containing $\boldsymbol{p}$, so they must approach $\boldsymbol{p}$ with $n \to \infty$. This contradiction completes the proof.

COROLLARY. *For all $\boldsymbol{p}$, such that $r(p_i,p_i) \neq r(p_j,p_j)$ for some choice of i and j, the ranking method does not respect quota. Indeed, the apportionment by ranking must tend arbitrarily far away from quota.*

Astonishing as it may seem, the method currently used by the United States does *not* satisfy this basic condition! However, the situation is not as dramatic as the Corollary implies if l depends upon n, although even here quota need not be attained. In this case the above arguments are modified in the obvious fashion to include the value n. Again, a bowl is defined; again, unless $r_n(p_i,p_i) = r_n(p_j,p_j)$ for $i \neq j$, at the nth step the center of the bowl is *not* at $\boldsymbol{p}$. However, in this case the center of the bowl may move with changing values of n. Indeed, for the method currently in use by the United States, $r_n(p_i,x) = p_i/(x(x+n^{-1}))^{1/2}$, it can be easily seen that the center of the bowl approaches $\boldsymbol{p}$ as $n \to \infty$. However, it is also an elementary exercise to see that the center of this bowl is at least a constant multiple of n^{-1} away from $\boldsymbol{p}$ where the constant depends on $\boldsymbol{p}$.

Not only does this "bowl" description show why some of the ranking methods deviate from quota, but it also can be used to supply information concerning any favoritism a given method might exhibit. For example, to see which ranking methods favor larger states over smaller states, one examines the relative position of the center of the bowl (for each n) with respect to various initial positions of $\boldsymbol{p} \in P_1$. If the center lies between $\boldsymbol{p}$ and a given vertex of P_1, then the state corresponding to this vertex is favored.

Can this be corrected by imposing a constraint $r(x,x) = 1$ for all $x \in (0,1)$ so that the center of the bowl agrees with p? It can, but then other problems may still crop up resulting from the positive step size of the iterates and the fact the direction (on P_1) must be on the line connecting the starting point with one of the vertices of P_1. It turns out for some choices of r that it is possible to choose an open set of population densities so that the apportionment for house size n respects quota but the iterate for $n+1$ does not. Here the edges of $B(\boldsymbol{p},\boldsymbol{x})$ are used; that is, those points $\boldsymbol{x}$ where more than one index satisfies the condition $B(\boldsymbol{p},\boldsymbol{x}) = r(p_i, x_i)$. Also, the $\boldsymbol{p}$'s are selected close enough to the boundary so that the direction vectors "tend" to be "almost" tangent to the sphere, with center $\boldsymbol{p}$ and radius determined by the distance from $\boldsymbol{p}$ to the nth iterate. However, the open sets seem to be small in measure, and the examples are somewhat contrived. Consequently, I don't view this as a serious issue; particularly since in the next section an apportionment method will be suggested which doesn't use a ranking function.

In BY (page 707) the ranking function approach is related to a divisor technique. For completeness, we describe and motivate this approach in terms of the geometry of $\mathbf{R}^s$. As before, this discussion doesn't necessarily correspond to the historical motivation.

One of the causes of the Alabama paradox is that $\boldsymbol{np}$ may be closer to an integer vector on $P_{n'}$ than any integer vector on P_n, where $n' < n$. Indeed, our earlier discussion relating $\boldsymbol{np}$ to a flow on a torus makes it clear that this can be a common occurrence. Therefore, instead of studying $\boldsymbol{np}$, perhaps attention should be focused on $\lambda\boldsymbol{p}$ where λ is some scalar. For example, we may choose λ to be such that $\lambda\boldsymbol{p}$ is at least as close to some integer vector of P_n as to any integer vector of $P_{n'}$ where $n' \neq n$. The vector selected is then one of the closest vectors to $\lambda\boldsymbol{p}$. Other conditions may be imposed upon $\lambda\boldsymbol{p}$. Another choice may be to choose λ sufficiently large so that the fractional part of the components are bounded by $1/2$ with equality in at least one component.

These types of conditions can be modeled in the following way. Let d be a monotonically increasing function on $[0,\infty]$ where $d(0) \geqslant 0$ and $d(y) \to \infty$ as $y \to \infty$. Extend d to be a mapping

from the positive orthant of $\mathbf{R}^s$ back into itself; that is, let $d(\boldsymbol{k})=(d(k_1),d(k_2),\ldots,d(k_s))$. The choice of d is determined by what characteristics we wish the system to possess. Each integer vector $\boldsymbol{k}\in P_n$ defines an s-dimensional rectangle with diagonal vertices $\boldsymbol{0}$ and $d(\boldsymbol{k})$. Let λ^* be the largest value of λ such that $\lambda\boldsymbol{p}$ is in the closure of some one of these rectangles defined by $\boldsymbol{k}\in P_n$ and d. $\boldsymbol{\lambda^* p}$ determines a rectangle and a diagonal vertex $d(k)$. The selected apportionment is some one $\boldsymbol{k}\in P_n$ corresponding to the vertex $d(\boldsymbol{k})$. If a consistent method is introduced to handle ties, the monotonicity of d implies that this is house monotone.

To see the problems encountered with this method, we relate it to the bowl description. Notice the choice of λ implies that $d(k_i)\geqslant\lambda^* p_i$ for each i where $\boldsymbol{k}=(k_1,\ldots,k_n)\in P_n$ is the selected vertex. Projecting onto P_1, we can describe the above by letting $r_n(p,y)=p/d(nx)$. The monotonicity of d means that r_n defines a bowl where the center is not necessarily at $\boldsymbol{p}$. $1/\lambda^*$ is then a restriction on how high up the side of the bowl the selected vertex of a $1/n$ copy of P_1 can be. Because the center of the bowl doesn't coincide with $\boldsymbol{p}$, the divisor techniques are subject to the same criticisms mentioned above.

4. Discussion. The causes of the Alabama paradox turned out to be related to consequences of the irrational flow on the torus. So, the norm-minimizing approaches were replaced with ranking methods, which turn out to be similar to placing a bowl over the simplex P_1 and defining a dynamic which will cause the iterates to tend to the center, or at least not slide too high up the sides. This may be acceptable if the bottom of the bowl corresponds to the population density—but it need not.

However, even if a "bowl" is selected with the bottom located at the correct place, all that is done is that a norm is replaced with some weaker form of measure or metric. Indeed, with the appropriate scaling and convexity assumption on $r(x,y)$, B *can* be expressed in terms of a metric. Also, choosing an r leads to a discussion concerning the rationale behind "this" choice for r. It should! Different choices of r can lead to different apportionments! It would be most desirable to have an apportionment method which doesn't need to use a ranking function r.

To design such a procedure, one must weaken either the norm-minimization requirement or the monotonicity condition, because they are mutually inconsistent on the quotient space $\boldsymbol{R}^s/\boldsymbol{Z}^s$. We shall describe a method which weakens the minimization requirement. We do this by using the basic ideas found in Section 2 to find a restricted set of apportionments which satisfy quota and which are house monotone. An apportionment for house size n would be some one apportionment from this set which minimizes the distance to $\boldsymbol{np}$. Notice that this approach of restricting the set of apportionments to obtain a "minimum" is in the spirit of Black's Single Peakness Condition in the theory of social choice [**6**].

The lower and upper quota values, $[np_i]$ and $[np_i]+1$, define a unit (hyper-) cube. Call any unit cube with non-negative integer vertices a quota cube if there is a positive integer value of λ such that $\lambda\boldsymbol{p}$ is in the interior of the cube. The quota cubes can be ordered in the obvious fashion. For the mth quota cube let $\lambda_e(m)$ and $\lambda_x(m)$ be, respectively, the minimum value and the maximum value of λ such that $\lambda\boldsymbol{p}$ is in this cube. Any face of this quota cube containing $\lambda_e(m)\boldsymbol{p}$ is called an entering face, while a face defined by $\lambda_x(m)\boldsymbol{p}$ is called an exiting face. It follows from the construction that should $m>1$, then the exiting face of the $(m-1)$st cube and the entering face of the mth cube always share some integer points. Call the vertices in the intersection of these two cubes and $P_{[\lambda_e(m)]}$ the *candidate-entering apportionments* for the mth cube. When these vertices are viewed as being on the $(m-1)$st cube, they will be called candidate-exiting apportionments. For the first cube the entering apportionment is $\boldsymbol{0}$.

The intersection of all entering faces is given by restricting the values of the components of $\boldsymbol{x}\in\mathbf{R}^s$ in the following fashion: If $\lambda_e(m)p_i$ is not an integer, then $[\lambda_e(m)p_i]\leqslant x_i\leqslant[\lambda_e(m)p_i]+1$. If $\lambda_e(m)p_j\in\boldsymbol{Z}$, then x_j is fixed at that value. A corresponding description holds for the exiting faces. The geometric positioning of adjacent quota cubes may fix the values of other compo-

nents of candidate-entering and candidate-exiting apportionments; namely, do the cubes share a face, or an edge, or a vertex? (Computationally, this only requires checking which upper quota values for the $(m-1)$st cube now become lower quota values for the mth cube.) For candidate-exiting apportionments, the components which are fixed in this fashion assume upper quota values; while for candidate-entering apportionments they assume lower quota values.

The function of these classes of vertices is to ensure that an apportionment is always on the correct quota cube at the correct house size. The problem is to determine whether on the mth quota cube it is possible to move in a house-monotone fashion from a candidate-entering to a candidate-exiting apportionment. Call such a choice of apportionments a path in the mth cube. It is obvious that there exists a path between a candidate-entering and a candidate-exiting apportionment iff each component of the latter is greater than or equal to the corresponding component of the former. This is because the number of seats to be apportioned in this cube equals the sum of the differences between the components; so, all the paths between the vertices would be given by the various permutations of adding unity to the appropriate components.

Call the candidate-entering apportionments *strong entering apportionments* if each candidate-entering apportionment is dominated by *all* candidate-exiting apportionments of that cube. This means there are paths from each entering apportionment to each candidate-exiting apportionment. A *sufficient* condition for this to occur in the mth cube is if the candidate-exiting apportionment is the upper quota point: a condition which is guaranteed if $\Sigma(\lambda_e(m+1)p_i)<1$. It follows from a simple modification of the argument in Section 2 that for any $\boldsymbol{p}$ this condition will be satisfied for an infinite number of cubes. (This is trivially true for the important case where $\boldsymbol{p}$ has rational entries.)

The strong entering apportionments will be used to define, in an iterative fashion, entering apportionments for all the cubes. If the mth cube does not have strong entering apportionments, then find the first cube after the mth which does, say the $(m'+1)$st. We define the entering apportionments for the m'th cube (and the exiting apportionments for the $(m'-1)$st cube) to be all candidate-entering apportionments which are dominated by at least one exiting apportionment. By continuing this backward iteration process, the entering and exiting apportionments are defined for all cubes. We shall show below that these sets are always non-empty. If for a given problem there is an additional side constraint of a fixed house size or a maximum house size n', say $n'=435$, then the candidate-entering apportionments of the quota cube following the one containing $n'\boldsymbol{p}$ can be treated as being strong entering apportionments.

Let $\mathfrak{S}$ be the set of all paths which can be defined by these entering and exiting apportionments. If at house size $n-1$, the selected apportionment is $\boldsymbol{k}^{(n-1)}$, then the eligibility set, $E(n)$, at house size n is the intersection of P_n with all paths in $\mathfrak{S}$ passing through $\boldsymbol{k}^{(n-1)}$. That is, $E(n)$ is all choices of $\boldsymbol{k}^{(n-1)}+\boldsymbol{e}_j$ which are dominated by some exiting apportionment. If there is some additional side constraint, such as requiring each state to have at least one seat, then $E(n)$ is the subset of house-monotone apportionments $\{\boldsymbol{k}^{(n-1)}+\boldsymbol{e}_j\}$ which are closest to $\mathfrak{S}$. In either case, the selected apportionment $\boldsymbol{k}^{(n)}$ is an element of $E(n)$ closest to $n\boldsymbol{p}$. (If $s=3$, all of this can be replaced by selecting a house-monotone apportionment which minimizes the distance to $n\boldsymbol{p}$.)

To show that the eligibility sets are always non-empty, we construct a family of house-monotone apportionments $\{\boldsymbol{q}^{(n)}\}$ which has a member in each eligibility set. At house size n, initially assign to each state its lower quota value, and assume that after this is done there are $l(n)$ seats left to be assigned. Order the states according to the values of $\lambda>n$ such that λp_i is an integer; that is, according to the ordering defined by when a state will leave a quota cube. According to this ordering, reassign the first $l(n)$ states their upper quota value. By construction, this family is house monotone, quota-preserving, and it has a representative in each set of candidate-entering apportionments. Therefore, this representative is also an entering apportionment.

A different house-monotone, quota-preserving method has been proposed on page 714 of BY. The easiest way to compare the two methods is to point out the differences in the philosophies

leading to their development. The construction of the quasi-norm minimizing method presented here is meant to strike a compromise among the constraints of finding the apportionment family closest to $\{np\}$ which also satisfies quota and which is house monotone. The BY Quota Method determines the eligibility set by using the requirements of quota-preserving, house-monotonicity, and an additional requirement they call consistency. This "consistency" requirement is a type of Independence of Irrelevant Alternatives condition [6]. The actual apportionment is then chosen by means of a ranking function, one which does not satisfy the diagonal condition stated in Section 3.

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A HISTORICAL SKETCH OF THE OLYMPIADS, NATIONAL AND INTERNATIONAL

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Preliminary remarks. It is natural to start this sketch with the sequence of events that led to the U.S.A. Mathematical Olympiad (USAMO) and the participation of our country in the International Mathematical Olympiad (IMO). That sequence began at the 1949 spring meeting of the Metropolitan New York Section of the Mathematical Association of America (MAA), when a committee was appointed to study the possibility of organizing a contest in high school mathematics. At the 1949 fall meeting of that Section, Professor W. H. Fagerstrom of City College was made chairman and director of the contest. The first competition, called the "Mathematical Contest," was held May 11, 1950, with over 6,000 students from 238 high schools taking part. The competition continued under the sponsorship of the Metropolitan New York Section and with the same title until, by 1956 or earlier, the word "Annual" was added. For the first five or six years, Professor Fagerstrom, by himself, did all the work of the competition in his home—except for the preparation of the examination questions.

Professor Turner has had a varied teaching career at all levels, from elementary school to university. Most of her mathematical training was at the University of Iowa. From 1946 to 1970 she taught at the State University of New York at Albany, where she is now Professor Emeritus. Her chief mathematical interest is in statistics. She played an important role in connection with starting the U.S.A. Mathematical Olympiad and getting this country involved with the International Mathematical Olympiad. Since 1975 she has had the official title of Advisor to the Olympiad Awards Ceremony.—*Editors*

Interest and participation in the project, within the Metropolitan New York Section, in the state of New York, and throughout the country, spread like wildfire. By 1957, participation had taken on such proportions that the Metropolitan Section asked the parent organization, the MAA, to make the activity one of its own. That it did—incidentally, before the advent of Sputnik.

The first examination of the competition on a national basis was held March 27, 1958, as an activity sponsored jointly by the MAA and the Society of Actuaries. Under nationalization, the competition was called the "Annual H.S. Mathematics Contest" until 1969; from then through 1976 it was called the "Annual H.S. Mathematics Competition." The 1977 Summary of Results and Awards was entitled, "Annual H.S. Mathematics Examination." Sponsors have been added from time to time; the competition is now sponsored jointly by the MAA, the Society of Actuaries, Mu Alpha Theta, the National Council of Teachers of Mathematics, and the Casualty Actuarial Society.

With the shift to a national basis, the activity spread by leaps and bounds, reaching across oceans. Many foreign countries and APO schools used the paper of the examination. The Netherlands was the first country to translate the questions into its native language. The United Kingdom was, perhaps, the first to make the most efficient use of it, both as a national competition and as a qualifying test for the British Mathematical Olympiad.

With the success of the British in this dual use, an effort was made in our country, after the International Congress of Mathematicians (ICM) in Moscow in 1966, to interest the Committee on High School Contests of the MAA in both the holding of a national Mathematical Olympiad and eventual participation in the IMO, an Eastern European originated and controlled affair. That effort failed. However, encouragement presented itself when an invitation was received from the Advisory Committee of the Guinness Awards for Science and Mathematics Teachers to bring a team of six secondary-school students from the upstate area of New York State to England to compete in mathematics with a similar group of British students from the London area (**[1]**). Thus the first involvement of the United States in international mathematical competition (**[2]**) took place on May 20, 1968, the date of the holding of the Fourth British Olympiad, at Imperial College, University of London, under the direction of Professor and Mrs. Walter Hayman (**[3]**).

I made an effort to promote further involvement by the United States in international mathematical competition at the ICM at Nice in 1970. Meeting with Professor Tiberiu Roman of Bucharest, Professor Helmut Bausch of the Academy of Sciences of the German Democratic Republic (DDR), and Professor Janos Surányi of the University of Budapest, I learned that: (1) an invitation had been sent to the United States by Roumania to participate in the Eleventh IMO in Bucharest in 1969, but a reply had been received to the effect that the invitation had come too late, that a time of nine months would be needed for preparation; (2) Hungary had not extended an invitation to the United States to the Twelfth IMO held in 1970; (3) but Czechoslovakia would send an invitation to both the United States and Canada for participation in the Thirteenth IMO to be held in Prague in 1971. Though I made concentrated attempts, I could not learn to whom in our country the 1969 and 1971 invitations had been sent.

Finally, interest in our holding a USAMO and participating in the IMO was aroused (**[4]**); our first USAMO was held May 9, 1972; and our first participation in the IMO was on July 8 and 9, 1974. Both initial steps were so successful that continuity was assured.

Professor Samuel L. Greitzer, chairman of the U.S.A. Mathematical Olympiad Committee, was instrumental in our obtaining an invitation to the Sixteenth IMO held in Erfurt, DDR. He and Professor Murray S. Klamkin, a member of the committee, have trained and led our outstandingly successful teams to the IMO's, with Professor Cecil C. Rousseau acting instead of Professor Klamkin in 1974.

The USAMO is an invitational affair; approximately one hundred top-ranking students in the Annual High School Mathematics Competition are asked to participate. The eight top-ranking students in the Olympiad are the declared winners and constitute our team to the IMO.

The eight winners have been honored each year, 1972–1977, in Washington, D.C., at a three-part Awards Ceremony: the bestowing of awards and the giving of the Olympiad Address in the Board Room of the National Academy of Sciences; the reception and dinner in the rooms of the Diplomatic Reception Area of the Department of State (with the exception of the second Awards Ceremony in 1973, when the reception and dinner were held in the Great Hall of the National Academy of Sciences). The addresses have been given by Emmanuel R. Piorè, Saunders Mac Lane, Lowell J. Paige, Peter D. Lax, Andrew M. Gleason, and Alan J. Hoffman, respectively.

National Mathematical Olympiads or their equivalents. Dr. Petar Kenderov of Bulgaria (recently, until November 1978, at the Goethe Institute, Frankfurt) encouraged me (in a letter to me dated September 15, 1977) to start collecting information about national and international Mathematical Olympiads.

In doing this, it has been found that "Olympiad" has been ambiguously used both as a word in an official title of a competition and as a synonym for competition; for example, it occurs in the listing of "years of first mathematical olympiads in various countries" in the ICMI Report on Mathematical Contests in Secondary Education (Olympiads) ([5]), regardless of the geographical area covered in the countries. In other words, there exists no standard interpretation of the word "olympiad." So, with help received from mathematicians in several countries who have been working closely with mathematical competitions, I shall attempt to clarify the situation.

Those countries having a mathematical competition officially called "Mathematical Olympiad" or "Mathematics Olympiad"—and with their names occurring in the title—are: Austria (1970), Bulgaria (1949), Canada (1969), Czechoslovakia (1951), Great Britain (1965), Israel (1968), the Netherlands (1962), Poland (1949/50), South Africa (1966), U.S.A. (1972), and the USSR (1967). (The dates in parentheses indicate when respective national Olympiads were first held.)

Other countries have competitions comparable or somewhat comparable to what are known officially as "Mathematical Olympiads." Cuba has one called "National Competition of Mathematics" (1971). The Federal Republic of Germany has a national competition named "National Mathematical Contest" (Bundeswettwerb Mathematik) (1970). Finland has held since 1965 a competition arranged by the teachers' association of that country—whether or not Finland has participated in the IMO. In France there exists a "General Competition of Mathematics." Students who participate belong to the highest grade of the high school and are selected by their professors. However, certain academies are organizing competitions of a broader scale. The German Democratic Republic has one named "Olympiad for Young Mathematicians" (Olympiade Junger Mathematiker) (1961). The Hungarian situation is probably the best known. The József Kürschák competition, the successor since 1949 to the Baron Lórand Eötvös competition started in 1894, and the National Secondary School Mathematical Contest that was started in 1923 are, perhaps, the primary ones. The Kürschák competition was for graduates of the secondary schools of the same year as the competition, who at the time were freshmen in the universities or in the Polytechnic Institute; now secondary-school students also are allowed to participate. The National Secondary School Mathematical Contest is only for students of the eleventh and twelfth grades. These competitions differ from our USAMO in that our participants are subject to an upper limit of twelfth-grade status but no lower limit. The Roumanian situation is interesting. The first mathematical contest for secondary-school students was organized in 1902 by the Gazeta Matematică and was continued until 1948. Since 1950 it has continued on a larger scale, in both type of school and range in age. The name was and is "Mathematical Contest" (Concursul de Matematică) but between 1954 and 1965 it was called "Mathematical Olympiad." Sweden holds a national competition called "Skolornas Matematiktävling."

Whatever the national competitions in Greece, Mongolia, Vietnam, and Yugoslavia are called, they were initiated in 1969, 1963, 1962, and 1962, respectively.

It is pertinent that the expression "Mathematical Olympiad" was first employed in 1934 for

the local Olympiad in Leningrad and next used in 1935 for the local Olympiad in Moscow. Might it not be of positive significance if those words were part of the official title of the national mathematical competitions held in each country?

The International Mathematical Olympiad. This international competition in mathematics originated in Roumania. The person responsible for its origin is Professor Tiberiu Roman of Bucharest who, fittingly, was called its "father" by Professor Endre Hódi of Hungary in a paper presented at the International Congress on Mathematical Education at Exeter in 1972. The title "International Mathematical Olympiad" was first used in January of 1959 when the then "Mathematical and Physical" Society of Roumania sent seven invitations to the first IMO; six were accepted. The competition program, held in Bucharest, extended from the 22nd to the 30th of July.

There is no organization to administer the IMO: no constitution, no central committee. During each IMO, held the first part of July, an international jury, composed of a president, who represents the host country, and n members, one from each of n participating countries, meet in plenary sessions to determine the questions of the IMO currently being held. An invitation for the next IMO is extended at the closing festivity of an IMO by the chief of a delegation in behalf of the Ministry of Education of his country. After several months, the Ministry of Education of the prospective host sends official invitations to the Ministries of the other countries.

The group of students and two accompanying adults from each country that has accepted an invitation to an IMO are considered a delegation; the two adults are known as leaders. The examination consists of six questions, three worked on each day, with four hours allotted for working each set.

Officially the IMO is a competition among students, not among teams, but team scores are generally published. Table 1 shows what countries have participated each year since the inception in 1959, the team rankings for each participating country for each year, and what countries have participated with incomplete teams (shown by asterisks). A complete team is composed of eight students. The table was compiled from data obtained primarily from Professor Roman and from the pamphlet "Problems and Solutions of the First Through Fifteenth International Mathematical Olympiad," published by the Council of the County of Leipzig, Education Division. Ties in team scores 1960–1973 were noted in the information from the DDR. Professor Roman may have considered a tie of no great consequence. Where there are no markings or asterisks, there was no participation. The country hosting the IMO in a given year is indicated by the underscoring of that country's team-ranking for that year.

Additional and explanatory comments. Though Brazil has neither had a national Mathematical Olympiad nor participated in the IMO, the Academia de Ciências de Estado de São Paulo sent Rosa Feldman as an observer to the Nineteenth IMO at Belgrade with the idea that in 1978 it might be possible for Brazil to send a group of students to the 20th IMO to be held in Bucharest.

As for China, "The first competitions were organized in 1956 by the Chinese mathematics society in Shanghai, Peking, Tsietsin and Hankow and it was not long before other cities followed suit. The era of 'antichampionism' in the sixties and The Great Cultural Revolution terminated the examination." [**6**]

Finland has had a difficult time in the efforts made by supporters to participate in the IMO. At the time of the seventh one, hosted by the DDR, the East Germans invited the teachers' association of Finland to send a team to the competition. All costs, including travel to and from the DDR, were paid by that country. Dr. Matti Lehtinen, University of Helsinki, was a member of that team put together hastily by the teachers in and about Helsinki. Having had no training and no advance knowledge of the competition, Finland finished last—tenth—but she has the distinction of being the first non-communist country to participate in the IMO. Since 1965, the teachers' association has arranged a competition to select members of the Finnish IMO team; but, with limited financial and manpower resources, that association has not met with great

TABLE 1: Team Rankings for Countries Participating in the IMO, 1959–1977

	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977
1. Algeria																			21
2. Austria												12	7	$7\frac{1}{2}$	8	6	6	$\underline{5}$	12
3. Belgium											13								19
4. Bulgaria	4	4	6	6	6	5	8	$\underline{6}$	$6\frac{1}{2}$	9	6	$7\frac{1}{2}$	11	10	$12\frac{1}{2}$	11	$\underline{7}$	4	6
5. Cuba													*	*	*	*		*	18
6. Czechoslovakia	3	1	4	$\underline{4}$	5	7	6	8	$6\frac{1}{2}$	7	8	$7\frac{1}{2}$	$\underline{9}$	9	7	12	11	$12\frac{1}{2}$	9
7. Fed. Rep. of Ger.																			7
8. Ger. Dem. Rep.	6	5	5	7	7	6	$\underline{5}$	3	2	1	2	$2\frac{1}{2}$	3	3	3	$\underline{4}$	2	8	8
9. Finland							10								14	15		16	16
10. France									*		$10\frac{1}{2}$	9	12		6	8	9	6	14
11. Great Britain									4	4	5	6	$5\frac{1}{2}$	5	5	9	5	2	$3\frac{1}{2}$
12. Greece																	14	17	
13. Hungary	2	$2\frac{1}{2}$	$\underline{1}$	1	2	2	2	2	3	3	1	$\underline{1}$	1	2	2	3	1	7	$3\frac{1}{2}$
14. Italy									*	11									20
15. Mongolia						8	9	9	11	12	9	14	13	13	15	16	15		17
16. Netherlands											14	13	10	12	$12\frac{1}{2}$	14	16	15	5
17. Poland	5		2	5	$\underline{8}$	4	4	4	10	5	$10\frac{1}{2}$	11	4	$\underline{6}$	4	13	13	9	11
18. Roumania	$\underline{1}$	$\underline{2\frac{1}{2}}$	3	3	3	3	3	5	5	8	$\underline{4}$	5	$5\frac{1}{2}$	4	9	7	8	11	15
19. Sweden									9	6	12	10	*	11	11	10	12	10	13
20. Soviet Union	*			2	1	$\underline{1}$	1	1	1	$\underline{2}$	3	$2\frac{1}{2}$	2	1	$\underline{1}$	1	4	1	2
21. U.S.A.																2	3	3	1
22. Vietnam																*	*	14	
23. Yugoslavia					4	9	7	7	$\underline{8}$	10	7	4	8	$7\frac{1}{2}$	10	5	10	$12\frac{1}{2}$	$\underline{10}$

success. Many teachers simply ignore the competition; some think that their students are not bright enough; some students think likewise; and many of those participating in the competition do not bother to report their solutions. Although the selection competition has been held annually, there was a long break in the Finnish participation in the IMO. The cost of sending a team abroad was too high for the teachers' association and official invitations did not always reach persons in the state bureaucracy who might have known what the invitations were about. Dr. Lehtinen states: "Unfortunately the prevalent attitudes in education in this country now stress equality in learning and in the name of democracy disapprove all competition which may place some person before some other."

Italy has no national mathematical competition. That country's lack of interest in the IMO—having participated for only a third time in 1977 after a lapse of eight years—is understandable when one recalls Professor de Finetti's remarks at the ICME at Exeter. He felt that it would have a better effect to have an international competition of an informal nature, one with problems that could be put in newspapers and that would appeal to the general public so that people would not consider mathematics as "something to be hated." Perhaps he had a point.

One notes from the chart that the Netherlands' record was far from outstanding until the Nineteenth IMO in 1977. Dr. Jan van de Craats of the University of Leiden has written that since the Netherlands had always had bad results it was decided to do some preparation. The program begun in the fall of 1976 not only proved successful, with the Netherlands' team ranking fifth in 1977, but resulted in the students' expressing a liking for the special mathematical training and being stimulated by it.

Attention might well be called to these facts: Of the five countries that have participated in all of the nineteen years, Roumania ranks third; her record for the first ten years put her second to Hungary, but since then her record has been slipping.

The question as to the participation of girls in mathematical competition frequently arises. With respect to the IMO's, Mr. Robert Lyness, who has been one of the British leaders for the past several years, recalls that since 1967 not more than four or five girls have participated per IMO. In 1976 a Russian girl made 39 out of 40 points.

At the opening of the celebration of the Ninth IMO in Yugoslavia in 1967, Mr. Vahasin Micunović, former Minister of Education, called mathematics a kind of science which is worthy of being chosen as a subject of international contest, since mathematics does not know borderlines and different ideologies. It might be added that participation in the IMO by a qualified student—such as a winner of the USAMO—is worthy of the interest, monetary expenditure, and recognition that his or her country might muster to make participation in an IMO possible.

I am indebted to mathematicians of various countries involved in national mathematical competitions and to the IMO for contributions of information used in this article. I am particularly indebted to Professor Tiberiu Roman, whose correspondence during the years since the ICM at Moscow in 1966 has provided the backbone of the article.

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MATHEMATICAL NOTES

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Beginning January 1, 1979, this section will be edited by Deborah Tepper Haimo and Franklin Tepper Haimo. Material for this department should be sent to Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.
R.P.B.

A SIMPLE IMPROVEMENT ON THE BINOMIAL SERIES

LEON GERBER

The purpose of this note is to show that the convergence of the partial sums of the binomial series for $(1+z)^a$ can be considerably speeded by dividing the last term by $1+z$. This improvement is most dramatic in the case $a=-1$ where the partial sum

$$1+z+z^2+\cdots+z^{k-1}+z^k$$

improves to the equality

$$(1+z)^{-1}=1+z+\cdots+z^{k-1}+\frac{z^k}{1+z}$$

THEOREM. *Let z be a complex number not on the ray $(-\infty,-1)$. Let s_k be the sum of the first k terms of the binomial series for $(1+z)^a$ whether or not it converges, let $r_k=(1+z)^a-s_k$ be the remainder after k terms, and let t_k be the* kth *term. Then we assert that*

$$t_{k+1}/(1+z)\sim r_k \quad (k\to\infty).$$

In fact

$$s'_{k+1}\equiv s_k+t_{k+1}/(1+z)=(1+z)^a+O(t_{k+1}/k).$$

Proof. We have

$$k!\,t_{k+1}=a(a-1)\cdots(a-k+1)z^k$$

and by Taylor's theorem

$$(k-1)!\,r_k=\int_0^z a(a-1)\cdots(a-k+1)(1+u)^{a-k}(z-u)^{k-1}\,du.$$

Thus

$$\begin{aligned} r_k/t_{k+1} &= kz^{-1}\int_0^k (1+u)^{a-k}(z-u)^{k-1}\,du \\ &= k(1+z)^a\int_0^1 x^{k-1}(1+xz)^{-1-a}\,dx \end{aligned}$$

where

$$x=\frac{z-u}{z(1+u)}.$$

Integrating by parts, we obtain

$$r_k/t_{k+1}=(1+z)^{-1}+(1+a)z(1+z)^a\int_0^1 x^k(1+xz)^{-2-a}\,dx.$$

Since $1+xz$ is bounded away from zero for $0\leqslant x\leqslant 1$, the second term is $O\left(\frac{1}{k}\right)$ as $k\to\infty$, so $(1+z)^a-s_k=r_k=t_{k+1}[(1+z)^{-1}+O(1/k)]$, which proves our assertion.

It follows that s_k' converges to $(1+z)$ whenever $t=O(k)$. Thus for $|z|=1$, s_k' converges for $\mathrm{Re}(a)>-2$ while it is known that s converges only for $\mathrm{Re}(a)>-1$. Further, it is an immediate corollary that $s_{k+1}=(1+z)^a+O(t_{k+1})$ so s_k' always converges more rapidly than s_k unless a is a positive integer. The improvement is greatest for negative exponents. In fact, for $(1+.96)^{-1.5}=.3644$, s_k alternates in sign for $k\leqslant 26$ while s_k' always has the correct sign, $s_2'=.44$ correct to one decimal and s_{26}' is correct to two decimals. It may be argued that exponents with negative real parts can be avoided by first taking reciprocals. However, improvement is great even for $\mathrm{Re}(a)>0$ as illustrated by the example $(1+.96)^{.5}=1.4000$.

k	s_k	r_k	s_k'	r_k'	$\lvert r_k/r_k'\rvert$
1	1.48	−.0800	1.2448	+.1551	.51
2	1.3648	+.0352	1.4212	−.0212	1.7
3	1.4201	−.0201	1.3930	+.0070	2.9
4	1.3869	+.0131	1.4032	−.0032	4.1
5	1.4092	−.0092	1.3983	+.0017	5.4
6	1.3932	+.0068	1.4010	−.0010	6.7
7	1.4053	−.0053	1.39934	+.00066	8.0
8	1.3958	+.0042	1.40045	−.00045	9.3
17	1.40096	−.00096	1.39995	+.00005	20.9
18	1.39915	+.00085	1.40004	−.00004	22.3
19	1.40075	−.00075	1.39997	+.00003	23.6
20	1.39933	+.00067	1.40003	−.00003	24.9

Note that s_{20} is not quite as accurate as s_7'.

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COUPLING AND THE RENEWAL THEOREM

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1. Introduction. The renewal theorem tells us about the expected number of visits of a random walk to a set. It has been of great importance in classical probability theory, and its proofs and ramification abound in the literature. Techniques used in various proofs include Fourier analysis, Tauberian theory, Banach algebras, harmonic functions, and Choquet's theorem; and all, to some degree, involve analysis of the "renewal equation," whose solution is the sought-after expectation. Such proofs are enlightening because they illustrate a pretty interplay between varied areas of analysis.

In recent years an interesting technique called "coupling" has led to direct and elegant proofs of the ergodic theorem for Markov chains. In this approach one lets a process evolve until it gets

close to a "nice" process, whose behavior is known, then couples the two processes in such a way that their individual marginal distributions are not disturbed. One thereby makes inferences about the original process. The idea can be found in Doeblin [3] and Vasershtein [9], was used by Ornstein [8], and was further developed by Griffeath [5]. It has already appeared in undergraduate textbook literature [6].

In view of the close relation between the ergodic and the renewal theorems, it is not surprising that the coupling argument should lend itself also to a direct proof of the renewal theorem, and it is the purpose of this note to give such a proof. The argument is intuitively natural, and quite accessible to a student who has had a good first course in probability.

A novelty of our proof is that it uses coupling and stationarity directly to yield the *two*-sided renewal theorem. Although we use the traditional "ladder variables," we do so in a simpler way than for the usual argument which extends the one-sided to the two-sided case.

We recently received a preprint of a coupling proof for the one-sided renewal theorem by Lindvall [7]. Our couplings and the organization of our arguments differ somewhat; his relies on the Hewitt–Savage 0–1 law, ours on the recurrence of a symmetric random walk.

2. Statement of result and prerequisites. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) non-arithmetic random variables, such that $E|X_1| < \infty$; $S_n = X_1 + \cdots + X_n$, $S_0 = 0$; $\mathbb{S} = (S_0, S_1, \ldots) =$ a random point set in R. For any set A, let $\mathcal{N}(A) =$ the number of points in $\mathbb{S} \cap A$. If A is an interval (a,b), $-\infty < a < b < \infty$, write $\mathcal{N}(a,b)$.

(2.1) BLACKWELL'S RENEWAL THEOREM. *If* $EX_1 = \mu > 0$, *then*

$$E\mathcal{N}(a+t, b+t) \to \begin{cases} \dfrac{b-a}{\mu} & \text{as } t \to \infty \\ 0 & \text{as } t \to -\infty. \end{cases}$$

The proof will be self-contained except for the following standard results, elementary proofs of which are readily accessible in textbooks (e.g., Chung [2], Feller [4], Breiman [1]).

(2.2) WALD'S EQUATION. If $EX_1 > 0$, and $N = \inf\{n : S_n > 0\}$, then $EN < \infty$ and $ES_N = EN \cdot EX_1$.

(2.3) TRANSIENCE. If $EX_1 \neq 0$ then $E\mathcal{N}(a,b) < \infty$.

(2.4) RECURRENCE. If $EX_1 = 0$ and $a < b$, then $(a,b) \cap \mathbb{S} \neq \varnothing$ w.p.1. The remainder of the article is a proof of (2.1). Thus, unless stated otherwise, $EX_1 = \mu > 0$.

3. Counting measures. We start by observing a few facts about $\mathcal{N}(A)$, where A is a bounded (Borel) set. Expressing $\mathcal{N}$ in terms of the indicator function $\chi_A(x) = 1$ if $x \in A$, $= 0$ if $x \in A$, we write

$$\mathcal{N}(A) = \sum_{n=0}^{\infty} \chi_A(S_n).$$

STEP 1. There exists a $\lambda < \infty$ such that

$$P\{\mathcal{N}(A-a) \geqslant k\} \leqslant P\{\mathcal{N}(-\lambda, \lambda) \geqslant k\}, \qquad -\infty < a < \infty, \qquad 0 \leqslant k < \infty.$$

Proof. Let $N_a =$ the first entry time of $S_n + a$ into A, i.e.,

$$N_a = \begin{cases} \inf(n : S_n \in A - a) \\ \infty \text{ if } S_n \notin A - a \text{ for all } n \geqslant 0. \end{cases}$$

Then

$$\mathcal{N}(A-a) = \begin{cases} 1 + \sum_{N_a+1}^{\infty} \chi_{A-a}(S_n) \text{ if } N_a < \infty \\ 0 \text{ if } N_a = \infty. \end{cases}$$

Thus for $k \geq 1$

$$P\{\mathfrak{N}(A-a) \geq k\} = P\left\{\sum_{N_a}^{\infty} \chi_{A-a}(S_n) \geq k,\, N_a < \infty\right\}$$
$$= P\left\{\sum_{N_a}^{\infty} \chi_{A-a-S_{N_a}}(S_n - S_{N_a}) \geq k,\, N_a < \infty\right\}$$
$$\leq P\left\{\sum_{N_a}^{\infty} \chi_{(-\lambda,\lambda)}(S_n - S_{N_a}) \geq k,\, N_a < \infty\right\} \leq P\left\{\sum_{n=0}^{\infty} \chi_{(-\lambda,\lambda)}(S_n) \geq k\right\}.$$

The inequality follows from the fact that $\{S_n - S_{N_a};\ n \geq N_a\}$ is independent of N_a, and is a copy (in distribution) of $\{S_n;\ n \geq 0\}$; and from the existence of a $0<\lambda<\infty$ such that $A-a-S_{N_a} \subset (-\lambda,\lambda)$. The latter is a consequence of the fact that $S_{N_a} \in A-a$ and A is bounded. □

REMARK 1. By (2.3) $E\mathfrak{N}(-\lambda,\lambda)<\infty$ for any $0<\lambda<\infty$. With Step 1 this implies that $\{\mathfrak{N}(A-a);\ a \in R\}$ is uniformly integrable.

REMARK 2. Since $S_n \to \infty$ w.p.l., and A is bounded, $\mathfrak{N}(A-a) \to 0$ w.p.l. as $a \to \infty$. By the uniform integrability $E\mathfrak{N}(A-a) \to 0$ as $a \to \infty$; proving the second part of (2.1). Let

$$\mu(a;A) = E\mathfrak{N}(A-a) \tag{3.1}$$

$$N = \inf(n : S_n > 0), \qquad Z = S_N > 0 \tag{3.2}$$

$$\nu(a;A) = E\left(\sum_{n=0}^{N-1} \chi_A(a+S_n)\right). \tag{3.3}$$

Thus $\nu(a)$ is the expected number of visits of the walk to the set $A-a$ *before* S_n becomes positive. (We will drop the fixed set A from the notation when the context is clear.)

STEP 2.

$$\mu(a) = \nu(a) + E\mu(a+Z). \tag{3.4}$$

Proof.

$$\sum_{0}^{\infty} \chi_A(a+S_n) = \sum_{0}^{N-1} \chi_A(a+S_n) + \sum_{N}^{\infty} \chi_A(a+Z+S_n-S_N). \tag{3.5}$$

Observe that $\{S_n - S_N : n = N, N+1, \ldots\}$ is independent of N and S_N, and has the same distribution as $\{S_n,\ n = 0,1,\ldots\}$. Thus, taking expectations of both sides of (3.5) yields (3.4). □

Let $T_k = Z_1 + \cdots + Z_k$, $k \geq 1$, $T_0 = 0$, where $Z_1, Z_2, \ldots$ are i.i.d. with the same distribution as S_N, and are *positive*.

STEP 3.

$$\mu(a) = E\left(\sum_{0}^{\infty} \nu(a+T_k)\right). \tag{3.6}$$

Proof. Iterate Step 2 to get for all $n>0$

$$\mu(a) = E\sum_{k=0}^{n-1} \nu(a+T_k) + E\mu(a+T_n). \tag{3.7}$$

By Step 1 $\mu(\cdot)$ is a bounded function. By remark 2 $\mu(a) \to 0$ as $a \to \infty$. Thus, since $T_n \to \infty$ w.p.l., $E\mu(a+T_n) \to 0$. Hence one can let $n \to \infty$ in (3.7) to get (3.6). □

Now let Z_0 be any positive random variable with distribution $F_0(\cdot), F(\cdot)$ be the distribution of

Z_i, $i \geqslant 1$, F^{*k} that of T_k, $F_0 * F^{*k}$ that of $Z_0 + Z_1 + \cdots + Z_k$, and $U_0(\cdot)$ be the associated "renewal measure"

$$U_0(t) = \sum_{k=0}^{\infty} (F_0 * F^{*k})(t). \tag{3.8}$$

Substitution into (3.6) then yields:

STEP 4.

$$E\mu(Z_0) = \int_0^{\infty} \nu(s)\, U_0(ds).$$

Proof.

$$E\mu(Z_0) = E \sum_{k=0}^{\infty} \nu(Z_0 + \cdots + Z_k) = \sum_{k=0}^{\infty} \int_0^{\infty} \nu(s)(F_0 * F^{*k})(ds). \quad \square$$

4. Stationarity. The function $U_0(t)$ equals the expectation of the number of visits $N(t)$ of the sequence

$$\{Z_0 + Z_1 + \cdots + Z_n : n \geqslant 0\}$$

to $[0,t]$, and direct substitution of (3.8) shows that it satisfies the "renewal equation"

$$U_0(t) = F_0(t) + \int_0^t U_0(t-y) F(dy). \tag{4.1}$$

It is the unique solution which is bounded on bounded sets. In particular, if we take

$$F_0(t) = \frac{1}{EZ_1} \int_0^t [1 - F(y)]\, dy, \tag{4.2}$$

then we can check that $U_0(t) = t/EZ_1$ is the solution.

Thus $U_0(\beta + t) - U_0(\alpha + t) = (\beta - \alpha)/EZ_1$ $(0 < \alpha < \beta)$, making the expected number of visits of $\{Z_0 + \cdots + Z_n\}$ to any interval depend only on its length (not its location). In this sense the one-sided renewal process $(Z_0 + \cdots + Z_n,\ n \geqslant 0)$ is called stationary.

In the two-sided case we similarly randomize the starting point by a random variable Z_0. Then we have analogously

STEP 5.1. If Z_0 has d.f. F_0 in (4.2), then

$$E\mu(Z_0; A) = \frac{1}{EZ_1} \int_0^{\infty} \nu(s; A)\, ds. \tag{4.3}$$

Proof. Step 4 and the fact that $U_0(t) = t/EZ_1$. More precisely:

STEP 5.2.

$$E\mu(Z_0; A) = \frac{1}{EZ_1} E\left(\sum_0^{N-1} m[(A - S_n) \cap R^+] \right). \tag{4.4}$$

Proof. Use the definition of ν in (4.3), and interchange the order of integration and summation.

Thus $E\mu$ is a complicated expression depending on the measure of the set swept out by $A - S_n$ before the walk S_n hits R^+. It is fortunate that, when $A \subset R^+$, (4.4) depends only on the Lebesgue measure of $A, m(A)$. (Note that $n < N$ implies $S_n \leqslant 0$ implies

$$A - S_n \subset R^+, \qquad m[(A - S_n) \cap R^+] = m(A).)$$

Thus by (4.4) and (2.2),

STEP 6. If $A \subset R^+$, then

$$E\mu(Z_0;A) = m(A)/\mu. \tag{4.5}$$

5. Coupling. Let $\xi_n = a + S_n$ and $\eta_n = Z_0 + \tilde{S}_n$, $n = 0, 1, \ldots$, where $\tilde{S}_k = \tilde{X}_1 + \cdots + \tilde{X}_k$, $\{X_n\}$ and $\{\tilde{X}_n\}$ are i.i.d., and independent of Z_0 (defined by (4.2)).

We know by Step 6 that for t sufficiently large

$$E\tilde{\mu}(Z_0;A+t) \equiv E\sum_0^\infty \chi_{A+t}(\eta_n) = \frac{m(A)}{\mu}. \tag{5.1}$$

We want to show that:

STEP 7.

$$\mu(a;A+t) \equiv E\sum_0^\infty \chi_{A+t}(\xi_n). \tag{5.2}$$

$$\to \begin{cases} m(A)/\mu & \text{as } t \to \infty \\ 0 & \text{as } t \to -\infty. \end{cases}$$

(m denotes Lebesgue measure.)

Proof. Since $E|X_i| < \infty$, it follows from (2.4) applied to the symmetric sequence $\{X_i - X_i'\}$ that given any $\varepsilon > 0$,

$$M \equiv \inf\{n : |\xi_n - \eta_n| < \varepsilon\} < \infty \quad \text{w.p.l.}$$

Set

$$\tilde{\xi}_n = \begin{cases} \xi_n & \text{when } n \leqslant M \\ \xi_M - \eta_M + \eta_n & \text{for } n > M. \end{cases}$$

Then $\{\tilde{\xi}_n\} \stackrel{D}{=} \{\xi_n\}$.

Hence

$$\mu(a;A) = E\sum \chi_A(\xi_n) = E\sum \chi_A(\tilde{\xi}_n) = E\sum_0^{M-1} \chi_A(\tilde{\xi}_n) + E\sum_M^\infty \chi_A(\Delta + \eta_n),$$

where $\Delta = \xi_M - \eta_M$, $|\Delta| \leqslant \varepsilon$. Since $E\tilde{\mu}(Z_0, A) = E\Sigma_0^\infty \chi_A(\eta_n)$, we have

$$\begin{aligned} &|\mu(a;A) - E\mu(Z_0;A)| \\ &\leqslant E\sum_0^{M-1} \chi_A(\tilde{\xi}_n) + E\sum_0^{M-1} \chi_{A-\Delta}(\eta_n) \\ &+ \left|E\left(\sum_0^\infty \chi_{A-\Delta}(\eta_n) - \sum_0^\infty \chi_A(\eta_n)\right)\right|. \end{aligned} \tag{5.3}$$

We now take $A = [a,b]$, $-\infty < a \leqslant b < \infty$. Then

$$A^- \equiv [a+\varepsilon, b-\varepsilon] \subset A - \Delta \subset [a-\varepsilon, b+\varepsilon] = A^+,$$

$$\sum_0^\infty \chi^{A^-}(\eta_n) \leqslant \sum_0^\infty \chi_{A-\Delta}(\eta_n) \leqslant \sum_0^\infty \chi^{A^+}(\eta_n),$$

and hence last term on the right of (5.3) is bounded by $2\varepsilon/\mu$. Replacing A by $A+t$ we see from (5.1) and (5.3) that for t sufficiently large

$$|\mu(a;A+t) - m(A)/\mu| \leqslant E\sum_0^{M-1} \chi_{A+t}(\tilde{\xi}_n) + E\sum_0^{M-1} \chi_{A-\Delta+t}(\eta_n) + \frac{2\varepsilon}{\mu}. \tag{5.4}$$

Now as $t \to \infty$ the first two terms $\to 0$, since the random variables under the expectations $\to 0$ w.p.l. as $t \to \infty$, and they are uniformly integrable in t by step 1. This proves the first part of

(5.2). (The second part is Remark 2). This completes the proof of the theorem.

For more general sets A one has the following:

(5.5) COROLLARY. *Let A be any bounded Borel set whose boundary has measure* 0. *Then*

$$\lim_{t\to\infty} E\,\mathfrak{N}(A+t)=\frac{m(A)}{\mu}.$$

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POLYNOMIALS WITH ZEROS UNIFORMLY DISTRIBUTED ON THE UNIT CIRCLE

PATRICK J. O'HARA AND RENE S. RODRIGUEZ

1. Introduction. For each $n=1,2,\dots$ let $x_{n1},\dots,x_{nn}$ be a finite sequence of points in $[0,1)$. For each $t\in[0,1]$ let $N_n(t)$ denote the number of terms from $x_{n1},\dots,x_{nn}$ that lie in $[0,t)$. Then the points $x_{n1},\dots,x_{nn}$, $n=1,2,\dots$ are said to be uniformly distributed if

$$\lim_{n\to\infty}\frac{N_n(t)}{n}=t, \tag{1}$$

for each $t\in[0,1]$. In particular, if for each n the points $x_{n1},\dots,x_{nn}$ are the first n terms of a given sequence x_1, $x_2,\dots$ in $[0,1)$, then this sequence is said to be uniformly distributed if (1) holds.

Beginning with the celebrated paper of Hermann Weyl [**12**] in 1916, the concept of uniform distribution and its relations to other areas in mathematics have been well developed. In [**8**] there is an excellent exposition and bibliography for much of this work; but as the authors indicate, certain topics had to be omitted. The purpose of this note is to present a proof of the following theorem, which gives several well-known equivalent versions of uniform distribution that are not mentioned in [**8**].

THEOREM 1. *For each $n=1,2,\dots$ let $x_{n1},\dots,x_{nn}$ be a finite sequence of points in* $[0,1)$ *and let P_n be the complex polynomial defined by*

$$P_n(z)=\prod_{k=1}^{n}\left(z-e^{i2\pi x_{nk}}\right).$$

Then the following statements are equivalent:

(i) *The points $x_{n1},\dots,x_{nn}$, $n=1,2,\dots$ are uniformly distributed.*

(ii) $\lim_{n\to\infty}\|P_n\|^{1/n}=1$, *where* $\|P_n\|=\max_{|z|=1}|P_n(z)|$.

(iii) $\limsup_{n\to\infty}|P_n(e^{i\theta})|^{1/n}\leqslant 1$ *for each* $\theta\in[0,2\pi]$.

(iv) $\lim_{n\to\infty}|P_n(z)|^{1/n}=1$ *uniformly on closed subsets of* $|z|<1$.
(v) $\lim_{n\to\infty}|P_n(z)|^{1/n}=|z|$ *uniformly on compact subsets of* $|z|>1$.

The proof of Theorem 1 will depend on two standard theorems from the literature on uniform distribution. In stating the first of these theorems, we shall use the following notations:

$F_1=\{f:f$ is a real-valued Riemann integrable function on $[0,1]\}$,
$F_2=\{f:f$ is a real-valued continuous function on $(-\infty,+\infty)$ with period $1\}$.

THEOREM 2. *The points* $x_{n1},\ldots,x_{nn}$, $n=1,2,\ldots$ *are uniformly distributed if and only if for every* $f\in F_1$ *or for every* $f\in F_2$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(x_{nk})=\int_0^1 f(x)\,dx. \tag{2}$$

A proof of this theorem is given in [**8**, p. 3]. Note that it is a routine argument to show that Theorem 2 remains valid if F_2 is replaced by any subclass of functions, D_2, whose finite linear combinations uniformly approximate the functions in F_2, i.e., for $\epsilon>0$ and $f\in F_2$ there exist functions $g_1,\ldots,g_m$ in D_2 and constants $c_1,\ldots,c_m$ such that

$$\max_{0\leqslant x\leqslant 1}|f(x)-\sum_{k=1}^{m}c_kg_k(x)|<\epsilon.$$

For example, the Weierstrass approximation theorem would allow D_2 to be the set of functions: $\sin(2\pi mx)$, $\cos(2\pi mx)$, $m=0,1,\ldots.$ This line of thought leads to the following well-known theorem of Weyl ([**8**, p. 7]).

THEOREM 3. (*Weyl's Criterion*). *The points* $x_{n1},\ldots,x_{nn}, n=1,2,\ldots,$ *are uniformly distributed if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\exp(2\pi imx_{nk})=0,\qquad m=\pm1,\pm2,\ldots.$$

2. Proof of Theorem 1. The implication (ii)→(iii) is obvious and so it suffices to show that (i)→(ii) and (iii)→(iv)→(v)→(i).

Proof of (i)→(ii). For each n choose θ_n so that $\|P_n\|=|P_n(e^{i2\pi\theta_n})|$. A straightforward computation shows that

$$\log\|P_n\|^{1/n}=\frac{1}{n}\sum_{k=1}^{n}\log|2\sin(\pi\alpha_{nk})|,$$

where $\alpha_{nk}=(x_{nk}-\theta_n)$ mod 1. If $0<\epsilon<1/6$ and $0<x<1$ then $\log|2\sin(\pi x)|\leqslant f_\epsilon(x)$ where

$$f_\epsilon(x)=\begin{cases}\log|2\sin(\pi x)|; & \epsilon\leqslant x\leqslant 1-\epsilon\\ 0; & 0\leqslant x<\epsilon \text{ or } 1-\epsilon<x\leqslant 1.\end{cases}$$

Since $\|P_n\|>|P_n(0)|=1$, it follows that for $0<\epsilon<1/6$,

$$0<\log\|P_n\|^{1/n}\leqslant\frac{1}{n}\sum_{k=1}^{n}f_\epsilon(\alpha_{nk}).$$

An easy application of Weyl's Criterion shows that the points $\alpha_{n1},\ldots,\alpha_{nn}$, $n=1,2,\ldots$ are uniformly distributed and so by Theorem 2,

$$\begin{aligned}0\leqslant\liminf_{n\to\infty}\log\|P_n\|^{1/n}&\leqslant\limsup_{n\to\infty}\log\|P_n\|^{1/n}\\ &\leqslant\int_0^1 f_\epsilon(x)\,dx=\int_\epsilon^{1-\epsilon}\log|2\sin(\pi x)|\,dx.\end{aligned}$$

As $\epsilon\to0$ the last integral above converges to zero (see [**10**, p. 135]) and so the proof is complete.

Proof of (iii)→(iv). Let f_n denote a branch of $P_n^{1/n}$ which is analytic on a simply connected domain containing the closed unit disc except for the zeros of P_n. If $f_n(z)$ is defined to be zero at the zeros of P_n, then f_n will be analytic on $|z|<1$ and continuous on $|z|\leqslant 1$. Let $g_n(z)=f_n(z)/f_n(0)$. Since $|g_n(z)|=|P_n(z)|^{1/n}\leqslant 2$, $|z|\leqslant 1$, $\{g_n\}$ is a normal family in $|z|<1$. It will suffice to show that every subsequence of $\{g_n\}$ which converges uniformly on compact subsets of $|z|<1$ must converge to 1. For then by applying a well-known property of normal families ([**10**, p. 53]) it will follow that $\{g_n\}$ converges uniformly to 1 on compact subsets of $|z|<1$. So let $\{g_{n_k}\}$ be a subsequence which converges uniformly on compact subsets of $|z|<1$ to a limit function G. Then for $0\leqslant r<1$,

$$1=G(0)\leqslant \frac{1}{2\pi}\int_0^{2\pi}|G(re^{i\theta})|d\theta=\lim_{k\to\infty}\frac{1}{2\pi}\int_0^{2\pi}|g_{n_k}(re^{i\theta})|d\theta$$

$$\leqslant \limsup_{k\to\infty}\frac{1}{2\pi}\int_0^{2\pi}|g_{n_k}(e^{i\theta})|d\theta\leqslant\frac{1}{2\pi}\int_0^{2\pi}\Big(\limsup_{k\to\infty}|g_{n_k}(e^{i\theta})|\Big)d\theta\leqslant 1.$$

The second inequality in the chain above follows from the fact that if f is analytic on $|z|<1$ and continuous on $|z|\leqslant 1$, then the integrals, $\int_0^{2\pi}|f(re^{i\theta})|d\theta$, are non-decreasing with r for $0\leqslant r\leqslant 1$. (This can be obtained, for example, by slightly extending the result in [**9**, p. 330].) The next to the last inequality above is an application of Fatou's lemma ([**9**, p. 22]), and the last inequality follows from the hypothesis. The proof is complete once it is observed that the conditions, $G(0)=1$ and $(1/2\pi)\int_0^{2\pi}|G(re^{i\theta})|d\theta=1$, $0\leqslant r<1$, hold only when $G(z)=1$ for $|z|<1$.

Proof of (iv)→(v). This implication is an immediate consequence of the identity,

$$\left|\frac{P_n(z)}{z^n}\right|=|P_n(1/\bar{z})|,\qquad z\neq 0.$$

Proof of (v)→(i). After some computation the hypothesis here can be rewritten (note that only pointwise convergence will be needed here):

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log|\exp(2\pi i x_{nk})-z|=\log|z|,\qquad |z|>1.$$

In other words, if f_z is defined by

$$f_z(x)=\log|\exp(2\pi i x)-z|,$$

then (2) holds for every function f_z with $|z|>1$. Therefore it suffices (see the remarks following Theorem 2) to establish the following lemma.

LEMMA. *Let $f\in F_2$ and $\sigma>0$ be arbitrary. There exist complex numbers $z_0,\ldots,z_m$ in $|z|>1$ and a real number c so that the function g, defined by*

$$g(x)=cf_{z_0}(x)+\sum_{k=1}^{m}f_{z_k}(x),$$

satisfies $|f(x)-g(x)|<\sigma$ for all x.

Proof. Let $\epsilon=\sigma/3$. By the Weierstrass approximation theorem there is a trigonometric polynomial, T, so that for every x,

$$|f(x/2\pi)-T(x)|<\epsilon. \tag{3}$$

Let Q be a complex polynomial such that Re $Q(e^{ix})=T(x)$. The function $\exp Q$ is entire and non-vanishing and hence $\lambda=\min_{|z|\leqslant 1}|\exp Q(z)|>0$. Choose $\delta>0$ so that $e^{-\epsilon}<1-\delta$ and $1+\delta<e^{\epsilon}$. There exists a complex polynomial, P (for example, a partial sum of the Taylor series for $\exp Q$) so that $|P(z)-\exp Q(z)|<\lambda\delta$ for $|z|\leqslant 1$. This last inequality implies that

$$e^{-\epsilon}<\left|\frac{P(z)}{\exp Q(z)}\right|<e^{\epsilon},|z|\leqslant 1, \tag{4}$$

and this inequality, with $z=e^{ix}$, can be rewritten:

$$\left|\log|P(e^{ix})|-T(x)\right|<\epsilon. \tag{5}$$

Let $P(z)=a\Pi_{k=1}^{m}(z-z_k)$ where by (4) the points z_k lie in $|z|>1$. In view of (3), (5), and the fact that

$$\log|P(e^{ix})|=\log|a|+\sum_{k=1}^{m} f_{z_k}(x/2\pi),$$

it only remains to show that there are constants c and z_0 with $|z_0|>1$ so that for all x,

$$|\log|a|-cf_{z_0}(x/2\pi)|<\epsilon. \tag{6}$$

This follows easily from the fact that

$$\lim_{|z|\to\infty} \log|a|\,\frac{\log|e^{ix}-z|}{\log|z|}=\log|a|.$$

In particular (6) will hold if z_0 is chosen with $|z_0|$ sufficiently large and $c=\log|a|/\log|z_0|$.

3. Remarks. The various implications in Theorem 1 can be found scattered about in the research literature, particularly in papers concerning the subject of complex polynomial interpolation. A more general version of (i)→(v) appeared in a famous paper of Fejér [**5**] in 1918. The converse, (v)→(i), was established by Kalmár [**7**] in 1926 (again in a more general form). In fact the proof presented here is essentially that of [**11**, pp. 168–170], in which Walsh credits the method to Kalmár. In a paper of Erdös and Turán [**3**, p. 165], the equivalence, (i)↔(iii), is credited to Fekete. Other pertinent papers include [**1**], [**4**], and [**6**].

As a final remark it is worth noting an interesting and apparently difficult unsolved problem due to Erdös [**2**]. Let $z_n=\exp(2\pi i x_n)$, $0\leqslant x_n<1$, $n=1,2,\ldots$ be any sequence of points on $|z|=1$. Prove (or disprove) that $\lim\sup_{n\to\infty}\|P_n\|=+\infty$. By using (iii)→(i) it can be seen that if $x_1, x_2,\ldots$ is not uniformly distributed then for some θ_0 and $\epsilon>0$, $|P_n(e^{i\theta_0})|>(1+\epsilon)^n$ for infinitely many n. Therefore only sequences $x_1,x_2,\ldots$ which are uniformly distributed need to be considered in this problem.

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CLASSROOM NOTES

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Beginning January 1, 1979, this section will be edited by Deborah Tepper Haimo and Franklin Tepper Haimo. Material for this department should be sent to Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St Louis, MO 63121.

A SHORT PROOF OF QUADRATIC RECIPROCITY

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Let $p=2p'+1$ and $q=2q'+1$ be odd primes, $p\neq q$. For a and q relatively prime, we have $a^{q-1}\equiv 1 \pmod q$ by Fermat's theorem, so $a^{q'}\equiv \pm 1 \pmod q$, and we define the Legendre symbol (a/q) to be $+1$ or -1 in the two cases. If $a\equiv x^2 \pmod q$, then $a^{q'}\equiv x^{2q'}\equiv 1 \pmod q$, so the q' squares, or "quadratic residues" (mod q) satisfy $(a/q)=1$ and the q' non-squares satisfy $(a/q)=-1$. The famous quadratic reciprocity theorem, discovered by Euler and Legendre and first proved by Gauss (Werke II, p. 3–8), states that for odd primes p, q,

$$(p/q)(q/p)=(-1)^{p'q'}, \text{ where } p'=\tfrac{1}{2}(p-1) \text{ and } q'=\tfrac{1}{2}(q-1). \tag{1}$$

As background for a short proof of this theorem, we note that for $1\leqslant k\leqslant q'$, division of kp by q maps k onto a least absolute remainder r_k defined by

$$kp=q([kp/q]+e_k)+(-1)^{e_k}r_k,\ 1\leqslant r_k\leqslant q', e_k=0 \text{ or } 1 \tag{2}$$

where $[kp/q]$ denotes the greatest integer $\leqslant kp/q$, and $[kp/q]+e_k$, with $e_k=0$ or 1, is the nearest integer to kp/q. From (2) we obtain

$$k\equiv[kp/q]+e_k+r_k \pmod 2 \tag{3}$$

$$kp\equiv(-1)^{e_k}r_k \pmod q \tag{4}$$

and for $1\leqslant m\leqslant q'$, congruence (4) implies

$$r_m-r_k\equiv \pm(m\pm k)p \pmod q. \tag{5}$$

Since $0<m+k<q$, the righthand member of (5) cannot vanish (mod q) unless $m=k$, so the mapping $k\to r_k$ is a permutation of the integers 1 to q'.

We now evaluate (p/q) as a product of $p'q'$ factors ± 1, which change sign when p and q are interchanged. The integer $hq-kp$ cannot vanish for $1\leqslant h\leqslant p'$ and $1\leqslant k\leqslant q'$, and is negative for $[kp/q]$ values of h. Hence by (3)

$$f_{pq}=\prod_{h=1}^{p'}\prod_{k=1}^{q'}\frac{hq-kp}{|hq-kp|}=\prod_{k=1}^{q'}(-1)^{[kp/q]}=\prod_{k=1}^{q'}(-1)^{k-r_k+e_k}. \tag{6}$$

Since the sum of $k-r_k$ vanishes and the product of k/r_k is 1, we have by (6) and (4)

$$f_{pq}=\prod_{k=1}^{q'}(-1)^{e_k}\equiv\prod_{k=1}^{q'}kp/r_k\equiv p^{q'} \pmod q. \tag{7}$$

Hence the product f_{pq} equals the Legendre symbol (p/q).

$$f_{pq}=(p/q), \qquad f_{qp}=(q/p) \tag{8}$$

$$(p/q)(q/p)=\prod_{h=1}^{p'}\prod_{k=1}^{q'}\frac{(hq-kp)(kp-hq)}{|hq-kp|\,|kp-hq|}=(-1)^{p'q'} \tag{9}$$

Equations (2) and (4) remain valid if $p=2$, and (7) becomes

$$\prod_{k=1}^{q'}(-1)^{e_k}\equiv\prod_{k=1}^{q'}2k/r_k\equiv 2^{q'}\ (\mathrm{mod}\ q). \tag{10}$$

For $p=2$ in (2), $[kp/q]=0$, and (3) becomes $e_k\equiv r_k$ (mod 2). Hence

$$(2/q)=\prod_{k=1}^{q'}(-1)^{e_k}=\prod_{k=1}^{q'}(-1)^{r_k}=(-1)^{q'(q'+1)/2}=(-1)^{(q^2-1)/8} \tag{11}$$

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MATHEMATICAL EDUCATION

EDITED BY PAUL T. MIELKE AND W. E. MASTROCOLA

Material for this department should be sent to W. E. Mastrocola, Department of Mathematics, Colgate University, Hamilton, NY 13346.

A MATHEMATICAL SCIENCES PROGRAM AT AN UPPER-DIVISION CAMPUS

FRANK J. SWETZ

1. Introduction. Since 1971, the Pennsylvania State University, Capitol Campus, has conducted a degree program in the mathematical sciences which has been well received by students, faculty, and employers and may be considered an academic success. This article considers the conception, objectives, contents, and limitations of this program.

2. Background and Conception. The Capitol Campus was founded in 1966 as an upper-division (junior-senior level) institution of the Pennsylvania State University. Its mission was to offer graduates of the state's two-year colleges an opportunity to continue to the bachelor's degree level, to serve the educational needs of the Harrisburg (state capitol) area, and to provide limited graduate work. The school's faculty was charged to be innovative and to develop flexible programs, relevant to the needs of a rapidly changing society. Subsequent development reflected these aims. Initially, principal programs were developed in the social sciences, the humanities, education, and engineering technology. A mathematics faculty was assembled to service the needs of these programs. At its maximum this faculty had nine mathematicians, the majority of whom held doctorates and participated in many kinds of mathematical activities. The group's experience spanned applications of mathematics and statistics in education, industry, social science, and actuarial work.

It soon became apparent that this nucleus of mathematicians formed a critical mass, one which needed a more adequate outlet for mathematical expression than that supplied solely by service work. Offering a degree in mathematics seemed to be the answer. Planning began in 1969. In the design of the degree program, several factors were considered:

1. Contemporary students sought a degree that was primarily an end in itself and not necessarily a prerequisite for graduate studies.

2. The course of study should be as flexible as possible, allowing for several career options [4].
3. The thrust of the degree had to complement the existing educational philosophies of Capitol Campus and to focus on real-life situations rather than theoretical considerations.

It was decided to offer a degree program in the mathematical sciences. The reactions of local industries and government agencies were favorable. The broad objectives of the program were to:

1. Provide a sound foundation in mathematical studies.
2. Instill awareness of the utility of mathematics, statistics, and computers.
3. Develop competence in the use of modern mathematical tools.
4. Nurture awareness that the growth of mathematics is a societal necessity—one in which the student can readily participate.

We foresaw the Bachelor of Science in Mathematical Sciences degree as preparing students for a variety of careers, including:

1. Managerial positions in government or industry.
2. Teaching mathematics and computer science at the secondary school level.
3. Entering government or industry as a mathematician [1].
4. Becoming an actuary.
5. Pursuing graduate study in areas that use mathematics, such as economics, business administration, operations research, computer science, or applied science.

3. The Program. The mathematical sciences program consists of 70 credits taken over two years. Each of our school years is divided into three ten-week units. A student entering the program from a two-year college is required to offer a minimum of 60 credits of collegiate work. Of these offerings a minimum of 15 credits in mathematics is required, including nine or more credits of calculus and six or more credits in courses such as college algebra, linear algebra, finite mathematics, analytic geometry, logic, and ordinary differential equations. However, entering students have a much wider variety of academic backgrounds and experiences; some already have a degree and seek a career reorientation, others transfer from four-year programs, and still others are older students, such as military veterans or housewives returning to work.

With this perspective we decided on a basic core of required courses in mathematical theory and techniques which we felt every working mathematician should know [2]. Various career goals could be built around this core, with an appropriate choice of electives. The mathematical sciences core is:

Ma.Sc. 301 Introduction to the Foundations of Mathematics
Ma.Sc. 321 Advanced Calculus I
Ma.Sc. 323 Ordinary Differential Equations
Ma.Sc. 331 Programming Techniques and Language for Digital Computers
Ma.Sc. 332 Mathematical Methods for Digital Computers
Ma.Sc. 342 Linear Algebra
Ma.Sc. 415 Mathematical Statistics and Applications I
Ma.Sc. 416 Mathematical Statistics and Applications II

Elective courses are selected from other mathematics offerings or other departments; a strong student-advisor relationship is maintained to help the students make good selections. Here are some typical elective schedules:

1. Managerial Interest
   Ma.Sc. 423 Application of Numerical Computer Techniques to Research Problems
   Ma.Sc. 461 Introduction to Operations Research
   Ma.Sc. 496 Independent Study
   Ma.Sc. 498 Selected Topics (Modeling)

Bus. 360 Management Decision Making
Bus. 380 Statistical Methods in Social and Managerial Sciences
Bus. 382 Applications of Quantitative Methods in the Social and Managerial Sciences
Bus. 483 Quantification IV
Econ. 310 Microeconomic Analysis and Policy
One additional social-humanistic course

2. Science Interest
Ma.Sc. 422 Advanced Calculus II
Ma.Sc. 423 Application of Numerical Computer Techniques to Research Problems
Ma.Sc. 461 Introduction to Operations Research
Ma.Sc. 473 Complex Analysis
Ma.Sc. 498 Selected Topics (Modeling)
Econ. 310 Microeconomic Analysis and Policy
Phil. 305 Logic
Phys. 351 Introduction to Modern Physics
Two additional social-humanistic courses

3. Education Interest
Ma.Sc. 433 Modern Algebra
Ma.Sc. 455 Euclidean Geometry and Transformations
Ma.Sc. 473 Complex Analysis
* Ed. 313 Field Observation
* Ed. 314 Learning Theory and Instructional Procedures
* Ed. 315 Social and Cultural Factors in Education
* Ed. 350 Student Teaching (triple course)
* Ed. 417 Approaches to the Teaching of Mathematics
Ma.Sc. 498 History of Mathematics
One additional social-humanistic course

* Course satisfies state requirements for teaching certification

4. Actuarial Interest
Ma.Sc. 423 Application of Numerical Computer Techniques to Research Problems
Ma.Sc. 461 Introduction to Operations Research
Ma.Sc. 496 Independent Study
Ma.Sc. 498 Selected Topics (Actuarial Science)
Bus. 330 Insurance I
Bus. 331 Insurance II
Phil. 305 Logic
Three additional social-humanistic courses

5. Computer Interest
Ma.Sc. 423 Application of Numerical Computer Techniques to Research Problems
Ma.Sc. 433 Modern Algebra
Ma.Sc. 461 Introduction to Operations Research
Bus. 442 Computer Systems and Programming
Bus. 443 Programming for Business Applications

Bus. 444 Data Structures and Information Processing
Bus. 445 File Processing and Report Preparation
Bus. 448 System Design Project
Two social-humanistic courses

This program has been in effect since 1971.

4. Results of the Program. Student evaluations of the program have been favorable, and feedback from graduates and employers is extremely encouraging. Our alumni have filled a variety of positions, much more diversified than we had originally imagined. A recent female graduate is working as an industrial engineer. Actuarial students take the first parts of their professional examinations while still undergraduates and enter their field as junior actuaries. In particular, graduates of the mathematics education option have reported how much better prepared they are to undertake the teaching of advanced placement and computer science work than their peers who are products of traditional mathematics or education programs. Two mathematics education majors stepped into junior college teaching positions immediately upon graduation. Several students have gone on to pursue graduate studies in mathematics and other fields.

Problems, or, rather, annoyances, the program has experienced result mainly from its being conducted in an upper-division school. These include:

1. Entering students have very diverse mathematical backgrounds. Those who have been away from school for many years are frequently required to take several prerequisites before being formally admitted to the program.
2. The program is small, averaging under forty students. Under the present academic climate dominated by financial exigency, administrators view small programs with suspicion. Thus it is difficult to justify new course offerings within the program; however, two courses (Ma.Sc. 498: Independent Study; and Ma.Sc. 496: Selected Topics) permit some flexibility in catering to special student needs and interests.

5. Conclusions. Despite these constraints, our experiences have shown that a mathematical sciences program in an upper-division academic institution offers a viable, and in many ways desirable, alternative to a traditional mathematics program. Some four-year institutions have adopted variations of our program, and it is seriously being considered by a new university of the Third World. The products of our program are knowledgeable users of mathematics and are readily employable. The *Realpolitik* of today's job market demands such persons. Can we afford not to supply them?

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PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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Beginning with January, 1979, this Department will be edited by A. P. Hillman.

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

No solutions (except those accompanying proposals) should be sent to Professor Hillman.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether or not you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood; for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and (if possible) submitted in duplicate, and should be mailed before April 30, 1979. Please enclose a self-addressed label or card (for acknowledgment).

E 2743. *Proposed by Peter Ungar, Courant Institute, New York University*

Find

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{(-1)^{i+j}}{i+j}.$$

E 2744. *Proposed by H. L. Montgomery, University of Michigan*

Let $a_n \geqslant 0$ and $a_{m+n} \leqslant a_m + a_n$ for $m,n = 1,2,\ldots$. Show that $\sum_{k=1}^{n} k^{-2}a_k \leqslant \frac{1}{4}n^{-1}a_n \log n$.

E 2745*. *Proposed by David Hammer, Santa Cruz, California*

Can every collection of non-overlapping pennies in the plane be colored with three colors so that no penny touches more than one penny with the same color?

E 2746*. *Proposed by George F. Shum, Ohio State University*

Let $A_1,\ldots,A_n$ be distinct non-collinear points in the plane. A circle with center P and radius r is called *minimal* if $A_kP \leqslant r$ for all k and equality holds for at least three values of k.

If $A_1,\ldots,A_n$ vary (n being fixed) what is the maximum number of minimal circles?

E 2747. *Proposed by H. L. Krall, Pennsylvania State University, and Emil Grosswald, Temple University*

Compute the determinant of the matrix $A=(a_{ij})$ where $0 \leqslant i,j \leqslant n-1$ and $a_{ij}=1/(i+j+1)!$.

E 2748. *Proposed by Lance Littlejohn, Pennsylvania State University*

If $f(x)=x^n \log x$ find

$$\lim_{n\to\infty} \frac{f^{(n)}\left(\frac{1}{n}\right)}{n!}.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

An Estimate for the Cardinality of a Set of Subsets

E 2666 [1977, 567]. *Proposed by Peter Frankl, Budapest, Hungary*

Let S be a finite set and let $\mathcal{P}$ be the set of all subsets of S. For $\mathcal{A} \subset \mathcal{P}$ and $\mathcal{B} \subset \mathcal{P}$ define $\mathcal{A} * \mathcal{B}$ to be the subset of $\mathcal{P}$ consisting of subsets $X \subset S$ such that $X \subset A \cup B$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

If $|\mathcal{A}|+|\mathcal{B}|>2^k$, prove that $|\mathcal{A}*\mathcal{B}| \geqslant 2^k$.

Solution by the University of South Alabama Problem Group. We say that $\mathcal{A}$ is hereditary if $X \subset Y \in \mathcal{A}$ implies $X \in \mathcal{A}$. It is clear that we can assume that both $\mathcal{A}$ and $\mathcal{B}$ are hereditary and that $\mathcal{A} \cap \mathcal{B} \neq \{\varnothing\}$. Thus there exists $a \in S$ such that $\{a\} \in \mathcal{A} \cap \mathcal{B}$. Let

$$\mathcal{A}'=\{X \in \mathcal{A} \mid a \notin X\}, \qquad \mathcal{B}'=\{X \in \mathcal{B} \mid a \notin X\},$$
$$\mathcal{A}''=\{X \in \mathcal{A}' \mid X \cup \{a\} \in \mathcal{A}\}, \qquad \mathcal{B}''=\{X \in \mathcal{B}' \mid X \cup \{a\} \in \mathcal{B}\}.$$

Clearly these four families are also hereditary. We have

$$(|\mathcal{A}'|+|\mathcal{B}''|)+(|\mathcal{A}''|+|\mathcal{B}'|)=|\mathcal{A}|+|\mathcal{B}|>2^k.$$

It follows that, say, $|\mathcal{A}'|+|\mathcal{B}''|>2^{k-1}$. By induction on k (case $k=0$ being true), we have $|\mathcal{A}'*\mathcal{B}''| \geqslant 2^{k-1}$. If

$$\mathcal{C}=\{X \cup \{a\} \mid X \in \mathcal{A}'*\mathcal{B}''\}$$

then $\mathcal{C} \subset \mathcal{A}*\mathcal{B}$ and thus $\mathcal{C}$ and $\mathcal{A}'*\mathcal{B}''$ are disjoint. Hence we have

$$|\mathcal{A}*\mathcal{B}| \geqslant |\mathcal{A}'*\mathcal{B}''|+|\mathcal{C}|=2|\mathcal{A}'*\mathcal{B}''| \geqslant 2^k.$$

REMARK. The result is the best possible. Indeed if $|S|=k$, $\mathcal{A}=\mathcal{B}=\mathcal{P}$ then $|\mathcal{A}|+|\mathcal{B}|=2^{k+1}$ and $|\mathcal{A}*\mathcal{B}|=2^k$.

Also solved by Dean Hoffman, and the proposer.

Digits in a Dyadic Expansion

E 2667 [1977, 567]. *Proposed by John R. Samborski, Hyattsville, Maryland*

If $\Sigma_{k=1}^{\infty} 2^{-n_k}$ is the binary expansion of $(\sqrt{5}-1)/2$, show that $n_k \leqslant 5\cdot 2^{k-2}-1$.

Solution by O. P. Lossers, Eindhoven University of Technology, Netherlands. Let $x=(\sqrt{5}-1)/2$ and write $n(k)=n_k$. Suppose $n(k+1)>2n(k)+2$ for some k. If

$$m=2^{n(k)}\sum_{i=1}^{k} 2^{-n(i)}$$

then $m2^{-n(k)}<1$ and $y=x-m\cdot 2^{-n(k)}$ satisfies $0<y<2^{-2n(k)-2}$. Since

$$1=x+x^2=m^2 2^{-2n(k)}+m\cdot 2^{-n(k)}+y(1+y+2m\cdot 2^{-n(k)}),$$

it follows that $y\cdot 2^{2n(k)}(1+y+2m\cdot 2^{-n(k)})$ is an integer. This is impossible since $y\cdot 2^{2n(k)}<1/4$, and $1+y+2m\cdot 2^{-n(k)}<4$.

Hence we can conclude that $n(k+1)\leqslant 2n(k)+2$ for all k. Since $n_1=1$, $n_2=4$, $n_3=5$, it follows easily that $n(k)\leqslant 5\cdot 2^{k-2}-1$. Moreover, we have $n(k)\leqslant 2^{k-1}-2$ for $k\geqslant 4$ since $n_4=6$.

Also solved by M. T. Bird, David Bloom, Bronx Community College Problem Group, C. T. Giel, A. A. Jagers (Netherlands), L. E. Mattics, David Montana, J. M. Stark, Allen Stenger, and the proposer.

Special Non-isosceles Triangles

E 2668 [1977, 568]. *Proposed by Ron Evans and I. Martin Isaacs, University of Wisconsin*

Find all non-isosceles triangles with two or more rational sides and with all angles rational (measured in degrees).

Solution by the proposer. We shall prove that such a triangle Δ has angles $\pi/6$, $\pi/3$, and $\pi/2$, where here and in the sequel angles are measured in radians. Let θ_1,θ_2 be angles of Δ each opposite a rational side and let $\alpha_j=\frac{1}{2}\pi-\theta_j=2\pi r_j/s_j$ where r_j, s_j are integers, $s_j>0$, $(r_j,s_j)=1$, and $|r_j/s_j|<1/4$. If $c_j=\cos\alpha_j$ then $c_j>0$ and $c_1/c_2\in\mathbf{Q}$ by the law of sines.

Case 1: $c_1\notin\mathbf{Q}$. Then c_j has degree $\phi(s_j)/2$ over $\mathbf{Q}$. Since $c_1/c_2\in\mathbf{Q}$ we must have $\phi(s_1)=\phi(s_2)$. Without loss of generality, let $s_2\geqslant s_1$. Then it follows that either $s_2=s_1$ or $s_2=2s_1$ with s_1 odd. If $s_1=s_2$ then clearly c_1 and c_2 are conjugate (over $\mathbf{Q}$). Now suppose that $s_2=2s_1$ where s_1 is odd. Since $(s_1+2,s_2)=1$, c_2 is conjugate to $\cos(2\pi r_2(s_1+2)/s_2)=-\cos(2\pi r_2/s_1)$. Hence c_2 is conjugate to $-c_1$. Thus in any case c_2 is conjugate to tc_2, where $t\in\mathbf{Q}$ and $t\neq\pm 1$ (recall that c_1/c_2 is a positive rational number $\neq 1$). Therefore, there is an automorphism σ of $\mathbf{Q}(c_2)$ such that $\sigma(c_2)=tc_2$. If n is the order of σ then $c_2=\sigma^n(c_2)=t^n c_2$. This gives $t^n=1$, $|t|=1$, a contradiction. Thus Case 1 cannot occur.

Case 2: $c_1\in\mathbf{Q}$. In this case $\phi(s_j)\leqslant 2$, so that $s_j\in\{1,2,3,4,6\}$. Since $|r_j/s_j|<1/4$, $r_j/s_j=0,\pm 1/6$. Thus $\theta_j=\pi/2$, $\pi/6$, or $5\pi/6$. It follows that θ_1 and θ_2 must be $\pi/2$ and $\pi/6$ in some order.

Also solved by L. E. Mattics.

Roundest Oval of Rademacher

E 2669 [1977, 568]. *Proposed by I. J. Schoenberg, University of Wisconsin*

Let $a>b>0$. For a given r, $0<r<b$ there is a unique $R>0$ such that the circle $(x-a+r)^2+y^2=r^2$ lies inside and touches the circle $x^2+(y-b+R)^2=R^2$. For which r is R/r minimal?

Solution by Lael F. Kinch, University of Louisville. Since the circles are tangent internally, we must have

$$(R-r)^2=(a-r)^2+(R-b)^2.$$

Thus

$$\frac{R}{r}=f(r)=\frac{a^2+b^2-2ar}{2r(b-r)}.$$

The function $f(r)$ has an absolute minimum value in the interval $(0,b)$ at

$$r_0=\frac{1}{2a}\left[a^2+b^2-(a-b)\sqrt{a^2+b^2}\,\right].$$

Also solved by Bronx Community College Problem Group, Paul Bruckman, Peter de Buda, Milton Eisner, Thomas Elsner, F. J. Flanigan, Landy Godbold, Michael Goldberg, Franklin Kemp, Jr., L. Kuipers (Switzerland), Carolyn MacDonald, Miami University Problems Group, Roger Nelsen, St. Olaf College Problem Group, August Sardinas, Harry Sedinger, Michael Skalsky, J. M. Stark, Edward Wong, Ken Yocom, and the proposer.

Proposer's comment. Let T be the point of tangency of the two circles. Let $O(r)$ be the oval symmetric with respect to both axes and whose part in the first quadrant consists of the arc AT of the first circle and the arc TB of the second circle (A and B are the points $(a,0)$ and $(0,b)$ respectively).

H. Rademacher showed (in his note *On the roundest oval*, Collected Works, M.I.T. Press, 1974, vol. 2, pp. 602–608) that the oval $O(r_0)$ is the "roundest oval" of semiaxes a,b, in the following sense: Let Γ be a smooth closed convex curve, symmetric with respect to both axes, $A,B\in\Gamma$, such that its radius of curvature ρ_P at the point P is non-decreasing along the arc AB. If among all such curves the ratio ρ_B/ρ_A is minimal, then we get the oval $O(r_0)$.

An Inequality for $(xe^{-x}-ye^{-y})/(e^{-x}-e^{-y})$

E 2670 [1977, 568]. *Proposed by Shyam Johari, Burroughs Corporation, and Stanley L. Sclove, University of Illinois at Chicago Circle*

Let

$$f(x,y)=\frac{xe^{-x}-ye^{-y}}{e^{-x}-e^{-y}}.$$

If $0<a<b<c<\infty$ and $0<x<y<z<\infty$, prove or disprove that

$$|f(a,b)+f(b,c)-f(x,y)-f(y,z)|\leqslant 2\max(|x-a|,|y-b|,|z-c|).$$

Solution by G. V. Ramanathan, University of Illinois at Chicago Circle. If $g(x)=x/(e^x-1)$, $h(x)=x/(1-e^{-x})$, then

$$f(x,y)=\frac{xe^{-x}-ye^{-y}}{e^{-x}-e^{-y}}=x-g(y-x)=y-h(y-x),$$

and

$$\begin{aligned}f(x,y)-f(a,b)&=x-a+g(b-a)-g(y-x)\\&=y-b+h(b-a)-h(y-x).\end{aligned}$$

Since g is decreasing and h is increasing in $(0,\infty)$, it follows that for $0<a<b<\infty$ and $0<x<y<\infty$ we have

$$\min(x-a,y-b)\leqslant f(x,y)-f(a,b)\leqslant\max(x-a,y-b).$$

The desired result follows.

Also solved by M. T. Bird, Bronx Community College Problem Group, Robert Bryant, Robert Breusch, Robert Geist, Walter Gerlach & Margaret Saunier, Lael Kinch, Beatriz Margolis (Venezuela), L. E. Mattics, and William Myers.

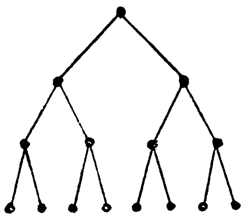
Labelings of Binary Trees

E 2671 [1977, 651]. *Proposed by Ibrahim Cahit, Nicosia, Cyprus*

Let $T=(V,E)$ be a k-level complete binary tree (see figure for $k=4$) with vertex set V and the edge set E. Thus $|V|=2^k-1$ and we put $N=\{1,2,3,\ldots,2^k-1\}$. For every bijection $f: V\to N$ define

$$W(f)=\sum_{\{i,j\}\in E}|f(i)-f(j)|.$$

Prove or disprove $\min_f W(f)=(k-1)2^{k-1}$ $(k\geqslant 2)$.



I. *Counter-example by Paul Vojta, student, University of Minnesota*. Assign values of f to a five-level tree as follows:

level 1: 16
level 2: 11,21
level 3: 4,12,20,28
level 4: 2,6,9,14,18,23,26,30
level 5: 1,3,5,7,8,10,13,15,17,19,22,24,25,27,29,31

(On each level the labels are listed from left to right.) Then $W(f)=60$ while $(k-1)2^{k-1}=64$.

II. *Solution by G. W. Peck, Massachusetts Institute of Technology*. If w_k is the minimum from the problem, then for $k>2$ we have

$$w_k=2^k+2\sum_{i=2}^{k-2}(w_i+2). \tag{1}$$

Indeed, if one chooses the path from 1 to 2^k-1, that accounts for at least 2^k-2 of $W(f)$. The remaining contributions come from at least two isolated arcs (contributions $\geqslant 2$) and pairs of i-level complete binary trees with an extra hook at the top in them. By induction, these contribute w_i+2 each, yielding the desired bound. (This bound is clearly realizable.) The argument suggests how one constructs a realization: one puts 1 and 2^k-1 on leaves (vertices of degree 1) and arranges the numbers in each region branching off the path between them consecutive according to the same plan (when a hook is present one puts the middle numbers in the group on it).

From (1) we obtain $w_{k+1}-w_k-2w_{k-1}=2^k+4$ $(k\geqslant 3)$ and it follows that

$$w_k=2^k\left(\frac{k}{3}+\frac{5}{18}\right)-2+(-1)^k\frac{2}{9}.$$

Counter-examples were found also by Anatole Beck, J. C. Binz (Switzerland), J. M. Brown & David Voss, Peter van der Helm (Netherlands), and Albert Nijenhuis. Nijenhuis has also computed w_k.

Editor's comment. The same problem appeared recently in SIAM Review 19 (1977), pp. 564–565, and its solution by F. R. K. Chung in 20 (1978). This should be consulted for further references.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to Professor R. C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before April 30, 1979.

6205 [1978, 282] **(correction).** *Proposed by Alan McConnell and Louis Shapiro, Howard University*

Let G be a group with no nontrivial elements of finite order, and let H be a cyclic subgroup of finite index in G. Show that G is itself cyclic.

6240. *Proposed by Mihai Eşanu, Bucharest University, Romania*

Let $a_n \neq 0$, $\lim_{n\to\infty} a_n = 0$. Prove that for every real number x there exist sequences $(\lambda_n), (\mu_n)$ of integers such that

$$x = \sum_{n=1}^{\infty} \lambda_n a_n = \prod_{n=1}^{\infty} \mu_n a_n.$$

6241. *Proposed by Robert Baillie, Computer-Based Education Research Laboratory, University of Illinois*

Prove

$$\text{(A)} \quad \sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \sum_{n=1}^{\infty} \left(\frac{\sin(n)}{n}\right)^2 = \frac{\pi-1}{2},$$

$$\text{(B)} \quad \sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^4} = \frac{(\pi-1)^2}{6}.$$

6242. *Proposed by Jan Mycielski, University of Colorado*

Let I be the interval $[0,1]$, λ the Lebesgue measure in I and μ a Borel measure in I. Suppose that $\lambda(A) = \frac{1}{2}$ implies $\mu(A) = \frac{1}{2}$ for every Borel set $A \subseteq I$. Prove that $\mu(B) = \lambda(B)$ for every Borel set $B \subseteq I$.

6243. *Proposed by Emil Grosswald, Temple University*

Find in closed form the sum S of the conditionally convergent series

$$\sum_{n=2}^{\infty} (-1)^n n^{-1} \log n.$$

6244. *Proposed by John Myhill, State University of New York at Buffalo*

Let f_i, $i = 0, 1, 2, \ldots$, be a sequence of (everywhere defined) real functions. Prove that there exist two functions ϕ, ψ such that each of the f_i can be obtained from ϕ and ψ by composition.

6245. *Proposed by C. L. Mallows, Bell Laboratories, Murray Hill, N.J.*

For $0 < a < 1$, $t \geqslant 0$, $b = 1 - a$, prove that

$$\frac{1}{\pi} \int_0^{\pi} \frac{(\sin u)^t}{(\sin au)^{at} (\sin bu)^{bt}} \, du = \frac{\Gamma(t+1)}{\Gamma(at+1)\Gamma(bt+1)}.$$

SOLUTIONS OF ADVANCED PROBLEMS

The Equation $\partial f/\partial x = \partial f/\partial y$

5871 [1972, 780; 1973, 1150; 1975, 530; 1976, 573]. *Proposed by P. R. Chernoff, University of California, Berkeley*

Let $f(x,y)$ be a real-valued function of two real variables which is separately differentiable. Assume that $\partial f/\partial x = \partial f/\partial y$ everywhere. Must there be a function g of one variable such that $f(x,y) = g(x+y)$? What if we assume a priori that f is jointly continuous?

IV. *Solution by the proposer*. The answer is affirmative. (An ingenious solution when f is assumed to be continuous was given by H. F. Royden [**1**].)

Interestingly, this problem turns out to have a history going back to Baire's thesis [**2**]. In Chapter 5, Baire considers the equation $\partial f/\partial x + \partial f/\partial y = 0$. By a rather long argument he shows that f must be a function of $x-y$, assuming a priori that f is jointly continuous. (His proof actually uses only the continuity of f when restricted to the lines $x-y=c$.) He remarks that he is unable to settle the general case. Subsequently, Montel considered the problem [**3**, pp. 297–298]. Unfortunately Montel's argument in itself does not establish even as much as Baire's. However, it contains a good idea: given f with $\partial f/\partial x = \partial f/\partial y$, try to show that $f\,dx + f\,dy$ is an exact differential. To carry this out with no additional hypotheses requires a strengthened version of Green's theorem.

The appropriate result was proved by G. P. Tolstoff in 1942 [**4**, Lemma 3]:

LEMMA. *Let P, Q be bounded, separately differentiable functions in a square K, and assume that $\partial P/\partial y = \partial Q/\partial x$ holds almost everywhere in K. Then $\int_{\partial R} P\,dx + Q\,dy = 0$ for every rectangle R in K with sides parallel to the coordinate axes.*

This is just what is needed to settle our problem, though Tolstoff himself doesn't seem to have considered it.

THEOREM. *Let f be a separately differentiable function on $\mathbf{R}^2$ (or on an open convex subset of $\mathbf{R}^2$). Assume that the relation*

$$\partial f/\partial x = \partial f/\partial y \tag{*}$$

holds almost everywhere. Then f is a function of $x+y$.

Proof. First, by replacing f with $\tan^{-1}(f)$, which also satisfies (*), we reduce to the case of bounded f. We then simply apply Tolstoff's lemma with $P=Q=f$. It follows in the usual way that there is a continuous, separately differentiable function F with $\partial F/\partial x = f = \partial F/\partial y$ everywhere. But then by Baire's or Royden's argument, F is a function of $x+y$.

However, there is one more wrinkle, by which the latter arguments may be avoided. Namely, we apply the argument above to F instead of f, deducing that there is a continuous G with $\partial G/\partial x = F = \partial G/\partial y$. Since F is continuous, G is C^1, which is enough to justify the obvious chain rule calculation showing that $G(x,y) = h(x+y)$. Hence $F(x,y) = \partial G/\partial x = h'(x+y)$ and $f(x,y) = \partial F/\partial x = h''(x+y)$.

References

1. H. F. Royden, Problem 5871, this MONTHLY, 82 (1975) 530–531.

2. R. Baire, Sur les fonctions de variables réelles, Ann. Mat. Pura Appl. (3), **3** (1899) 1–123.

3. P. Montel, Sur les suites infinies de fonctions, Ann. Sci. École Norm. Sup. (3), **24** (1907) 233–334.

4. G. P. Tolstoff, Sur les fonctions bornées vérifiant les conditions de Cauchy–Riemann, Mat. Sbornik, **52** (1942) 79–85.

Even Perfect Numbers

6036 [1975, 671; 1977, 225]. *Proposed by Carl Pomerance, University of Georgia*

If n is a natural number, we let $\sigma(n)$ denote the sum of the divisors of n, $S(n)$ the set of prime divisors of n, and $\omega(n)$ the cardinality of $S(n)$. Clearly if n is an even perfect number, then $S(n)=S(\sigma(n))$ and $\omega(n)=2$. Prove the converse.

III. *Solution by the University of British Columbia Problems Group.* Let $n=p^r q^s$ where $p<q$ are primes and suppose that

$$\sigma(n)=q^{s_1}p^{r_1}=\left(\frac{p^{r+1}-1}{p-1}\right)\left(\frac{q^{s+1}-1}{q-1}\right).$$

We show that $p=2$. Note that if $r+1=jk$ then

$$q^{s_1}=\left(\frac{p^{jk}-1}{p^k-1}\right)\left(\frac{p^k-1}{p-1}\right)$$

so that $m=p^{k-1}q^s$ satisfies the hypotheses. Thus we may take $r+1$ and $s+1$ to be prime.

Now $p^{r+1}\equiv 1 \bmod q$ and $p^{r_1}\equiv 1 \bmod q$. Since $r+1$ is prime and $p\not\equiv 1 \bmod q$, $r+1$ divides r_1. If $q\not\equiv 1 \bmod p$, then $s+1$ divides s_1 so that n divides $\sigma(n)$. But

$$\sigma(n)<\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)n\leqslant 3n.$$

Thus, if $q\not\equiv 1 \bmod p$, $\sigma(n)=2n$ and so $p=2$. Hence we take $q\equiv 1 \bmod p$.

(1) $p^{r_1}=1+q+\cdots+q^s\equiv(s+1)\bmod p$. Since $s+1$ is prime, $s+1=p$.

(2) Let $\dfrac{r_1}{r+1}=t$. Then

$$\left(\frac{q^{s+1}-1}{q-1}\right)=p^{r_1}=(p^{r+1})^t=(q^{s_1}(p-1)+1)^t.$$

Expanding yields that q^{s_1} divides $q+q^2+\cdots+q^s$ so that $s_1=1$ and $q=1+p+\cdots+p^r$.

(3) p^{r_1} divides $q^{s+1}-1=(1+p+\cdots+p^r)^p-1$. If $p\neq 2$, then expanding the right-hand side yields that p^{r_1} divides p^2+dp^3. Thus $r_1=2$, $r=1$ and hence $1+p=q$. Thus p odd implies the contradiction $p=2$.

Let n be any number satisfying the hypotheses. By (3), $n=2^r q^s$. Let $\sigma(n)=2^{r_1}q^{s_1}$.

(4) Suppose $q^v+1=2^u$ where $u\geqslant 2$. Then $q^v\equiv 3 \bmod 4$ so that $q\equiv 3 \bmod 4$ and v is odd. However $q^{2v}\equiv 1 \bmod 2^u$. Hence since v is odd, $q^2\equiv 1 \bmod 2^u$ so that $q\equiv -1 \bmod 2^u$. Since $q^v=2^u-1$, $v=1$.

(5) Since $q^{s_1}=2^{r+1}-1$, (4) shows that $s_1=1$. Thus $q=2^{r+1}-1$ is a Mersenne prime and it suffices to show $s=1$.

(6) $2^{r_1}=\left(\dfrac{q^{(s+1)/2}-1}{q-1}\right)(q^{(s+1)/2}+1)$. Since $s+1$ is even, $q^{(s+1)/2}+1=2^u$, whence $s=1$ from (4).

Editor's comment. The proposer pointed out that the original solutions make the unproved assertion that if n and $\sigma(n)$ had the same prime factors, then $n|\sigma(n)$. Clearly this could not be the case without additional proof. For example, if $n=A^xB^y$ with x very large and y small.

The Differences of the Partition Function

6137 [1977, 141]. *Proposed by I. J. Good, Virginia Polytechnic Institute and State University*

Let $p(n)$ denote the number of partitions of n $(n=1,2,\ldots)$, and let k denote an integer greater than 3. Prove that $\Delta^k p(n)$ $(n=1,2,\ldots)$ is a sequence of alternating terms.

I. *Solution by Reinhard Razen, University of New South Wales, Australia.* The given statement

does not hold since there are counterexamples at $\Delta^4 p(69)$ and $\Delta^5 p(135)$. In fact, I believe that $\Delta^k p(n)$ is eventually positive for every positive integer k.

II. *Comments by the proposer.* We have calculated $\Delta^k p(n)$ for $k=0,1,2,\ldots,11$ and n going beyond 1000, by means of Euler's recurrence relation for $p(n)$. Using a program prepared by Dr. J. F. Crook in quadruple-precision arithmetic, we find that $\Delta^k p(n)$, for each fixed $k \geqslant 1$, alternates in sign for $k \leqslant n \leqslant g(k)$ and that $\Delta^k p(n) \geqslant 0$ when $n \geqslant g(k)$ where $g(k)$, is given in Table 1, and where $f(k)=6(k-1)(k-2)+\frac{1}{2}k^3$ is also tabulated. There are of course no negative terms when $k=0$ or $k=1$, and it can be shown that there are none when $k=2$.

Table 1. Values of $g(k)$ and $f(k)$

k	1	2	3	4	5	6	7	8	9	10	11
$g(k)$	1	2	26	68	134	228	352	510	704	934	1204
$f(k)$	$\frac{1}{2}$	4	$25\frac{1}{2}$	68	134	228	$351\frac{1}{2}$	508	$700\frac{1}{2}$	932	$1205\frac{1}{2}$

Another problem could be set: Prove that (i) $g(k)$ always exists; (ii) $g(k)$ is an increasing function of k: and decide whether (iii) $g(k)$ is always well approximated by $f(k)$.

Finitely Axiomatizable Properties in a First-Order Predicate Calculus

6139 [1977, 221]. *Proposed by D. P. Munro, University of Sydney, Australia*

Consider a first-order predicate calculus, and all the relational structures appropriate to that calculus.

(a) Let $P_1,\ldots,P_k$ be a finite collection of mutually exclusive and exhaustive axiomatizable properties (so every relational structure has exactly one of the properties P_i). Must any of the P_i be in fact axiomatizable, and if so, how many?

(b) As for (a), but with a countably infinite collection of mutually exclusive and exhaustive axiomatizable properties.

Solution by the proposer. (a) We first show that, if P_1 and P_2 are two mutually exclusive and exhaustive axiomatizable properties, then each is finitely axiomatizable. For suppose that P_1 is not finitely axiomatizable. Let $P_1=\{\varphi_i | i=1,2,\ldots\}$ and $P_2=\{\psi_i | i=1,2,\ldots\}$, where the φ_i and ψ_i are sentences. (Strictly we should write "let $\{\varphi_i | i=1,2,\ldots\}$ be an axiomatization of P_1, etc.") For any n, $\{\varphi_1,\ldots,\varphi_n\} \cup P_2$ has a model, since P_1 is not finitely axiomatizable. So from the compactness theorem it follows that $P_1 \cup P_2$ has a model, which is a contradiction.

Now if P and Q are axiomatizable properties, then so is (P or Q). For if $P=\{\varphi_i | i=1,2,\ldots\}$ and $Q=\{\psi_j | j=1,2,\ldots\}$, a structure is a model of P or of Q if and only if it is a model of $\{\varphi_i \vee \psi_j | i,j=1,2,\ldots\}$. Hence if $P_1,P_2,\ldots,P_k$ are mutually exclusive and exhaustive axiomatizable properties, then for each i, the same is true of P_i and (P_1 or P_2 or $\cdots$ or P_{i-1} or P_{i+1} or $\cdots$ or P_k); thus, by our initial observation, each P_i is finitely axiomatizable.

(b) Let the properties be P_1, P_2, etc.

(1) Not all of the P_i can be finitely axiomatizable. For if the single sentence p_i is an axiomatization of P_i, then every finite subset of $Q=\{\neg p_i : i=1,2,\ldots\}$ has a model. So Q has a model, which contradicts the exhaustiveness of the P_i.

(2) The result in (1) is the only restriction on the P_i. We give examples demonstrating this which are based on the theory of vector spaces, as axiomatized in J. L. Bell and A. B. Slomson, *Models and Untraproducts*, North-Holland, Amsterdam, 1969, p. 97. Let V be the property of being a vector space of dimension $\geqslant 1$, $D(n)$ the property of being a vector space of dimension n, $D(\infty)$ the property of being a vector space of infinitely many dimensions, $F(p)$ the property of being a vector space over a field of characteristic p. Then V, $D(n)$ and $F(p)$ for $p>0$ are finitely axiomatizable, and $D(\infty)$ and $F(0)$ are axiomatizable but not finitely axiomatizable (see Bell and Slomson, pp. 96–99).

(3) An example where all but one of the properties are finitely axiomatizable:

$$\neg V, D(1), D(2), \ldots, D(\infty).$$

(4) An example where none of the properties is finitely axiomatizable:

$$\neg V \vee (D(\infty) \wedge F(0)), D(1) \vee (D(\infty) \wedge F(2)), \ldots, D(n) \vee (D(\infty) \wedge F(P_n)), \ldots$$

where P_n is the nth prime. Note that by the initial remark all these properties are in fact axiomatizable.

(5) Examples for the remaining cases (finitely many properties finitely axiomatizable; infinitely many finitely axiomatizable and infinitely many not; finitely many not finitely axiomatizable) may be similarly constructed.

Also solved by Xavier Caicedo, Dryden Cope, and Peter Winkler.

Note. An example similar to (4) but in another context was provided by Winkler (with similar examples by Caicedo and Cope): Let L be the first-order predicate calculus with a single unary predicate symbol U. Let A_n, $n=0,1,2,\ldots$, be the sentence in L which states that there are exactly n elements in U; let B_n say that there are exactly n elements *not* in U. Let $T=\{\neg A_j \wedge \neg B_j : j \geqslant 0\}$, and for each $i \geqslant 0$ let $T_i = \{A_i \vee B_i\} \cup \{\neg A_j : j \neq i\}$. It is easily verified that the corresponding properties P, P_0, P_1, etc., are mutually exclusive and exhaustive, and none can be finitely axiomatized.

Caciedo proves also that if every P_i is axiomatized by a *complete* theory, then infinitely many P_i must be finitely axiomatizable. He notes that examples may be given in which the number of non-finitely axiomatizable P_i is infinite or is any finite number except 0. He notes further that the answers to (a) and (b) remain true when stated in terms of "finitely axiomatizable with respect to T," where T is a theory; for example, if T is equivalent to a finite disjunction of mutually inconsistent axiomatizable properties, then all of them must be finitely axiomatizable with respect to T.

Convolution Products on Functions N→C

6145 [1977, 300]. *Proposed by Michael Barr, Eidgenössische Technische Hochschule, Zürich, Switzerland*

Let $\mathbf{N}=\{0,1,2,\ldots\}$ be the natural numbers, $\mathbf{C}^*$ the nonzero complex numbers. Suppose $\rho: \mathbf{N} \times \mathbf{N} \to \mathbf{C}^*$ is a "kernel function" with the property that the convolution product defined on functions $\mathbf{N} \to \mathbf{C}$ by the formula

$$(f *_\rho g)(n) = \sum_{i+j=n} \rho(i,j) f(i) g(j)$$

is associative. Show that there is a function $\sigma: \mathbf{N} \to \mathbf{C}^*$ such that $f *_\rho g = \sigma^{-1}(\sigma f * \sigma g)$, where an unadorned $*$ denotes the usual convolution with respect to the kernel which is identically 1. Note that this implies that $f *_\rho g = g *_\rho f$ and ultimately that ρ is symmetric.

Solution by J. G. Mauldon, Amherst College. The required function is uniquely defined by

$$\sigma(0) = \rho(0,0), \quad \sigma(n+1) = \sigma(n)/\rho(1,n) \quad n \geqslant 0, \tag{*}$$

so that in fact $\sigma(n) = \rho(0,0)/\Pi_{j=0}^{n-1}\rho(1,j)$. (Note then that $\rho(1,0)=\rho(0,0)$ and consequently $\sigma(1)=1$.)

Proof. The basic functions $f_i: \mathbf{N} \to \mathbf{C}^*$ are defined by $f_i(n) = \delta_{in}$ and the identity $(f_i *_\rho f_j) *_\rho f_k = f_i *_\rho (f_j *_\rho f_k)$ yields

$$\rho(i,j)\rho(i+j,k) = \rho(i,j+k)\rho(j,k).$$

Taking first $j=k=0$ and then $i=j=0$, we find $\rho(i,0)=\rho(0,0)=\rho(0,k)$. Taking $i=k=p$ and $j=q-p$ (where $q \geqslant p \geqslant 0$) and using induction on q, we next find the more general result $\rho(p,q)=\rho(q,p)$. Finally, taking $j=1$, we have

$$\rho(1,i)\rho(1+i,k) = \rho(1,k)\rho(i,k+1),$$

from which, with (*), we immediately deduce

$$\frac{\sigma(i+1)\sigma(k)}{\rho(i+1,k)} = \frac{\sigma(i)\sigma(k+1)}{\rho(i,k+1)}.$$

Keeping $i+j$ constant, using induction on i, and observing that $\rho(i+j,0)=\sigma(0)$, we further deduce that $\sigma(i)\sigma(j)/\rho(i,j)=\sigma(i+j)$.

Further computation now yields

$$f_i *_\rho f_j = \rho(i,j) f_{i+j} = (\sigma(i)\sigma(j)/\sigma(i+j)) f_{i+j} = \sigma^{-1}(\sigma f_i * \sigma f_j),$$

from which the required more general conclusion follows by linearity.

Also solved by R. C. Lyndon, Arnaldo Mandel (Canada), L. E. Mattics, Alan Schwartz, G. D. Williams (England), and the proposer.

Note. Williams considers the products as $\mathbf{N}\to\mathbf{C}$ (as the proposer intended) rather than $\mathbf{N}\to\mathbf{C}^*$ (as erroneously printed) so that they may be well defined for the associative law. He also notes that the result naturally is to be regarded as the vanishing of a certain cohomology group.

Lyndon and Mandel note that $\mathbf{C}$ may be replaced by any field. Lyndon notes further that if $\mathbf{N}$ is replaced by $\mathbf{Z}$, the problem asserts that every abelian extension of the multiplicative group $\mathbf{C}^*$ by $\mathbf{Z}$ splits.

Did Bacon Write Shakespeare's Plays?

6146 [1977, 300]. *Proposed by Edward J. Wegman, University of North Carolina, and Anton Glaser, Pennsylvania State University, Abington Campus*

The year 1623 marked not only the publication of Shakespeare's *First Folio*, but also Sir Francis Bacon's *De Augmentis Scientiarum*, in which Bacon proudly explained his "biliteral" cipher for writing *omnia per omnia*, or anything by anything. Bacon assigned the 24 letters of the alphabet (j and u were absent) to the first 24 5-bit strings from 00000 to 10111. (Actually he used the 5-letter words AAAAA to BABBB, but this is a non-essential difference.) The word Bacon would appear as

$$00001 \quad 00000 \quad 00010 \quad 01101 \quad 01100$$

and this in turn could be hidden in a covertext of at least 25 letters, such as

00	00	10	000	00	00	1001	10	101	100
↓↓	↓↓	↓↓	↓↓↓	↓↓	↓↓	↓↓↓↓	↓↓	↓↓↓	↓↓↓
To	be,	*or*	not	to	be,	*th*a*t*	*is*	*the*	*q*uestion.

Here, "0" was replaced by one typestyle (in this case Roman) and "1" by another (in this case *Italic*). Thus the 4,500,000 letters of the *First Folio* may be interpreted as a string of 4,500,000 binary digits.

What is the probability that the message "Bacon wrote this" appears in the *First Folio* "by accident"

(a) if the probability of a letter being Roman is $1/2$?

(b) if the probability of the letter being Italic is $1/10$?

Solution by L. E. Clarke, University of East Anglia, England. After encipherment, the message "Bacon wrote this" appears as the following string of 70 binary digits:

$$0000100000000100110101100101001000001101100100010010010001110100010001 \qquad (1)$$

consisting of 46 0's and 24 1's, and has the particular property that if all digits are shifted i places to the right, where i is any one of $1,2,\ldots,69$, at least one of the shifted digits will be in a position where it differs from the binary digit originally occupying that position.

Let A_i be the event that the string (1) appears in the *First Folio*, regarded as a string of $N=4{,}500{,}000$ binary digits, with the first digit of (1) coinciding with the ith digit of the *First Folio*. Thus the possible values of i are $1,2,\ldots,N-69$.

Let p be the probability that a particular letter is Roman, i.e., that the corresponding digit is 0. Then

$$P(A_i)=p^{46}(1-p)^{24}=P, \text{ say.}$$

Let also S_r denote the sum of the probabilities

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) \quad (i_1 < i_2 < \cdots < i_r).$$

Then the desired probability, i.e., the probability that at least one of the A's occurs, is

$$S_1 - S_2 + S_3 - \cdots$$

(see W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, 3rd ed., p. 99). Because of the particular property of the string (1), $P(A_i \cap A_j) = 0$ whenever $0 < |i-j| < 70$. Thus S_r reduces to a sum of $\binom{N-69r}{r}$ terms each equal to p^r, and the desired probability is

$$\sum (-1)^{r-1} \binom{N-69r}{r} p^r = \pi(p), \text{ say,}$$

where the limits of summation are $r=1$ and $r=[N/70]$.

By Bonferroni's inequalities (see Feller, p. 110), $\pi(p)$ lies between S_1 and $S_1 - S_2$. Moreover, $S_2 < \frac{1}{2} S_1^2$. Thus S_1 is a sufficiently good approximation to $\pi(p)$, and we obtain the following answers (correct to three significant figures):

$$\text{(a)} \quad \pi\left(\tfrac{1}{2}\right) = 3.81 \times 10^{-15}; \quad \text{(b)} \quad \pi\left(\tfrac{9}{10}\right) = 3.53 \times 10^{-20}.$$

Note further that

$$\pi' = \frac{P'}{P} \sum (-1)^{r-1} r S_r,$$

and the sum is the probability that exactly one of the A's occurs (see Feller, p. 106), and is therefore >0 for $0<p<1$. Therefore π is a maximum when P is a maximum, i.e., when $p=46/70$. Thus under the most favorable circumstances, the probability of the message appearing by accident is

$$\pi(46/70) = 1.28 \times 10^{-13}.$$

Also solved by Marguerite Gerstel, L.E. Mattics, Lajos Takács, J. G. Wendel, and the proposers.

Note. Let $\nu(N)=k$ if the message appears exactly k times, where N is the number of digits in the decipher. Takács shows that $E[\nu(N)] = (N-69)P$, $\operatorname{var}\{\nu(N)\} = (N-69)P - 139NP^2 + 1442P^2$ and

$$\lim_{N\to\infty} P\left\{ \frac{\nu(N) - NP}{\sqrt{nP(1-139P)}} \leqslant x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

Takács continues: Let $\tau(k)=n$ if exactly n digits are required to be printed until the message appears precisely k times. Then similar probability statements are possible; in particular, the expectation of the number of digits required for the message to appear for the first time is $1/p = 3.51 \times 10^{19}$ if $p=46/70$, $=1.18 \times 10^{21}$ for $p=1/2$, $=1.27 \times 10^{26}$ for $p=9/10$.

UNSOLVED PROBLEMS

This list supplements lists printed on page 711 of the June–July 1969 issue of this MONTHLY, on page 1033 of the November 1971 issue, and on pages 529–530 of the May 1974 issue, in addition to pages 76–89 of the August–September 1957 Supplement (The Otto Dunkel Memorial Problem Book). The editors would welcome solutions or discussions.

5872 [1972, 913] A convex polytope
5881 [1972, 1044] Sign of a Jacobian
5884 [1972, 1140] Expectations for sequences of random variables
5888 [1972, 1141] Function not continuous on a dense set
5889 [1973, 82] Isomorphic topological groups

5893	[1973, 83]	Topological group
5910	[1973, 441]	Euclidean rings
5917	[1973, 697]	Upper bound of a length
5927	[1973, 814]	Convex subsets of C
5954	[1974, 176]	Measurable subsets
5956	[1974, 176]	Homeomorphism Z^+ to Z^+
5980	[1974, 670]	Probability of an arithmetic progression
5991	[1974, 910]	Algebraic proof of a theorem by Levi
6005	[1974, 1121]	Product of deficient numbers
6024	[1975, 409]	Uniformly distributed sequence
6028	[1975, 410]	Polynomial bijections
6029	[1975, 410]	Bornological linear space
6076	[1976, 141]	A smoothing function
E 2289	[1971, 405]	A problem in logic
E 2432	[1973, 943]	Representing numbers by $n\,n$'s
E 2438	[1973, 1057]	Squares beginning with specified digits
E 2462	[1974, 281]	Extension of an Erdös inequality
E 2494	[1974, 902]	A problem in sets
E 2521	[1975, 169]	Selecting test items
E 2530	[1975, 400]	Partition of the + integers
E 2594	[1976, 379]	Probability of a self-intersecting polygon
E 2596	[1976, 379]	Building a square with Cuisenaire rods
E 2608	[1976, 567]	Exploration in a train

For the following problems only comments or partial solutions have been printed. Complete solutions are solicited.

5861	[1975, 767]	Increasing polynomials on an ordered field
5871	[1976, 573]	The equation $\partial f/\partial x = \partial f/\partial y$
5886	[1974, 294]	Inserting a 3-dimensional cube in a tesseract
5890	[1974, 295]	A minimum value
5973	[1976, 142]	The strip
5983	[1976, 293]	Rational approximation to $\sqrt{2}$ and π
5989	[1976, 748]	An integer sequence from the harmonic series
6001	[1976, 493]	Remainder term in MacLaurin's series
6016	[1976, 820]	A large modified factorial
6020	[1977, 65]	Friendly integers
6051	[1977, 492]	Extending a sublinear map
6060	[1978, 390]	Combinatorics in finite space
E 2427	[1974, 780]	Bounds for Egyptian fraction partition of unity
E 2468	[1977, 59]	When $2^n - 2^m$ divides $3^n - 3^m$
E 2476	[1974, 516]	Tetratangent spheres kissing precisely
E 2539	[1976, 742]	A known unsolved problem in disguise
E 2555	[1977, 135]	Indefinite quadratic form in a box
E 2569	[1977, 296]	Stack of pancakes

Problems listed earlier as unsolved, for which solutions have now been received and published, are the following:

3834	[1958, 47]	5413	[1975, 85]	E 966	[1977, 568]
3951	[1958, 371]	5415	[1969, 948]	E 1030	[1975, 1010]
4003	[1966, 421]	5427	[1975, 673]	E 1073	[1976, 135]
4052	[1975, 1016]	5437	[1976, 818]	E 1075	[1976, 54]
4306	[1962, 421]	5575	[1975, 674]	E 1150	[1971, 405]
4444	[1962, 173]	5589	[1976, 141]	E 1255	[1975, 661]
4538	[1966, 675]	5608	[1978, 500]	E 1298	[1975, 661]
4555	[1961, 68]	5643	[1975, 677]	E 1822	[1977, 569]
4638	[1963, 1015]	5670	[1975, 677]	E 1847	[1970, 523]
4664	[1968, 415]	5687	[1976, 572]	E 2331	[1975, 1012]
4744	[1966, 89]	5723	[1976, 62]	E 2344	[1975, 937]
5124	[1976, 662]	5773	[1975, 942]	E 2349	[1976, 54]
5314	[1975, 672]	E 435	[1976, 813]	E 2384	[1976, 285]
5385	[1976, 662]	E 570	[1976, 285]	E 2392	[1977, 58]
5405	[1975, 1017]	E 585	[1976, 133]	E 2401	[1976, 198]

MISCELLANEA

AN INTERESTING LETTER

18. By a strange sequence of events, an undated letter has come to light, asserted to be from Fermat to Descartes. Although the provenience of the letter is clouded, we feel that it may be of interest to readers of the MONTHLY. (cf. A. Orenstein, Int. Logic Rev. 6 (Dec. 1975))

M. René Descartes:

You have argued cogently that he who thinks, is, without regard for the nature of these thoughts. Reflecting upon this, I have found another use for my "method of descent" which I think will interest you.

Consider: Most people think of themselves from time to time, but we may suppose that there are some selfless people who never think of themselves. Let us hypothesize that I am a person whose sole thoughts are of each of the selfless persons. I will argue that I cannot exist, even though I have thoughts!

For, either I must be selfless or not selfless. If I am selfless, then at some time my thoughts must turn to myself as one of the selfless persons; but by doing so, I reveal that I am not selfless! On the other hand, if I am not selfless, then I will sometime think of myself. However, since the only object of my thoughts are selfless beings, I myself must have been selfless!

From this dilemma, I can only conclude that it is inconceivable that I exist.

I draw the conclusion that my existence depends not only on the fact that I think, but also upon the content of my thoughts.

May I suggest that you pass this letter on to young Blaise Pascal. He has a bright mind and wide interests. Perhaps he can clarify the implications of this for both God and Reality.

Pierre de Fermat

(Translated and communicated by R. C. Buck, who comments: "It would be interesting if this letter were authentic, and preceded Descartes' 1647 visit to Pascal and the latter's subsequent drastic change in interests.")

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, L. *Dictionary of Logical Terms and Symbols.* Carol Horn Greenstein. Van N-Rein, 1978, xiii + 188 pp, $11.95. [ISBN: 0-442-22834-1] More like a multilanguage reference book than a dictionary. Offers handy simultaneous translation of the basic logical concepts and argument forms into the notational systems most commonly used by logicians, computer scientists and engineers. Includes glossaries of abbreviations and of logical terms. Absence of a list of symbols makes it difficult to look up unfamiliar symbols. Marred by a number of typographical errors. GHM

GENERAL, T(14-16: 2). *Fundamental Concepts of Elementary Mathematics.* Lawrence A. Trivieri. Har-Row, 1977, ix + 630 pp, $13.95. [ISBN: 0-06-046675-8] Designed to meet the mathematical needs of elementary school teachers, it is also suitable as an introduction to mathematical systems for liberal arts students. Concentrates on number systems, but also deals with logic, sets, relations, matrices, general algebraic systems, geometry, linear programming, probability and statistics. The presentation tends to be formal and requires some mathematical sophistication. RSK

GENERAL, S**(13-16), P, L***. *International Mathematical Olympiads, 1959-1977.* Samuel L. Greitzer. New Math. Lib., V. 27. MAA, 1978, xi + 204 pp, $6.50 (P). [ISBN: 0-88385-627-1] "Our aim in the present collection is not only to help the high school student satisfy his curiosity by presenting solutions with tools familiar to him, but also to instruct him in the use of more sophisticated methods and different modes of attack by including explanatory material and alternate solutions." A welcome addition to the NML series. LCL

BASIC, T(13), S. *Mini-Courses in Elementary Statistics: Descriptive Statistics and Probability Distributions; Mini-Courses in Mathematics: Trigonometry; Mini-Courses in Mathematics: Algebra.* Gordon MacLeod. Educulture. A collection of audio-tutorial modules covering the introductory courses in algebra (19 modules), trigonometry (9 modules), and statistics (7 modules, 9 more in preparation). Each module consists of a response manual of 48-52 pages and two audio cassettes. The response manuals are closely coordinated with the audio portions so that there is a constant interaction between the student and the narrator. The modules are targeted especially for students with weak backgrounds or abilities in mathematics who would have difficulty reading a mathematics text. Single modules in the algebra and trigonometry programs sell for $29.50; $37.50 for the statistics (this includes one dozen response manuals and two audio-cassettes). Reduced rates for the purchase of complete programs. A Teacher's Guide with diagnostic tests, post tests, and worked-out solutions is also available. LCL

PRECALCULUS, T*(13: 1), S. *Basic Mathematics for Calculus.* Dennis G. Zill, Jacqueline M. Dewar, Warren S. Wright. Wadsworth, 1978, x + 510 pp, $12.95. [ISBN: 0-534-00569-1] This book succeeds in covering only those topics which will be of direct use in calculus, and also previews those uses. The text relies heavily on lots of worked examples and problems. It assumes minimal high school background. CEC

PRECALCULUS, T(13: 1), S. *Intermediate Algebra, Second Edition.* John H. Minnick. P-H, 1978, x + 404 pp, $13.95. [ISBN: 0-13-469569-0] This typical intermediate algebra book includes polynomials, algebraic expressions, equations and inequalities, graphs, systems of equations, sequences and series, exponential and logarithmic functions and a brief introduction to complex numbers. An abundance of worked examples and problems. CEC

PRECALCULUS, T*(13: 1). *Algebra and Trigonometry with Analytic Geometry, Fourth Edition.* Earl W. Swokowski. Prindle, 1978, vii + 600 pp, $15.50. [ISBN: 0-87150-255-0] The usual precalculus course with the addition of a chapter on analytic geometry. Set notation has been almost eliminated from this edition. This book is strong on examples, figures and problems. Graphical techniques are emphasized. CEC

EDUCATION, T(15: 1), S, P, L. *Mathematics for Elementary School Teachers.* James E. Schultz. Merrill, 1977, x + 421 pp, $13.50. [ISBN: 0-675-08509-8] A mathematics book covering topics that elementary school teachers should see, as opposed to books which include teaching methods. The main emphasis is on set theory and number systems with a brief section on geometry. The collection of exercises is outstanding. CEC

EDUCATION, P, L. *Mathematics Library: Elementary and Junior High School, Fourth Edition.* Margariete Montague Wheeler, Clarence Ethel Hardgrove. NCTM, 1978, viii + 53 pp, $4 (P). [ISBN: 0-87353-126-4] This publication is an updated version of the annotated bibliography formerly prepared by Hardgrove and Herbert F. Miller. It includes over 400 titles and recommends grade levels. It suggests to teachers and librarians a selection of books that may serve to enrich instructional programs. CEC

EDUCATION, T*(14: 1), S, L. *The Third "R": Mathematics Teaching for Grades K-8.* Gerald R. Rising, Joseph B. Harkin. Wadsworth, 1978, xvi + 332 pp, $13.95. [ISBN: 0-534-00567-5] This book consists of four parts: an overview of mathematics and the general psychological principals of teaching it, the content of elementary school mathematics (including logic), teaching styles, and the management of the classroom. The authors have succeeded in communicating a broader view of school mathematics and have provided a rich variety of teaching ideas. The exercises are outstanding. CEC

EDUCATION, S(16-18), P. *Epistemology and Psychology of Functions.* Jean Piaget, *et al.* Reidel, 1977, xiv + 205 pp, $34. [ISBN: 90-277-0804-5] Translation of 1968 French edition. Studies (based on experiments with children) of the role of the function as a precursor of the mental operations in cognitive thinking. Defends thesis that thinking in functions is the beginning of understanding. Includes essay by J.-B. Grize on history and development of notion of function in mathematics from an epistemological viewpoint. GHM

FOUNDATIONS, S(15-16), L. *Foundations of Analysis: Landau Revisited.* C.J. Mozzochi. Exposition Pr, 1976, vii + 69 pp, $7.50. [ISBN: 0-682-48511-X] A rewriting of Landau's *Grundlagen der Analysis* from the naive set-theoretic point of view. Construction of integers, rationals, reals, complex numbers from Peano's axioms. RBK

FOUNDATIONS, T?(15-17: 1), S, L. *Lattice Theory.* Helmuth Gericke. Frederick Ungar Pub, 1966, 185 pp. Translation of 1963 German edition. Introduction at undergraduate level to lattice theory and the theory of relations. Progresses as far as the basic decomposition and embedding theorems. No exercises. GHM

FOUNDATIONS, P. *Modern Uses of Multiple-Valued Logic.* Ed: J. Michael Dunn, George Epstein. Episteme, V. 2. Reidel, 1977, 338 pp, $39.50. [ISBN: 90-277-0747-2] Invited papers from 1975 International Symposium on multiple-valued logic. Includes 155 page bibliography of many-valued logic from 1966-1975, arranged chronologically, by author, and by topic. Includes editor's introduction and prefatory remarks for each paper. Stresses the variety of uses of multiple-valued logic in algebraic logic, computer science, recursive functions, linguistics, fuzzy logic, and information processing. GHM

FOUNDATIONS, T(15-17: 1). *Mathematical Logic: An Introduction to Model Theory.* A.H. Lightstone. Math. Concepts and Methods in Sci. and Eng., V. 9. Plenum Pr, 1978, xiii + 338 pp, $22.50. [ISBN: 0-306-30894-0] Begins with leisurely development of propositional calculus. Book's chief distinction is its development of the unorthodox notion of "semantical system" for predicate calculus, more general than usual "relational systems" of model theory and tailored to needs of nonstandard analysis (which receives 35 pages). Briefly touches on Löwenheim-Skolem theorem and complete theories, with longer discussion of axioms of set theory. Modest number of exercises, chiefly simple verifications of formal properties. GHM

COMBINATORICS, P. *Lecture Notes in Economics and Mathematical Systems-153: Kombinatorische Entscheidungsprobleme: Methoden und Anwendungen.* Ed: Thomas M. Liebling, Max Rössler. Springer-Verlag, 1978, 206 pp, $12.40 (P). [ISBN: 0-387-08540-8; 3-540-08540-8] A collection of lectures from a course given at the Institut für Operations Research of the ETH in Zürich. JAS

COMBINATORICS, P. *Matroids and Linking Systems.* A. Schrijver. Math. Centre Tracts, No. 88. Math Centrum, 1978, vii + 125 pp, Dfl. 15 (P). [ISBN: 90-6196-154-8] This monograph includes an introduction to graphs, matroids and linking systems, the induction of matroids by linking systems, the structure of linking systems via bipartite graphs, operations on linking systems, polymatroids and polylinking systems, and an algorithm for solving matroid and linking system problems. No problems. A substantial bibliography. CEC

LINEAR ALGEBRA, T(14-17: 1), S. *Mathematical Tools for Applied Multivariate Analysis, Student Edition.* Paul E. Green, J. Douglas Carroll. Acad Pr, 1978, xiii + 376 pp, $13.50 (P). [ISBN: 0-12-297552-9] Paperback reprint of the 1976 hardcover edition (TR, January 1977). RSK

LINEAR ALGEBRA, T**(14: 1), S*, L*. *Linear Algebra.* Larry Smith. Springer-Verlag, 1978, vii + 280 pp, $12. [ISBN: 0-387-90235-X; 3-540-90235-X] A text designed to lead sophomores from single to multivariable calculus. The topics are more limited than in some texts ("a more or less direct path" to the principal axis theorem) but the presentation offers both more mathematical depth and insight into "what's really going on" than most competing books. However, all the core material for a sophomore linear algebra course is present, enhanced by fine exposition involving definitions and what the defined ideas are good for. JAS

ALGEBRA, S*(13), L. *Algebraic Equations of Arbitrary Degrees.* A.G. Kurosh. Trans: V. Kisin. MIR (Imported by: Imported Pub, Inc., 320 W. Ohio St., Chicago, IL 60610), 1977, 35 pp, $1.25 (P). The elementary theory of algebraic equations, written for high school students taking part in the Mathematics Olympiad at Moscow State University. LCL

ALGEBRA, P. *Lecture Notes in Mathematics-634: Die Approximationseigenschaft lokaler Ringe.* Herbert Kurke, *et al.* Springer-Verlag, 1978, iv + 204 pp, $12.40 (P). [ISBN: 0-387-08656-0; 3-540-08656-0] Certain rings, including local rings, have the property that in them special approximate solutions to polynomials imply the actual existence of solutions. These results extend work of M. Grunberg and M. Artin, and are used to study certain rings. JAS

ALGEBRA, P. *Representation Theory of Algebras: Proceedings of the Philadelphia Conference.* Ed: Robert Gordon. Lect. Notes in Pure and Appl. Math., V. 37. Dekker, 1978, vii + 463 pp, $35 (P). [ISBN: 0-8247-6714-4] Proceedings of the conference held at Temple University's Sugar Loaf facility, May 24-28, 1976. Approximately half the papers were presented by Maurice Auslander. JAS

ALGEBRA, T(14: 1). *Modern Algebra.* Reuben Sandler, L. Sheila Foster. Har-Row, 1978, vi + 217 pp, $15.95. [ISBN: 0-06-045718-X] A gentle introduction to groups, rings and fields. Attentive to motivation, the authors discuss the aims of each section and provide many examples before new concepts are tried. Groups appear first and account for more than half the text. Few challenging exercises. TRS

ALGEBRA, P. *Lecture Notes in Mathematics-632: Schêma de Picard Local.* Jean-François Boutot. Springer-Verlag, 1978, ix + 165 pp, $9 (P). [ISBN: 0-387-08650-1; 3-540-08650-1] A construction of a local Picard functor for a local k-algebra R "under" k where k is the residue field of R. JAS

ALGEBRA, T(16-17: 2), S, L. *Introduction to Modern Algebra.* Marvin Marcus. Pure and Appl. Math., V. 47. Dekker, 1978, xii + 489 pp, $19.50. [ISBN: 0-8247-6479-X] An advanced introductory algebra text oriented toward the applications of algebra to other branches of mathematics and to science. Unusually detailed and broad coverage. In addition to standard topics, extensive space is devoted to permutation groups, Pólya counting theory, polynomial theory, canonical forms, differential equations, and group representations. Many exercises. TRS

ALGEBRA, P. *Topics in Group Theory and Computation.* Ed: Michael P.J. Curran. Acad Pr, 1977, xiii + 118 pp, $11.35. [ISBN: 0-12-200150-8] The invited papers by John H. Conway, John Leech, Marshall Hall, Jr., and Peter M. Newmann from the July 16-23, 1973 Summer School on Group Theory and Computation at the Royal Irish Academy at University College, Galway. JAS

FINITE MATHEMATICS, T(13: 1-2). *Mainstreams of Finite Mathematics with Applications.* Chris P. Tsokos. Merrill, 1978, xiv + 496 pp, $13.95. [ISBN: 0-675-08436-9] Topics: logic, sets, counting, probability through the binomial and normal distributions, matrices with applications to Markov chains and to linear programming, game theory, computers. For the nonscience-oriented student with only high school mathematics. TRS

CALCULUS, T(13: 2-3). *Basic Technical Mathematics with Calculus.* Ralph H. Hannon. Saunders, 1978, xi + 547 pp, $15.95. [ISBN: 0-7216-4497-X] This book covers precalculus, trigonometry, differentiation, integration and first and second order differential equations. It is a cookbook approach which lacks depth. The emphasis is on worked examples and long lists of problems. CEC

CALCULUS, T*(13-14: 2, 3). *Calculus with Analytic Geometry, Sixth Edition.* Richard E. Johnson, Fred L. Kiokemeister. Allyn, 1978, 10 + 860 pp, $19.95. [ISBN: 0-205-05917-1]; *Study Guide,* vii + 273 pp, $5.95 (P); *Instructor's Supplement,* vi + 227 pp, (P). This edition preserves the basic features of the previous editions. Additional material provides more intuitive background for the presentation of difficult topics. The section on infinite series has been extensively rewritten. New sections on related rates, path-independent line integrals, complex numbers and Euler's formula. (*Fifth Edition,* TR, October 1974.) CEC

CALCULUS, T(13-14: 2, 3). *Calculus: One and Several Variables with Analytic Geometry, Part One, Third Edition.* S.L. Salas, Einar Hille. Wiley, 1978, xii + 645 pp, $17.50. [ISBN: 0-471-03285-9] This new edition has essentially the same content as previous editions (*Second Edition,* TR, October 1974). All figures have been redrawn and the longer chapters have been split. CEC

CALCULUS, T(14-15: 1, 2). *Vector and Tensor Analysis.* Eutiquio C. Young. Pure and Appl. Math., V. 48. Dekker, 1978, ix + 526 pp, $37.50. [ISBN: 0-8247-6671-7] Even at the special adoption price of $24.50 each for five or more copies, this typescript text seems overpriced. The vectors are all three or two dimensional, and the presentation is very much oriented towards classical elementary physics. The treatment of tensors at this earthy level is unusual but may be just what some students need. However, the result is a very substantial quantity of information presented via complex formulas without the clarity provided by the conceptual framework of linear algebra (vector spaces and linear transformations are not defined) and its geometric offshoots. JAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-633: Les Courants Résiduels Associes à une Forme Méromorphe.* Nicolas R. Coleff, Miguel E. Herrera. Springer-Verlag, 1978, x + 210 pp, $12.40 (P). [ISBN: 0-387-08651-X; 3-540-08651-X] A presentation of a theory of residues of meromorphic forms in arbitrary codimension which generalizes codimension one results of Dolbeault, Bungart, and Grothendieck. JAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-593: Abstract Analytic Function Theory and Hardy Algebras.* Klaus Barbey, Heinz König. Springer-Verlag, 1977, vii + 260 pp, $11 (P). [ISBN: 0-387-08252-2; 3-540-08252-2] Theory of Hardy algebras beginning with concrete situation of harmonic and holomorphic functions in the unit disk and ending with applications to function algebras on compact planar sets. RBK

DIFFERENTIAL EQUATIONS, S(17). *Differential Equations: Geometric Theory, Second Edition.* Solomon Lefschetz. Dover, 1977, x + 390 pp, $5 (P). [ISBN: 0-486-63463-9] Reprint of the 1963 edition. Originally published in 1957, investigates non-linear equations of second order. After studying the basic concepts, the author then concentrates on point stability and two dimensional systems. TLS

DIFFERENTIAL EQUATIONS, T*(16-18: 1, 2), S, P, L. *Modern Methods in Partial Differential Equations: An Introduction.* Martin Schechter. McGraw, 1977, xv + 245 pp, $28.50. [ISBN: 0-07-055193-6] Not a book about the Laplace, heat and wave equations. Rather, this is an introduction to the powerful methods that have been used in the study of linear partial differential equations of arbitrary order over the last 30 years. Ideas from functional analysis are developed when needed. Exercises. Prerequisite: a very good background in advanced calculus. TRS

DIFFERENTIAL EQUATIONS, T(14-15: 1), S, L. *Introduction to Ordinary Differential Equations.* Rodney D. Driver. Har-Row, 1978, xi + 340 pp, $12.95. [ISBN: 0-06-041738-2] Text contains usual topics of first course: elementary solution methods, linear equations and systems, power series solutions, Laplace transform techniques. An introductory chapter on delay differential equations is the text's most distinguishing feature--the attack on these equations is at a very elementary level, emphasis being placed on simple methods which work in practical examples. Good exercise sets. Presumes only calculus. TRS

DIFFERENTIAL EQUATIONS, P. *Moving Boundary Problems.* Ed: D.G. Wilson, Alan D. Solomon, Paul T. Boggs. Acad Pr, 1978, x + 329 pp, $15. [ISBN: 0-12-757350-X] Proceedings of the symposium/workshop held in Gatlinburg, Tennessee, September 26-28, 1977. Sections of the volume are devoted to theory, methods, and applications. JAS

NUMERICAL ANALYSIS, P, *Lecture Notes in Mathematics-631: Numerical Treatment of Differential Equations*. Ed: R. Bulirsch, R.D. Grigorieff, J. Schröder. Springer-Verlag, 1978, ix + 219 pp, $12.40 (P). [ISBN: 0-387-08539-4; 3-540-08539-4] Proceedings of the conference held at Oberwolfach, July 4-10, 1976. The emphasis is on initial value problems for ordinary differential equations and fast methods for solving difference equations for elliptic boundary value problems. JAS

NUMERICAL ANALYSIS, P, *Lecture Notes in Mathematics-630: Numerical Analysis*. Ed: G.A. Watson. Springer-Verlag, 1978, xii + 199 pp, $12.40 (P). [ISBN: 0-387-08538-6; 3-540-08538-6] The invited papers from the Biennial Conference held at Dundee, June 29-July 1, 1977. JAS

NUMERICAL ANALYSIS, P, *Numerische Behandlung von Differentialgleichungen mit besonderer Berücksichtigung freier Randwertaufgaben*. J. Albrecht, *et al.* Birkhäuser, 1978, 279 pp, Sfr. 44 (P). [ISBN: 3-7643-0986-5] Proceedings of the May 1-7, 1977 conference at Oberwolfach. JAS

OPTIMIZATION, T(13-14: 1), S, L, *An Introduction to Linear Programming and Matrix Game Theory*. M.J. Fryer. Halsted Pr, 1978, vi + 120 pp, $7.95 (P). [ISBN: 0-470-99327-8] A "semi-programmed" format intended for social science students. Linear programming, simplex method, game theory, duality, all at an introductory manipulative level. RWN

OPTIMIZATION, S(17-18), *Einführung in die Nichtlineare Optimierung*. Habil K.-H. Elster, *et al.* B.G. Teubner, 1977, 299 pp, 29M (P). An introduction to the theory of nonlinear optimization intended for mathematicians, engineers and economists. Assumes linear algebra and calculus. Bibliography, but no exercises. JD-B

OPTIMIZATION, T?(16), S, L, *Mathematics for Operations Research*. W.H. Marlow. Wiley, 1978, xv + 483 pp, $19.95. [ISBN: 0-471-57233-0] The mathematics necessary to study operations research, except for probability and statistics. Includes linear algebra, multivariable calculus, some complex variable, and linear differential and difference equations. Lots of problems. CEC

OPTIMIZATION, T*(15-17: 1, 2), L, *Introduction to Operations Research Techniques*. Hans G. Daellenbach, John A. George. Allyn, 1978, xvii + 603 pp, $17.95. [ISBN: 0-205-05755-1]; *Solutions Manual*, 157 pp, free (P). [ISBN: 0-205-05756-X] A down-to-earth survey of standard topics (LP, networks, PERT, dynamic programming, Markov decision processes, queues, inventory control, integer and nonlinear programming, simulation, heuristics), illustrated with many realistic, nontrivial examples and exercises. Designed as a "springboard to the more advanced literature," the volume concludes with an appendix listing bibliographies, journals and OR texts; each chapter, moreover, concludes with an annotated list of special references. LAS

ANALYSIS, P, *Probabilistic Analysis and Related Topics, V. 1*. Ed: A.T. Bharucha-Reid. Acad Pr, 1978, ix + 239 pp, $25. [ISBN: 0-12-095601-2] The first of several proposed volumes of collected short monographs on the analysis and applications of random functions. Topics treated in this volume include differential and integrodifferential equations, Gaussian measures, and stochastic Riemannian geometry. The authors are S.D. Chatterji, Pao-liu Chow, Ruth Curtain, D. Kannan, V. Mandrekar, and Mark Pinsky. JAS

ANALYSIS, S(18), P, *On Axiom A Diffeomorphisms*. Rufus Bowen. CBMS Reg. Conf. in Math., No. 35. AMS, 1978, vii + 45 pp, $6 (P). [ISBN: 0-8218-1685-3] An expository paper on developments since Smale's 1967 paper. Discusses several topics (entropy, ergodic theory, etc.) by examining basic problems and examples. Also looks at some open problems and work on problems closely associated with the Axiom A diffeomorphisms. Extensive bibliography. A worthwhile introduction to the theory. TLS

ANALYSIS, P, *Lecture Notes on Nil-Theta Functions*. Louis Auslander. CBMS Reg. Conf. in Math., No. 34. AMS, 1977, vii + 96 pp, $5.60 (P). [ISBN: 0-8218-1684-5] Presentation of classical results in a manner that unites in a single theory A. Weil's proof of the Plancherel Theorem, his new treatment of Abelian varieties, and the Weil-Brezin map. LAS

ANALYSIS, P, *Oeuvres*. Juliusz Pawel Schauder. PWN, 1978, 487 pp. Complete works (with the exception of a Polish summary of his dissertation and two brief notes) covering Schauder's work on fixed points and bases for normed spaces and other work in analysis. JAS

ANALYSIS, P, *General Inequalities 1*. Ed: E.F. Beckenbach. Int. Ser. Num. Math., V. 41. Birkhäuser, 1978, xv + 332 pp, Sfr. 52. [ISBN: 3-7643-0972-5] Proceedings of the First International Conference on General Inequalities, Oberwolfach, May 10-14, 1976. Papers are divided into five sections: mean values and classical inequalities, approximations and probabilistic inequalities, functional inequalities, differential and integral inequalities, and geometric and topological inequalities. The volume closes with "Remarks and Problems." JAS

ANALYSIS, T(17-18), *Valós Függvénytan és Ortogonális Sorok*. Mikolás Miklós. Tankönyvkiadó, 1978, 393 pp. [ISBN: 963-17-2204-X] An exposition (without exercises) of the basics of modern real analysis--Riemann and Lebesgue theory, measures, orthogonal series in Hilbert space--for graduate students of mathematics, physics, and engineering. In Hungarian. JAS

ANALYSIS, P, *Convolution Integral Equations with Special Function Kernels*. H.M. Srivastava, R.G. Buschman. Halsted Pr, 1977, 164 pp, $9.75. [ISBN: 0-470-99050-3] Definitions, general properties, methods and examples of convolution equations with special function kernels. Half of book is a table of inversion formulas. RBK

DIFFERENTIAL GEOMETRY, T(18: 1), P, *Lecture Notes in Mathematics-640: Curvature and Characteristic Classes*. Johan L. Dupont. Springer-Verlag, 1978, ix + 175 pp, $10 (P). [ISBN: 0-387-08663-3; 3-540-08663-3] Lecture notes developing Chern-Weil theory for arbitrary Lie groups. Begins with careful but relatively elementary proof of deRham theorem. The techniques developed here are then reapplied throughout the text. Extensive problem sets; many problems are quite involved and referred to later. Bibliography quite ordinary. TLS

GEOMETRY, S(10-14). *Tessellations: The Geometry of Patterns.* Stanley Bezuszka, Margaret Kenney, Linda Silvey. Creative Pub, 1977, vi + 169 pp, $6.50 (P). Using tessellations as its theme and construction as its method, this book introduces many geometric concepts in a novel way. The student is encouraged to design progressively more complicated tessellations, then study why they were or were not possible. The ideas of interior angles, Euclidean motions, coloring problems, duality, etc., are thus introduced. Also encourages an artistic approach to geometry. This workbook provides many sheets for constructing tessellations. TLS

GEOMETRY, P. *Beiträge zur Geometrischen Algebra.* Hans J. Arnold, Walter Benz, Heinrich Wefelscheid. Math. Reihe, B. 21. Birkhäuser, 1977, 383 pp, Sfr. 118. [ISBN: 3-7643-0908-3] Proceedings of a symposium on geometric algebra and the foundations of geometry, held at Duisburg in 1976. JD-B

GEOMETRY. *Greek Geometry from Thales to Euclid.* George Johnston Allman. Arno Pr, 1976, xii + 237 pp, $14. [ISBN: 0-405-07287-2] A reprinting of an 1889 edition. JNC

GEOMETRY, P. *Lectures on Closed Geodesics.* Wilhelm Klingenberg. Grund. math. Wissenschaften, B. 230. Springer-Verlag, 1978, xi + 227 pp, $32.50. [ISBN: 0-387-08393-6; 3-540-08393-6] A presentation of the major results on the existence of closed geodesics on closed Riemannian manifolds: the Lusternik-Schnirelmann theorems, the Gromoll-Meyer conditions for existence, the theory of three closed geodesics for compact, simply-connected manifolds. SG

GEOMETRY, T(16-17: 1, 2), L. *Differential Geometry.* Heinrich W. Guggenheimer. Dover, 1977, x + 378 pp, $6 (P). [ISBN: 0-486-63433-7] An unabridged, corrected, and slightly reduced (photographically) reprint of the original 1963 McGraw-Hill edition. JAS

DIFFERENTIAL TOPOLOGY, P. *Lecture Notes in Mathematics-638: The Atiyah-Singer Index Theorem: An Introduction.* Patrick Shanahan. Springer-Verlag, 1978, v + 224 pp, $12.40 (P). [ISBN: 0-387-08660-9; 3-540-08660-9] The Index Theorem, "one of the fundamental mathematical discoveries of recent decades," is a very general result relating the topological characteristics of manifolds to the dimensions of spaces of related differential operators. This sophisticated expository account includes an outline of the proof and numerous applications to various other parts of mathematics. LAS

TOPOLOGY, P. *A Product Formula for Surgery Obstructions.* John W. Morgan. Memoirs No. 201. AMS, 1978, xiv + 90 pp, $7.60 (P). [ISBN: 0-8218-2201-2] A calculation of the surgery obstruction of the product of a degree one normal map and a closed manifold in terms of the obstruction of the original normal map and invariants of the closed manifold. JAS

TOPOLOGY, T*(18: 1, 2), S*, P. *K-Theory: An Introduction.* Max Karoubi. Grund. math. Wissenschaften, B. 226. Springer-Verlag, 1978, xviii + 308 pp, $39. [ISBN: 0-387-08090-2; 3-540-08090-2] An exposition of (topological) K-theory including some applications. The presentation is intended to be self-contained for the mathematician acquainted with the introductory homotopy theory of projective spaces and classical vector bundles (although this material is reviewed quickly). Numerous exercises, and good indices make this an attractive introduction to an important generalized homology theory.JAS

TOPOLOGY, S(18), P. *Infinite Loop Spaces.* J.F. Adams. Princeton U Pr, 1978, x + 214 pp, $14; $5.50 (P). Survey lectures given by the author as Hermann Weyl Lectures at the Institute for Advanced Study, Princeton in May and April of 1975. JAS

TOPOLOGY, T(15-16: 1, 2). *TP et TD de Topologie Générale, Second Edition.* Alain Faisant. Hermann (US Distr: SMPF, 14 E. 60th St., NY 10022), 1977, 304 pp, 62F (P). [ISBN: 2-7056-5745-2] Moore method text. Brief sections give basic definitions, then an abundance of exercises, followed by answers. One drawback seems the limited number of examples generated in the problems. TLS

TOPOLOGY, T?(16-17). *Topology and Maps.* Taqdir Husain. Math. Concepts and Methods in Sci. and Eng., V. 5. Plenum Pr, 1977, xx + 337 pp, $29.50. [ISBN: 0-306-31005-8] Six chapters of introductory topology presented rather formally followed by four chapters on function spaces and various sorts of maps. The strong emphasis on modern analysis is reflected in the early definition of categories and functors, and the extensive study of uniform spaces while the term "connected" does not seem to appear even once. JAS

TOPOLOGY, T(15-17: 1). *Undergraduate Topology.* Robert H. Kasriel. Krieger, 1977, xiv + 285 pp, $12. [ISBN: 0-88275-444-0] A reprint of the 1971 Saunders edition. (Original edition, TR, April 1971; ER, June 1972.) JAS

TOPOLOGY, T(18: 1, 2), L. *Set-Theoretic Topology with Emphasis on Problems from the Theory of Coverings, Zero Dimensionality and Cardinal Invariants.* Gregory Naber. U Microfilms Intern, 1977, xv + 706 pp, $37.25. [ISBN: 0-8357-0257-X] A sophisticated "introduction" to topology consisting of a very quick presentation (review) of the basic concepts of point set topology followed by a thorough development of three special areas. The aim is to teach topology as an alluring subject, introducing the student to some exciting, recent work leading to unanswered questions. The three areas are: the theory of coverings (metrization problems among others); zero dimensionality and N-compactness; and cardinal invariants and Alexandroff's problem. Substantial problem sets occur at the end of each section; bibliography and index are also substantial. JAS

TOPOLOGY, S(17-18), P, L. *Treelike Spaces and Related Connected Topological Spaces.* A.E. Brouwer. Math. Centre Tracts, No. 75. Math Centrum, 1977, iv + 109 pp, Dfl. 14 (P). [ISBN: 90-6196-132-7] A study of the structure of "easily disconnected" spaces. Examples include: V spaces in which each connected subset has at most one non-cut point and treelike spaces in which any two distinct points can be separated by a third point. JEG

TOPOLOGY, P. *Lecture Notes in Mathematics-628: Obstruction Theory.* Hans J. Baues. Springer-Verlag, 1977, xi + 387 pp, $14.30 (P). [ISBN: 0-387-08534-3; 3-540-08534-3] A systematic presentation

of obstruction theory in the homotopy classification of maps, which attempts to unify and integrate the different approaches found in the literature. This is the first self-contained exposition on the subject to appear. JEG

TOPOLOGY, P. *Unraveling the Integral Knot Concordance Group*. Neal W. Stoltzfus. Memoirs No. 192. AMS, 1977, iv + 91 pp, $7.20 (P). [ISBN: 0-8218-2192-X] A study of the algebraic structure of the group of concordance classes of high dimensional homotopy spheres knotted in codimension two in a standard sphere. JAS

TOPOLOGY, P. *Normal Structures and Bordism Theory, with Applications to MSp_**. Nigel Ray, Robert Switzer, Larry Taylor. Memoirs No. 193. AMS, 1977, ix + 66 pp, $7.20 (P). [ISBN: 0-8218-2193-8] Three coordinated papers. The first discusses the problem of enumerating the bordism classes carried on a fixed manifold by means of varying its normal structure. The second develops the additional machinery needed to apply these results to the study of S_p structures on Alexander's family of manifolds. This study is carried out in the third. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-643: Multiaxial Actions on Manifolds*. Michael Davis. Springer-Verlag, 1978, vi + 141 pp, $9 (P). [ISBN: 0-387-08667-6; 3-540-08667-6]. Notes for a series of five lectures from the transformation group seminar at the Institute for Advanced Study, examining smooth actions which resemble linear representations. JEG

TOPOLOGY, T*(16-17: 1), S, L. *Basic Concepts of Algebraic Topology*. Fred H. Croom. Springer-Verlag, 1978, x + 177 pp, $14.80. [ISBN: 0-387-90288-0; 3-540-90288-0] For the student who has a good undergraduate knowledge of point set topology and of group theory, this book presents an unusual opportunity to learn some algebraic topology with a sense of both geometry and history. This is genuinely an introduction to simplicial homology (no cohomology), and to homotopy theory (Hurewicz theorem stated without proof). Exact sequences occur rarely, categories not at all, but the Euler-Poincaré theorem and the Borsuk-Ulam theorem are treated for their content (rather than proofs). JAS

PROBABILITY, P. *Lecture Notes in Mathematics-624: Marches Aléatoires sur les Groupes de Lie*. Yves Guivarc'h, Michael Keane, Bernard Roynette. Springer-Verlag, 1977, vii + 292 pp, $11.50 (P). [ISBN: 0-387-08526-2; 3-540-08526-2] An analytic and probabilistic study of random walks on Lie groups. JAS

PROBABILITY, S(16-18), P. *Random Allocations*. Valentin F. Kolchin, Boris A. Sevast'yanov, Vladimir P. Chistyakov. Trans: A.V. Balakrishnan. V.H. Winston, 1978, xi + 262 pp, $19.95. [ISBN: 0-470-99394-4]

PROBABILITY, P*. *Linear Least-Squares Estimation*. Ed: Thomas Kailath. Benchmark Papers in Elec. Eng. and Comp. Sci., V. 17. Dowden, Hutchinson & Ross, 1977, xiii + 318 pp, $25. [ISBN: 0-470-99238-7] Collection of 20 classical papers which provide the foundations for modern least-squares estimation of random processes. Following a survey article on linear filtering theory the papers are divided into three groups: early mathematical foundations of prediction theory, Wiener-Hopf equations and optimum filters, and state-space models and recursive filters. RSK

PROBABILITY, P*. *Algorithmic Methods in Probability*. Ed: Marcel F. Neuts. TIMS Stud. in Management Sci., V. 7. North-Holland, 1977, vi + 294 pp, $26.75 (P). [ISBN: 0-444-85049-X] Seventeen papers dealing with algorithmic analysis of stochastic models. Three papers discuss general algorithms, two concern the methodology of simulation, three deal with some different statistical problems, and the remaining nine pertain to a large number of problems in the theory of queues. RSK

PROBABILITY, P. *Decomposition of Superpositions of Density Functions and Discrete Distributions*. Pál Medgyessy. Halsted Pr, 1977, 308 pp, $27.50. [ISBN: 0-470-15017-3] Very readable, well-motivated survey of methods (developed primarily by the author) for treating the problems indicated in the title. Illustrated by numerous applications in spectroscopy, nuclear physics, biology, statistics, and systems identification. GHM

PROBABILITY, P, L. *Ergodic and Information Theory*. Ed: Robert M. Gray, Lee D. Davisson. Dowden, Hutchinson & Ross, 1977, xiii + 387 pp, $30. [ISBN: 0-87933-300-6] A collection of 33 papers and editors' comments on the development of ergodic theory and the notion of entropy from the viewpoint of information theory. JEG

PROBABILITY, T(14-16: 1), S, L. *Urn Models and Their Application: An Approach to Modern Discrete Probability Theory*. Norman L. Johnson, Samuel Kotz. Wiley, 1977, xiii + 402 pp, $21.95. [ISBN: 0-471-44630-0] Develops much of discrete probability theory using simple urn models. Includes fundamental definitions, discrete distributions, occupancy problems, models with stochastic replacements, and limit distributions. A large chapter on applications, e.g., genetics, capture-recapture, learning, sampling and decision theory. Numerous exercises. RWN

PROBABILITY, P. *Stochastic Systems: Modeling, Identification and Optimization*. Ed: Roger J.-B. Wets. North-Holland, 1976. *V. I*, ix + 243 pp [ISBN: 0-7204-0569-6]; *V. II*, x + 263 pp, $19.75 each (P). [ISBN: 0-7204-0570-X] Most of the proceedings (a few papers were not prepared for publication) of the symposium held in Lexington, Kentucky in June 1975. The first volume contains papers dealing with modeling and identification; the second with models involving control of stochastic process. These two volumes are volumes five and six of the "Mathematical Programming Study" series. JAS

PROBABILITY, P. *Transactions of the Seventh Prague Conference on Information Theory, Statistical Decision Functions, Random Processes and of the 1974 European Meeting of Statisticians, V. A*. Ed: Jaroslav Kožešnik. Reidel, 1977, 602 pp, $52. Eleven of seventeen invited papers and 117 of 147 "communications" appear in two volumes A and B. Volume A contains papers (approximately one-half of those published) concerning probability theory and stochastic processes. Volume B is to contain those concerning statistics and information theory. JAS

PROBABILITY, T(16-18: 1, 2), S, P. *The Asymptotic Theory of Extreme Order Statistics.* Janos Galambos. Wiley, 1978, xiii + 352 pp, $24.95. [ISBN: 0-471-02148-2]

PROBABILITY, T(15-17), S. *An Introduction to Stochastic Processes with Special Reference to Methods and Applications, Third Edition.* M.S. Bartlett. Cambridge U Pr, 1978, xvii + 388 pp, $26.50. [ISBN: 0-521-04116-3] This new edition has additional material on extinction probabilities for variable environments, multi-dimensional diffusion equations, and the use of likelihood functions. (First Edition, TR, October 1967.) FLW

STATISTICS, T*(13-14: 2). *An Introduction to the Statistical Analysis of Data.* T.W. Anderson, Stanley L. Sclove. HM, 1978, xvi + 704 pp, $14.50. [ISBN: 0-395-15045-0] First ten chapters were published in 1974 under the title *Introductory Statistical Analysis* (TR, December 1974). New chapters are on inferences concerning variances, contingency tables, analysis of variance, simple regression, multiple regression, and sampling techniques. A well-written, attractively printed text. RSK

STATISTICS, T(13-15: 1). *Statistical Survey Techniques.* Raymond J. Jessen. Wiley, 1978, vii + 520 pp, $24.95. [ISBN: 0-471-44260-7] In the Wiley Series in Probability and Mathematical Statistics. A practical self-contained introduction to the design and analysis of surveys, amply illustrated with real data, much from the author's own work. Each chapter concludes with a summary, extensive review illustrations, references and exercises. Prerequisites in mathematics and statistics are minimal, with most derivations included in separate mathematical notes sections. RSK

STATISTICS, P. *SIMPLE: A Software Handbook of Statistical Techniques.* J.R.F. Alonso. Sterling Swift Pub, 1978, 261 pp, $10.95 (P). [ISBN: 0-88408-098-6] A collection of interactive statistical programs, together with a few business and mathematics programs, written in Basic for use either on a mini-computer or as an elementary statistics package on a large computer. Includes a description of each technique, a complete listing and a sample run. RSK

STATISTICS, T*(16-17: 2), L. *Introduction to Mathematical Statistics, Fourth Edition.* Robert V. Hogg, Allen T. Craig. Macmillan, 1978, x + 438 pp, $15.95. [ISBN: 0-02-355710-9] Revision of the authors' well-known text (*Third Edition,* TR, April 1970), primarily involving a change in the order of presentation of some topics. Chapters 1-5 contain some small changes but still cover basic distribution theory. Chapters 6-9 now contain primarily standard material on estimation and tests of statistical hypotheses, while sufficient statistics are covered in Chapter 10, and some less standard topics in statistical inference have been put in Chapter 11. Chapter 12, on further normal distribution theory, remains the same. RSK

STATISTICS, P*. *Applications of Statistics.* Ed: Paruchuri R. Krishnaiah. North-Holland, 1977, xiv + 543 pp, $45. [ISBN: 0-444-85034-1] Contains all the invited papers presented at the Symposium on Applications of Statistics sponsored by the Air Force Flight Dynamics Laboratory and held at Wright State University, Dayton, Ohio in June, 1976. Some of the areas of applications covered include acoustics, cancer research, cluster analysis, communication, econometrics, hydrology, meterology, model building, pattern recognition, pharmacokinetics, psychometrics, reduction of dimensionality, reliability, stability of structures, and turbulence. RSK

STATISTICS, P*. *Contributions to Survey Sampling and Applied Statistics: Papers in Honor of H.O. Hartley.* Ed: H.A. David. Acad Pr, 1978, xxvii + 318 pp, $32. [ISBN: 0-12-204750-8] Interesting collection of papers by associates of Hartley as a tribute to him on the occasion of his 65th birthday. Includes reminiscences by three of his co-workers and a list of his publications. There are 8 papers on sampling, 5 on the linear model, 1 on time series, 4 on outliers, robustness, and censoring, and 2 on mathematical programming and computing. RSK

STATISTICS, T(13-15: 1, 2). *Introductory Statistics for Sociology.* Judith D. Handel. P-H, 1978, xi + 387 pp, $14.95. [ISBN: 0-13-503060-9] Presupposes only high school algebra. Attempts considerable discussion of multivariate descriptive statistics along with the more standard topics. FLW

STATISTICS, T(13-14: 1, 2). *Statistical Reasoning in Psychology and Education, Second Edition.* Edward W. Minium. Wiley, 1978, xii + 564 pp, $14.95. [ISBN: 0-471-60828-9] Presupposes high school algebra. Standard topics with examples from the fields indicated in the title. FLW

STATISTICS, T(16-18: 1, 2), S, P, L. *Applied Time Series Analysis, V. 1: Basic Techniques.* Robert K. Otnes, Loren Enochson. Wiley, 1978, xiv + 449 pp, $25. [ISBN: 0-471-24235-7] Emphasizes software, programming languages, floating point calculations, and flexibility. FLW

STATISTICS, T(13-17: 1, 2), S, P, L*. *Fundamentals of Decision Analysis.* Irving H. LaValle. HR&W, 1978, xiii + 626 pp, $22.50. [ISBN: 0-03-085408-3] Emphasizes the general applicability of the decision-analytic model. Shows how "classical" inference and stochastic simulation are encompassed. Most of the material requires no calculus. FLW

STATISTICS, T*(15-18: 1), S, L. *Fixed Effects Analysis of Variance.* Lloyd Fisher, John McDonald. Probl and Math. Stat. Acad Pr, 1978, xiii + 177 pp, $16. [ISBN: 0-12-257350-1] For a one term analysis of variance course that presupposes some linear algebra and calculus-based statistics. FLW

STATISTICS, T(13: 1). *Basic Statistics for Nurses.* Rebecca Grant Knapp. Wiley, 1978, viii + 308 pp, $9.50 (P). [ISBN: 0-471-03545-9] Brief (occasionally very brief) introduction to the use of the most common statistical formulas and techniques, with illustrations drawn from nursing and allied health professions. Detailed solutions to all exercises. LCL

STATISTICS, T(13: 1, 2), S. *Statistics for Management.* Richard I. Levin. P-H, 1978, 568 pp, $15.95. [ISBN: 0-13-845305-5] Presupposes no college mathematics. The usual topics plus subjective probabilities, time series, index numbers, and decision theory. FLW

STATISTICS, T(13-14: 1). *Business Statistics, Concepts and Applications.* William J. Stevenson. Har-Row, 1978, xviii + 518 pp, $14.95. [ISBN: 0-06-046445-3] Probability, estimation and hypothesis testing, regression and correlation, analysis of variance and nonparametrics; plus a chapter each on index numbers and time series. Leisurely and informal; a complete absence of theorems, proofs and derivations. Workbook and Study Guide available, making the text suitable for a self-paced approach. Presumes only minimal high school algebra. LCL

STATISTICS, T(15-16: 1-3). *Probability and Statistics for Engineers and Scientists, Second Edition.* Ronald E. Walpole, Raymond H. Myers. Macmillan, 1978, xii + 580 pp, $16.95. [ISBN: 0-02-424110-5] This new edition contains additional material on nonparametric tests, tolerance intervals, linear regression, and sequential model selection. (First Edition, TR, August-September 1972.) FLW

STATISTICS, T(15-16: 1), S. *Introductory Applied Statistics in Science.* Sung C. Choi. P-H, 1978, x + 278 pp, $15.95. [ISBN: 0-13-501619-3] Presupposes some calculus. The usual topics with quite a few numerical examples and a short chapter on computer analysis. FLW

STATISTICS, T(13: 1), S. *Business Statistics, Second Edition.* Donald J. Koosis. Wiley, 1978, viii + 296 pp, $4.95 (P). [ISBN: 0-471-03426-6] A programmed text that could be used for self-study or to supplement a standard text. Presupposes only elementary algebra. (First Edition, TR, June-July 1973.) FLW

STATISTICS, T(13-16: 1), S, L*. *A Sampler on Sampling.* Bill Williams. Wiley, 1978, xv + 254 pp, $15.95. [ISBN: 0-471-03036-8] Numerical (rather than theoretical) exposition of the principles of statistical sampling "for the non-mathematically inclined." Carefully chosen, fully discussed examples convince the reader of the potential hazards of indiscriminant treatment of sample data, and illustrate correct use of basic concepts. Many exercises, including open-ended, discussion-type questions. LAS

STATISTICS, T(15-16: 1, 2). *Probability and Statistics.* William L. Quirin. Har-Row, 1978, xi + 488 pp, $12.95. [ISBN: 0-06-045293-5] Presupposes calculus. The standard topics in probability and statistics, plus Markov chains. FLW

STATISTICS, P. *Statistics of Random Processes II, Applications.* R.S. Liptser, A.N. Shiryayev. Appl. of Math., No. 6. Springer-Verlag, 1978, x + 339 pp, $29.80. [ISBN: 0-387-90236-8; 3-540-90236-8]

COMPUTER PROGRAMMING, T(13-14: 1), L. *Understanding FORTRAN.* Michel Boillot. West Pub, 1978, xi + 490 pp, $11.50 (P). [ISBN: 0-8299-0205-8] A slowly paced, examples-first approach to Fortran programming. RWN

COMPUTER PROGRAMMING, T(13-14: 1). *FORTRAN Programming Using Structured Flowcharts.* Richard E. Haskell. SRA, 1978, xii + 280 pp, $9.95 (P). [ISBN: 0-574-21135-7] A moderately-paced introduction to Fortran with an emphasis on concepts from structured programming. Subroutines are introduced surprisingly late. Several types of statements are relegated to an appendix. RWN

COMPUTER PROGRAMMING, T. *COBOL, A Pragmatic Approach.* Robert T. Grauer, Marshal A. Crawford. P-H, 1978, xvii + 394 pp, $13.95 (P). [ISBN: 0-13-139097-X] Introductory text. Aims to teach commercial data processing, not just textbook Cobol. After covering ANS Cobol, discusses debugging, JCL, programming style, file processing, and BAL. TH

COMPUTER PROGRAMMING, T. *Understanding BASIC in Business.* Michel Boillot. West Pub, 1978, vii + 276 pp, $8.95 (P). [ISBN: 0-8299-0206-6] Introductory text. Each chapter begins with a complete programming example to introduce new statements. TH

COMPUTER PROGRAMMING, T(13). *Basic BASIC, Second Edition: An Introduction to Computer Programming in BASIC Language.* James S. Coan. Hayden, 1978, 269 pp, $8.95 (P); $9.95. [ISBN: 0-8104-5106-9; 0-8104-5107-7] An elementary level introduction, this second edition includes a discussion of strings and data files. Some mathematical topics covered are: polynomials, trigonometric functions, complex numbers, probability, and matrices. SG

COMPUTER PROGRAMMING, T(13-14: 1), L. *Computers, Their Impact and Use: Structured Programming in PL/1.* Robert E. Lynch, John R. Rice. HR&W, 1978, ix + 452 pp, $10.95 (P). [ISBN: 0-03-088527-2] PL/1 version of introductory text *Computers, Their Impact and Use: Basic Language* (TR, April 1977). RWN

COMPUTER SCIENCE, T(13), L. *Beginning Computer Science.* James L. Poirot, David N. Groves. Sterling Swift Pub, 1978, iii + 290 pp, $9.95 (P). [ISBN: 0-88408-106-0] Substantial overlap with the authors' *Computer Science for the Teacher* (TR, October 1976). Brief history, overview of applications, flowcharting, introduction to Basic, computer arithmetic, calculators and minicomputers. Recommended for high school students. RWN

COMPUTER SCIENCE, S. *Elemente der Informatik.* Klaus Menzel. Teubner, Stuttgart, 1978, 224 pp, DM 22,80 (P). [ISBN: 3-519-02708-9] An introduction to algorithms and their uses, written for prospective secondary-school mathematics teachers. JD-B

COMPUTER SCIENCE, T(15-16: 1). *Structured Concurrent Programming with Operating Systems Applications.* R.C. Holt, *et al.* A-W, 1978, 262 pp, $10.50 (P). [ISBN: 0-201-02937-5] Uses a concurrent structured programming language CSP/k (an extension of SP/k which is a subset of PL/1) to develop and demonstrate concurrent programming techniques for operating systems. Presumes knowledge of a high-level language and some experience with using operating systems. RWN

COMPUTER SCIENCE, T(15-16: 1), S, L. *Computer Data Structures.* John L. Pfaltz. McGraw, 1977, xi + 446 pp, $17.95. [ISBN: 0-07-049743-5] Representation of information using linked lists, tree and graphs. Algorithms for operating on these structures. Storage allocation techniques. Special file structures. Includes applications to computer graphics. RWN

COMPUTER SCIENCE, T(16-17: 1, 2). *Theory & Design of Switching Circuits.* Arthur D. Friedman, Premachandran R. Menon. Computer Sci Pr, 1975, xii + 581 pp, $19.95. [ISBN: 0-914894-52-8] Combinatorial circuits, synchronous sequential circuits, asynchronous circuit design, decomposition, physical design problems, and applications to problems in logical design related to magnetic bubbles. RWN

COMPUTER SCIENCE, T(15-18: 1, 2), S, L. *An Introduction to the General Theory of Algorithms.* Michael Machtey, Paul Young. North-Holland, 1978, vii + 264 pp, $19.95; $9.95 (P). [ISBN: 0-444-00226-X; 0-444-00227-8] Begins with a development of the evidence which supports the assertion that the class of partial recursive functions is the class of algorithmically computable functions. Reductions; coding of RAM programs; classical examples of algorithmically unsolvable problems. Recursive function theory and applications to mathematical logic. Computational complexity; exponentially and superexponentially difficult problems; complete problems. Exercises. References. Glossary. Index. RJA

COMPUTER SCIENCE, S(13-18), P. *Programmable Calculators: How to Use Them.* Charles J. and Roger J. Sippl. Matrix Pub, 1978, xxiii + 526 pp, $14.95 (P). [ISBN: 0-916460-08-8] Thorough survey of programmable calculators of all sizes and capabilities. Provides in depth descriptive, technical, and programming details for the various machines included. Some comparative treatment with other types of calculating devices. History of developments in the field. Many application examples of uses of specific models and of general engineering, scientific, and data processing examples. Appendix of problem-solving programs, formulas, and procedures. Index. RJA

COMPUTER SCIENCE, T(15-18: 1, 2), S, P, L. *The Environment for Systems Programs.* Frederic G. Withington. A-W, 1978, xv + 324 pp, $16.95. [ISBN: 0-201-14459-X] Intended for developers of systems programs. Describes users of such programs and the requirements needed to interact successfully with such software. Discusses various types of managers, systems analysts, computer system operators, and organizational behavior patterns that affect design of systems software. Batch, interactive, transaction-processing, and combined modes. Description of existing provider environments and provider's interests. History. Annotated bibliography and indices. RJA

COMPUTER SCIENCE, T(17-18), S, P, L. *Automata-Theoretic Aspects of Formal Power Series.* Arto Salomaa, Matti Soittola. Springer-Verlag, 1978, x + 171 pp, $16.50. [ISBN: 0-387-90282-1; 3-540-90282-1] The formalism of power series is capable of unifying and generalizing known results in theoretical computer science. This book is the first to appear which develops a theory of formal power series in noncommuting variables, the main emphasis being on results applicable to automata and formal language theory. LCL

COMPUTER SCIENCE, S(15-18), P. *Automated Theorem Proving: A Logical Basis.* Donald W. Loveland. Fund. Stud. in Comp. Sci., V. 6. North-Holland, 1978, xiii + 405 pp, $43.50. [ISBN: 0-7204-0499-1] Exposition of major techniques and conceptual advances in automated (computerized) theorem proving through the late 1960's. No specific implementations given. Focus is on the logical basis of the resolution method and its many refinements. Complete proofs of most theorems. Occasional exercises. This valuable compilation is marred only by an infelicitous style, crowded typography and stiff price. GHM

COMPUTER SCIENCE, T(13-14: 1), L. *Business Data Processing.* William S. Davis. A-W, 1978, xxii + 442 pp, $13.95. [ISBN: 0-201-01116-6] A very gentle introduction that should compensate for any student qualms about computers. Nevertheless the book introduces most of the major concepts of computer language, programming, and machine design. The presentation is current and not childish, though the mathematical level is very low. JAS

COMPUTER SCIENCE, T(15-18: 1, 2), S, P, L. *Computer Data Security.* Harry Katzan, Jr. Van N-Rein, 1973, viii + 223 pp, $12.95. [ISBN: 0-442-24258-1] Begins with a description of the data security problem. Then considers computer systems, computer software, data management, data communications, threats to security and countermeasures, methods for implementing security measures, descriptions of actual systems having security provisions, and cryptographic techniques used in data security. Bibliography. Name index. Subject index. RJA

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-2: New Trends in Systems Analysis.* Ed: A. Bensoussan, J.L. Lions. Springer-Verlag, 1977, vii + 759 pp, $22.60 (P). [ISBN: 0-387-08406-1; 3-540-08406-1] Proceedings of the International Symposium, Versailles, December 13-17, 1976. Includes special sections on a number of topics including industrial robotics and applications of microprocessors, applications of control theory to energy, economics, and pollution. JAS

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-1: Distributed Parameter Systems: Modelling and Identification.* Ed: A. Ruberti. Springer-Verlag, 1978, v + 458 pp, $18.50 (P). [ISBN: 0-387-08405-3; 3-540-08405-3] Proceedings of the IFIP Working Conference held in Rome, Italy, June 21-24, 1976. JAS

SYSTEMS THEORY, S(14-18), P, L. *Decisions with Multiple Objectives: Preferences and Value Tradeoffs.* Ralph L. Keeney, Howard Raiffa. Wiley, 1976, xxviii + 569 pp, $21.50. [ISBN: 0-471-46510-0] "On cardinal utility analysis with multiple conflicting objectives: the case of individual decision making under uncertainty from the prescriptive point of view--with special emphasis on applications but with a little theory thrown in for spice." FLW

SYSTEMS THEORY, S(16), P, L. *Systems: Approaches, Theories, Applications.* Ed: William E. Hartnett. Episteme, V. 3. Reidel, 1977, x + 202 pp, $32. [ISBN: 90-277-0822-3] A collection of papers from the Eighth George Hudson Symposium on general systems theory, ranging from specific technical results to general approaches to theory formulation. JEG

SYSTEMS THEORY, T(15-17: 1), S, L. *System Simulation, Second Edition.* Geoffrey Gordon. P-H, 1978, xii + 324 pp, $18.95. [ISBN: 0-13-881797-9] An updating of the popular 1969 edition. System

models, continuous and discrete simulation techniques, elements of probability theory, the use of GPSS and SIMSCRIPT. RNW

SYSTEMS THEORY, P*. *Distributed Parameter Systems: Identification, Estimation, and Control.* Ed: W. Harmon Ray, Demetrios G. Lainiotis. Control and Systems Theory, V. 6. Dekker, 1978, x + 597 pp, $49.50. [ISBN: 0-8247-6601-6] "Concerned with the dynamic behavior of processes distributed in space as well as evolving in time." Part 1 develops the fundamentals of the theory, while Part 2 discusses several important applications. A comprehensive treatment by leaders in this new field. RSK

SYSTEMS THEORY, P. *Nonlinear Systems, Stability Analysis.* Ed: J.K. Aggarwall, M. Vidyasagar. Dowden, Hutchinson & Ross, 1977, xiii + 380 pp, $32.50. [ISBN: 0-470-99044-9] A collection of twenty-six papers reproduced from assorted journals and provided with editorial introductions. This is volume sixteen in the series Benchmark Papers in Electrical Engineering and Computer Science. JAS

APPLICATIONS, P. *Transactions of the Twenty-Third Conference of Army Mathematicians.* US Army Research, Durham, NC, 1977, xv + 654 pp, (P). About three-quarters of the papers presented at the title conference. These papers represent mostly classical applied mathematics in industrial, academic, and military contexts. JAS

APPLICATIONS, T*(15: 1), S*, L. *Introduction to the Laplace Transform.* Peter K.F. Kuhfittig. Math. Concepts and Methods in Sci. and Eng., V. 8. Plenum Pr, 1978, x + 205 pp, $19.50. [ISBN: 0-306-31060-0] A very readable introduction to the Laplace transform and its basic applications. It is aimed primarily at engineering students, but it is mathematically honest. It includes a brief sketch of complex variable theory, and lots of examples and exercises. CEC

APPLICATIONS (BIOLOGY), T(16-18: 1), S, P, L*. *Lectures on Nonlinear-Differential-Equation Models in Biology.* J.D. Murray. Clarendon Pr, 1977, xiii + 370 pp, $24.50. [ISBN: 0-19-853350-0] The author describes some deterministic models from biology (e.g., enzyme kinetics, facilitated diffusion, olfactory communication, biological oscillators, developmental biology), discusses the mathematical techniques, and compares the results with experiment. Sufficient description of the biology for the non-biologist. Requires solid experience with differential equations but no probability or statistics. An excellent book for a seminar on applied differential equations. TRS

APPLICATIONS (CONTROL THEORY), P. *Control System Design for Pole-Zero Assignment.* Ed: F. Fallside. Acad Pr, 1977, x + 239 pp, $22.50. [ISBN: 0-12-248250-6] Papers from a "Working Party" held at Cambridge University in September 1974. Some of these papers have appeared elsewhere. JAS

APPLICATIONS (CYBERNETICS), T(14-15: 1), S, L. *The Foundations of Cybernetics.* F.H. George Gordon, 1977, xiv + 286 pp, $24.50. [ISBN: 0-677-05340-1] A brief introductory survey, ranging from logic and animal behavior to automata and artificial intelligence. 18 chapters, each beginning with an "argument" (i.e., a thesis), and concluding with a summary and survey of recent advances. Many references, but no exercises, problems or projects. LAS

APPLICATIONS (DIFFERENTIAL GAMES), P. *Lecture Notes in Control and Information Sciences-3: Differential Games and Applications.* Ed: P. Hagedorn, H.W. Knobloch, G.J. Olsder. Springer-Verlag, 1977, xii + 236 pp, $11.50 (P). [ISBN: 0-387-08407-X; 3-540-08407-X] The main lectures from the workshop at Enschede, Netherlands, March 16-25, 1977. JAS

APPLICATIONS (ECONOMICS), T(16-18: 1), S, P, L. *Oligopoly and the Theory of Games.* James W. Friedman. North-Holland, 1977, xiii + 311 pp, $26.75. [ISBN: 0-7204-0505-X] Basic coverage of oligopoly theory and n-person nonzero sum game theory. FLW

APPLICATIONS (ECONOMICS). *Mathematical Economics and Operations Research: A Guide to Information Sources.* Joseph Zaremba. Gale Res, 1978, xi + 606 pp, $18. [ISBN: 0-8103-1298-0] An annotated bibliography of 1,600 English language books (up to early 1975), divided into eighteen subject categories. A very useful reference. LAS

APPLICATIONS (ECONOMICS), T(15-18: 1), S, P, L. *Dynamic Linear Economic Models.* James L. Kenkel. Gordon, 1974, xvii + 380 pp, $29.50. [ISBN: 0-677-04950-1] "A fairly rigorous and detailed treatment of the theory of difference equations and their applications in the constructions and analysis of dynamic economic models." FLW

APPLICATIONS (ECONOMICS), T(13-14). *Mathematics of Finance, Fifth Edition.* Robert Cissell, Helen Cissell, David C. Flaspohler. HM, 1978, xiii + 578 pp, $14.95. [ISBN: 0-395-25807-3] This edition has additional examples and exercises, and new review exercises. (*Third Edition,* TR, April 1969; *Fourth Edition,* TR, March 1973.) FLW

APPLICATIONS (ENGINEERING), T(15-18: 1, 2), S, P, L. *The Finite Element Method.* O.C. Zienkiewicz. McGraw, 1977, xv + 787 pp, $22.50. [ISBN: 0-07-084072-5] Begins with preliminaries on discrete physical systems, including elasticity examples of finite element approximations. General definitions of the finite element method follow. Stress analysis; bending of thin plates; shells; steady-state field, nonlinear dimension; flow of viscous fluids; computer techniques. Chapter references. Appendices. Author and subject indices. RJA

APPLICATIONS (ENGINEERING), T(16-17: 1, 2). *Analytical Dynamics of Discrete Systems.* Reinhardt M. Rosenberg. Math. Concepts and Methods in Sci. and Eng., V. 4. Plenum Pr, 1977, xvii + 424 pp, $25. [ISBN: 0-306-31014-7] An outgrowth of the author's senior level course in mechanical engineering at the University of California, Berkeley, this book presents a second course in dynamics which emphasizes Lagrangian mechanics. The presentation is informal and takes time for historical and "cultural" observations. JAS

APPLICATIONS (PHYSICS). *Mathematics and the Universe: An Interpretation Based on the Theory of Relativity.* E.T. Lawrence. Vantage Pr, 1977, 106 pp, $8.95. [ISBN: 533-02537-0] A mathematically unsophisticated (no references, no calculus) philosophical treatise. JAS

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-70: Wave Propagation and Underwater Acoustics.* Ed: Joseph B. Keller, John S. Papadakis. Springer-Verlag, 1977, viii + 287 pp, $11.50 (P). [ISBN: 0-387-08527-0; 3-540-08527-0] This collection of lecture notes consists of six papers from a workshop held in Mystic, Connecticut in November 1974: taken together, these papers constitute a general survey of the mathematical theory of underwater sound propagation. CEC

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-73: Invariant Wave Equations.* Ed: Giorgio Velo, Arthur S. Wightman. Springer-Verlag, 1978, 416 pp, $18.50 (P). [ISBN: 0-387-08655-2; 3-540-08655-2] Collected lecture notes from the session of the International School of Mathematical Physics on invariant wave equations which was held in Erice, Sicily, June 27-July 9, 1977. JAS

APPLICATIONS (PHYSICS), P, *Mathematical Foundations of Quantum Theory.* Ed: A.R. Marlow. Acad Pr, 1978, x + 372 pp, $22. [ISBN: 0-12-473250-X] Proceedings of the conference held at Loyola University, New Orleans, June 2-4, 1977 wherein mathematicians and physicists discussed and studied intensively the mathematical and logical foundations of quantum theory. JAS

APPLICATIONS (PHYSICS), S(18), P, *Interdisciplinary Mathematics, V. XIX: Yang-Mills, Kaluza-Klein, and the Einstein Program.* Robert Hermann. Math Sci Pr, 1978, xii + 198 pp, $20 (P). [ISBN: 0-915692-25-2] Exposition and development of ideas of the author, Yang, and others. The work studies differential geometric ideas and questions of quantization of general relativity. JAS

APPLICATIONS (PHYSICS), P, *International Conference on Mathematical Problems of Quantum Field Theory and Quantum Statistics, Part 1: Axiomatic Quantum Field Theory.* Ed: V.S. Vladimirov. Proc. of Steklov Inst. of Math., No. 135. AMS, 1978, iv + 260 pp, $50 (P). [ISBN: 0-8218-3035-X] Translation of Volume 1935 (1975) containing some of the papers from the 1972 conference named in the title. LAS

APPLICATIONS (PHYSICS), T(14-16: 1), S, *Vibrations and Waves in Physics.* Iain G. Main. Cambridge U Pr, 1978, xiv + 356 pp, $37.50; $8.95 (P). [ISBN: 0-521-21662-1] A text for a physics course. Mathematical models of idealized prototype systems provide the theory's basis and the text's unity. Order-of-magnitude estimates of controlling quantities are used to show why glass is transparent, metals are shiny and the sky is blue, why microwave grills are efficient and tidal waves are destructive. Fourier methods. Exercises. TRS

APPLICATIONS (PHYSICS), S(16-17), P, *Unidirectional Wave Motions.* H. Levine. Appl. Math. and Mech., V. 23. North-Holland, 1978, xii + 502 pp, $69.75. [ISBN: 0-444-85043-0] An up-to-date, comprehensive survey of efficient methods in the analysis of unidirectional wave motions. Some 87 accessible sections constitute the organizational framework of the book and cover such topics as Green's functions, matched expansions, the Doppler effect, scattering matrices, dispersion relations, variational principles, periodic and random configurations, and stability. Exercises. TRS

APPLICATIONS (PHYSICS), T(17-18: 1, 2), S, P, *A Course on the Application of Group Theory to Quantum Mechanics.* Irene Verona Schensted. Neo Pr, 1976, vii + 342 pp, $10; $5 (P). [ISBN: 0-911014-24-1] A major (five fold) extension of the "Short Course on the Applications of Group Theory to Quantum Mechanics" which was presented as a set of four lectures at the University of Michigan ten years ago. The presentation is informal (in style and typography) and aimed at the graduate student preparing to be a mathematical physicist. Relatively broad coverage of subjects, inclusion of heuristics, and a reasonable index make it look interesting--at a very reasonable price. JAS

APPLICATIONS (PHYSICS), T(17:1), S, P, *An Introduction to Thermomechanics.* Hans Ziegler. Appl. Math. and Mech., V. 21. North-Holland, 1977, xi + 308 pp, $36.75. [ISBN: 0-444-11080-1] A unified introduction to the fields of continuum mechanics and thermodynamics. Presupposes vector algebra and analysis, the basic laws of mechanics and thermodynamics, n-dimensional geometry, the theory of functions, and the notion of convexity. Includes problems and a bibliography. CEC

APPLICATIONS (PHYSICS), T?(18: 1), P, *Navier-Stokes Equations: Theory and Numerical Analysis.* Roger Temam. Stud. in Math. and its Appl., V. 2. North-Holland, 1977, x + 500 pp, $45. [ISBN: 0-7204-2840-8] This book surveys and describes the present state of the mathematical theory of Navier-Stokes equations of viscous incompressible fluids and examines many recently developed methods of numerical solution of these equations at moderate Reynolds numbers. There are no exercises but there is an extensive bibliography. CEC

APPLICATIONS (PHYSICS), P, *Dilations of Irreversible Evolutions in Algebraic Quantum Theory.* D.E. Evans, J.T. Lewis. Dublin Inst. for Adv. Stud., 1977, v + 104 pp, £3.15 (P). Notes, designed to be self-contained, from a seminar at Dublin in 1975-76 in which Brian Davies' 1972 paper "Some Contraction Semigroups in Quantum Probability" was studied. JAS

APPLICATIONS (SYSTEMS ANALYSIS), T(17: 1, 2), P, *Dynamical Systems and Their Applications: Linear Theory.* John L. Casti. Math. in Sci. and Eng., V. 135. Acad Pr, 1977, xv + 240 pp, $19.50. [ISBN: 0-12-163450-7] Mathematical aspects of linear systems analysis using the frequency response approach. Controllability, readability, observability, constructibility, realizability and stability. Transformation groups on systems and their invariants. Optimal selection of inputs. Bibliography. Exercises. RWN

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Jay E. Goldfeather, Carleton; Timothy Hoel, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D.C. 20036.

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1225 Connecticut Avenue., N.W., Washington, D.C. 20036.

PERSONAL ITEMS

College of the Pacific: Professor William Topp has been appointed Chairman of the Mathematics Department. Associate Professor Roland di Franco has been promoted to Professor. Assistant Professor David Hughes has been promoted to Associate Professor. Dr. Larry Gassman has been appointed Assistant Professor.

Clemson University: Dr. Harold B. Reiter, University of North Carolina, Charlotte, is a Visiting Associate Professor. Ms. Betty B. Walker, University of North Carolina, Charlotte, is a Visiting Instructor. Professor Calvin T. Long, Washington State University, is visiting on an NSF Science Faculty Professional Development Award.

University of Alabama, Huntsville: Associate Professor J. Hoomani has been promoted to Professor. Instructor Helen H. James has been promoted to Assistant Professor.

Southern Methodist University: Instructor Carolyn Shull has been promoted to Assistant Professor. Associate Professor Richard K. Williams has been promoted to Professor. Professor David W. Starr and Assistant Professor Leon Tisdale have retired.

University of Akron: Ms. Judith A. Palagallo has been appointed Assistant Professor. Assistant Professor Phillip H. Schmidt has been promoted to Associate Professor. Associate Professor Louis D. Rodabaugh has retired with the title of Associate Professor Emeritus.

San Jose State University: Dr. Max Agoston, Wesleyan University, has been appointed Assistant Professor. Assistant Professor Michael Burke has been promoted to Associate Professor. Associate Professors Frederick Stern, Marjorie Fitting, and C. Kenneth Bradshaw have been promoted to Professors. Associate Professor John Mitchem has been promoted to Professor and Chairman of the Mathematics Department.

University of Illinois: Professor Joseph L. Doob has retired. Dr. Carl Pomerance, University of Georgia, is a Visiting Associate Professor. Professor Emden R. Gansner, Massachusetts Institute of Technology, is a Visiting Lecturer. Associate Professor Stephen V. Ullom has been promoted to Professor.

Assistant Professor E. D. McCune, Stephen F. Austin State University, has been promoted to Associate Professor.

Associate Professor Bernard McDonald, University of Oklahoma, has been promoted to Professor.

Professor Carlton E. Lemke, Rensselaer Polytechnic Institute, has been awarded (jointly with John F. Nash, Jr.) the John von Neumann Prize, a distinguished service award in the field of operations research.

Assistant Professor Melvin A. Nyman, Manchester College, Indiana, has been promoted to Associate Professor.

Professor Harold Scott Macdonald Coxeter, University of Toronto, has been elected an Honorary Member of the London Mathematical Society in recognition of his outstanding work in geometry and group theory.

Professor Madan L. Puri, Indiana University, is a Visiting Professor at the University of Washington.

Professor Carroll O. Wilde, Naval Postgraduate School, has been awarded the "Rear Admiral John Jay Schieffelin Award for Excellence in Teaching". It is the first time the award has been given to a department chairman.

Professor Ralph Mansfield, Chicago City Colleges, has retired with the title of Professor Emeritus. He lives in Esporlas, Mallorca, Spain, and invites MAA members to visit him when they are in the Mediterranean Area.

Professor F. D. Parker, St. Lawrence University, is an exchange Professor at Plymouth Polytechnic (England) for the 1978-79 academic year.

Professor C. R. Wylie, Jr., Furman University, has retired with the title of William R. Kenan, Jr. Professor of Mathematics Emeritus.

Associate Professor Donald E. Johnson, North Central College, Naperville, Illinois, has been promoted to Professor and appointed Chairman of the Natural Sciences and Mathematics Division.

Professor Wai-Kai Chen, Ohio University, has been awarded the Distinguished Professor award. He is the twenty-second Ohio University faculty member to be given that title.

Michael W. Ecker, Ph.D., City University of New York, has been appointed Assistant Professor at the Pennsylvania State University Dunmore campus.

WOMEN SCIENTISTS ROSTER

Last year the National Science Foundation supported a pilot Visiting Women Scientists Program in which 40 women scientists visited 110 high schools across the country. Based on the success of the pilot program, a number of schools have requested lists of women scientists who might be willing to meet with their students. Women scientists who wish to be included in a roster to be released to schools should send the following information to Ms. Carol Place, Research Triangle Institute, Box 12194, Research Triangle Park, North Carolina 27709 by January 31, 1979:

(1) name
(2) mailing address;
(3) telephone number;
(4) type of science (biological, physical, engineering, mathematics, social science);
(5) specific science field (e.g., bacteriology, mechanical engineering);
(6) highest degree earned;
(7) type of employment (academic, non-profit organization, profit-making organization, government);
(8) race or ethnic background.

Respondents should omit any information they do not wish to have released.

REQUEST FOR INFORMATION ON IMPLEMENTATION OF CUPM PROGRAM

The Department of Mathematics at Illinois State University is considering the implementation of the CUPM "Recommendations for an Undergraduate Program in Computational Mathematics" which appears in the book, *A Compendium of CUPM Recommendations, Vol. II*. The department is seeking information from anyone who has experience in the development and the teaching of all or part of the program.

The program consists of:

(i) five mathematics courses (calculus, linear algebra, advanced calculus)
(ii) four computational mathematics courses (Computational Model and Problem Solving, Introduction to Numerical Computation, Combinatorial Computing, Differential Equations and Numerical Methods)
(iii) three computer science courses (Introduction to Computing, Computer Organization and Programming, Programming Languages and Data Structure)

This program provides valuable interplay between mathematics and computing and meets student needs for scientific programming. The students are exposed to a wide variety of computer applications from the biological, social, behavioral, and physical sciences.

Would anyone with information on the computational mathematics program (suggested program modifications, syllabi, available texts and computer programs, marketability of program graduates) please contact Dr. Orlyn Edge, Department of Mathematics, Illinois State University, Normal, Illinois 61761.

COMMUNITY AND JUNIOR COLLEGE STAFFING CENTER

The American Association of Community and Junior Colleges maintains a Career Staffing Center for its member institutions and those individuals who would like to be considered for staff positions at more than 900 member colleges. Write for details to AACJC Career Staffing Center, P.O. Box 298-C, Alexandria, Virginia 22314.

REQUEST FOR INFORMATION ON PROBLEM SOLVING COURSES

The Mathematics Department at Hamilton College would like information on courses offered on mathematical problem solving at the college level. The Department intends to compile a report on such courses (including tips on what does and what does not work, good problem sources, etc.) and perhaps offer a symposium at a future MAA meeting. If you teach a course in problem solving or know of someone who does, please contact Alan H. Schoenfeld, Mathematics Department, Hamilton College, Clinton, New York 13323.

NEW JOURNAL OF CALCULATOR-DEMONSTRATED MATH INSTRUCTION

Didactic Programming has just come off the press with its first issue. This periodical is dedicated, in particular, to the exchange of new ideas in the use of the hand held calculator as a teaching aid.

At present the publication is being edited by Professor Arthur David Snider, Department of Mathematics, Center for Mathematical Services, University of South Florida, Tampa. The editors invite authors to contribute articles relating to the implementation of calculator techniques in the classroom. Since much has already been written on the use of calculators in teaching calculus, they will be more receptive to papers dealing with other subjects. The first issue, for example, discusses

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

CUPM ANNOUNCEMENT

The Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America has established a Panel to consider the question: "What should every graduate of an American college or university know of mathematics?" It is hoped that the ultimate recommendations of this Panel will provide welcome guidance to colleges, universities, and such bodies as state boards of education, many of which are already actively considering such questions in the current wave of renewed interest in core curricula and general education.

The Panel, relying extensively on surveys of informed opinion, wishes to arrive at a list of minimum mathematical competencies for all college graduates, where "mathematical" is meant to include statistics, computing, etc., as well as mathematics in the narrow sense. Its report should contain, besides this list, a reasoned statement about why every college graduate should have acquired some understanding of mathematical thought; suggestions about courses in which the minimal competencies might be acquired; and general observations about such matters as interinstitutional coordination in furthering mathematical literacy.

The Panel has begun to collect information and opinions on the problem and will welcome contributions from readers of this announcement. Facts about other local, regional, or national efforts (past or present) in the area of the Panel's charge, personal views about the general issue or specific aspects, and copies of or references to pertinent documents are among the things the Panel would be glad to receive. They may be sent to the Panel in care of its chairman, D. Bushaw, Department of Pure and Applied Mathematics, Washington State University, Pullman, Washington 99164.

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The Southern California Section of the MAA held its fifty-ninth meeting at California State University, Fullerton on March 11, 1978. One hundred twenty-five attended the meeting. The Chairman of the Section, Professor John Todd of the California Institute of Technology presided.

At the business meeting the results of the elections of officers for 1978-79 were announced: Chairman: James Murphy, California State College, San Bernardino; 1st Vice-Chairman: John McGhee of California State University Northridge; 2nd Vice-Chairman: Dena L. Patterson of Santa Monica City College; Program Chairman: Susan Montgomery, University of Southern California. John Todd becomes Past Chairman and Edmund Deaton, San Diego State University continues as Secretary-Treasurer.

Professor Charles DePrima of California Institute of Technology was the luncheon speaker. He discussed his experiences with Professor R. Courant during their time together in New York. Rollin Sandberg was in charge of local arrangements.

The following program was presented:

The changing concept of change: the derivative from Fermat to Wierstrass, Judith Grabiner, California State College, Dominguez Hills.

Panel discussion: The MAA-NCTM recommendation for the college preparatory mathematics curriculum in high schools. Gerald Marley, CSU, Fullerton (moderator), James Caballero, Santa Monica High School, David Cohen, UCLA, Henry Mansfield, El Camino College, Les Winters, Kennedy High School, Los Angeles.

The Carmichael Lamda Function and Aut(Z_n), Henry Bray, San Diego State University.

Reflection and their applications in geometry, Computer Graphics, Linear Algebra and Numerical Analysis. Paul Yale, Pomona College.

The complete statistician, Leo Breiman, TSC

Interdisciplinary approach to prediction-based decision, Peter Bright, CSU, Northridge.

Mathematics instruction and student reasoning skills, Michael Clapp and Dave Pagni, both CSU-Fullerton.

Edmund I. Deaton, *Secretary-Treasurer*

TWO VOLUME COLLECTION OF CHAUVENET PAPERS

This two-volume collection of the twenty-four prize-winning Chauvenet Papers contains the finest collection of expository articles in mathematics ever assembled!

The Chauvenet Prize for special merit in mathematical exposition was first awarded by the Mathematical Association of America in 1925 to G.A. Bliss. Since that time, twenty-four Chauvenet Prizes have been awarded and the Prize has become the Association's most prestigious award for mathematical exposition. The list of authors is a veritable *WHO's WHO* of mathematical expositors, and contains some of the more prominent mathematicians of the past fifty years. Clearly written, well organized, expository in nature, these papers are the jewels of mathematical writing during our times. They were selected by peer juries of experts judged for their presentation as well as their content. James C. Abbott is the editor.

some demonstrations for optimization theory and numerical analysis.

Other types of articles having interest for educators will also be appreciated. For example, an exposition of the CORDIC algorithms employed by the calculators to evaluate functions, and a comprehensive discussion of the round-off procedures used in the various devices would be welcomed. Book reviews of the texts by Ball, Eisberg, Henrici, McCarty, Smith, and others would be appropriate.

Further information may be obtained from the Editor's office.

VOLUNTEERS SOUGHT FOR EMPLOYMENT REGISTER JOB COUNSELING

The AMS-MAA-SIAM Joint Committee on Employment Opportunities has long recognized the desire of applicants using the Employment Register at annual meetings for counseling and encouragement regarding many phases of job seeking, not only in the area of research mathematics, but in special areas of employment such as government and industrial work, actuarial mathematics, positions in foreign countries, etc. The Joint Committee and the Society's Committee on Employment and Educational Policy seek volunteers from the mathematical community to serve as counselors in conjunction with the Register scheduled January 25-27, 1979 in Biloxi, Mississippi. Persons planning on attending the Biloxi meeting who are interested in participating in this program for one or two hours should write to Dr. William J. LeVeque, Executive Director, American Mathematical Society, P.O. Box 6248, Providence, Rhode Island 02940.

SUMMER INSTITUTE OF 1980

Suggestions of topics for a summer institute for 1980 are being received by the Committee on Summer Institutes, prior to January 15 for consideration at the Biloxi meeting. The institute is intended to provide an understandable presentation of the state of the art in an active field of research in pure mathematics (there being other provisions for presentations in applied mathematics). Optimally, the suggestions should include the proposed members of the organizing committee (or at least its chairpersons) and a two- or three-page detailed outline of what are the subjects to be covered including suggested principal speakers.

Recent topics have been Automorphic Forms, Representations and L-functions (1977), and Harmonic Analysis and Euclidean Spaces (1978); the topic for 1979 is Finite Group Theory. There have in fact been very few suggestions in recent years. Suggestions may be sent to any member of the committee which consists of Daniel Gorenstein (Rutgers University), Robion C. Kirby (University of California, Berkeley), Peter E. Ney (University of Wisconsin, Madison), Ralph E. Showalter (University of Texas at Austin), Harold M. Stark, Chairman (Massachusetts Institute of Technology), and Joseph L. Taylor (University of Utah).

MATHEMATICAL ASSOCIATION OF AMERICA

THE FIFTY-EIGHTH SUMMER MEETING OF THE ASSOCIATION

The Fifty-Eighth Summer Meeting of the Mathematical Association of America was held at Brown University, Providence, Rhode Island, from Tuesday, August 8 to Thursday, August 10, 1978, in conjunction with meetings of the American Mathematical Society, the Association for Women in Mathematics, and Pi Mu Epsilon. There were registered 963 persons including 578 members of the MAA.

Sessions of the Association were held on Tuesday morning and afternoon, Wednesday morning, and Thursday afternoon. Each of the sessions was dual with one session emphasizing the subject matter of mathematics and the other emphasizing some classroom aspects. The latter sessions were organized by Professor Helen B. Siner of Staten Island Community College and were held in the Barus and Holley Building. The other sessions were held in Meehan Auditorium.

Presiding officers at the three Earle Raymond Hedrick Lectures by Professor Richard K. Guy were President Henry L. Alder, President-Elect Dorothy L. Bernstein, and Second Vice-President Howard E. Zink; at the lecture by Professor Eugene M. Kleinberg, Professor Donald M. Hill; at the lecture by Professor Benedict Gross, Professor Ernest C. Schlesinger; at the lecture by Professor Theodore A. Sundstrom, Professor Helen B. Siner; at the lecture by Professor Alar Toomre, Professor Donald J. Albers; at the lecture by Professor Stanley Friedlander, Professor Roland Lamberson; at the lecture by Professor Philip J. Davis, Professor Grattan Murphy; at the lecture by Dr. Frank R. Buianouckas, Professor Henry L. Africk; at the lecture by Professor Ernst Snapper, Professor Donald L. Kreider; at the lecture by Dr. Neil J. A. Sloane, Professor Vera S. Pless; at the lecture by Professor Robert Burghardt, Professor Helen B. Siner.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: *Winning Ways: Lectures on Combinatorial Games,* Lecture I: *What is a Game?,* by Professor Richard K. Guy, University of Calgary.

Conway's definitions of numbers and games were illustrated by several examples. The game of Hackenbush and Berlekamp's Hackenbush number system. Cool and hot games illustrated by Col and Snort. Simplifying games by the elimination of dominated and reversible options. Toads-and-Frogs, Cutcake, Domineering, Poker Nim and Northcott's game.

Non-Standard Calculus: The Wave of the Future?, Professor Eugene M. Kleinberg, Massachusetts Institute of Technology.

Why, even after the elimination of rigor, does calculus remain far more difficult for students to learn than elementary algebra? Probably because, until now, the subject could not be taught without some mention of limits, a concept which usually takes years to absorb.

Limits can now be eliminated entirely. By borrowing some notation from mathematical logic, it is possible to construct, precisely and rigorously, infinitesimal numbers. As with the reals, freshmen are not usually presented with the actual construction here, but the infinitesimals themselves are as easy and natural to work with as reals, and using them it becomes incredibly easy to develop the concepts of calculus. Derivatives become literally quotients, and integrals literally sums. The entire subject is now very algebraic, and, based on initial reports, far easier for students to learn.

250 Years of the Gamma Function, Professor Benedict Gross, Princeton University.

This talk sketched the history of Euler's gamma function. Euler began with the problem of interpolating $n!$ to a real valued function; some fifty years later he noticed the connection between $\Gamma(x)$ and certain abelian integrals. Both points of view are reflected in recent developments in number theory: the problem of interpolation has a p-adic analog, and the connection with abelian integrals yields an interesting interpretation of the values of $\Gamma(x)$ at rational arguments. Some examples were discussed.

College IV: One College's Experience with a Self-Paced Curriculum, Professor Theodore A. Sundstrom, Grand Valley State College.

College IV has developed a competency-based curriculum with emphasis on professional programs. The mathematics curriculum at College IV is being redesigned to meet the needs of the professional programs. The discussion focused on two items: the mathematics competency which is related to a course emphasizing statistical reasoning and a new calculator-based algebra course entitled Algebra for Statistics; and the mathematics curriculum which includes precalculus, calculus, and statistics. Many of the mathematics courses are taught on a self-paced basis supported by a learning laboratory consisting mainly of locally produced videotapes.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II: *Impartial Games,* by Professor Richard K. Guy.

Impartial games are games in which the same options are available to both players. The prototype, Nim, is well known, but the general Sprague-Grundy theory, although 40 years old, is less so. This is illustrated by a new series of coin-turning games based on an idea of Lenstra. There are surprising connexions with many parts of mathematics. Turning Turtles and the Mock Turtle theorem. Moebius, Mogul and M_{24}. Turnips, the ternary scale and nim-multiplication. Acrostic games, the uglification theorem and deriders of zero. Sparring, Boxing, Fencing and the elementary symmetric functions.

Interacting Galaxies, Professor Alar Toomre, Massachusetts Institute of Technology.

Some bizarre shapes -- severe warps, long tails, or even bright rings -- are seen occasionally among galaxies with close neighbors. This review begins with photographs illustrating several such examples and it goes on to explain, with various other slides and a computer-made movie, how surprisingly easy it has proved to imitate those oddities as tidal damage left over from recent encounters. The dynamics remains more puzzling, of course, in a few other cases where one can point to no plausible partners.

Panel Discussion: *Bayesian vs. non-Bayesian Statistics.*

A panel discussion with presentations by Professor John W. Pratt, Harvard University, and Professor Lawrence Brown, Rutgers University. Moderated by Professor I. R. Savage, Yale University.

Professor Pratt said that statistical inference in a scientific context should be objective and interpretable. Objectivity includes agreement among analysts, which is attainable only by convention. Interpretability includes describing what we know, what the data tell us, with what uncertainty, about parameters or future observations. Insistence on "objective" probability only has led to oblique inference statements; difficulties are partially hidden but anomalies abound and problems cannot be fully posed, let alone resolved. Bayesian inference statements are directly interpretable, the difficulties can be openly recognized and dealt with, and the appropriateness of the inevitable approximations and arbitrariness can be discussed in relation to a simple, well-defined, clear, ideal form of inference.

An Application of Descartes' Rule of Signs, by Professor Stanley Friedlander, Bronx Community College.

Descartes' Rule of Signs leads to a geometric interpretation and proof of a well-known set of symmetric inequalities. The approach taken is at a community college precalculus level.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III: *Three New Theories for Partizan Games,* by Professor Richard K. Guy.

Compounds of games may be played in various ways. In the usual "sum" players move in just *one* component, in the "join" in *all* components, and in the "union" in *some* of them. Steinhaus and C.A.B. Smith gave theories for impartial games involving "remoteness" and "suspense" functions. Since the appearance of Conway's book "On Numbers and Games", corresponding theories for partizan games, in which different option sets are available to players, have been developed. These will be illustrated by All-the-King's-Horses, Cutcakes, Eatcakes, Hotcakes and Falada. The third theory is for "loopy" games in which the "ending condition" is dropped, infinite play is possible and games may be drawn.

Business Meeting of the Association: Presentation of Carl B. Allendoerfer, Lester R. Ford, and George Pólya Awards.

Regions Admitting Quadrature Formulas, by Professor Philip J. Davis, Brown University.

Let B be a domain in the complex $z = x + iy$ plane. Let

$$L(f) = a_1 f^{(n_1)}(z_1) + \dots + a_m f^{(n_m)}(z_m), \quad z_j \in B.$$

If, for the class of integrable analytic functions defined on B one always has $\iint_B f\,dx\,dy = L(f)$, then B is said to admit a quadrature formula. Results relative to the the existence, uniqueness, and representation of such quadrature identities were surveyed. The work of Aharonov and H. S. Shapiro, Davis, Davis and Pollak, Gautier and Stenger, Gustafson, and Kratz was mentioned.

What a Difference a Dimension Makes, by Dr. Frank R. Buianouckas, Bronx Community College.

An heuristic account of some novel difficulties arrived at when attempting to model Walrasian Economics using topological dynamics. In short, if supply = demand holds at equilibrium then, for finitely many commodities, stable or unstable equilibrium implies an even or an odd number of commodities. This peculiar state of affairs may be an harbinger of a fly in somebody's ointment. Is it the mathematician's or the economist's? Odds anyone?

FOURTH SESSION OF THE ASSOCIATION

The Three Crises in Mathematics: Logicism, Intuitionism, and Formalism, by Professor Ernst Snapper, Dartmouth College.

Logicism tries to identify mathematics with logic; intuitionism tries to rebuild mathematics from the bottom up using only constructive methods; formalism identifies mathematics with formalized theories. These three schools all enriched mathematics with important, new mathematics, but they all failed in their attempt to give mathematics a firm foundation, acceptable to all mathematicians. All three schools are rooted in philosophy. The philosophical roots of logicism surface when it defines "logic"; those of intuitionism when it defines "constructive methods"; and those of formalism when it defines "finitary reasoning."

Panel Discussion: *CUPM's Proposed General Mathematical Sciences Program.*

A panel discussion with Professor Richard A. Alo, Lamar University, and Professor William F. Lucas, Cornell University. Moderated by Professor Alan C. Tucker, State University of New York, Stony Brook.

The CUPM Panel on a General Mathematical Science Program is revising and augmenting past CUPM curriculum recommendations for the mathematics major to form a model mathematical sciences major. The standard mathematics major at many institutions had already evolved into a mathematical sciences major. The panel members discussed the background of their revision effort, cooperation with computer science curriculum groups, preliminary curriculum suggestions, impact on traditional pure mathematics training, and the challenge of teaching new applied courses which still develop the timeless values of rigorous mathematical thinking. Some experiences of institutions with mathematical sciences programs were mentioned.

Error Correcting Codes and Secret Codes, by Dr. Neil J. A. Sloane, Bell Telephone Laboratories.

This survey focused on four extremely appealing new ideas in cryptography: (i) codes for a channel with an eavesdropper (Wyner), (ii) product ciphers - roll the dough then fold it (Shannon, Feistel et al.), (iii) public key encryption systems (Hellman, Rivest et al.), and (iv) codes for detecting deception (Gilbert, MacWilliams, Sloane).

Humor in Mathematics, Professor Robert Burghardt, Rockland Community College, State University of New York.

This talk illustrated the way that one person has been able to enliven the mathematics presentation in the classroom. Examples were presented of the stories used to increase the understanding of mathematical concepts, such as "Trichotomy Axiom and Goldilocks"; "δ-ε and Joe the Machinist." The lecture demonstrated how current events and fads can be woven into mathematical presentations and testing situations. The use of cartoons to dramatize a point were shown and songs spoofing mathematics were sung.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in Meehan Auditorium on Tuesday evening at 7:00 P.M. The following films were shown:

7:00-7:22 P.M.	Adventures in Perception (Maurits Escher)
7:25-7:39 P.M.	Area Under a Curve (Rickart) - a film of the MAA Calculus Film Project
7:41-8:06 P.M.	Conic Sections: a BBC Broadcast as part of the Open University Foundations Course in Mathematics
8:07-8:32 P.M.	The Gauss-Bonnet Theorem (Allendoerfer)
8:34-8:38 P.M.	The Seven Bridges of Konigsberg (Cornwell)
8:40-9:06 P.M.	Space Filling Curves (Max) - a film of the Topology Films Project
9:10-9:20 P.M.	Geometric Introduction to Partial Derivatives (Myers)
9:24-9:45 P.M.	Shapes of the Future: Some Unsolved Problems in Geometry-Three Dimensions, Part II (Klee)

On Wednesday evening at 7:00 P.M., Dr. Charles M. Strauss of Brown University and Charles M. Strauss, Incorporated gave an illustrated lecture on interactive computer graphics in mathematical research. Computer generated films were shown which have been produced by Dr. Strauss in collaboration with Professor Thomas F. Banchoff. A demonstration of the graphics equipment was given at the conclusion of the program.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Monday at 9:00 A.M. in the Ballroom of the Marriott Inn with thirty-eight members present. Among the items of business transacted were the following:

The Board elected the following persons as Associate Editors:

Monthly (for the term extending through 1981):

Richard K. Guy, University of Calgary
Raoul Hailpern, Mathematical Association of America
Deborah T. Haimo, University of Missouri, St. Louis
Franklin Haimo, Washington University
Roger C. Lyndon, University of Michigan
M. Susan Montgomery, University of Southern California
Seymour Schuster, Carleton College
J. Arthur Seebach, St. Olaf College
Ivar Stakgold, University of Delaware
Lynn A. Steen, St. Olaf College.

Two-Year College Mathematics Journal (for the term extending through 1983):

Gerald L. Alexanderson, University of Santa Clara
Glenn D. Allinger, Montana State University
William G. Chinn, City College of San Francisco
Ronald M. Davis, Northern Virginia C. C.
Howard W. Eves, University of Maine
Stanley Friedlander, Bronx C. C.
Thomas M. Green, Contra Costa College
Samuel A. Greenspan, Bronx C. C.
Raoul Hailpern, Mathematical Association of America
Ross A. Honsberger, University of Waterloo
Harold R. Jacobs, Grant High School
Erwin Just, Bronx C. C.
Bruce W. King, Schenectady C. C.
Norman E. Ladd, College of the Redwoods
Roland H. Lamberson, Des Moines Area C. C.
William A. Leonard, California State University, Fullerton
Peter A. Lindstrom, Genesee C. C.
David A. Logothetti, University of Santa Clara
Warren Page, New York City C. C.
George Pólya, Stanford University
Edward B. Wright, Linn-Benton C. C.

Mathematics Magazine (for the term extending through 1980):

James E. Hall, University of Wisconsin

The Board elected Professor Richard K. Guy of the University of Calgary to fill the unexpired portion of the term of Professor Thomas Hull as Governor-at-Large.

The Board approved revisions of the *By-Laws* of the New Jersey and Missouri Sections.

The Board voted to accept the following grants:

1. A grant of $10,000 from the International Business Machines Corporation for support of *Women and Mathematics* during 1978-79.

2. A grant of $8500 from the International Business Machines Corporation for support of the U.S.A. Mathematical Olympiad Awards Ceremony and Dinner in 1978.

3. A grant of approximately $9000 from the Army Research office for support of travel to the 1978 International Mathematical Olympiad in Rumania in 1978.

4. A grant of $6430 from the Office of Naval Research for support of the 1978 Olympiad Training Session held at the United States Naval Academy.

The Board received the report of Professor Malcolm W. Pownall as Chairman of the Committee on Visiting Lecturers and Consultants. It was reported that the MAA had sponsored 95 visits by Visiting Lecturers and 11 visits by Consultants in 1977-78.

The Board approved meeting in San Francisco during the period January 9-11, 1981.

The Board discussed at length the proposed Cabinet-level Department of Education and approved a statement in opposition to the proposed transfer of the NSF science education programs to the proposed Department of Education. The statement also indicated that the Board is not convinced by available evidence that creation of a Department of Education as currently proposed would benefit education in the United States. The Secretary was instructed to prepare copies of the Board resolution for distribution to MAA members in attendance at the Providence meeting.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting was held at 10:00 A.M. in Meehan Auditorium with President Alder presiding.

President Alder first noted that the past year had seen the passing of two of the most distinguished members of the Association; Professors Kenneth Ownsworth May and Edward Griffith Begle. He called upon Dr. Alfred B. Willcox who read a tribute to Professor May prepared by Alfred L. Putnam and upon Professor Leonard Gillman to read a tribute to Professor Begle. These follow:

KENNETH OWNSWORTH MAY

It is with sadness that the Association records the death, during the past year, of Kenneth May of the University of Toronto.

Kenneth May loved mathematics and learning. Characteristically, whatever he did he approached with spirit and a keenly disciplined mind. Nowhere was he more himself than in working with mathematics to prompt others, and especially undergraduates, to gain direct experience with mathematical thinking. His *Elementary Analysis* in 1952 was among the early published texts in which fundamental concepts of college mathematics were given a formulation in set terms. Because he aimed to prompt students to learn to use those crucial concepts well, he avoided what he considered unhelpful foundational niceties in his explication while striving for a tightly articulated mathematical development. *Delta Epsilon*, the Carleton College undergraduate journal of which he was the first editor, gave him much satisfaction in stimulating student work in mathematics. When his professional interests moved to the history of mathematics, and he moved with them to Toronto, there came for him the culmination of his very personal commitment to mathematics. In that commitment to history he set for himself and others demanding standards of scholarship and exposition, and he deepened and embellished the understanding of mathematics. *Historia Mathematica*, which he was instrumental in establishing, has been a most lively and valuable organ of the history of mathematics.

Kenneth Ownsworth May was born in 1915 in Portland, Oregon. He was a distinguished undergraduate in mathematics at the University of California, Berkeley. After graduation he had some years abroad; then, following Army service in the War, he received his Ph.D. from Berkeley in 1946. That same year he accepted an appointment in mathematics at Carleton College. During his chairmanship of the department from 1951 to 1961 he developed a distinguished undergraduate faculty in mathematics which gave mathematics there a very particular vigor. In 1966 he received an appointment in mathematics at the University of Toronto, where in 1973 he became Director of the Institute for History and Philosophy of Science and Technology.

Honors and recognition which came his way Kenneth May accepted with grace. He served the Association as Governor twice, from 1953 to 1956 and 1964 to 1966, and as Associate Editor of the *Monthly* from 1963 to 1970. His championship of the rights of human beings was reasoned and firm. As an acquaintance, he was engaging and thought-provoking. As a friend, he was wise and true. Kenneth May enriched the life of mathematics for us all.

Alfred Putnam

EDWARD GRIFFITH BEGLE

The Mathematical Association of America records with profound sorrow the death of Professor Edward Griffith Begle of Stanford University.

Ed Begle was born in Saginaw, Michigan in 1914. He obtained his Bachelors and Masters degrees at the University of Michigan and his Ph.D. at Princeton. In 1942, after a year as National Research Fellow, he joined the faculty at Yale, where he remained until moving to Stanford in 1961.

Ed Begle was a member of the Association for 42 years. No one can match him in the range of his mathematical activities and the friends he won along the way. He made a mark in research (particularly for his extension of the Vietoris mapping theorem), he served six years as Secretary of the American Mathematical Society, he is listed on the honor plaque at the headquarters of the

National Council of Teachers of Mathematics, and he won the MAA Award for Distinguished Service to Mathematics.

In 1958 Ed Begle became Director of the newly formed School Mathematics Study Group, a post he held throughout the 14 years of SMSG. In this very important work he showed consummate skill--marked by a basic warmth that easily shone though his make-believe gruff exterior, penetrating insight, and infinite patience--in harmonizing a cacophonous collection of mathematicians and teachers from the grade schools, high schools, colleges, and universities.

Shortly after Ed Begle died, we sent out an appeal for funds to establish a conference room in his memory at the new MAA Headquarters. The response was overwhelming. I have never seen such an outpouring of friendship and affection. Gifts came from retired people, widows, students, and people out of a job; many of the contributors were not even members of MAA. And people sent heart-warming letters. "What you are doing in memory of Ed Begle is beautiful." "Bless you for permitting some of the countless people whose lives Ed affected to be part of a significant tribute to our beloved Ed."

The Edward G. Begle Conference Room has now been established, a significant and lasting memorial to a great and wonderful man. And, indeed, what could be more appropriate in honor of Ed Begle than a conference room!

Leonard Gillman

President Alder next presented the second set of Carl B. Allendoerfer Awards, the thirteenth set of Lester R. Ford Awards, and second set of George Pólya Awards. Present to accept these Awards were Professors David A. Smith, Branko Grünbaum, Ralph P. Boas, Allen H. Holmes, Walter Sanders, and Frieda Zames.

President Alder then called upon Professor Ralph P. Boas, Editor of the *Monthly*, who addressed the Business meeting as follows:

"Emory P. Starke was first listed in the *Monthly* as an editor of Problems in 1946; he became the over-all problem editor in November, 1964 (but he must have started work earlier than these dates). He will retire, at his own request, at the end of his current appointment, that is, at the end of 1978, after some 32 years of service as editor and 17 years after retiring as Professor and Department Chairman at Rutgers. In 1973 he had already served longer than any member of the editorial board in the previous history of the *Monthly*, and was presented with an Honorary Life Membership in our Association in recognition of his distinguished service. Most of us are aware of his unfailing good taste in the selection of problems and the meticulous care with which he has edited both problems and solutions, but probably only a few of us realize how hard he has worked at the job. He has had to make decisions about several times more submissions than can possibly be used, while coping with the individual prejudices of a long series of editors; he has edited the solutions, and prepared printer's copy of both problems and solutions; he has carried on an extensive editorial correspondence; and for many years he has done all this without a secretary and without even a Xerox machine.

The masthead of the *Monthly* is not going to look right without Emory's name. I am happy to report that the Board has reappointed him with the title of Associate Editor Emeritus. It is my hope that he will still be willing to give us editorial advice from time to time, and that he will return to his former practice of submitting interesting problems himself."

President Alder next announced that Professor Nura D. Turner, SUNY at Albany, had been voted the Association's Certificate of Merit in recognition of her outstanding contributions to the U.S.A. Mathematical Olympiad. He read a citation prepared for this occasion and presented the Certificate to Professor Turner.

The Secretary presented his report announcing first some of the actions taken by the Board of Governors. The Secretary also announced that the team of eight high school students who represented the United States at the International Mathematical Olympiad in July, 1978 had finished second to the host country Romania. He thanked the coaches of the team, Professors Samuel L. Greitzer and Murray S. Klamkin, and gratefully acknowledged grants from the Army Research Office, IBM, and the Office of Naval Research which had been made in support of participation by a U.S. team. The Secretary also thanked officials of the U.S. Naval Academy, especially Professor Theodore J. Benac and Lt. Dan Cunha, for their work on behalf of the three-week training session held at the U.S. Naval Academy.

The Secretary announced that Professor Edmund Gregory Lee had been selected as the first AMS-MAA-SIAM Congressional Fellow. It was also announced that the Board of Governors had continued its funding of this position for the 1979-80 academic year.

The Secretary reported on the results of the Building Fund solicitation started in early March, 1978. As of early July, this Fund totalled about \$346,000 on pledges, contributions, and dues advances from approximately 1600 persons. The Secretary noted that if only those gifts not in excess of \$5000 are taken into account and each dues advance is recorded as a contribution of \$50, then the average gift is approximately \$90.

The MAA purchased its new headquarters at 1527-29 Eighteenth St., N.W. on July 10. The Secretary recalled that, prior to the decision to purchase a building, Professor Mary P. Dolciani and the Vaughn Foundation of Tyler, Texas had declared their support for such an undertaking. He said that had it not been for this support, the idea may have been abandoned. In recognition of this generous support, the officers of the Association have approved naming 1529 Eighteenth Street in honor of the Vaughn family and the entire headquarters complex in honor of the Dolciani family.

The Secretary thanked Leonard Gillman for his hard work in connection with the Building Fund. He noted that Professor Gillman had been central to the preparation of the brochure on the headquarters and MAA activities and had conducted a solicitation in memory of Professor Edward G. Begle. He also said that Leonard Gillman had first contacted the Vaughn Foundation and had himself been a generous donor to the Building Fund. The Secretary said Leonard should be thanked for his good suggestions, humor, and hard work and that much of the success of the Building Fund solicitation was due to Professor Gillman's efforts.

The Secretary expressed thanks to the members of the Committee on Arrangements. He also called attention to the excellent program and thanked the members of the Program Committee: Donald L. Kreider, *Chairman*; Thomas F. Banchoff, Grattan P. Murphy, Ernest C. Schlesinger, Helen B. Siner, Richard P. Stanley.

MEETING OF SECTION OFFICERS

The meeting of representatives of the Sections was held on Tuesday, August 8, 1978, in the Ballroom of the Providence Marriott Inn. Professor Samuel W. Hahn presided. Forty-four persons were present, representing twenty-seven of the twenty-nine Sections. Members of the Committee on Sections present were Professors James C. Bradford, Louis A. Guillou, Jacqueline C. Moss, Dr. Alfred B. Willcox, and Professor Hahn.

President Henry L. Alder first addressed the Section Officers as follows:

"It is a great pleasure to welcome all of you to this meeting of Section Officers. As you are well aware, the Sections are considered the lifeblood of the Association. The more effective the Sections are, the better the Association is able to accomplish its mission. The meeting of Section Officers is designed to give you, the Section Officers, ideas how to carry out most effectively and efficiently the responsibilities of the Sections. I hope very much that you will find this meeting profitable and be able to carry back many valuable ideas to your Sections for implementation.

Having had the privilege of visiting many Sections recently, I have been deeply impressed with the excellent way in which several of the Sections are organized and how effectively they carry out their mission. I have been particularly pleased to see that parts of Section meetings are devoted to topics which are of major current interest in the MAA; such as the pamphlet *Recommendations for the Preparation of High School Students for Collegiate Mathematics Courses*, the CLEP Examinations, and many other items.

I have found it particularly encouraging to note that, during the past year, the number of Sections starting a newsletter has increased substantially. My understanding is that of the 29 Sections, 17 have now a newsletter in operation. It is clear that this is a very valuable service which the members value highly.

Finally, let me encourage you to let us, the national officers, know of anything we can do to help you in any way as Section Officers. I want to assure you that any suggestions you have will be given the most careful consideration."

Professor Hahn indicated that he was presiding at this meeting in the absence of the Chairman of the Committee of Sections, Dean Lester H. Lange. Dean Lange is currently traveling abroad. Professor Hahn also noted the absence of Professor E. K. McLachlan who is convalescing after a recent illness. All present wished Professor McLachlan a speedy recovery.

Section Officers were requested by the Committee on the Putnam Prize Competition to include in the report of meetings published in the *Monthly* special awards or special recognition of any sort accorded high ranking Putnam contestants. Such information can also be sent directly to Professor Leonard F. Klosinski.

Attention was called to a collection of Section materials available at the meeting. These materials are available as an exchange of ideas among Sections.

Professor Hahn called attention to the loose leaf binder, *Handbook for Section Officers*. It contains a file of information that may be of use in planning meetings and Section activities. This *Handbook* has been mailed to all Section Chairmen and Section Secretaries and it is intended that it be passed to subsequent occupants of these offices.

Professor Guillou reported on the revision of the booklet, *Guidelines for Sections*. The revision should be available in August, 1979. It is planned to incorporate in the revision additional ideas for Section Secretaries as well as ideas for newletters.

In response to a question from Professor David W. Ballew, Dr. Willcox said that he would investigate whether Sections must file income tax reports.

Professor Hahn recalled that, on January 5, 1978, the Board of Governors had referred to the Committee on Sections consideration of collection of Section dues by the national organization. In response to this charge and certain other concerns, the Committee on Sections had decided to request each Section to file an annual report. Professor Hahn reported that the first of these reports has now been filed by the Section Secretaries and thanked them for their prompt responses.

A summary of the responses to the first annual report was circulated. This summary indicated the number of MAA members residing in each Section, the number of meetings and attendance, whether the Section has a newsletter and the number of issues, whether the Section sponsors a visiting lecturer program and the number of visits made, whether the Section sponsors a workshop, whether the Section collects registration fees and/or dues, and whether the income of the Section is sufficient to support the present activities of the Section. Also included is an indication of whether proposed activities had not been undertaken due to a lack of available funds and an identification of what are felt to be the strongest features of the activities of each Section as well as the most serious problems now facing each Section. (Copies of this summary are available upon request from Dr. Alfred B. Willcox, Dolciani Mathematics Center, 1529 Eighteenth St., N. W., Washington, D. C. 20036.)

Professor Hahn reported the following recommendation to the Board of Governors: "The Committee on Sections recommends that the National Office not become involved in collection of Section dues at this time; primarily for the following reasons:

1. The impact of the recent increase in the annual subvention to Sections on the solution of the financial problems of Sections has yet to be assessed.

2. At the present time, only four Sections charge dues.

3. Most Sections report that their income is adequate to support their present activities.

4. It would cost an estimated $1000 for program modification in the Washington office in order to initiate collection of Section dues.

5. National collection might have an adverse effect on the current headquarters building fund drive."

Professor Hahn noted that this recommendation is subject to change at a later date. The recommendation is only that there not be national collection of Section dues at the <u>present</u> time.

Leonard Gillman, Treasurer of the MAA, called attention to the availability of grants for Section activities. He said that there had been six $300 grants made in 1977-78 and that an amount sufficient to fund additional grants had been budgeted. He encouraged additional applications to help fund whatever new activities Sections might wish to undertake.

Professor Yousef Alavi commented on the finances of the Michigan Section. He noted that the Dean of Michigan State University had underwritten a $3 per registrant fee at a recent Section meeting held there. He wondered how it was that the other Sections found their income sufficient. Professor Alavi noted that a check-off for local dues is available on the national dues forms of other organizations of which he is a member.

Professor Hahn noted that a voluntary check-off for Section dues was what had been discussed by the Committtee on Sections.

In response to a comment by Professor Violet Larney of the Seaway Section, President Alder said that the national organization currently makes a payment to each Section and that, by action voted by the Board in January, 1978, this payment had been approximately doubled.

It was noted that colleges and universities support both Section activities and activities of the national organization. In particular, Professor Ballew of the Rocky Mountain Section said that the host institutions for meetings of his Section routinely help support the meeting. Professor Donald M. Hill of the Florida Section noted that the host institution for meetings of that Section ususally make funds available for one of the invited speakers. Professor Hahn pointed out that it is sometimes possible to schedule an MAA Visiting Lecturer for a visit near a Section meeting and invite this person to lecture at the meeting. Finally, Professor Robert Bumcrot of the Metropolitan New York Section noted that the Actuary Club of New York makes a donation of $600 to their Section.

Professor Kenneth R. Rebman commented that availability of funds might decrease the ingenuity now used to approach the problem of finances. He also said that it was important that there be proper plans outlined for the use of funds in advance of money being accumulated. In particular, he felt that a small bank balance did not do harm unless there were activities not being undertaken on account of nonavailability of funds.

Professor Ballew said that many Ph.D.-granting institutions do not participate in Section activities. Indeed, he pointed out that certain such institutions will not pay travel expenses to Section meetings while such funds are made available for attendance at other meetings. Professor Hahn replied that poor participation in Section activities by faculty at Ph.D.-granting institutions, by two-year college faculty, and by representatives of industry had been the problems most often identified on the annual reports.

President Alder agreed with Professor Ballew that non-participation in Section activities by persons from Ph.D.-granting institutions was a serious problem. He said he thought having Section officers from such institutions might help alleviate the problem in that these persons might cause their colleagues to become involved.

There followed a discussion of the problem of attendance at Section meetings during which the following suggestions were made:

1. Joint AMS-MAA meetings.

2. Joint SIAM-MAA meetings.

3. Nominating Committee might choose persons from constituencies not well represented at meetings.

4. The MAA needs to advertise its activities such as its publications program.

5. The MAA needs to stress that education of undergraduates is a concern of <u>all</u> colleges.

6. That Section Officers should invite, by telephone, persons from Ph.D.-granting institutions.

7. That Ph.D.-granting institutions can be told that Section meetings are a good place to meet with prospective graduate students.

8. That Section newsletters might be mailed more widely than just to MAA members (the postage problem was noted).

9. Strengthened and parallel programs.

10. Concurrent meetings of department chairmen and sessions for student papers.

11. Choice of non-mathematicians, such as Professor Edward Teller, as speakers has caused more interest in Section meetings.

12. All speakers should be persons who give excellent <u>expository</u> talks.

13. Departmental Representatives should help increase interest within their departments by announcing meetings and soliciting papers for presentation. Also, Governors might host a meeting, perhaps a breakfast, for Departmental Representatives.

Professor Bradford pointed out that Professor James N. Younglove, Governor of the Texas Section, consults Departmental Representatives on items of interest to the Section. For example, these persons have been consulted with regard to a revision of the Section By-Laws. Professor E. Allan Davis advocated making attendance at Section meetings be a condition of appointment for Departmental Representatives. Dr. Willcox advocated meetings for these Representatives with an agenda of current MAA activities prepared by the Governor.

Professor Hahn suggested that the Committee on Sections should outline a list of duties of Departmental Representatives. Dr. Alavi said that Departmental Representatives should provide a communications link with their department. He felt that there should not be additional duties described for these persons.

Professor John W. Kenelly said that tradition and a good program has resulted in good participation by Ph.D.-granting institutions in the Southeastern Section. He also called attention to the office of Section Lecturer. This is a person who is elected by the Section one year to give a major lecture at the subsequent meeting.

Professor Rebman said that the MAA offers many services and attendance at Section meetings is just one such service. Many members will wish to take advantage of one or another of these services but perhaps not all.

Dr. Willcox indicated that it was planned to provide annually for each Section Secretary an alphabetized list of names and addresses of each Section member. He said that it would also be possible to provide more information about each Section member if that is desired by the Section Secretary.

Professor Hahn suggested an informal meeting of the Section Officers in January, 1979 in conjunction with the meeting in Biloxi. A show of hands indicated that quite a number of those present were planning to attend the meeting.

Professor Alavi said that members of the Michigan Section enjoyed browsing through the MAA library on display at their Section meetings. He suggested that other Sections might wish to purchase this library.

It was noted that both the Intermountain and Kansas Sections have better than 100% attendance at their meetings. In each case, it was explained that this is because of the attendance of secondary school teachers.

Professor Ronald M. Davis reported on the summer seminars and the conference on lower division mathematics sponsored by the Maryland-District of Columbia-Virginia Section.

Professor William F. Lucas requested that Section Secretaries send to him any available information about short courses sponsored by the Section. He said that this information is of interest to the MAA Committee on Continuing Education.

DINNER IN HONOR OF THIRTY-YEAR MEMBERS

On Wednesday, the Association held a dinner in honor of its thirty-year members. This dinner was held at 7:00 P.M. in the Ballroom of the Marriott Inn. The dinner was designed as a pleasant occasion to recall the services of the senior members of the MAA and to inform them of current activities and future plans. Professor G. Baley Price served as toastmaster and President Henry L. Alder brought greetings from the MAA. The speakers were Dr. Alfred B. Willcox, *The New Headquarters Building*; Professor G. Baley Price, *A World War II History Project*; Professor Nura D. Turner, *Careers of Olympiad Students*; and Dr. Henry O. Pollak, *A Report on PRIME 80*.

Fifty-seven persons attended the dinner.

MEETINGS OF OTHER ORGANIZATIONS

The Pi Mu Epsilon Fraternity held its sessions for contributed papers on Wednesday and Thursday at 3:00 P.M. The Banquet was held at 6:30 P.M. on Wednesday in the Canal Room of the Marriott Inn. Following the Banquet, the J. Sutherland Frame Lecture was delivered by Professor Herbert E. Robbins. The title of this lecture was *The Statistics of Incidents and Accidents*.

The Association for Women in Mathematics sponsored a panel discussion, *Women Mathematicians Before 1950*, at 8:00 P.M. on Wednesday. Professor Patricia C. Kenschaft served as moderator.

The American Mathematical Society held sessions from Wednesday, August 9, through Saturday, August 12. There were scheduled the following nine one-hour addresses:

WEDNESDAY

1:00 P.M. *Myopic Economic Agents*
Donald J. Brown, Cowles Foundation for Research in Economics, Yale University

2:15 P.M. *On Recent Developments in Yang-Mills Theory*
Raoul H. Bott, Harvard University.

THURSDAY

10:00 A.M. *Fostering and Hindering Creativity in Mathematics*
Professor Hassler Whitney, Institute for Advanced Study

11:00 A.M. *The Limits of Effectiveness in Classical Mathematics*
Anil Nerode, Cornell University

FRIDAY

11:00 A.M. *Algebraic Topology of Smooth Algebraic Varieties*
John W. Morgan, Columbia University

1:00 P.M. *Automorphic Forms and the Trace Formula*
James G. Arthur, Duke University

2:15 P.M. *Global Problems in Singularities Theory*
Boris Moishezon, Columbia University

SATURDAY

11:00 A.M. *On the Geometry of Algebraic Cycles*
Spencer Bloch, University of Chicago

1:00 P.M. *The Scattering of Sound Waves*
James V. Ralston, University of California, Los Angeles

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements consisted of Walter F. Freiberger, Jonathan D. Lubin, *Co-Chairmen;* Raymond G. Ayoub, *ex officio,* Philip J. Davis, Clement L. DeMayo, Wendell H. Fleming, Jack K. Hale, Richard A. Howland, William J. LeVeque, *ex officio,* John J. McLaughry, Lewis I. Pakula, and David P. Roselle, *ex officio.*

Registration headquarters were located in the lobby of the Barus and Holley Building. Books and educational material were displayed near the registration area on Tuesday, Wednesday, and Thursday. The Mathematical Sciences Employment Register operated on an informal basis throughout the meeting.

On Thursday evening, there was an authentic New England Clambake on the grounds of the Francis Farm in Rehoboth, Massachusetts. The AMS headquarters, located at 201 Charles Street, was open for tours throughout the meeting.

David P. Roselle
Secretary

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This volume is based on the papers presented at the Bicentennial Program of the Association on January 24–26, 1976. In addition to the major historical addresses, the papers cover the following panel discussions: Two-Year College Mathematics in 1976; Mathematics in Our Culture; The Teaching of Mathematics in College; A 1976 Perspective for the Future; The Role of Applications in the Teaching of Undergraduate Mathematics.

The following is a list of the Panelists and the Authors: Donald J. Albers, Garrett Birkhoff, J. H. Ewing, Judith V. Grabiner, W. H. Gustafson, P. R. Halmos, R. W. Hamming, I. N. Herstein, Peter J. Hilton, Morris Kline, R. D. Larsson, Peter D. Lax, Peter A. Lindstrom, R. H. McDowell, S. H. Moolgavkar, Shelba Jean Morman, C. V. Newsom, Mina S. Rees, Fred S. Roberts, R. A. Rosenbaum, S. K. Stein, Dirk J. Struik, Dalton Tarwater, W. H. Wheeler, A. B. Willcox, W. P. Ziemer.

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CALENDAR OF FUTURE MEETINGS

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.

Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21–23, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, Johnson County Community College, Overland Park, April 7, 1979.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday–Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Adelphi University, May 5, 1979.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, early May and early November.

NORTH CENTRAL, end of April and October. Deadline for papers April 1 and October 1.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Miami University, Middletown, April 20–21, 1979.

OKLAHOMA–ARKANSAS, Oklahoma State University, Stillwater, March 30–31, 1979.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, University of Denver, Denver, spring 1979.

SEAWAY, Saturday in late April and first Saturday in November. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN, University of Tennessee, Chattanooga, spring 1979.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Houston, Texas, January 3–8, 1979.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, Biloxi, Mississippi, January 24–27, 1979.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Sheraton Park Hotel, Washington, D.C., December 4–6, 1978.

ASSOCIATION FOR SYMBOLIC LOGIC, Biloxi, Mississippi, January 24–25, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

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INSTITUTE OF MATHEMATICAL STATISTICS, Washington, D.C., August 13–16, 1979.

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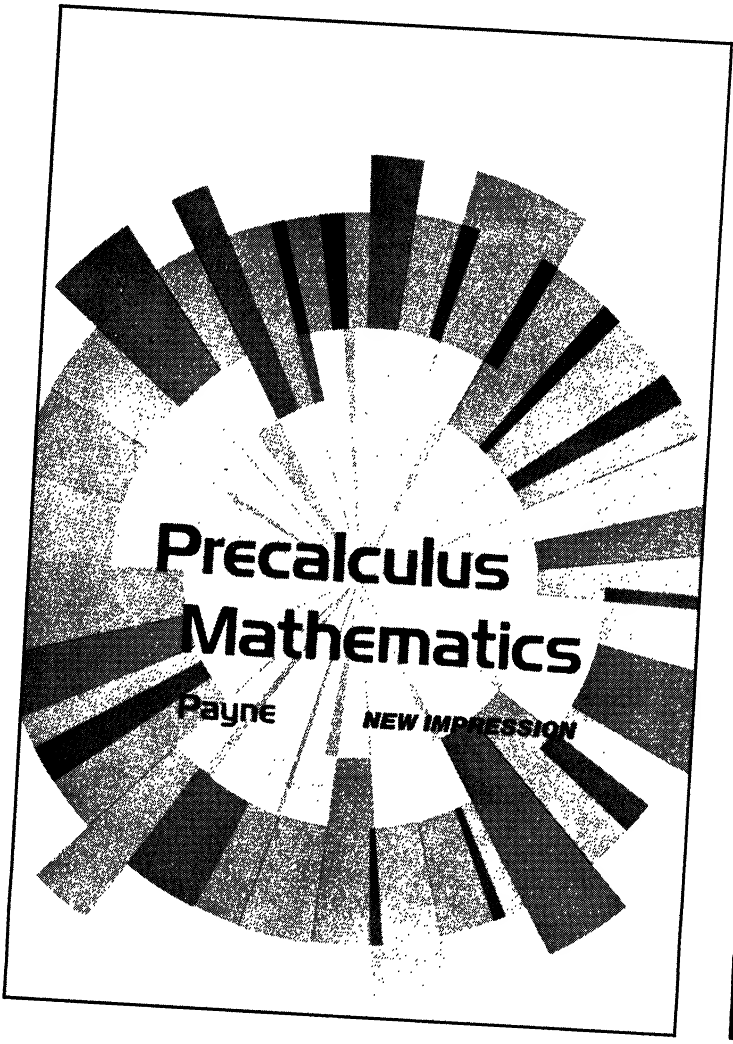
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Hyatt Regency Hotel, New Orleans, Louisiana, April 29–May 1, 1979.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlbach, Kansas City, Missouri, November 8–10, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS



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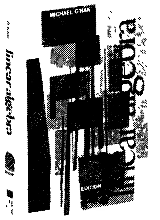

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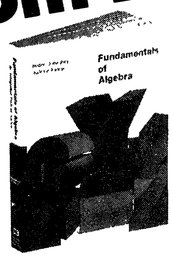

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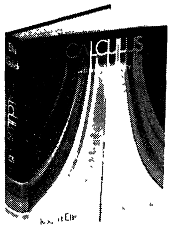

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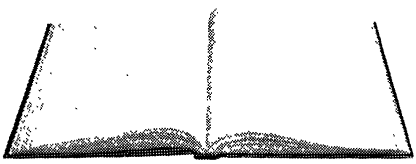

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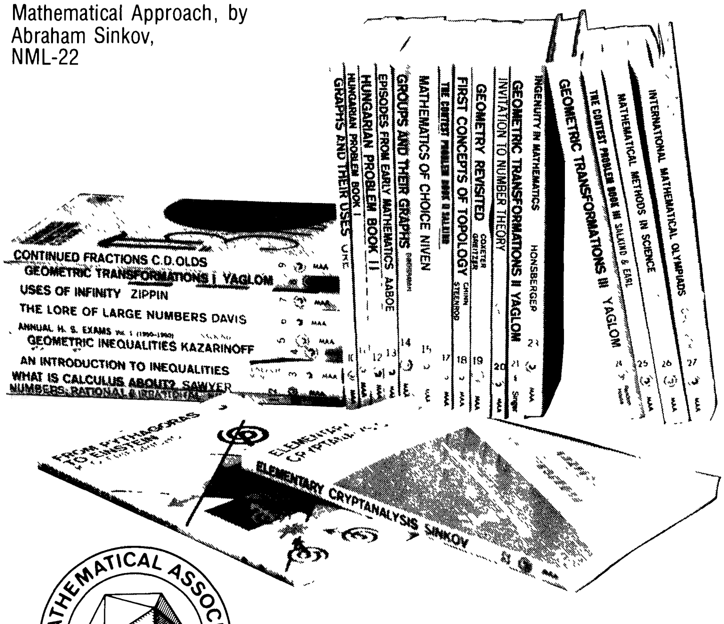

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